Examples of Type IV unprojection

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Abstract

I show that $\mathbb{P}(2,3)$ has an embedding $\mathbb{P}(2,3) \cong \Gamma \subset \mathbb{P}(4,5,6,9)$ whose image Γ is contained in a quasismooth K3 hypersurface $X_{24} \subset \mathbb{P}(4,5,6,9)$. The pair $\Gamma \subset X_{24}$ unprojects to the codimension 4 K3 surface $Y \subset \mathbb{P}(4,5,5,6,7,8,9)$ with

$$\begin{aligned} \text{Basket} &= [\,\tfrac{1}{2}(1,1), \tfrac{1}{5}(1,4), \tfrac{1}{5}(2,3), \tfrac{1}{9}(4,5)\,]. \\ \text{Numerator} &= 1 - t^{12} - t^{13} - 2t^{14} - 2t^{15} - 2t^{16} - t^{17} \\ &\qquad\qquad + t^{19} + 2t^{20} + 3t^{21} + 4t^{2} + 3t^{23} + \cdots \end{aligned}$$

(Altınok4(111) in the Magma K3 database). The local coordinates at the third centre $P_3 = \frac{1}{5}(2,3)$ of Y are of weight 7 and 8 (rather than 2 and 3), so both are eliminated by the projection from P_3. Together with other examples, this gives substance to Type IV unprojections. Several more cases of Type IV unprojections are known up to codimension 5 or 6. The paper also contains some Magma programming routines suitable as exercises for babies.

1 Introduction

Unprojection makes new Gorenstein rings out of old. The idea is to use the adjunction formula in Serre–Grothendieck duality to construct rational functions with poles along a given divisor. Adjoining these functions is an analog of Castelnuovo's contractibility criterion in minimal models of surfaces. Type I unprojections were introduced by Kustin and Miller [KM] and later by Papadakis and Reid [PR]. They also figure prominently in birational geometry, see Corti, Pukhlikov and Reid [CPR] and [Ki], and in that context can be traced back to work on the Cremona group in the late 19th century (Geiser involutions, see [CPR], 2.6.3.). Unprojections become progressively more complicated as the divisor to be unprojected gets further from projectively Gorenstein, so that several unprojection variables have to be adjoined at the same time. Type II projections first occurred in [CPR], 4.10–4.12 and 7.3 (in which context they are related to Bertini involutions), and are studied in more detail in [Ki], Section 9. The typical case of Type III projections is the classical projection of a Fano 3-fold $V_{2g-2} \subset \mathbb{P}^{g+1}$ from a line (compare [Ki], Section 9 and Example 9.16).

This paper introduces Type IV unprojections. I don't intend to give a formal definition here. The defining feature of Type IV is that the coordinate ring of the divisor to be unprojected is a ring $A = \mathbb{C}[\Gamma]$ whose normalisation $\widetilde{A}$ is Gorenstein, and such that the quotient $\widetilde{A}/A$ is an Artinian module needing 2 linearly independent generators over A. An ideal case would be

$$A = \mathbb{C}[\Gamma] := \mathbb{C}[u^2, uv, v^2, u^3, u^2v, uv^2, v^3], \quad \text{with} \quad \widetilde{A} = \mathbb{C}[u, v]. \tag{1.1}$$

Here $\widetilde{A}$ needs $1, u, v$ as generators over A, and u, v are linearly independent. I do not treat this case here, because any example with it would have codimension ≥ 4 unprojecting to codimension ≥ 7; see however Section 6

The Type IV unprojections studied in this paper add 3 new variables as generators to the ambient ring, so potentially increase the minimal number of generators by 3 (hence also the codimension). But this increase is masked if some of the old variables become expressible in terms of the new; this frequently happens in small codimension. The K3 database in Magma (export 2.8, [Ma]) reports 5 families of K3 surfaces in codimension 4 having a Type IV unprojection to codimension 1 (see 7.3 and compare [ABR]); there are also 6 families in codimension 5 having a Type IV unprojection to codimension 2 and several similar things in codimension 6.

2 Embedding of Γ and quasismoothness of X_{24}

Write u, v for coordinates on $\mathbb{P}(2,3)$ and x, y, z, t for $\mathbb{P}(4,5,6,9)$. Up to change of coordinates, the general map $i\colon \mathbb{P}(2,3) \to \mathbb{P}(4,5,6,9)$ is

$$i : (u, v) \mapsto (u^2, uv, u^3 + v^2, u^3v + v^3). \tag{2.1}$$

Choosing the 4th entry $t = u^3v + v^3$ (so that $t = vz$) tidies up later calculations. It is an embedding because monomials in $(u^2, uv, u^3 + v^2, u^3v + v^3)$ span the vector space $H^0(\mathbb{P}(2,3), \mathcal{O}(i))$ for all $i \gg 0$ (in fact for every $i \geq 13$, see 7.1).

