Elliptic quintic $E \subset \mathbb{P}^4$ with TO_5 action

Abstract

This is mostly a preliminary sketch for the Campedelli 7×7 construction. Using the regular representation V^{reg} of TO_5 we construct 5×5 skew matrices with entries in V^{reg}, with invariance properties that ensure its ideal of 4×4 Pfaffians is TO_5-invariant, so generates the ideal of a quintic elliptic curve $E \subset \mathbb{P}^4$ with a TO_5 action.

1 Order of service

(1) We recall TO_5 over the base ring $B = \mathbb{Z}[S,t]/(St^4 + 5)$ and what we need from its representation theory.

(2) We list 10 skew 5×5 matrices N_{ij} and verify their invariance.

(3) We reduce everything modulo the ideal $(S,t) = (S,t,5)$ and verify that a general linear combination of $\overline{N}_{ij}$ has Pfaffians defining a nonsingular quintic elliptic curve $E_5 \subset \mathbb{P}^4$ over $\mathbb{F}_5$. It has a free action of $\boldsymbol{\alpha}_5$ by construction. Nonsingularity is an open property, and E has a flat deformation over B with fibres having action by all the other group schemes of order 5.

(4) Finally, we explain several derivations of the matrices N_{ij}. This is "just" linear algebra, but with a characteristic flavour arising from the base ring B.

2 TO_5 and its representations

Write $P = St^4 + 5$ and choose the base ring $B = \mathbb{Z}[S,t]/(P)$. This is just the subring $\mathbb{Z}[t][S] \subset \mathbb{Q}(t)$ where $S = \frac{5}{t^4}$. It has prime elements t and S over the prime 5 of $\mathbb{Z}$. We can picture $\operatorname{Spec} B$ as a surface fibred over $\operatorname{Spec}\mathbb{Z}$, with reducible fibre $St^4 = 0$ over the prime 5. Next, set $A = B[x]/(F)$ where

$$F = x^5 - S(t^3x^4 + 2t^2x^3 + 2tx^2 + x). \tag{2.1}$$

Check that $(1+tx)^5 - 1$ is in the ideal (P, F), so A contains the 5 roots of unity $(1+tx)^i$ for $i = 0, 1, \dots, 4$, distinct where t is invertible.

Then $\mathrm{TO}_5 = \operatorname{Spec} A$ is a group scheme over $\operatorname{Spec} B$, with multiplication map $m\colon x_1, x_2 \mapsto x_1 + x_2 + tx_1x_2$. As already described, we view TO_5 as the closed subscheme of the affine x-line $\mathrm{TO}_5 \subset \mathbb{A}^1_{\langle x\rangle,B}$, and m as the restriction of the polynomial map $m\colon \mathbb{A}^1 \times_B \mathbb{A}^1 \to \mathbb{A}^1$ given by $x_1, x_2 \mapsto x_1 + x_2 + tx_1x_2$, or (if we really need the Hopf algebra or comultiplication), we can take the induced map $m^*\colon A \mapsto B[x_1, x_2] = A \otimes_B A$ on coordinate rings $A = B[x]$ to be $x \mapsto x_1 + x_2 + tx_1x_2$ (or $x \otimes 1 + 1 \otimes x + tx \otimes x$).

The regular representation V^{reg} of TO_5 is its coordinate ring, the free B-module generated by $1, x, x^2, x^3, x^4$. It is therefore the 4th symmetric power $\operatorname{Sym}^4(1, x)$ of the 2-dimensional representation on the free module $(1, x)$ with the affine representation

$$\begin{pmatrix} 1 & . \\ x & 1+tx \end{pmatrix}. \tag{2.2}$$

We write u_0, u_1, u_2, u_3, u_4 for the basis of V^{reg} corresponding to $1, x, x^2, x^3, x^4$. Then the action of TO_5 is given by the matrix

$$D_u = \begin{pmatrix} 1 & . & . & . & . \\ x & T & . & . & . \\ x^2 & 2xT & T^2 & . & . \\ x^3 & 3x^2T & 3xT^2 & T^3 & . \\ x^4 & 4x^3T & 6x^2T^2 & 4xT^3 & T^4 \end{pmatrix}, \tag{2.3}$$

with $T = 1 + tx$ for brevity.

