Fun in codimension 4

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Abstract

I discuss some graded ring constructions of algebraic varieties, mostly motivated by work on algebraic surfaces by Horikawa and his followers. My aim, insofar as possible, is to see the geometric constructions as pullbacks of key varieties. The ideal would be to lift case divisions such as Horikawa's I, II_a and II_b out of the geometry of surfaces and into general theory of codimension 4 Gorenstein ideals, Tom and Jerry unprojections, key varieties and so on.

1 The Horikawa quintics revisited

Horikawa's famous paper [H1] studies canonical surfaces with $p_g = 4$, $K^2 = 5$, with the case division

Type I $|K|$ is free and embeds to a quintic $X_5 \subset \mathbb{P}^3$;

Type II $|K|$ has a transversal base point P, and, after blowing it up, φ_K defines a double cover to a quadric $Q \subset \mathbb{P}^3$, which may be of rank 4 (Type II_a) or 3 (Type II_b).

For details, see [H1], [G], [R2].

1.1 The curve and the choice of rendition

In Type II, the curve section $C \in |K_X|$ is a genus 6 hyperelliptic curve with a marked Weierstrass point $P \in C$, polarised by a half-canonical divisor $A = 5P = P + 2g^1_2$. In coordinates t_1, t_2 on $\mathbb{P}^1$, with $P = (0,1)$ and $P_2, \dots, P_{14}$ given by $f_{13}(t_1, t_2) = 0$, the ring $R(C, A) = R(C, P)^{[5]}$ is generated by

$$\begin{array}{ll} \text{in degree 1} & x_1 = ut_1^2,\ \ x_2 = ut_1t_2,\ \ x_3 = ut_2^2, \\ \text{in degree 2} & y_2 = t_2^5, \\ \text{in degree 3} & z_1 = vt_1,\ \ z_2 = vt_2, \end{array} \tag{1}$$

where $u^2 = t_1$ and $v^2 = f_{13}(t_1, t_2)$, and related by

$$\bigwedge^2 N = 0 \quad \text{where } N = \begin{pmatrix} x_1 & x_2 & x_3^2 & z_1 \\ x_2 & x_3 & y_2 & z_2 \end{pmatrix}, \quad \text{and} \quad \begin{aligned} z_1^2 &= [t_1^2 f_{13}], \\ z_1 z_2 &= [t_1 t_2 f_{13}], \\ z_2^2 &= [t_2^2 f_{13}]. \end{aligned} \tag{2}$$

The square brackets mean I *render* forms in t_1, t_2 of degree divisible by 5 (in this case, 15) as weighted forms in x_1, x_2, x_3, y_2 (in this case, of degree 6). The main point is that different possible renditions give rise to different deformation families of $R(C, A)$ and components of the moduli space of X, growing out of the apparently harmless substitution $x_1x_3 \mapsto x_2^2$. The identity $x_1x_3 = x_2^2$ in $R(C, P)$ becomes a relation in $R(C, A)$, but, once we deform the ring, will only be a congruence modulo deformation parameters.

Every monomial in $t_1^i t_2^j$ of degree 13 has $i \geq 3$ or $j \geq 8$ (with a bit to spare), so I can write f_{13} in the form

$$f_{13}(t_1, t_2) = A_{10} t_1^3 - B_5 t_2^8. \tag{3}$$

Fix once and for all some rendition α_4, b_2 of A_{10}, B_5; for example, do

$$x_1x_3 \mapsto x_2^2, \quad x_1y_2 \mapsto x_2x_3^2, \quad x_2y_2 \mapsto x_3^3 \tag{4}$$

repeatedly to remove all occurrences of x_1x_3, x_1y_2, x_2y_2. Then $(t_1^2, t_1t_2, t_2^2)f_{13}$ in (2) render as:

$$\text{(A)} \quad \begin{matrix} ax_1 - bx_3^4 \\ ax_2 - bx_3^2 y_2 \\ ax_3 - by_2^2 \end{matrix} \quad \text{with } a = \alpha x_1\text{; or} \quad \text{(B)} \quad \begin{matrix} \alpha x_1^2 - bx_3^4 \\ \alpha x_1 x_2 - bx_3^2 y_2 \\ \alpha x_2^2 - by_2^2 \end{matrix} \tag{5}$$

the only difference being $\alpha x_1 x_3 \mapsto \alpha x_2^2$ in the last line. Case A will correspond to Horikawa's Types II_b and I, whereas Case B will correspond to Types II_b and Type II_a.

1.2 Case B

There is not too much to say about Case B. The roll $x_1^2 \mapsto x_1x_2 \mapsto x_2^2$ in (5) is quadratic in the rows, which allows me to replace N in (2) by a general matrix, and the last 3 equations as a general quadratic expression evaluated on its rows. The 9 equations are in rolling factors format:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2' & y_1 & z_1 \\ x_2 & x_3 & y_2 & z_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{aligned} z_1^2 &= \alpha x_1^2 - by_1^2, \\ z_1 z_2 &= \alpha x_1 x_2 - by_1 y_2, \\ z_2^2 &= \alpha x_2^2 - by_2^2, \end{aligned} \tag{6}$$

with $x_2' = x_2$ and $y_1 = x_3^2$. This format allows the quadric to deform to rank 4 (when x_2' becomes an independent variable). This construction with general N is an anticanonical divisor in a weighted form of Segre $\mathbb{P}^1 \times \mathbb{P}^3$; it is the main bulk construction of codimension 4 Gorenstein ideals having no known interpretation as a Kustin–Miller unprojection.

1.3 Case A

Case A in (5) allows me to roll factors $x_1 \mapsto x_2 \mapsto x_3$ without putting in terms that are explicitly quadratic in the rows of N; this depends on the coincidence $n_{12} = n_{21} = x_2$ in N, which therefore obstructs deforming the quadric $x_1x_3 - x_2^2$ to rank > 3. The variable x_1 appears linearly in 4 equations multiplying x_3, y_2, z_2, a, and not in the others, so we can eliminate it, and treat the ring as a Kustin–Miller unprojection from the Pfaffians of

$$M = \begin{pmatrix} 0 & x_2 & x_3^2 & z_1 \\ & x_3 & y_2 & z_2 \\ & & z_2 & -by_2 \\ & & & -a \end{pmatrix} \quad \text{of weights} \quad \begin{matrix} 0 & 1 & 2 & 3 \\ & 1 & 2 & 3 \\ & & 3 & 4 \\ & & & 5 \end{matrix} \qquad (7)$$

with unprojection ideal the codimension 4 c.i. $I = (x_3, y_2, z_2, a)$. The weight 0 of the entry $m_{12} = 0$ is noteworthy. Apart from $m_{13} = x_2$ and $m_{15} = z_1$, every entry of M is in I, so we can treat it in the bigger families of Tom_1 or Jerry_{24} unprojections.

Jer$_{24}$ This case depends on *both* the coincidences $x_2 = x_2$ and $x_3 \mid n_{13}$, and does not lead to anything new. For Horikawa surfaces, it only gives deformations inside Type II_b.

The Jerry format (7) requires $m_{12} \in I$, so M keeps its 0. It also requires $m_{14} = x_3^2$ to remain in the ideal $I = (x_3, y_2, z_2, A)$, thus only allowing x_3^2 to change by adding multiples of x_2x_3 or y_2; these can be nullified by column operations, so this entry also does not change. The entry $m_{35} = -by_2$ is a free entry, so treat it as a token B. After this, the pivot $m_{24} = y_2$ can be projected out, and the deformation family calculated as a parallel unprojection. It is a variant on rolling factors:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3^2 & z_1 \\ x_2 & x_3 & y_2 & z_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{aligned} z_1^2 &= ax_1 - x_2x_3B, \\ z_1z_2 &= ax_2 - x_3^2B, \\ z_2^2 &= ax_3 - y_2B. \end{aligned} \qquad (8)$$

Tom$_1$ This is where the special hyperelliptic case II$_b$ deforms up to a general quintic hypersurface. The original ring and its general Tom$_1$ deformation are the Pfaffians of the following extrasymmetric 6×6 matrixes:

$$M_0 = \begin{pmatrix} 0 & x_3^2 & x_1 & x_2 & z_1 \\ & y_2 & x_2 & x_3 & z_2 \\ & & z_1 & z_2 & a \\ & & & 0 & bx_3^2 \\ & & & & by_2 \end{pmatrix} \quad \mapsto \quad M_\lambda = \begin{pmatrix} \lambda & y_1 & x_1 & x_2 & z_1 \\ & y_2 & x_2 & x_3 & z_2 \\ & & z_1 & z_2 & a \\ & & & \lambda b & by_1 \\ & & & & by_2 \end{pmatrix}, \tag{9}$$

where λ, y_1, b, a are indeterminates of weight $0, 2, 2, 5$. Write

$$\mathcal{CV} \subset \mathbb{A}^9_{\langle x_{1\dots3}, y_1, y_2, b, z_1, z_2, a\rangle} \times \mathbb{A}^1_\lambda \tag{10}$$

for the key variety defined by the 4×4 Pfaffians of M_λ, the affine cone over the weighted projective variety $\mathcal{V} \subset \mathbb{P}(1^3, 2^3, 3^2, 5) \times \mathbb{A}^1$.