The image $\Gamma = i(\mathbb{P}(2,3))$ is defined by

$$\begin{array}{llll} f_{18}: & x^3z + xyt - z^3 + t^2, & q_{21}: & x^3t - xyz^2 + y^3z, \\ g_{19}: & x^2yz - xzt + y^2t, & q_{22}: & -xt^2 + y^2z^2, \\ h_{20}: & x^5 - x^2z^2 + 2xy^2z - y^4, & q_{23}: & -x^2zt + yz^3 - yt^2 \end{array} \tag{2.2}$$

(see 7.1). For reasons of degree, only the first 3 of these can take part in the equation of X_{24}: set

$$F_{24} = xh_{20} + yg_{19} + zf_{18}. \tag{2.3}$$

One checks by brute force (see 7.1) that F defines a quasismooth hypersurface $X_{24} \subset \mathbb{P}(4,5,6,9)$.

3 Resolution of $\mathbb{C}[u,v]$ as a $\mathbb{C}[x,y,z,t]$ module

Write $A = \mathbb{C}[x,y,z,t]$ for the homogeneous coordinate ring of $\mathbb{P}(4,5,6,9)$ and $\mathbb{C}[\Gamma] = A/I_\Gamma$ for that of $\Gamma \subset \mathbb{P}(4,5,6,9)$; by construction, its normalisation is the coordinate ring $\mathbb{C}[u,v]$ of $\mathbb{P}(2,3)$. Thus the embedding i of (2.1) makes $\mathbb{C}[u,v]$ a module over the polynomial ring $\mathbb{C}[x,y,z,t]$, with

$$\begin{gathered} x\cdot 1 = u^2, \quad y \cdot 1 = uv, \quad z\cdot 1 - x\cdot u = v^2, \quad x \cdot u = u^3, \\ x\cdot v = u^2v, \quad y\cdot v = uv^2, \quad z\cdot v - x^2\cdot 1 = v^3, \quad \text{etc.} \end{gathered} \tag{3.1}$$

This already shows that $\mathbb{C}[u,v]$ is generated as a $\mathbb{C}[x,y,z,t]$-module by the 3 elements $(1,u,v)$. The relations between these generators are

$$(v,u,1)M = 0, \tag{3.2}$$

where

$$M = \begin{pmatrix} -x & -y & -z & -t & 0 & 0 \\ y & z & 0 & -xz & t & x^2 \\ 0 & -x^2 & t & z^2 & -yz & -xz+y^2 \end{pmatrix} \tag{3.3}$$

(see 7.2). The 3×6 matrix M is homogeneous with entries of weight

$$\begin{pmatrix} 4 & 5 & 6 & 9 & 8 & 7 \\ 5 & 6 & 7 & 10 & 9 & 8 \\ 7 & 8 & 9 & 12 & 11 & 10 \end{pmatrix} \tag{3.4}$$

and satisfies

$$MJ\,{}^tM = 0, \quad \text{where} \quad J = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}. \tag{3.5}$$

Now $\mathbb{C}[u,v]$ is a Gorenstein module of codim 2 over $\mathbb{C}[x,y,z,t]$, and one checks that its resolution is

$$\mathbb{C}[u,v] \longleftarrow L_0 \xleftarrow{M} L_1 \xleftarrow{J\,{}^tM} L_2 \leftarrow 0, \tag{3.6}$$

where

$$
\begin{aligned}
L_0 &= A \oplus A(-2) \oplus A(-3),\\
L_1 &= A(-7) \oplus A(-8) \oplus A(-9) \oplus A(-12) \oplus A(-11) \oplus A(-10),\\
L_2 &= A(-16) \oplus A(-17) \oplus A(-19).
\end{aligned}
$$

4 Maps between complexes

I find the unprojection variables by comparing the resolutions of $\mathbb{C}[X] = A/(F_{24})$ and $\mathbb{C}[u,v]$. For this, consider first the diagram including part of the resolution of $\mathbb{C}[\Gamma]$:

$$
\begin{array}{ccccccccc}
\mathbb{C}[X] & \leftarrow & A & \leftarrow & A(-24) & \leftarrow & 0 & & \\
\downarrow & & \| & & \downarrow & & & & \\
\mathbb{C}[\Gamma] & \leftarrow & A & \leftarrow & K_1 & \leftarrow & \cdots & & \\
\cap & & \cap & & \downarrow & & & & \\
\mathbb{C}[u,v] & \leftarrow & L_0 & \leftarrow & L_1 & \leftarrow & L_2 & \leftarrow & 0
\end{array}
\tag{4.1}
$$