Our aim is to write out TO_5 equivariant maps $V^{\mathrm{reg}} \to \bigwedge^2 V^{\mathrm{reg}}$. The eventual aim is to write out 5×5 skew matrices N having entries linear forms in the u_i, so that its ideal of 4×4 Pfaffians is TO_5 invariant, and defines an elliptic curve scheme $E \subset \mathbb{P}^4_B$ with a free action of TO_5, that is nonsingular over the closed point $S = t = 0$ of $\operatorname{Spec} B$.

The second exterior power $\bigwedge^2 V^{\mathrm{reg}}$ is the free B-module with basis $w = \{w_{ij}\}$ for $0 \le i < j \le 4$, with $w_{ij} = u_i \wedge u_j$. We order the basis as

$$w_{01}, w_{02}, w_{03}, w_{04}, w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}. \tag{2.4}$$

The action of TO_5 is given on skew matrices by $D_u N^t D_u$, which in this basis works out as the 10×10 matrix $D_w =$

$$\begin{pmatrix}
T & . & . & . & . & . & . & . & . & . \\
2xT & T^2 & . & . & . & . & . & . & . & \\
3x^2T & 3xT^2 & T^3 & . & . & . & . & . & . & \\
4x^3T & 6x^2T^2 & 4xT^3 & T^4 & . & . & . & . & . & \\
x^2T & xT^2 & 0 & 0 & T^3 & . & . & . & . & \\
2x^3T & 3x^2T^2 & xT^3 & 0 & 3xT^3 & T^4 & . & . & . & . \\
3x^4T & 6x^3T^2 & 4x^2T^3 & xT^4 & 6x^2T^3 & 4xT^4 & T^5 & . & . & . \\
x^4T & 2x^3T^2 & x^2T^3 & 0 & 3x^2T^3 & 2xT^4 & 0 & T^5 & . & . \\
2x^5T & 5x^4T^2 & 4x^3T^3 & x^2T^4 & 8x^3T^3 & 8x^2T^4 & 2xT^5 & 4xT^5 & T^6 & . \\
x^6T & 3x^5T^2 & 3x^4T^3 & x^3T^4 & 6x^4T^3 & 8x^3T^4 & 3x^2T^5 & 6x^2T^5 & 3xT^6 & T^7
\end{pmatrix}$$

The skew matrices N that we need represent TO_5-linear homomorphism $V^{\mathrm{reg}} \to \bigwedge^2 V^{\mathrm{reg}}$. We write them in terms of 10×5 matrices with 50 unknown entries m_{ijk}, that represent the 10 entries w_{ij} of the matrix written out as linear forms $w_{ij} = \sum_0^4 m_{ijk} u_k$ in the u_k. Invariance under the group scheme action means the 50 linear conditions on M written as $D_w M = M D_u$. We need to solve for M in that (see Subsection **??**). The tricky point is not so much the size of the problem, but that we are doing linear algebra over a funny ring, so equals means congruence modulo (P, F).

Remark (provisional) We can solve this set of equations by hook or by crook. There are probably better methods, based on the fact that the 10 basic solutions give rise to invariant skew matrices N_{ij}, determined by having leading term u_0 in the single entry n_{ij}, and no other occurrence of u_0.

3 The 10 matrices N_{ij}

We write a skew 5×5 matrix as its 10 upper-triangular entries $w_{01}, \dots, w_{34}$. (Please write out the zeros down the diagonal and the lower-triangular entries $u_{ji} = -u_{ij}$ if you are confused at first.)

Using the quantities

$$\begin{aligned}
v_0 &= u_0,\\
v_1 &= u_0 + tu_1,\\
v_2 &= u_0 + 2tu_1 + t^2u_2,\\
v_3 &= u_0 + 3tu_1 + 3t^2u_2 + t^3u_3,\\
v_4 &= u_0 + 4tu_1 + 6t^2u_2 + 4t^3u_3 + t^4u_4,
\end{aligned} \tag{3.1}$$

simplifies the notation and clarifies the logic. They are eigenforms of D_u with eigenvalues $(1+tx)^i$. We can also invert the above equations as

$$\begin{aligned}
u_0 &= v_0,\\
tu_1 &= v_1 - v_0,\\
t^2u_2 &= v_2 - 2v_1 + v_0,\\
t^3u_3 &= v_3 - 3v_2 + 3v_1 - v_0,\\
t^4u_4 &= v_4 - 4v_3 + 6v_2 - 4v_1 + v_0
\end{aligned} \tag{3.2}$$

so the v_i also base the regular representation where t is invertible. The alternative basis v_i is important because it makes the calculations very easy when t is invertible (the reductive case), and one strategy to solve our relations is to successively cancel powers of t.