The fibre $\mathcal{CV}_{\lambda\neq0}$ is just a copy of $\mathbb{A}^5_{\langle x_{1\dots3}, y_1, y_2\rangle}$ cunningly set up to degenerate as $\lambda \to 0$ to the codimension 4 variety $\mathcal{CV}_0$ given by the Pfaffians of M_0.

Proposition 1.1 *Assume first that λ is invertible; then $\mathcal{CV}_\lambda$ is the graph over $\mathbb{A}^5_{\langle x_{1\dots3}, y_1, y_2\rangle}$ of the functions b, z_1, z_2, a defined by four of the Pfaffians $\mathrm{Pf}_{12.ij}$:*

$$\begin{aligned} -\lambda z_1 &= x_1 y_2 - x_2 y_1, \quad & -\lambda z_2 &= x_2 y_2 - x_3 y_1, \\ -\lambda a &= y_2 z_1 - y_1 z_2 \quad \textit{and} \quad & \lambda^2 b &= x_1 x_3 - x_2^2; \end{aligned} \tag{11}$$

after this, the remaining Pfaffian equations hold as identities. Therefore the fibre $\mathcal{CV}_{\lambda\neq0} \cong \mathbb{A}^5$ and $\mathcal{V}_{\lambda\neq0} \cong \mathbb{P}^4(1^3, 2^2)$.

When $\lambda = 0$, the variety $\mathcal{CV}_0$ is given by

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & y_1 & z_1 \\ x_2 & x_3 & y_2 & z_2 \end{pmatrix} = 0 \quad \textit{and} \quad \begin{aligned} z_1^2 &= ax_1 - by_1^2 \\ z_1 z_2 &= ax_2 - by_1 y_2 \\ z_2^2 &= ax_3 - by_2^2 \end{aligned} \tag{12}$$

It is an affine Gorenstein codimension 4 variety that can be viewed in an obvious way as an anticanonical divisor in a scroll.

Remark 1.2 The affine cone $\mathcal{CV}_0$ also has a parametric form that displays it as a simple birational transformation away from the quotient of $\mathbb{A}^5_{\langle b, s_1, s_2, u, v\rangle}$

by the μ_2 action $\frac{1}{2}(0,1,1,1,1)$. Indeed, the quantities

$$\begin{array}{lll} x_1 = s_1^2, \quad x_2 = s_1 s_2, \quad x_3 = s_2^2 & & \\ \quad y_1 = s_1 u, \quad y_2 = s_2 u, & \text{and} & a = v^2 + bu^2, \\ \quad z_1 = s_1 v, \quad z_2 = s_2 v, & & \end{array} \tag{13}$$

satisfy the relations (12) identically. In straight projective space $x_1, \dots, z_2$ would be coordinates on the scroll $\mathbb{P}_{\mathbb{P}^1}(2,1,1)$, the blowup of $\mathbb{P}^1 \subset \mathbb{P}^3$ or the projection of $v_2(\mathbb{P}^3)$ from a conic.

1.4 Embedded degeneration of quintic curves

The key variety $\mathcal{CV}$ (10) describes degenerations of quintics hypersurfaces, starting from the degeneration of a plane quintic curve to the nonsingular hyperelliptic curve $C_0 \subset \mathbb{P}(1,1,1,2,3,3)$ as described in Griffin [G]. The simplifying feature here is that it is contained as a complete intersection inside the key variety $\mathcal{V}$.

Consider the complete intersection

$$(b = B_2, y_1 = Y_2 a = A_5) \subset \mathcal{V} \subset \mathbb{P}(1^3, 2^3, 3^2, 5) \times \mathbb{A}^1, \tag{14}$$

where $B_2(x_{1..3}, y_2)$, $Y_2(x_{1..3}, y_2)$, $A_5(x_{1..3}, y_2)$ are general forms in x_i, y_2 of the stated weights. For $\lambda \neq 0$, the fibre $\mathcal{V}_\lambda$ is $\mathbb{P}(1,1,1,2,2)$, and the two quadratic equations in (14) eliminate y_1, y_2 (because B_2 contains y_2), so C_λ is a general plane quintic $C_5 \subset \mathbb{P}^2_{\langle x_{1\dots3}\rangle}$. For $\lambda = 0$, the equations of $\mathcal{V}_0$ with the specialisation of y_1, b, a are essentially the hyperelliptic equations we started from in (1).

1.5 Generalities on regular pullbacks

The above *embedded* treatment inside $\mathcal{V}$ works for plane quintic curves, but not for higher dimensional quintic hypersurfaces, for example because $h^0(V_\lambda, \mathcal{O}(1)) = 3$ whereas we need $h^0(X, A) = n + 2$. The more general notion is *regular pullback* from a key variety; I explain this briefly for completeness.

By definition, a *key variety* is an affine variety $W \subset \mathbb{A}^N$ that I wish to treat as a key variety (in other words, it is a psychological state); as usual, write $k[W] = k[x_{1\dots n}]/I_W$ for its affine ideal and coordinate ring. W might be, say, the affine cone $a\,\mathrm{Grass}(2,5) \subset \bigwedge^2 \mathbb{C}^5$ over the Plücker embedding of

Grass(2, 5) of [CR], with equations the 4×4 Pfaffians of a generic 5×5 skew matrix, or the extrasymmetric variety $\mathcal{CV} \subset \mathbb{A}^9 \times \mathbb{A}^1_\lambda$ of (10), or the origin $0 \in \mathbb{A}^n$ defined by the regular sequence $x_{1\dots n}$.

Given an ambient ring R (either regular local or polynomial and graded in positive degrees), and a morphism $\varphi\colon \operatorname{Spec} R \to \mathbb{A}^N$ to the ambient space of W, take the pullback or scheme theoretic inverse image $\varphi^{-1}W \subset \operatorname{Spec} R$, and require it to be a regular pullback in the sense of Proposition 1.3. The morphism φ specifies values $\varphi^*(x_i) = X_i \in R$; the pullback is then defined by the ideal $\varphi^* I_W \subset R$, in other words, by substituting elements $X_i \in R$ for x_i in the equations of W. It is the same thing as the intersection with the graph of φ

$$\Gamma_\varphi \cap (\operatorname{Spec} R \times W) \subset \operatorname{Spec} R \times \mathbb{A}^N; \tag{15}$$

the graph Γ_φ is of course the complete intersection cut out by the equations $X_i = x_i$ for $i = 1, \dots, n$.

Proposition 1.3 *Equivalent conditions:*

(i) $X_i - x_i$ *for* $i = 1, \dots, n$ *form a regular sequence for* $\operatorname{Spec} R \times W$.

(ii) The resolution complex of W *remains exact on pulling back to* $\operatorname{Spec} R$.

Assume also that W *is Cohen–Macaulay; then (i) and (ii) are equivalent to*

(iii) $\varphi^{-1}W$ *has the expected dimension, that is,* $\operatorname{codim} \varphi^{-1}W = \operatorname{codim} W$.

In my case (10), I substitute specific values $X_1, \dots, A \in R$ for the variables $x_1, \dots, a$ of $\mathcal{CV}$ into the extrasymmetric matrix M_λ of (9), and use the resulting Pfaffians to generate an ideal of R.

Even though I am mainly interested in projective varieties and graded rings, the construction itself works on the level of affine cones: φ is usually homogeneous (equivariant for appropriate $\mathbb{G}_m$ actions), but the induced map $\varphi\colon \operatorname{Proj} R \dashrightarrow \mathbb{P}(W)$ need not be a morphism.

1.6 Application to Horikawa quintics

We saw that the halfcanonical linear system $A = g^2_5$ of a nonsingular plane quintic C_5 can acquire a base point and become $A = P + 2g^1_2$. The extrasymmetric format (9) also allows the polarising $|\mathcal{O}(1)|$ of a quintic n-fold $V^n_5 \subset \mathbb{P}^{n+1}$ to acquire a transverse base point and $\varphi_{\mathcal{O}(1)}$ to degenerate to a double cover of a rank 3 quadric Q, while V_5 remains nonsingular in codimension 2.