Here $K_1 = A(-18) \oplus \cdots \oplus (-23)$ is the free module corresponding to the 6 generators (2.2) of the ideal I_Γ. The first downarrow $A(-24) \to K_1$ is the matrix $(z, y, x, 0, 0, 0)$ that expresses F_{24} as the combination (2.3) of f_{18}, g_{19}, h_{20}. For any $f \in I_\Gamma$, the column vector $(0, 0, f) \in \ker\{L_0 \to \mathbb{C}[u,v]\}$, and is therefore hit by some element of L_1. The second downarrow is the 6×6 matrix expressing the 6 generators (2.2) of K_1 as linear combinations of the 6 columns of M. This matrix is presumably implicit in the Gröbner basis calculation that eliminates u, v. Because F_{24} is a linear combination of only the first three generators (2.3), I only need the corresponding three columns:

$$
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ f_{18} & g_{19} & h_{20} \end{pmatrix} = MN, \tag{4.2}
$$

where

$$N = \begin{pmatrix} 0 & 0 & x^2y+xt \\ -xz & 0 & -x^3 \\ t+xy & 0 & 0 \\ -z & 0 & -x^2 \\ 0 & -x^2 & -xy \\ 0 & t & -y^2 \end{pmatrix}. \tag{4.3}$$

Composing gives

$$\begin{array}{ccccccccc} \mathbb{C}[X] & \leftarrow & A & \leftarrow & A(-24) & \leftarrow & 0 & & \\ \downarrow & & \cap & & \downarrow & & & & \\ \mathbb{C}[u,v] & \leftarrow & L_0 & \leftarrow & L_1 & \leftarrow & L_2 & \leftarrow & 0, \end{array} \tag{4.4}$$

where the downarrow takes the basis F_{24} to

$$N \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} x^3y + x^2t \\ -x^4 - xz^2 \\ xyz + zt \\ -x^3 - z^2 \\ -2x^2y \\ -xy^2 + yt \end{pmatrix} \in L_1(24). \tag{4.5}$$

This allows me to start writing down the unprojection Y of Γ in X. The theory is similar to that of Kustin and Miller [KM], and Papadakis and Reid [PR] and [Ki]: the adjunction formula $\omega_\Gamma = \mathcal{E}xt^1_X(\mathcal{O}_\Gamma, \omega_X)$ gives an exact sequence

$$0 \to \omega_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_\Gamma, \omega_X) \to \omega_\Gamma \to 0. \tag{4.6}$$

Note that I abuse notation by writing X, Γ in place of the punctured affine cones over them. The modules and Homs between them are really the Serre modules of sheaves on X, Γ. For example, $\Gamma \cong \mathbb{P}^1$, but as $\mathbb{P}(2,3)$ it has $\omega_\Gamma = \mathcal{O}(-5)$, since on the punctured affine cone over Γ, the dualising sheaf twisted by $\mathcal{O}(5)$ is based by $\mathrm{d}u \wedge \mathrm{d}v$.

Moreover since $\mathbb{C}[\Gamma] \hookrightarrow \mathbb{C}[u,v]$ is an isomorphism outside the origin, the dualising module $\omega_{\mathbb{C}[\Gamma]} = \omega_{\mathbb{C}[u,v]}$ is isomorphic to $\mathbb{C}[u,v]$, so is free of rank 1 as a $\mathbb{C}[u,v]$ module; however, as a module over $\mathbb{C}[\Gamma]$ or $\mathbb{C}[X]$ or $\mathbb{C}[x,y,z,t]$, it needs 3 generators, corresponding to $1, u, v$. I used the fact that $\mathbb{C}[u,v]$ is a Gorenstein module to derive the resolution (3.6); it is best not to enquire too closely about the resolution of $\mathbb{C}[\Gamma]$.

Now the homogeneous coordinate ring $\mathbb{C}[Y]$ of the unprojected variety Y is obtained from $\mathbb{C}[X]$ by adjoining rational functions s_0, s_1, s_2 with poles along Γ, viewed as homomorphisms $\mathcal{I}_\Gamma \to \omega_X \cong \mathcal{O}_X$. Here $s_0 \in \mathcal{H}om(\mathcal{I}_\Gamma, \omega_X(5))$ is the rational form with pole along Γ whose Poincaré residue is a basis of $\omega_\Gamma(5) \cong \mathcal{O}_{\mathbb{P}(2,3)}$. As in the proof of Castelnuovo's criterion, the graded ring with s_0 adjoined already contracts Γ to a point. The elements s_1, s_2 that map to u, v times the generator of $\omega_\Gamma(5)$ under $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_\Gamma, \omega_X) \twoheadrightarrow \omega_\Gamma$ are required to make the image projectively normal.