What makes the matrices below a bit complicated is the relation F, that replaces x^5 by a sum of 4 terms involving S. The first four solutions having u_0 in the south-east corner are simpler, as they have a derivation using only the relation $T^5 = 1$.

The 10 basic solutions to our linear algebra problem are the skew matrices

$$N_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & u_0 + 2tu_1 + t^2u_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & v_2 \end{pmatrix}$$

$$N_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & u_0 + tu_1 \\ & & & 3(u_1 + tu_2) \end{pmatrix},$$

$$N_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & u_0 & 4u_1 \\ & & & 6u_2 \end{pmatrix}, \quad N_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & u_0 \\ & & 0 & 2u_1 \\ & & & 3u_2 \end{pmatrix}$$

From here on, things get more complicated:

$N_{13} =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & u_0 + 4tu_1 + 6t^2u_2 + 4t^3u_3 + t^4u_4 & -16u_1 - 24tu_2 - 16t^2u_3 - 4t^3u_4 \\ & & -8u_1 - 12tu_2 - 8t^2u_3 - 2t^3u_4 & -8St^3u_1 + 48u_2 + 32tu_3 + 8t^2u_4 \\ & & & 8St^2u_1 + 8St^3u_2 - 32u_3 - 8tu_4 \end{pmatrix}$$

Working modulo P, we can read its entries as v_4, $-2(v_4-v_0)/t$, $-4(v_4-v_0)/t$, $8(v_4+v_1-2v_0)/t^2$ and $-8(v_4-v_2+3v_1-3v_0))/t^3$.

$$N_{04} = \begin{pmatrix} 0 & 0 & 0 & u_0 + 4tu_1 + 6t^2u_2 + 4t^3u_3 + t^4u_4 \\ & 0 & 0 & -4u_1 - 6tu_2 - 4t^2u_3 - t^3u_4 \\ & & 0 & -St^3u_1 + 6u_2 + 4tu_3 + t^2u_4 \\ & & & St^2u_1 + St^3u_2 - 4u_3 - tu_4 \end{pmatrix}$$

with entries $v_4, -(v_4-v_0)/t, (v_4+v_1-2v_0)/t^2, -(v_4-v_2+3v_1-3v_0))/t^3$.

$N_{12} =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ & v_3 & 3u_1 + 9tu_2 + 9t^2u_3 + 3t^3u_4 & 6St^3u_1 - 54u_2 - 42tu_3 - 12t^2u_4 \\ & & 3St^3u_1 - 27u_2 - 21tu_3 - 6t^2u_4 & -16St^2u_1 - 24St^3u_2 + 88u_3 + 24tu_4 \\ & & & 18Stu_1 + 24St^2u_2 + 18St^3u_3 - 24u_4 \end{pmatrix}$$

with entries v_3, $3(v_4-v_3)/t$, $3(-2v_4+v_3+v_0)/t^2$, $6(-2v_4+v_3+v_0)/t^2$, $8(3v_4-v_3+v_1-3v_0)/t^3$, $6(-4v_4+v_3+v_2-4v_1+6v_0)/t^4$.

$N_{03} =$

$$\begin{pmatrix} 0 & 0 & u_0 + 3tu_1 + 3t^2u_2 + t^3u_3 & 4u_1 + 12tu_2 + 12t^2u_3 + 4t^3u_4 \\ & 0 & u_1 + 3tu_2 + 3t^2u_3 + t^3u_4 & 4St^3u_1 - 36u_2 - 28tu_3 - 8t^2u_4 \\ & & St^3u_1 - 9u_2 - 7tu_3 - 2t^2u_4 & -8St^2u_1 - 12St^3u_2 + 44u_3 + 12tu_4 \\ & & & 9Stu_1 + 12St^2u_2 + 9St^3u_3 - 12u_4 \end{pmatrix}$$

with entries: v_3, $(v_4-v_3)/t$, $(-2v_4+v_3+v_0)/t^2$, $4(v_4-v_3)/t$, $4(-2v_4+v_3+v_0)/t^2$, $4(3v_4-v_3+v_1-3v_0)/t^3$, $3(-4v_4+v_3+v_2-4v_1+6v_0)/t^4$.