Definition 1.4 A numerical quintic is a projective n-fold X with at worst terminal singularities, polarised by an ample Cartier divisor A, such that

$$K_{X_0} = (3-n)A, \quad A^n = 5, \quad \text{and} \quad h^0(A) = n+2. \tag{16}$$

Theorem 1.5 *Let $R = k[x_{1\dots n+2}, y_2, z_1, z_2]$ be the graded polynomial ring with $\operatorname{wt} x_i, y_2, z_i = 1, 2, 3$. Let $b = B_2, y_1 = Y_{1,2}, a = A_5$ be general forms in x_i, y_2 of the stated weights, and write*

$$\mathcal{X} \subset \operatorname{Proj} R \times \mathbb{A}^1_\lambda = \mathbb{P}^{n+4}(1^{n+2}, 2, 3^2) \times \mathbb{A}^1_\lambda \tag{17}$$

for the variety defined by the 4×4 Pfaffians of the extrasymmetric matrix M_λ of (9). It is a flat family X_λ of projectively Gorenstein codimension 4 varieties parametrised by λ, and the fibre $X_{\lambda\neq 0}$ is projectively equivalent to a general quintic in $\mathbb{P}^{n+1}$, lifted to $\mathbb{P}(1^{n+2}, 2, 3^2)$ by the forms b, z_1, z_2 of (11) (b contains y_2).

When $\lambda = 0$, the Pfaffians take the form (12) with $y_1 = Q$; the subscheme $X_0 \subset \mathbb{P}^{n+4}(1^{n+2}, 2, 3^2)$ defined by these equations is a numerical quintic with singular locus of dimension $n-3$ (empty if $n \le 2$). The linear system $|A|$ has a single transverse base point $P \in \operatorname{NonSing} X_0$.

The rational map $\varphi_{|A|}$ blows up P and defines a generically 2-to-1 morphism $\widetilde{\varphi}\colon \widetilde{X}_0 \to Q_3 \in \mathbb{P}^{n+1}$ where $Q_3 : (x_1x_3 = x_2^2)$.

Geometry of $F \subset \mathbb{P}(1^{n+2}, 2)$ Consider the involution that acts by -1 on λ, z_1, z_2 and fixes x_i, y_2 and b, y_1, a. Each Pfaffian of M_λ is $\pm$ invariant (compare (9–12)), and this induces an involution on $\mathcal{X}$ that restricts to a "hyperelliptic" involution on X_0. The quotient morphism $X_0 \to F \subset \mathbb{P}(1^{n+2}, 2)$ given by the free linear system $|\mathcal{O}(2)|$ is a finite double cover of the codimension 2 determinantal n-fold F given by

$$\bigwedge^2 N = 0 \quad \text{where} \quad N = \begin{pmatrix} x_1 & x_2 & y_1 \\ x_2 & x_3 & y_2 \end{pmatrix} \tag{18}$$

(and $y_1 = Y_{1,2}(x_i)$ general). This F is singular exactly where $N = 0$, together with the quasismooth point $P_{y_2} \in F$, an isolated $\frac{1}{2}$ orbifold point.

The reader new to all this should concentrate on the surface case $n = 2$, which is familiar from [H1], and relates closely to the relative 2-canonical morphism of a genus 2 fibration at a 2-disconnected fibre as described in [CP]: the 1-canonical image is then the quadric $Q : (x_1x_3 = x_2^2) \subset \mathbb{P}^3_{\langle x_{1\dots 4}\rangle}$.

The blown up base point of $|K_S|$ maps to the x_3-axis $L : (x_1 = x_2 = 0)$, and has two marked point $Y_{1,2} = 0$ (typically, $x_3^2 - x_4^2 = 0$) that are the essential singularities of the branch locus in Horikawa's treatment.

Write $Q : (x_1x_3 = x_2^2) \subset \mathbb{P}^{n+1}_{\langle x_{1...n+2}\rangle}$ for the n-fold quadric of rank 3, the image of X_0 under $\varphi_{\mathcal{O}(1)}$. The birational map $\beta\colon Q \dashrightarrow F$ is given by quadratic forms on Q allowed poles on the fibre $L = \mathbb{P}^{n-1} : (x_1 = x_2)$ but required to vanish on $L \cap Y_{1,2}$, giving $y_2 = x_2Y_1/x_1 = x_3Y_1/x_2$ in addition to quadratic forms in $x_{1...n+2}$. Expressed in birational geometry, β first blows up the vertex $x_1 = x_2 = x_3$ to make the n-fold scroll $\mathbb{F}(2, 0^{n-1})$, then blows up the nonsingular quadric $Y_1 = 0$ in the fibre L, and finally contracts L to a $\frac{1}{2}$ orbifold point at P_{y_2}. The career of the locus $x_1 = x_2 = x_3 = 0$ is also interesting: it starts life as the vertex $\mathbb{P}^{n-2}$ of the quadric, is blown up to the negative locus $E = \mathbb{P}^1 \times \mathbb{P}^{n-2}$ of the scroll. At the fibre L it meets the nonsingular quadric $L \cap Y_1$, and after the blowup of Y_1, is contracted to the $\frac{1}{2}$ orbifold point P_{y_2} of the divisor $\mathbb{P}^{n-1}(1^{n-1}, 2)_{\langle x_{4...n+2}, y_2\rangle} \subset F$, given also by $x_1 = x_2 = x_3 = 0$.

The branch locus of the double cover $X_0 \to F$ consists of the divisor

$$D : (ax_1 = bY_1^2,\ \ ax_2 = bY_1y_2,\ \ ax_3 = by_2^2), \tag{19}$$

together with the $\frac{1}{2}$ point P_{y_2}; these are disjoint because $y_2 \in b = B_2(x_i, y_2)$. To prove Theorem 1.5, I only need to establish that D is nonsingular outside the singularities of F.

Conjecture *The Type A family is a generic hypersurface if $\lambda \neq 0$. When $\lambda = 0$ it is a birational double cover of a quadric of rank 3, and is singular at a conic in the vertex (e.g. 2 points if $n = 3$). The Type B family when $x_2 \neq x_2'$ is a double cover of a quadric of rank 4, and is singular at a conic in the vertex (e.g., nonsingular if $n = 3$, singular at 2 points if $n = 4$). It would be interesting to know the relation between the topology of the general Type A and the general Type B, e.g., for the Calabi–Yau case. Can do by computer algebra. Should be easy by Bertini. Hypersurface question on affine pieces.*

1.7 Comparison with Horikawa's treatment

Horikawa divides surfaces with $p_g = 4$, $K^2 = 5$ into three families Type I, II_b and II_a, where I and II_a are the irreducible components of moduli, and II_b is in the closure of both, in codimension 1 in each.

My three cases are

Case A with $\lambda \neq 0$ giving Horikawa's family I,

Case A with $\lambda = 0$ giving II_b,

Case B with $x_2 = x_2'$ also giving II_b,

Case B with $x_2 \neq x_2'$ giving II_a.

When $n = 1$ the last 3 cases coincide; they form the hyperelliptic locus, which is a codimension 1 subvariety of family I.

Jul 2011, Feb 2012 I understand this better, but the proof is not written. Regular pullback of my key variety $\mathcal{CV}$ of (10) gives a nonsingular n-fold quintic hypersurface $Y_\lambda \subset \mathbb{P}^{n+1}$ when $\lambda \neq 0$ degenerating to a codimension 4 n-fold $Y_0 \subset \mathbb{P}(1^{n+2}, 2, 3, 3)$ when $\lambda = 0$; in general Y_0 has a nonsingular point that is a base point of $|\mathcal{O}(1)|$, and its blowup is a double cover of a quadric of rank 3; Y_0 has ordinary double points over a hyperplane section of the vertex of the quadric, that is, a codimension 3 locus. On the other hand, the format (B) gives rise to n-folds that are in general nonsingular in codimension 4.

In the case $n = 3$ we have a transition from a nonsingular quintic hypersurface Y_5 to a nonsingular $Y_2 \subset \mathbb{P}^8(1^5, 2, 3, 3)$ that is birationally a double cover of a quadric of rank 4. The transition passes through a singular $Y_2' \subset \mathbb{P}^8(1^5, 2, 3, 3)$ that has two ordinary nodes amd is birationally double cover of a quadric of rank 3.