I choose s_0 to map to $1 \in \mathbb{C}[u,v]$, and s_1, s_2 to $u \cdot 1, v \cdot 1$ respectively; since $\omega_{\mathbb{P}(2,3)} = \mathcal{O}(-5)$ and $\omega_X = \mathcal{O}_X$,

$$\deg s_0 = 5, \quad \text{and} \quad \deg s_1, s_2 = 7, 8. \tag{4.7}$$

The *linear* relations between s_0, s_1, s_2 come from (4.4). Namely (essentially as in [KM], see also [PR], end of Section 1), s_0, s_1, s_2 correspond to the generators of L_2 or the rows of M, and $J\,{}^tM \colon L_2 \to L_1$ must map them to the image of F_{24} under the downarrow. This gives the equations

$$J\,{}^tM \begin{pmatrix} s_2 \\ s_1 \\ s_0 \end{pmatrix} = N \begin{pmatrix} z \\ y \\ x \end{pmatrix},$$

or spelled out:

$$\begin{pmatrix} -t & -xz & z^2 \\ 0 & t & -yz \\ 0 & x^2 & -xz+y^2 \\ x & -y & 0 \\ y & -z & x^2 \\ z & 0 & -t \end{pmatrix} \begin{pmatrix} s_2 \\ s_1 \\ s_0 \end{pmatrix} = \begin{pmatrix} x^3y + x^2t \\ -x^4 - xz^2 \\ xyz + zt \\ -x^3 - z^2 \\ -2x^2y \\ -xy^2 + yt \end{pmatrix}. \tag{4.8}$$

I write $R_1, \dots, R_6$ for these relations. The equations f, g, h of Γ appear naturally 3×3 minors of $J\,{}^tM$ (or 2×2 minors of defective 2×3 blocks with zeros down one column) so that $R_1, \dots, R_6$ can be solved by linear algebra to give expressions for (f, g, h) times s_2, s_1, s_0. For example

$$\begin{aligned} fs_0 &= -3x^3yz + x^2y^3 - x^2zt - yt^2, \\ gs_0 &= x^6 + x^3z^2 + xyzt + zt^2, \\ hs_0 &= -x^5y + 2x^2yz^2 - xy^3z + xz^2t - y^2zt. \end{aligned}$$

This makes explicit that s_0 is a homomorphism $\mathcal{I}_\Gamma \to \mathcal{O}_X$. But it is much more useful to use $R_1, \dots, R_6$ as they are, without taking determinants.

5 The quadratic relations for s_1^2, s_1s_2, s_2^2

Quadratic relations in s_1, s_2 must exist for several reasons. One is that the ring extension $\mathbb{C}[X] \subset \mathbb{C}[Y]$ can be obtained by adjoining s_0 first, followed by the normalisation $\mathbb{C}[X][s_0] \subset \mathbb{C}[Y]$; the second step is integral, and one guesses that $\mathbb{C}[Y]$ is generated by $1, s_1, s_2$ as a module over $\mathbb{C}[X][s_0]$. Or, since s_1, s_2 correspond in some sense to us_0 and vs_1 on taking Poincaré residue to Γ, and since, restricted to Γ, $x = u^2, y = uv, z = v^2 + xu$, there must be relations S_0, S_1, S_2 saying that

$$\begin{aligned} s_1^2 - xs_0^2 &\in \left\langle xys_0, ts_0, zs_2, x^2z, xy^2, yt \right\rangle, \\ s_1s_2 - ys_0^2 &\in \left\langle xzs_0, y^2s_0, x^2s_1, xyz, y^3, zt \right\rangle, \\ s_2^2 - zs_0^2 + xs_0s_1 &\in \left\langle yzs_0, xys_1, ts_1, x^4, xz^2, y^2z \right\rangle, \end{aligned} \tag{5.1}$$

where the right-hand side just lists alll monomials of degree 14, 15 and 16.

To find these relations explicitly, I proceed as follows. Since $MJ\,{}^t\!M = 0$ and the first two rows of MN are zero by (4.2), the first two rows of M provide two syzygies between the 6 rows $R_1, \dots, R_6$ of (4.8). I massage the second very slightly to make them both 4-term syzygies:

$$\begin{aligned} xR_1 + yR_2 + zR_3 + tR_4 &\equiv 0, \quad \text{and} \\ yR_1 + z(R_2 - xR_4) + tR_5 + x^2R_6 &\equiv 0. \end{aligned} \tag{5.2}$$

This suggests realising the relations as 4×4 Pfaffians of 5×5 skew matrixess, as follows:

$$\begin{pmatrix} x & y & z & t \\ & z & s_1 & ys_0 - xz \\ & & s_2 + x^2 & -xs_1 + zs_0 \\ & & & x^3 \end{pmatrix}, \quad \begin{pmatrix} y & z & x^2 & t \\ & s_0 + y & s_1 & s_2 \\ & & s_2 + x^2 & -xy \\ & & & xs_1 - zs_0 \end{pmatrix}. \tag{5.3}$$

My convention is to write only the upper triangular terms $a_{12}, a_{13}, \dots, a_{45}$ of a skew matrix. The Pfaffians are defined by $\mathrm{Pf}_{ij.kl} = a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk}$ (up to $\pm$).