$N_{02} =$

$$\begin{pmatrix} 0 & u_0 + 2tu_1 + t^2u_2 & 3u_1 + 6tu_2 + 3t^2u_3 & 6u_2 + 12tu_3 + 6t^2u_4 \\ & u_1 + 2tu_2 + t^2u_3 & 3u_2 + 6tu_3 + 3t^2u_4 & 6St^2u_1 + 12St^3u_2 - 54u_3 - 18tu_4 \\ & & 2St^2u_1 + 4St^3u_2 - 18u_3 - 6tu_4 & -15Stu_1 - 25St^2u_2 - 20St^3u_3 + 30u_4 \\ & & & 18Su_1 + 27Stu_2 + 21St^2u_3 + 6St^3u_4 \end{pmatrix}$$

with entries: v_2, $(v_3 - v_2)/t$, $3(v_3 - v_2)/t$, $3(v_4 - 2v_3 + v_2)/t^2$, $2(-3v_4 + 3v_3 - v_2 + v_0)/t^3$, $6(v_4 - 2v_3 + v_2)/t^2$, $6(-3v_4 + 3v_3 - v_2)/t^3$, $5(6v_4 - 4v_3 + v_2 + v_1 - 4v_0)/t^4$, $3S(2v_4 - v_3 + v_1 - 2v_0)/t$.

$N_{01} =$

$$\begin{pmatrix} u_0 + tu_1 & 2u_1 + 2tu_2 & 3u_2 + 3tu_3 & 4u_3 + 4tu_4 \\ & u_2 + tu_3 & 2u_3 + 2tu_4 & 3Stu_1 + 6St^2u_2 + 6St^3u_3 - 12u_4 \\ & & Stu_1 + 2St^2u_2 + 2St^3u_3 - 4u_4 & -8Su_1 - 14Stu_2 - 12St^2u_3 - 4St^3u_4 \\ & & & -2S^2t^3u_1 + 16Su_2 + 13Stu_3 + 4St^2u_4 \end{pmatrix}$$

with entries: v_1, $2(v_2 - v_1)/t$, $(v_3 - 2v_2 + v_1)/t^2$, $3(v_3 - 2v_2 + v_1)/t^2$, $2(v_4 - 3v_3 + 3v_2 - v_1)/t^3$, $-(4v_4 - 6v_3 + 4v_2 - v_1 - v_0)/t^4$, $4(v_4 - 3v_3 + 3v_2 - v_1)/t^3$, $-3(4v_4 - 6v_3 + 4v_2 - v_1 - v_0)/t^4$, $2S(-2v_4 + 2v_3 - v_2 + v_0)/t$, $S(4v_4 - 3v_3 + v_2 + v_1 - 3v_0)/t^2$.

3.1 Invariance property

To state the invariance property the matrices N_{ij} enjoy, introduce the polynomial algebra $A[u] = A[u_0, u_1, u_2, u_3, u_4]$, and the A-algebra homomorphism $D\colon A[u] \to A[u]$ that takes

$$\begin{aligned} u_0 &\mapsto u_0, \\ u_1 &\mapsto xu_0 + Tu_1, \\ u_2 &\mapsto x^2u_0 + 2xTu_1 + T^2u_2, \\ u_3 &\mapsto x^3u_0 + 3x^2Tu_1 + 3xT^2u_2 + T^3u_3, \\ u_4 &\mapsto x^4u_0 + 4x^3Tu_1 + 6x^2T^2u_2 + 4xT^3u_3 + T^4u_4, \end{aligned} \quad \text{so} \quad \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \mapsto D_u \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{3.3}$$

where $T = 1 + tx$ and D_u is as in (2.3). Write $D(N)$ for the matrix obtained by applying D to each of the entries of N.

Proposition 3.1 *The above matrices N_{ij} each satisfies*

$$D_u N\, {}^t\!D_u = D(N). \tag{3.4}$$

That is, the skew homomorphism defined by $N \colon V^{\mathrm{reg}} \to V^{\mathrm{reg}}$ defines a TO_5*-linear* $\mathrm{Hom}(V^{\mathrm{reg}}, \bigwedge^2 V^{\mathrm{reg}})$*.*

It follows that any linear combination $N = \sum c_{ij} N_{ij}$ with $c_{ij} \in B$ defines a TO_5*-equivariant complex*

$$P_0 \leftarrow P_1 \stackrel{N}{\leftarrow} P_2 \leftarrow P_3 \leftarrow 0 \tag{3.5}$$

where $P_0 = A[u]$, $P_1 = 5A[u](-2)$, $P_3 = A[u](-5)$, $P_1 = \mathrm{Hom}(P_1, P_3)$

4 Reduction modulo (S, t) to the $\boldsymbol{\alpha}_5$ case