These 3-folds have different Betti numbers B_2, so are not topologically equivalent.

In the 3-fold case, I want general case B corresponds to Horikawa fake quintic Y_2, with a single base point and double cover of quadric of rank 4. Questions: can Y_2 be nonsingular, and is it diffeo to $Y_1 =$ quintic in $\mathbb{P}^4$. Maybe $\operatorname{Pic} Y_2 = 2\mathbb{Z}$?

1.8 Other applications

The restriction to $y_1 = Y_{1,2}$, that is, only one new variable y_2 in degree 2, was motivated by the application to quintic hypersurfaces in straight $\mathbb{P}^n$. There are many other interesting cases, starting with natural degenerations of hypersurfaces $V_5 \subset \mathbb{P}(1^n, 2)$ to codimension 4.

The key variety $\mathcal{CV}$ of (11) has a 3-parameter family of $\mathbb{C}^\times$ actions with weights:

$$\begin{array}{lll} x_1 \mapsto n-l & & & \\ x_2 \mapsto n & y_1 \mapsto m & z_1 \mapsto n+m & b \mapsto 2n \\ & y_2 \mapsto m+l & z_2 \mapsto n+m+l & a \mapsto n+2m+l \\ x_3 \mapsto n+l & & & \end{array} \tag{20}$$

The determinantal $\bigwedge^2 N = 0$ and its double cover given by (12) apply in other cases. In particular, the same tricks give natural degenerations of K3 and Fano hypersurfaces $V_5 \subset \mathbb{P}(1^n, 2)$ to codimension 4.

To finish.

1.9 The obstruction

The two deformation families of Case A and Case B are incompatible already at the first infinitesimal level: you can deform to the extrasymmetric format (9) with $\lambda \neq 0$, or you can deform the rank 3 quadric $x_1x_3 - x_2^2$ to rank 4, but you can't do both. Compare [R2], Section 5, which calculates Horikawa's obstruction to deformation as $\lambda(x_2 - x_2') = 0$.

2 On the BCP construction

Extending Horikawa's work on surfaces with $p_g = 4$, $K^2 = 6$, Bauer, Catanese and Pignatelli [BCP] study deformations of the ring $R(C, \frac{3}{2}P)$, where C is a hyperelliptic curve of genus 3 and $P \in C$ a Weierstrass point viewed as a $\frac{1}{2}$ orbifold point. Start from the hypersurface

$$R(C, \tfrac{1}{2}P) = k[a,b,c]/(c^2 = f_7(a^4, b)) \tag{21}$$

with $\mathrm{wt}(a,b,c) = 1, 4, 14$. Its Proj $C_{28} \subset \mathbb{P}(1,4,14)$ has a $\frac{1}{2}$ orbifold point at $P = (1,0,0)$ and ample divisor $A = \frac{1}{2}P$ with $K_{C,\mathrm{orb}} = 9A = 2g_2^1 + \frac{1}{2}P$. The ring $R(C, \frac{3}{2}P)$ is the third Veronese truncation $R(C, \frac{1}{2}P)^{[3]}$, and is Gorenstein codimension 4 with generators

$$x = a^3, \quad y = a^2b, \quad z = ab^2, \quad u = b^3, \quad v = ac, \quad w = bc$$

with $\mathrm{wt}(x,y,z,u,v,w) = 1,2,3,4,5,6$ and relations

$$\bigwedge^2 \begin{pmatrix} x & y & z & v \\ y & z & u & w \end{pmatrix} = 0 \quad \text{and} \quad \begin{array}{rcl} v^2 &=& [a^2 f], \\ vw &=& [abf], \\ w^2 &=& [b^2 f], \end{array} \tag{22}$$

where the square brackets render a form of weight $3d$ in a, b into a form of weight d in x, y, z, u. There are 2×2 different renditions of (22), that express our ring in four ways as sections of key varieties. The choices are at the two ends of the binary form $f_7(a^4, b)$: terms with high powers of a roll as

$$a^3, a^2b, ab^2 \mapsto x, y, z \qquad \text{or as} \qquad a^6, a^5b, a^4b^2 \mapsto x^2, xy, y^2 \tag{23}$$

and at the other end, terms with high powers of b roll as

$$a^2b^4, ab^5, b^6 \mapsto z^2, zu, u^2 \qquad \text{or as} \qquad a^2b, ab^2, b^3 \mapsto y, z, u \tag{24}$$

Every monomial $a^{28-4j}b^j$ in $f_7(a^4, b)$ has $j \ge 4$ or $28 - 4j \ge 4$, so choosing renditions at the two ends of f gives:

$$\begin{array}{rcl} v^2 &=& Ax + Cy \\ vw &=& Ay + Cz \\ w^2 &=& Az + Cu \end{array} \quad \text{or} \quad \begin{array}{l} Ax - Dz^2 \\ Ay - Dzu \\ Az - Du^2 \end{array} \quad \text{or} \quad \begin{array}{l} Bx^2 + Cy \\ Bxy + Cz \\ By^2 + Cu \end{array} \quad \text{or} \quad \begin{array}{l} Bx^2 + Dz^2 \\ Bxy + Dzu \\ By^2 + Du^2 \end{array} \tag{25}$$

where $A = A_9$, $B = B_8$, $C = C_8$, $D = D_4$ are forms of the stated weights in x, y, z, u. The four cases in (25) are called (I)–(IV).

Case I This is a double Jerry, see [TJ], Section 8. Two projections eliminate x and u to the codimension 2 c.i. ideal

$$yw = zv, \quad vw = Ay + Cz \tag{26}$$

in the product of the ideals $I_x = (z, w, A)$ and $I_u = (y, v, C)$. The relations (26) deform to the apparently more general form

$$\begin{pmatrix} z & w & A \end{pmatrix} M \begin{pmatrix} y \\ v \\ c \end{pmatrix} = 0, \quad \begin{pmatrix} z & w & A \end{pmatrix} N \begin{pmatrix} y \\ v \\ c \end{pmatrix} = 0 \tag{27}$$

with M of weights $\begin{smallmatrix} 3 & 0 & -3 \\ 0 & -3 & -6 \\ -3 & -6 & -9 \end{smallmatrix}$ and N of weights $\begin{smallmatrix} 6 & 3 & 0 \\ 3 & 0 & -3 \\ 0 & -3 & -6 \end{smallmatrix}$. However, there are not many deformation entries of positive degree in these matrixes, and they can be absorbed by coordinate changes. For example, in $yw - zv + m_{11}zw$, the m_{11} is absorbed by $w \mapsto w + m_{11}z$ or $v \mapsto v - m_{11}y$.

So the variety $W_{8,11} \subset \mathbb{P}(1, 2, 3, 4, 5, 6, 8, 9)_{\langle y,z,v,w,C,A\rangle}$ given by (26) and its double unprojection $V \subset \mathbb{P}(1, 2, 3, 4, 5, 6, 8, 9)_{\langle x,y,z,u,v,w,C,A\rangle}$ is rigid in these degrees.

Case II Unproject from x to get 5 equations fitting together as the Pfaffians of

$$\begin{pmatrix} 0 & y & z & v \\ & z & u & w \\ & & w & -Du \\ & & & A \end{pmatrix}, \quad \text{of weights} \begin{matrix} 0 & 2 & 3 & 5 \\ & 3 & 4 & 6 \\ & & 6 & 8 \\ & & & 9 \end{matrix} \tag{28}$$

with unprojection ideal $I_x = (z, u, w, A)$. The entry $m_{12} = 0$ has weight 0. Except for m_{13}, m_{15} all the entries are in I_x, so we can view this as Tom_1 or Jerry_{23}.

The Tom_1 equations give the extrasymmetric format

$$\begin{pmatrix} 0 & z & x & y & v \\ & u & y & z & w \\ & & v & w & A \\ & & & 0 & Dz \\ & & & & Du \end{pmatrix}, \tag{29}$$

We can deform the zero entries m_{12} and m_{45} in (29) to λ and λD (with λ a variable of degree 0), and the entry $m_{24} = z$ to an independent variable t of weight 3, leading to the matrix

$$\begin{pmatrix} \lambda & z & x & y & v \\ & u & y & t & w \\ & & v & w & A \\ & & & \lambda D & Dz \\ & & & & Du \end{pmatrix} \tag{30}$$

When $\lambda \neq 0$ the equations eliminate v, w, A, D to give affine space $\mathbb{A}^5_{\langle x,y,z,t,u\rangle}$ or $\mathbb{P}(1,2,3,3,4)_{\langle x,y,z,t,u\rangle}$. Putting back the values of the tokens A, D gives the surface codimension 2 c.i.

$$S_{4,9} : (\lambda^2 D = xt - y^2, \lambda A = zw - uv) \subset \mathbb{P}^4(1,2,3,3,4)_{\langle x,y,z,t,u\rangle}, \tag{31}$$

where $v = yz - xu$ and $w = zt - yu$. Since D has the same weight as the variable u, we can think of D as $\mu u + D'$, and for $\mu \neq 0$, this is a general K3 surface $S_9 \subset \mathbb{P}(1,2,3,3)$ with a built-in degeneration.