The second matrix has first four Pfaffians $R_5, R_6, R_2 - xR_4, R_1$, and the fifth is one of the required quadratic relations:

$$S_2: \; s_2^2 - zs_0^2 + xs_0s_1 = yzs_0 - 2xys_1 - x^2s_2. \tag{5.4}$$

The first matrix gives R_1, R_2, R_3, R_4 and

$$\mathrm{Pf}_{23.45} = xs_1^2 - zs_0s_1 + ys_0s_2 - x^2ys_0 - xzs_2, \tag{5.5}$$

which is not quite what we want. However, subtracting s_0R_5 gives a relation that is identically divisible by x, and taking out the factor gives

$$S_0 : \ s_1^2 - xs_0^2 = xys_0 + zs_2. \tag{5.6}$$

The equation for s_1s_2 can be found by constructing an explicit syzygy, for example by taking $s_1R_4 + yS_0 + zR_5$, that has xs_1s_2 as one term, and all terms of which are divisible by x. However, this is again a 4-term syzygy, once again best understood by realising the relations as Pfaffians:

$$\begin{pmatrix} x & y & z & s_1 \\ & z & s_1 & s_2 \\ & & s_2 + x^2 & -xs_0 - 2xy \\ & & & -s_0^2 - ys_0 \end{pmatrix}. \tag{5.7}$$

Thus

$$S_1 : \ s_1s_2 - ys_0^2 = (y^2 - xz)s_0 - x^2s_1 - 2xyz. \tag{5.8}$$

In fact, all the relations $R_1, \dots, R_5, S_0, S_1, S_2$ (but, annoyingly, not R_6) can be written together as the 4×4 Pfaffians of the following 6×6 matrix:

$$\begin{pmatrix} x & y & z & s_1 & t \\ & z & s_1 & s_2 & ys_0 - xz \\ & & s_2 + x^2 & -xs_0 - 2xy & -xs_1 + zs_0 \\ & & & -s_0^2 - ys_0 & x^3 \\ & & & & s_0s_2 + x^2s_0 + x^2y + xt \end{pmatrix}. \tag{5.9}$$

6 Another example

Consider the following two families of K3 surfaces:

(a) The codimension 2 complete intersection $X_{6,6} \subset \mathbb{P}(2,2,2,3,3)$, known to its friends as Fletcher2(14), with $9 \times \frac{1}{2}(1,1)$ singularities on the plane $\mathbb{P}(2,2,2) = \mathbb{P}^2$.

(b) The codimension 5 K3 surface $Y \subset \mathbb{P}(2,2,2,2,3,3,3,3)$ in symmetric determinantal format, with 10 nodes. Write $x_1, \dots, x_4, y_1, \dots, y_4$ for coordinates, and let M be a general 4×4 symmetric matrix of linear forms in $x_1,\dots,x_4$. The ideal of Y is generated by the 14 equations

$$M\begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\end{pmatrix}=0,\quad y_iy_j=(M^\dagger)_{ij};$$

here $M^\dagger$ is the adjoint matrix of signed maximal minors. Y is the determinantal hypersurface $Y:(\det M=0)\subset\mathbb{P}^3=\mathbb{P}(2,2,2,2)$, and the symmetric matrix M is the resolution of an ample divisorial sheaf $\mathcal{O}_Y(A)$ (singular at the 10 nodes, where $\operatorname{rank} M=2$), such that the $x_i\in H^0(\mathcal{O}_Y(2A))$, $y_i\in H^0(\mathcal{O}_Y(3A))$.

Then Y has a projection of Type IV from any node P to a codimension 2 complete intersection $X_{6,6}\subset\mathbb{P}(2,2,2,3,3)$.