Case III Eliminating u gives the ring as the unprojection of the ideal of Pfaffians

$$\begin{pmatrix} 0 & x & y & v \\ & y & z & w \\ & & v & C \\ & & & -Bx \end{pmatrix} \quad \text{of weights} \quad {\scriptstyle \begin{matrix} -1 & 1 & 2 & 5 \\ & 2 & 3 & 6 \\ & & 5 & 8 \\ & & & 9 \end{matrix}} \tag{32}$$

in the c.i. ideal $I = (x, y, v, C)$, with the entry $m_{12} = 0$ of weight -1. Except for m_{24}, m_{25}, every entry of the matrix is in I, so the ring can be viewed either as a Tom_2 or Jerry_{13} unprojection.

Because of the -1, these formats do not allow to lose the 2×3 minors.

Case IV Every variable appears quadratically in the equations, so there is no naturally occuring unprojection. The deformation family is the matrix format.

3 Divisor of odd degree in $v_2(\mathbb{P}^2)$

As part of part of the trigonal dichotomy, Castelnuovo, Petri and Mukai tell us that the canonical model $C_{10} \subset \mathbb{P}^5$ of a nonhyperelliptic, nontrigonal curve of genus 6 is either a quadric section of a del Pezzo surface S_5 in the general case, or is the second Veronese embedding $v_2(C_5 \subset \mathbb{P}^2)$ of a plane quintic. In either case there are 6 quadric relations; in the general case these are 5×5 Pfaffians intersect a quadric hypersurface, leading to a 6×10 resolution. In the plane quintic case, C needs 3 further cubic equations. The equations of C are $\bigwedge^2 M = 0$ and $(A_1, A_2, A_3)M = 0$ where

$$M = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} \tag{33}$$

corresponds to the Veronese embedding $x_1 = u^2$, $x_2 = uv$, $x_3 = uw$, $x_4 = v^2$, $x_5 = vw$, $x_6 = w^2$, and the three cubic equations are renditions $[uf_5]$, $[vf_5]$, $[wf_5]$ where $f_5(u, v, w)$ defines $C_5 \subset \mathbb{P}^2$.

As before, the key is a seemingly trivial trick with this rendition: observing that every monomial in u, v, w of degree 5 is divisible either by u, or by v^3 or w^3, I write the equation of C_5 as $f_5 = uA + v^3B + w^3C$ (with A quadratic in the x_i and B, C linear) and the renditions as

$$\begin{aligned} uf_5 &= x_1A + x_2x_4B + x_3x_6C \\ vf_5 &= x_2A + x_4^2B + x_5x_6C \\ wf_5 &= x_3A + x_4x_5B + x_6^2C \end{aligned} \tag{34}$$

Now the set of all 9 equations defining C can be written as 4×4 Pfaffians of

$$M = \begin{pmatrix} 0 & 0 & x_1 & x_2 & x_3 \\ & 0 & x_2 & x_4 & x_5 \\ & & x_3 & x_5 & x_6 \\ & & & x_6C & -x_4B \\ & & & & A \end{pmatrix} \tag{35}$$

The promising appearance of this as a 6×6 extrasymmetric matrix is a deception: the top left-hand block cannot become nonzero while preserving the format. I work instead by projecting out x_1. Then

$$M = \begin{pmatrix} 0 & x_2 & x_4 & x_5 \\ & x_3 & x_5 & x_6 \\ & & x_6C & -x_4B \\ & & & A \end{pmatrix} \mapsto \begin{pmatrix} \lambda & x_2 & x_4 & x_5 \\ & x_3 & x_5 & x_6 \\ & & x_6C & -x_4B \\ & & & A \end{pmatrix} \tag{36}$$

is a Jerry_{45} with unprojection ideal (x_4, x_5, x_6, A). In this format, the three top left entries are free, so I can replace $0 \mapsto \lambda$; since A is a token, I can also project him out to get the equations

$$\begin{pmatrix} x_3 & -x_2 & \lambda C \\ \lambda B & -x_3 & x_2 \end{pmatrix} \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0 \implies x_1 \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x_2^2 - \lambda x_3 C \\ x_2x_3 - \lambda^2 BC \\ x_3^2 - \lambda x_2 B \end{pmatrix}. \tag{37}$$

Thus the equations without A are

$$\begin{pmatrix} x_1 & x_2 & x_3 & \lambda B \\ & \lambda C & x_2 & x_3 \\ & & x_4 & x_5 \\ & & & x_6 \end{pmatrix}. \tag{38}$$

The final *long equation* is

$$x_1A + x_2x_4B + x_3x_6C - \lambda BCx_5 = 0. \tag{39}$$

I do not how to write it as a Pfaffian in any useful way. This is characteristic of Jerry.

It is interesting to understand how the deformation $\lambda \neq 0$ allows the 9 equations defining the special g^2_5 curve to pass to just 6 equations defining the general curve. The effect of the λ in (38) is to give 5 general equations defining a del Pezzo surface S_5; the other quadratic equation $f = \lambda A - x_4x_6 + x_5^2$ defines the general curve C as quadratic section of S_5. Having done this, the three cubic equations $x_i f$ for $i = 1, 2, 3$ become combinations of these 6.

Wenfei's case: $v_2(C_{15} \subset \mathbb{P}(1,3,5))$ **deforms to** $C_{6,6} \subset \mathbb{P}(1,2,3,3)$ The general curve $C_{15} \subset \mathbb{P}(1,3,5)_{\langle u,v,w\rangle}$ is nonsingular, with $K_C = 6P$ where $P : (v^5 + w^3 = 0) \in \mathbb{P}^1(3,5)$. Polarising the same curve by $2P$ gives the second Veronese $v_2(C_{15} \subset \mathbb{P}(1,3,5))$; as before, write $x = u^2$, $y = uv$, $z_1 = uw$, $z_2 = v^2$, $s = vw$ and $t = w^2$ for the generators, dividing degrees by 2 so that $\operatorname{wt} x, y, z_1, z_2, s, t = 1, 2, 3, 3, 4, 5$; also write $f_{15} = uA_1 + vA_2 + wA_3$, and render the A_i (temporarily) as forms in x, y, z_1, z_2, s, t of weights $7, 6, 5$. Then as before, the ring $R(C, 2P)$ is related by $\bigwedge^2 M = 0$ and $(A_1, A_2, A_3)M = 0$, giving 9 relations of weights $4, 5, 6,\ 6, 7, 8,\ 8, 9, 10$. I can write them as the 4×4 Pfaffians of

$$\begin{pmatrix} 0 & 0 & x & y & z_1 \\ & 0 & y & z_2 & s \\ & & z_1 & s & t \\ & & & A_3 & -A_2 \\ & & & & A_1 \end{pmatrix} \quad \text{of weights} \quad \begin{matrix} -1 & 0 & 1 & 2 & 3 \\ & 1 & 2 & 3 & 4 \\ & & 3 & 4 & 5 \\ & & & 5 & 6 \\ & & & & 7 \end{matrix} \tag{40}$$

This extrasymmetric format does not as it stands allow me to deform C to the c.i. $C_{6,6} \subset \mathbb{P}(1,2,3,3)$. As before, a rendition trick is the key: the monomials of $f_{15}(u,v,w)$ not divisible by v are $u^{15}, u^{10}w, u^5w^2, w^3$; therefore, every monomial in f is divisible by u^3 or v or w^3, giving the rendition

$$f_{15} = Du^3 + Bv + Ew^3 \quad \text{with} \quad \operatorname{wt} D, B, E = 6, 6, 0. \tag{41}$$

The fact that E is a nonzero scalar is the thing that will express s, t as functions of x, y, z_1, z_2 when $\lambda \neq 0$.

Now rewrite the equations not involving z_2 as the Pfaffians of

$$\begin{pmatrix} 0 & x & y & z_1 \\ & z_1 & s & t \\ & & Et & -B \\ & & & Dx \end{pmatrix} \mapsto \begin{pmatrix} \lambda & x & y & z_1 \\ & z_1 & s & t \\ & & Et & -B \\ & & & Dx \end{pmatrix}. \tag{42}$$

This is a Jerry$_{35}$ matrix: the entries in Rows 3 and 5 are in the unprojection ideal (x, z_1, t, B). Almost the same calculation as above give the unprojection equations for z_2 as the Pfaffians of

$$\begin{pmatrix} \lambda E & x & y & z_1 \\ & y & z_2 & s \\ & & s & t \\ & & & \lambda D \end{pmatrix} \quad \text{of weights} \quad \begin{smallmatrix} 0 & 1 & 2 & 3 \\ & 2 & 3 & 4 \\ & & 4 & 5 \\ & & & 6 \end{smallmatrix} \tag{43}$$

together with a long equation for Bz_2. When $\lambda E \neq 0$, these equations eliminate s, t

Full set of equations

$$\begin{array}{cc} y^2 - xz_2 + \lambda Es & 4 \\ yz_1 - xs + \lambda Et & 5 \\ xt - z_1^2 + \lambda B & 6 \\ z_1z_2 - ys + \lambda^2 DE & 6 \\ z_1s - yt + \lambda Dx & 7 \end{array} \qquad \begin{array}{cc} Dx^2 + By + Ez_1t & 8 \\ s^2 - z_2t + \lambda Dy & 8 \\ Bz_2 + Dxy + Est + \lambda DEz_1 & 9 \\ Dxz_1 + Bs + Et^2 & 10 \end{array} \tag{44}$$

Notice that if $\lambda = E = 1$ (which I can take wlog) then the first 4 equations express s, t, B, D as simple polynomial expressions in x, y, z_1, z_2, and one checks that the remaining 5 equations then hold identically, so that the variety defined by these equation is just the graph over $\mathbb{A}^4_{\langle x,y,z_1,z_2\rangle}$ of s, t, B, D. Substituting general sextics in x, y, z_1, z_2 for B, D defines a complete intersection $C_{6,6} \subset \mathbb{P}(1, 2, 3, 3)$.

Scrap K3 example: take the hypersurface $X_5 \subset \mathbb{P}(1, 1, 1, 2)$; it is a hypersurface with a $\frac{1}{2}$ orbifold point. If you want to treat it by resolving, you get nonsingular K3 S with fractional divisor $D = B + \frac{1}{2}\Gamma$, where B is ample and $B^2 = 2$, so defines a double cover $S \to \mathbb{P}^2$, and $B \cdot \Gamma = 1$, so maps to a bitangent line.

The case $|2D|$ is in the literature as part of the trigonal dichotomy – the curves in $|2D|$ have a g^2_5, so the image of φ_{2D} is contained in the Veronese cone $\mathbb{P}(1, 1, 1, 2) = P * v_2(\mathbb{P}^2) \subset \mathbb{P}^6$. Instead of being an intersection of quadrics, its ideal contains the 6 quadric cones through $v_2(\mathbb{P}^2)$ and three cubics corresponding to the rendered products $[x_i g_5]$.

The ring $R(X, 2D)$ is the following codimension 4 structure with 9×16 resolution. Take the symmetric matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} \quad \text{and set} \quad \bigwedge^2 M = 0 \quad \text{and} \quad \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} M = 0. \qquad (45)$$

If you give x_i, y_i weight one, this is a projectively Gorenstein 4-fold in straight $\mathbb{P}^8$ contained in the cone $\mathbb{P}^2 * v_2(\mathbb{P}^2)$. It can be viewed as the degeneration $\lambda \to 0$ of the extrasymmetric Pfaffian variety

$$\begin{pmatrix} \lambda y_3 & -\lambda y_2 & x_1 & x_2 & x_3 \\ & \lambda y_1 & x_2 & x_4 & x_5 \\ & & x_3 & x_5 & x_6 \\ & & & y_3 & -y_2 \\ & & & & y_1 \end{pmatrix}. \qquad (46)$$

Setting y_i to be quadratic in the x_i gives the canonical curve $C \subset \mathbb{P}^5$ or the second Veronese of the K3 $X_5 \subset \mathbb{P}(1, 1, 1, 2)$. The degeneration with $\lambda \to 0$ in (38) does not explain how to deform it to a general canonical curve or K3 surface. For this, we need to factorise the y_i some more.

Let $f_5(u_1, u_2, u_3)$ be the equation of a nonsingular plane quintic. Every monomial in f contains one of u_1^2, u_2^2, u_3^2.

So we can render $u_1 f, u_2 f, u_3 f$ as

$$x_1 A + x_2 B + x_3 C$$

A similar weighted structure should handle many 2nd Veronese embedding of a hypersurface in $\mathbb{P}(a, b, c)$ or $\mathbb{P}(a, b, c, d)$ with three variables a, b, c of odd weight.

4 Horikawa Dicks case

Surfaces with $p_g = 3$, $K^2 = 4$. Family II_a: assume the general $C \in |K_S|$ is a hyperelliptic curve of genus 5 polarised by the halfcanonical divisor $A = P_0 + g^1_2 + P_\infty$, where P_0, P_∞ are Weierstrass points. Take coordinates on $\mathbb{P}^1$ so that $P_0, P_\infty \mapsto 0, \infty$ and $f_{10}(t_1, t_2)$ gives the other 10 branch points. Write

$u\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_0+P_\infty)$ and $v\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_1+\cdots+P_{10})$ for the constant sections, so $u^2=t_1t_2$ and $v^2=f_{10}(t_1,t_2)$. Then $R(C,A)$ is generated by

$$\begin{aligned} x_1=ut_1,\ \ x_2=ut_2 &\quad \text{in degree 1,}\\ y_1=t_1^4,\ \ y_2=t_2^4 &\quad \text{in degree 2,}\\ z_1=vt_1,\ \ z_2=vt_2 &\quad \text{in degree 3,} \end{aligned} \tag{47}$$

and related by

$$\operatorname{rank}\begin{pmatrix} y_1 & x_1 & x_2^2 & z_1\\ x_1^2 & x_2 & y_2 & z_2 \end{pmatrix} \le 1 \quad \text{and} \quad \begin{aligned} z_1^2 &= [t_1^2f_{10}],\\ z_1z_2 &= [t_1t_2f_{10}],\\ z_2^2 &= [t_2^2f_{10}], \end{aligned} \tag{48}$$

where, as before, the brackets [ ] render the right-hand side as sextics in x_i,y_i. The point is to understand the different ways of doing this.

Remark 4.1 Note that $A=g^1_4$ on a curve of $g=5$ has Brill–Noether number 1, so imposes 1 condition on the moduli of C, and C,A has 11 moduli. The hyperelliptic guy has $2g-1=9$ moduli, and the trigonal guy with $K_C=2(g^1_3+P_\infty)$ has 10 moduli. The result for curves is that the two fixed points imposes transversal nonsingular divisorial conditions on C,A.

4.1 Deforming away the base point P_0

The curve C deforms to lose the fixed point P_0, so that $A=P_0+g^1_2+P_\infty\mapsto g^1_3+P_\infty$. It seems elegant to treat this deformation first in terms of the following bigger variety

$$V\subset \mathbb{A}^{10}_{\langle x_1,x_2,c,y_1,y_2,D,z_1,z_2,a,\beta\rangle} \tag{49}$$

defined by

$$\bigwedge^2\begin{pmatrix} y_1 & x_1 & D & z_1\\ cx_1 & x_2 & y_2 & z_2 \end{pmatrix}=0 \quad \text{and} \quad \begin{aligned} z_1^2 &= Ay_1+bD^2,\\ z_1z_2 &= Acx_1+bDy_2,\\ z_2^2 &= Acx_2+by_2^2. \end{aligned} \tag{50}$$

This corresponds to choosing a rendition, and tokenising the features that make possible the subsequent deformation (massage based on hindsight). In detail, any monomial in f_{10} is divisible either by t_1^2 or t_2^6, giving $f_{10}=At_1^2+bt_2^6$, with $A=A_8(t_1,t_2)$ and $b=b_4(t_1,t_2)$. I multiply by t_1^2, t_1t_2, t_2^2, then substitute

$$(t_1^4,t_1^3t_2,t_1^2t_2^2)\mapsto(y_1,x_1^2,x_1x_2) \tag{51}$$

in the first summand, and

$$(t_1^2 t_2^6, t_1 t_2^7, t_2^8) \mapsto (x_2^4, x_2^2 y_2, y_2^2) \tag{52}$$

in the second. After this, I tokenise x_1^2 as cx_1 and $x_2^2 = D$.