Working top down from Y, I choose coordinates so that the node $P=(1,0,0,0)$ in $\mathbb{P}(2,2,2,2)$, and y_1,y_2 are local coordinates at P (compare [CPR], 3.4 for local coordinates on quotient singularities and their fractional divisor of zeros on a blowup). The geometric meaning of projection is that I blow up P to give an exceptional -2-curve E, and consider the blown up surface polarised by the new Weil divisor $A-\frac{1}{2}E$. This eliminates x_1 because it does not vanish on E; moreover, it eliminates y_1,y_2 because these have weight 3 in the graded ring, but vanish on E with multiplicity exactly $\frac{1}{2}$ (see [CPR], 3.4). This is the mark of a Type IV projection. It is an exercise to make the appropriate preliminary deduction about M at P, and to do the elimination to get two equations of degree 6 in the remaining variables.

Now work bottom up from X. As in Section 2, one checks that

$$\mathbb{P}^1\to\mathbb{P}(2,2,2,3,3)\quad\text{given by}\quad(u,v)\mapsto(u^2,uv,v^2,u^3,v^3)\tag{6.1}$$

(or a more general pair of cubics if you prefer) is an embedding, that the image is contained in the obvious conic $q=x_1x_3-x_2^2$ and in the 3 sextics $f=x_1^3-y_1^2$, $g=x_1x_2x_3-y_1y_2$, $h=x_3^3-y_2^2$, and that two general sextic linear combinations of q,f,g,h define a quasismooth K3 surface $X_{6,6}$.

I can resolve $\mathbb{C}[u,v]$ as a module over $\mathbb{C}[\underline{x},\underline{y}]$ as in (3.6), obtaining

$$\mathbb{C}[u,v]\longleftarrow L_0\xleftarrow{P}L_1\xleftarrow{M}L_2\xleftarrow{{}^tP}L_3\leftarrow 0,\tag{6.2}$$

where

$$\begin{aligned}
L_0 &= A \oplus 2A(-2),\\
L_1 &= 4A(-3) \oplus 5A(-4),\\
L_2 &= 5A(-6) \oplus 4A(-7),\\
L_3 &= 2A(-9) \oplus A(-10).
\end{aligned}$$

Here M is a skew 9×9 matrix with generic rank 6. Next, following the argument of (4.1–4.4), I can compose two maps between complexes to map the Koszul complex of X to

$$\begin{array}{ccccccccc}
A & \leftarrow & A(-6) \oplus A(-6) & \leftarrow & A(-12) & \leftarrow & 0 & & \\
\cap & & \downarrow & & \downarrow & & & & \\
L_0 & \leftarrow & L_1 & \leftarrow & L_2 & \leftarrow & L_3 & \leftarrow & 0,
\end{array} \tag{6.3}$$

where the final downarrow defines an element col $\in L_2(12)$ analogous to (4.5). The 3 unprojection variables x_4, y_3, y_4 in degrees $2, 3, 3$ correspond to the basis of L_3, and as in (4.8), the linear relations between them are in the Kustin–Miller form ${}^tP \begin{pmatrix} y_4 \\ y_3 \\ x_4 \end{pmatrix} = \text{col}$. I believe that the quadratic relations for y_3^2, y_3y_4, y_4^2 can be worked out as in Section 5.

This example is instructive: although both families of K3 surfaces are well understood, the equations describing the birational map between them are very subtle, and would require a lot of computation to elucidate.

7 Magma routines

7.1 Embedding Γ and quasismoothness of X_4

The first thing is to make the polynomial ring $S = \mathbb{Q}[u, v, x, y, z, t]$ and define the ideal of the graph of the embedding $\mathbb{P}(2,3) \to \mathbb{P}(4,5,6,9)$:

```
> Q := Rationals();
> S<u,v,x,y,z,t> := PolynomialRing(Q,[2,3,4,5,6,9]);
> I := Ideal([-x+u^2,-y+u*v,-z+v^2+u^3,-t+u^3*v+v^3]);
```

(you can print I by doing "I;" at the prompt, and similarly throughout).

Now eliminate the first two generators to get the ideal of the image Γ:

```
> J := EliminationIdeal(I,2);
> [WeightedDegree(Basis(J)[i]) : i in [1..#Basis(J)]];
```

answer: $[20, 18, 19, 21, 22, 23]$. This says that the 1st, 2nd and 3rd basis elements have weight $20, 18, 19$. (At this point I used the output of "J;" to write (2.2).) I make their generic linear combination of weight 24 by hand:

```
> h := Basis(J)[1]; g := Basis(J)[3]; f := Basis(J)[2];
> F0 := x*h + y*g + z*f; IsHomogeneous(F0);
```

answer: true.

Remark 7.1 Taking generic values such as $1, 1, 1$ for coefficients in a linear system and jiggling them if necessary is a computer algebra substitute for Bertini's theorem. Quasismooth is an open condition, so if it ever holds, we would be infinitely unlucky to happen on a singular guy. In fact both $F_0 = xh + yg - zf$ and $xh + yg$ give singular X_{24}.