The resulting set of 9 equations has the following two interpretations as unprojections, where I introduce a deformation parameter λ:

$$y_1 \cdot (x_2, y_2, z_2, A) \quad \text{and Tom}_1 \text{ matrix} \quad M_1 = \begin{pmatrix} \lambda & x_1 & D & z_1 \\ & x_2 & y_2 & z_2 \\ & & z_2 & -by_2 \\ & & & Ac \end{pmatrix} \tag{53}$$

and

$$x_2 \cdot (y_1, D, z_1, A) \quad \text{and Tom}_2 \text{ matrix} \quad M_2 = \begin{pmatrix} \lambda c & y_1 & D & z_1 \\ & cx_1 & y_2 & z_2 \\ & & z_1 & -bD \\ & & & A \end{pmatrix}. \tag{54}$$

The two sets of Pfaffians overlap in two equations for $y_2 z_1$ and $z_1 z_2$; putting them together and coloning out a or d or y_2 or z_1 or z_2 gives the "long equation"

$$x_2 y_1 = cx_1^2 + \lambda^2 bc. \tag{55}$$

Thus the 9 equations are

$$\begin{array}{lll} x_1 y_2 = Dx_2 + \lambda z_2, & x_1 z_2 = x_2 z_1 - \lambda b y_2, & z_1^2 + Ay_1 + bD^2, \\ x_2 y_1 = cx_1^2 + \lambda^2 bc, & y_1 z_2 = cx_1 z_1 - \lambda bcD, & z_1 z_2 + Acx_1 + bDy_2, \\ y_1 y_2 = cDx_1 + \lambda c z_1, & y_2 z_1 = Dz_2 - \lambda Ac, & z_2^2 + Acx_2 + by_2^2. \end{array} \tag{56}$$

When $\lambda \neq 0$ these eliminate z_2, leaving the codimension 3 variety generated by the Pfaffians of

$$\begin{pmatrix} 0 & x_2 & c & y_2 \\ & x_1^2 + \lambda^2 b & y_1 & \lambda z_1 + Dx_1 \\ & & Dx_1 - \lambda z_1 & \lambda^2 A \\ & & & -D^2 \end{pmatrix} \quad \text{of weights} \quad \begin{matrix} 0 & 1 & 1 & 2 \\ & 2 & 2 & 3 \\ & & 3 & 4 \\ & & & 4 \end{matrix} \tag{57}$$

4.2 Moving the base point P_∞

A parallel interpretation of the original nine equations (48) allows the other base point P_∞ to move. I keep $x_1^2 = C$ and $y_2^2 = B$ as tokens (instead of factoring them as cx_1 and by_2), but factor the quantities D and A as $D = dx_2$ and $A = ay_1$. This gives

$$\bigwedge^2 \begin{pmatrix} y_1 & x_1 & dx_2 & z_1 \\ C & x_2 & y_2 & z_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{aligned} z_1^2 &= ay_1^2 + Bdx_1, \\ z_1z_2 &= aCy_1 + Bdx_2, \\ z_2^2 &= aC^2 + By_2. \end{aligned} \tag{58}$$

The μ deformation comes from the unprojection interpretations:

$$y_2 \cdot (x_1, y_1, z_1, B) \quad \text{and Tom}_2 \text{ matrix} \quad M_1 = \begin{pmatrix} \mu & y_1 & x_1 & z_1 \\ & C & x_2 & z_2 \\ & & z_1 & -Bd \\ & & & ay_1 \end{pmatrix} \tag{59}$$

and

$$x_1 \cdot (y_2, C, z_2, B) \quad \text{and Tom}_1 \text{ matrix} \quad M_2 = \begin{pmatrix} \mu d & y_1 & dx_2 & z_1 \\ & C & y_2 & z_2 \\ & & z_2 & -B \\ & & & aC \end{pmatrix}. \tag{60}$$

The two sets of Pfaffians overlap in two equations for y_1z_2 and z_1z_2; coloning out gives the "long equation"

$$x_1y_2 = dx_2^2 + \mu^2 ad. \tag{61}$$

Thus the 9 equations are

$$\begin{matrix} x_1y_2 = dx_2^2 + \mu^2 ad, & x_1z_2 = x_2z_1 + \mu ay_1, & z_1^2 + ay_1^2 + Bdx_1, \\ x_2y_1 = Cx_1 + \mu z_1, & y_1z_2 = Cz_1 - \mu dB, & z_1z_2 + ay_1C + Bdx_2, \\ y_1y_2 = dCx_2 + \mu dz_2, & y_2z_1 = dx_2z_2 - \mu adC, & z_2^2 + aC^2 + By_2. \end{matrix} \tag{62}$$

4.3 Putting together the λ and μ deformations

My λ and μ deformation families depend on choices and assumptions that are a priori incompatible if f_{10} has a nonzero term in $t_1^5t_2^5$. Ignoring this for the moment, assume that $f_{10} = a_4t_1^6 + bt_2^6$. With a little trial and error, one checks that the λ and μ deformations (56) and (62) fit together, somewhat miraculously, with only a single $\lambda\mu$ term in the z_1z_2 equation:

$$\begin{aligned}
x_1y_2 &= dx_2^2 + \lambda z_2 + \mu^2 ad,\\
x_2y_1 &= cx_1^2 + \lambda^2 bc + \mu z_1,\\
y_1y_2 &= cdx_1x_2 + \lambda cz_1 + \mu dz_2,\\
x_1z_2 &= x_2z_1 - \lambda by_2 + \mu ay_1,\\
y_1z_2 &= cx_1z_1 - \lambda bcdx_2 - \mu bdy_2,\\
y_2z_1 &= dx_2z_2 - \lambda acy_1 - \mu acdx_1,\\
&z_1^2 + ay_1^2 + bdx_1y_2 - \lambda bdz_2,\\
&z_1z_2 + acx_1y_1 + bdx_2y_2 - \lambda\mu abcd,\\
&z_2^2 + acx_2y_1 + by_2^2 - \mu acz_1.
\end{aligned} \tag{63}$$

I assert that setting μ or λ to zero gives back the known λ and μ deformation families, and that these equations define a flat deformation over $\mathbb{A}^2_{\langle\lambda,\mu\rangle}$. To check flatness, it is enough to check that the 16 syzygies (66) hold (with $e=0$).

Finally, I deal with the missing term in $a_5t_1^5t_2^5$ in $f_{10}(t_1,t_2)$ by setting $f_{10} = a_4t_1^6 + et_1t_2 + b_4t_2^6$ where $e = a_5y_1y_2$, and render it as $L_7 \mapsto L_7 + ex_1^2$, $L_8 \mapsto L_8 + ex_1x_2$, $L_9 \mapsto L_9 + ex_2^2$ or

$$\begin{aligned}
t_1^2f_{10} &= ay_1^2 + ex_1^2 + bdx_1y_2,\\
t_1t_2f_{10} &= acx_1y_1 + ex_1x_2 + bdx_2y_2,\\
t_2^2f_{10} &= acx_2y_1 + ex_2^2 + by_2^2.
\end{aligned} \tag{64}$$

The equations become

$$\begin{aligned}
L_1 &: x_1y_2 = d(x_2^2 + \mu^2 a) + \lambda z_2,\\
L_2 &: x_2y_1 = c(x_1^2 + \lambda^2 b) + \mu z_1,\\
L_3 &: y_1y_2 = cdx_1x_2 + \lambda cz_1 + \mu dz_2 - \lambda\mu e\\
&\qquad \equiv c(dx_1x_2 + \lambda z_1) + \mu(dz_2 - \lambda e)\\
&\qquad \equiv d(cx_1x_2 + \mu z_2) + \lambda(cz_1 - \mu e),\\
L_4 &: x_1z_2 = x_2z_1 - \lambda by_2 + \mu ay_1,\\
L_5 &: y_1z_2 = (cz_1 - \mu e)x_1 - bd(\lambda cx_2 + \mu y_2),\\
L_6 &: y_2z_1 = (dz_2 - \lambda e)x_2 - ac(\lambda y_1 + \mu dx_1),\\
L_7 &: \quad z_1^2 + ay_1^2 + ex_1^2 + bdx_1y_2 - \lambda b(dz_2 - \lambda e) = 0,\\
L_8 &: z_1z_2 + acx_1y_1 + ex_1x_2 + bdx_2y_2 - \lambda\mu abcd = 0,\\
L_9 &: \quad z_2^2 + acx_2y_1 + ex_2^2 + by_2^2 - \mu a(cz_1 - \mu e) = 0.
\end{aligned} \tag{65}$$