To work in the polynomial subring $R = k[x, y, z, t]$, the type checking in Magma insists that I force the elements x, y, z, t into it explicitly (otherwise it has no way of knowing that the user wants to identify the variables in R and S just because they have the same human names). One way is to set up a homomorphism $\varphi\colon S \to R$ taking the generators $u, v \mapsto 0, 0$ and $x, y, z, t \mapsto X, Y, Z, T$:

```
> R<X,Y,Z,T> := PolynomialRing(Q,[4,5,6,9]);
> fie := hom< S -> R | 0,0,X,Y,Z,T >;
> F := fie(F0);
```

Now I define the hypersurface $X_{24} : (F = 0) \subset \mathbb{P}(4, 5, 6, 9)$ and ask the big question:

```
> PP:=Proj(R); X:=Scheme(PP,F);
> IsNonSingular(X);
```

answer: true.

For completeness, I prove that $\mathbb{P}(2, 3) \to \Gamma$ is an isomorphism. For this, I check that monomials of weight i in $\mathbb{C}[X, Y, Z, T]$ map to polynomials in u, v that span the vector space of polynomials of weight i:

```
> RR<U,V> := PolynomialRing(Q,[2,3]);
> psi := hom< R -> RR | U^2, U*V, U^3+V^2, U^3*V+V^3 >;
> for i in [12..24] do
>   K := ideal< RR | psi(MonomialsOfWeightedDegree(R,i)) >;
>   [m in K : m in MonomialsOfWeightedDegree(RR,i)];
> end for;
```

the last line returns the list of Boolean value (is $\psi(m) \in K$?) for every monomial in X, Y, Z, T of degree $i \in [12, \dots, 24]$. The answer [true, false, false], [true, true], etc., says that the 2 monomials U^3 and V^2 in degree 12 fail the test, and every other monomial passes.

7.2 Resolution of $\mathbb{C}[u, v]$

The problem is to treat $\mathbb{C}[u, v]$ as a module over the ring $\mathbb{C}[x, y, z, t]$, where the action is as in (3.1). The following is based on a Magma routine written by Allan Steel and Gavin Brown. We treat $\mathbb{C}[u, v]$ as the quotient

$$S/(x - u^2, y - uv, z - u^3 - v^2, t - u^3 v - v^3), \tag{7.1}$$

where $S = \mathbb{C}[u, v, x, y, z, t]$ (that is, the coordinate ring of the graph of i). It is certainly generated by $1, u, v$ over the big ring S (in fact, by 1 only), and it is trivial to calculate the matrix of relations:

$$\begin{pmatrix} u & -1 & 0 \\ v & 0 & -1 \\ x - u^2 & 0 & 0 \\ y - uv & 0 & 0 \\ z - u^3 - v^2 & 0 & 0 \\ t - u^3 v - v^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ u \\ v \end{pmatrix} = 0. \tag{7.2}$$

Define $L \subset F$ to be the submodule of the free module $F = S^3$ generated by the rows of this matrix, and Gröbnerise it. In Magma Gröbner is a transitive verb that takes as its object a submodule of a free module.

```
> Q := Rationals(); // Omit this line at your peril.
> S<u,v,x,y,z,t>:=PolynomialRing(Q,[2,3,4,5,6,9]);
> Free := Module(S,3);
> L:=sub< Free | [ [u, -1, 0], [v, 0, -1], [-u^2+x, 0, 0],
>     [-u*v+y, 0, 0], [-u^3-v^2+z, 0, 0],
>     [-u^3*v-v^3+t, 0, 0] ] >;
> Groebner(L);
```

Gröbnerising does many row operations on the matrix in (7.2), to put the dependence on u at the top and on the right, then the same for v. It gives the new basis of L:

$$\text{Basis}(L) = \begin{pmatrix} -y & 0 & u \\ -x & u & 0 \\ u & -1 & 0 \\ -z & x & v \\ -y & v & 0 \\ v & 0 & -1 \\ -xz+y^2 & x^2 & 0 \\ x^2 & -z & y \\ -z^2 & xz & t \\ 0 & xt^2-y^2z^2 & 0 \\ xt & -yz & 0 \\ 0 & -y & x \\ -z^2t^2 & y^2z^3 & t^3 \\ yz & -t & 0 \\ -t & 0 & z \end{pmatrix} \tag{7.3}$$

The rows not involving u, v generate the submodule $N \subset F = S^3$ consisting of all linear relations not involving u, v in the coefficients.

```
> L1 := [f : f in Basis(L) | &and[Degree(f[i],u) le 0 and
>   Degree(f[i],v) le 0 : i in [1..3] ]];
> N := sub< Free | L1 >;
> MinimalBasis(N);
```

answer: the minimal basis of N is the matrix (3.3), up to cosmetic changes.