This set of equations comes neatly from $I_0 = (L_1, L_2, L_4, L_8)$ (unchanged from (63) except for the unsurprising term ex_1x_2 in L_8) by coloning out

$x_1x_2y_1y_2$; its syzygy matrix M is

$$\begin{array}{ccccccccc}
y_1 & dx_2 & -x_1 & -\mu d & \lambda & . & . & . & . \\
cx_1 & y_2 & -x_2 & \lambda c & . & \mu & . & . & . \\
. & -z_1 & \lambda b & -y_1 & x_1 & . & \mu & . & . \\
\lambda bc & -z_2 & . & -cx_1 & x_2 & . & . & \mu & . \\
-z_1 & \mu ad & . & dx_2 & . & x_1 & . & \lambda & . \\
-z_2 & . & \mu a & y_2 & . & x_2 & . & . & \lambda \\
. & -ay_1 & . & z_1 & . & -\lambda b & -x_2 & x_1 & . \\
by_2 & . & . & z_2 & \mu a & . & . & -x_2 & x_1 \\
. & \mu ae & . & acy_1 + ex_2 & . & -by_2 & -\mu ac & z_2 & -z_1 \\
\lambda be & . & . & -bdy_2 - ex_1 & -ay_1 & . & -z_2 & z_1 & -\lambda bd \\
acy_1 + ex_2 & . & . & -\mu acd & . & z_2 & . & y_2 & -dx_2 \\
-ex_1 & . & -ay_1 & dz_2 - \lambda e & . & -z_1 & -y_2 & . & dx_1 \\
. & -ex_2 & -by_2 & -cz_1 + \mu e & -z_2 & . & cx_2 & . & -y_1 \\
. & bdy_2 + ex_1 & . & \lambda bcd & z_1 & . & -cx_1 & y_1 & . \\
-cz_1 + \mu e & . & z_2 & cdx_2 & -y_2 & . & . & . & -\mu d \\
. & dz_2 - \lambda e & -z_1 & cdx_1 & . & y_1 & \lambda c & . & .
\end{array} \tag{66}$$

(Or ad lib, apply opposite row operation to Rows 1–8 and Rows 9–16, or swap Rows i and $i+8$.) One checks that it satisfies ${}^tMJM = 0$ where $J = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ is the standard quadratic form, so that essentially the same matrix M also provides the second syzygies, giving the projective resolution

$$\mathcal{O}_X \leftarrow \mathcal{O} \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow P_4 \leftarrow 0, \tag{67}$$

with

$$\begin{aligned}
P_1 &= 2\mathcal{O}(-3) \oplus 2\mathcal{O}(-4) \oplus 2\mathcal{O}(-5) \oplus 3\mathcal{O}(-6), \\
P_2 &= 2\mathcal{O}(-5) \oplus 4\mathcal{O}(-6) \oplus 4\mathcal{O}(-7) \oplus 4\mathcal{O}(-8) \oplus 2\mathcal{O}(-10), \\
P_4 &= \mathcal{O}(-14) \quad \text{and} \quad P_3 = \mathcal{H}om(P_2, P_4) = P_2^\vee \otimes \mathcal{O}(-14).
\end{aligned} \tag{68}$$

4.4 Set $\lambda \neq 0$ and eliminate z_2

If λ is invertible, L_1 gives $z_2 = ((x_2^2 + \mu^2 a)d - x_1y_2)/\lambda$, and (65) boil down to the Pfaffians of

$$\begin{pmatrix}
\mu & x_2 & c & y_2 \\
 & x_1^2 + \lambda^2 b & y_1 & \lambda z_1 + dx_1x_2 \\
 & & z_1 & -a(\lambda y_1 + \mu dx_1) \\
 & & & \lambda e - dz_2
\end{pmatrix}, \tag{69}$$

the first 4 of which give

$$\begin{aligned}
&c(x_1^2+\lambda^2 b) - x_2y_1 + \mu z_1,\\
&\mu a(\lambda y_1+\mu dx_1) + x_2(\lambda z_1 + dx_1x_2) - (x_1^2+\lambda^2 b)y_2,\\
&\mu\lambda e - \mu dz_2 - \lambda cz_1 - cdx_1x_2 + y_1y_2,\\
&\lambda ex_2 - dx_2z_2 + ac(\lambda y_1+\mu dx_1) + y_2z_1,
\end{aligned} \tag{70}$$

whereas

$$\mathrm{Pf}_{23.45} = \mu adx_1y_1 + dx_1x_2z_1 - \lambda^2 bdz_2 - dx_1^2z_2 + \lambda ay_1^2 + \lambda z_1^2 + \lambda e(x_1^2+\lambda^2 b). \tag{71}$$

After subtracting dx_1L_4, this is divisible by λ and gives
$L_7 = \mu^2abd^2 + bd^2x_2^2 + ay_1^2 + z_1^2 + \lambda^2 be + ex_1^2$.

Similarly if μ is invertible, set $z_1 = ((x_1^2+\lambda^2 b)c - x_2y_1)/\mu$

$$\begin{pmatrix}
\lambda & x_1 & d & y_1\\
 & x_2^2+\mu^2 a & y_2 & \mu z_2 + cx_1x_2\\
 & & z_2 & -b(\mu y_2+\lambda cx_2)\\
 & & & \mu e - cz_1
\end{pmatrix} \tag{72}$$

The two matrixes have a common Pfaffian 12.45, and (after cancelling λ and μ judiciously), their Pfaffians together generate the ideal (63). Check that

$$-\lambda L[4] = x_1 \,\mathrm{Pf}_{12.34}(M_\lambda) + \mathrm{Pf}_{12.35}(M_\mu), \tag{73}$$

$$\mu L[4] = x_2 \,\mathrm{Pf}_{12.34}(M_\mu) + \mathrm{Pf}_{12.35}(M_\lambda), \tag{74}$$

$$\lambda\mu L[7] = \lambda z_1 \,\mathrm{Pf}_{12.34}(M_\mu) + (\lambda^2 b + x_1^2)\,\mathrm{Pf}_{12.45}(M_\lambda) - (y_1+dx_1)\,\mathrm{Pf}_{12.35}(M_\lambda). \tag{75}$$

`TO BE CONTINUED`

References

[BCP] Ingrid Bauer, Fabrizio Catanese and Roberto Pignatelli, The moduli space of surfaces with $K^2 = 6$ and $p_g = 4$. Math. Ann. **336** (2006) 421–438

[TJ] Gavin Brown, Michael Kerber and Miles Reid, Fano 3-folds in codimension 4, Tom and Jerry. Part I, Compositio Math **148** (2012) 1171–1194, preprint arXiv:1009.4313, 33 pp.

[CP] Fabrizio Catanese and Roberto Pignatelli, Fibrations of low genus, I. Ann. Sci. École Norm. Sup. (4) **39** (2006) 1011–1049

[CR] Alessio Corti and Miles Reid, Weighted Grassmannians, in Algebraic geometry, de Gruyter, Berlin (2002), pp. 141–163

[D] Duncan Dicks, Surfaces with $p_g = 3, K^2 = 4$ and extension-deformation theory, 1988, Warwick PhD thesis

[G] Ed Griffin, Families of quintic surfaces and curves, Compositio Math. **55** (1985) 33–62

[H1] Eiji Horikawa, On deformations of quintic surfaces Invent. Math. **31** (1975) 43–85

[H2] Eiji Horikawa, Algebraic surfaces of general type with small c_1^2. I. Ann. of Math. **104** (1976) 357–387. II. Invent. Math. **37** (1976) 121–155. III. Invent. Math. **47** (1978) 209–248. IV. Invent. Math. **50** (1978/79) 103–128 V. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981) 745–755

[R2] Miles Reid, Infinitesimal view of extending a hyperplane section—deformation theory and computer algebra, in Algebraic geometry (L'Aquila, 1988), Springer LNM **1417**, pp. 214–286

[R3] Miles Reid, Graded rings and birational geometry, in Proc. of algebraic geometry symposium (Kinosaki, Oct 2000), K. Ohno (Ed.), 1–72, get from my website + Algebraic geometry links, 3-folds

[R4] Miles Reid, Gorenstein in codimension 4 - the general structure theory, submitted to Algebraic Geometry in East Asia (Taibei Nov 2011), 17 pp. Advanced Studies in Pure Mathematics, Vol 51 or 52, 2012. Get from

`www.warwick.ac.uk/staff/Miles.Reid/3folds/structure.pdf`