7.3 The K3 Database in Magma

Export 2.8 of the computer algebra system Magma [Ma] includes the first public version of Gavin Brown's K3 surface database, which is largely a reworking in computer language of the calculations of Anthony Iano-Fletcher and Selma Altınok's Warwick theses [Fl], [A]. It lists families of polarised K3 surfaces in searchable form. Further mathematical presentation, and more computer documentation will be available soon (see [ABR]). Here I only illustrate how to use the database to find the projections of Type IV mentioned in the introduction.

```
> D:=K3Database("t");
> #D;
```

answer: the database D contains 391 elements. The first line defines D to be the K3Database, and requests that Magma refers to the variable in Hilbert series by its human name t, rather than its preferred internal representation \$.1. (You can put empty brackets () if you like, but if you omit the brackets, Magma will just sulk and refuse to play.)

```
> #[ X : X in D | Codimension(X) eq 4 ];
```

answer: 142. Note the logic of Magma's very useful and flexible sequence construction [ ]. Here it constructs the sequence of $X \in D$ such that $\operatorname{codim} X = 4$. It runs through all $X \in D$ (the entry after the colon), evaluates the Boolean value of the statement following the bar | (is $\operatorname{codim} X = 4$?), and returns X (the entry before the colon) if the answer is yes. We could ask it to return any function of X by putting $f(X)$ before the colon. Asking for the number of items in a list #[ ] is usually preferable to asking for the list itself, which may be huge and uninformative. An alternative is to give the list a name, and ask to see just a bit of it:

```
> cod4 := [ X : X in D | Codimension(X) eq 4 ];
> cod4[1..5];
```

To play with projections, we first ask Magma to calculate the possible projection of Type I, II and IV of each element of the database D from each of its singularities:

```
> Centres(~D);
> Cod4Type4 := [ X : X in D | Codimension(X) eq 4 \
>  and &or[ p[2] eq [4] : p in Centres(X)] ];
```

The last line here defines Cod4Type4 to be the sequence of $X \in D$ having $\operatorname{codim} X = 4$ and such that at least one of the centres p of X has Type IV. The internal [ ] constructs a sequence of Boolean values (for p a singularity of X, is the projection from p of Type IV?) and the &or[ ] evaluates "or". To write this code you need to know a little about how the database stores its data. You can find out by interrogating it. For example,

```
> X := D[293]; Centres(X);
```

tells you that No. 293 in D has a singularity of type $\frac{1}{5}(2,3)$ with a projection of Type I, etc., and you figure out that each centre p is stored as a sequence of entries of which the second $p[2]$ is the type of the projection. I ask how many elements does Cod4Type4 contain, and what are the first two elements?

```
> #Cod4Type4;
> Cod4Type4[1..2];
```

answer: 5 elements. For the first, see below. The second is the example treated in Sections 2–5 above. All five of these K3s are from Selma Altınok's thesis. For historical reasons, the database lists the codimension 4 K3s as Altinok4(−24) through to Altinok4(121), missing Nos. 0, 76, 81, 107. The 5 elements of Cod4Type4 are listed as Altinok4(84), (111), (53), (−11) and (−6).

Example 7.2 The first element of the above list is Altinok4(84), the codimension 4 K3 surface $Y \subset \mathbb{P}(5,5,6,7,8,9,11)$ with basket $2\times\frac{1}{5}(2,3)$, $\frac{1}{11}(5,6)$ and Hilbert numerator

$$1 - t^{14} - t^{15} - 2t^{12} - t^{17} - 2t^{18} - t^{19} - t^{20} + \cdots$$

The database says that this has a projection of Type IV to a surface having the same numerical type as $X_{66} \subset \mathbb{P}(5,6,22,33)$, but containing a copy of $\mathbb{P}(2,3)$. This could happen because X_{66} is actually a hypersurface of $\mathbb{P}(5,6,22,33)$ containing the image of a general embedding $\mathbb{P}(2,3) \hookrightarrow \mathbb{P}(5,6,22,33)$; or X_{66} could itself be a bit degenerate, for example, with the curve $\mathbb{P}(2,3)$ as a base locus of some monogonal linear system. It would be interesting to see this and the other cases in the list worked out in the spirit of this paper.

Remark 7.3 We expect subsequent exports of Magma to contain substantial improvements to the internal structures of the K3 database, its contents, and its user interface. Whereas at present we use Magma mainly as a bookkeeping device to handle large quantities of combinatorial data, we hope that in future we can use it to automate many of the computations in commutative and homological algebra involved in the geometry of K3 surfaces and Fano 3-folds, including the calculations above and those in [CPR] and [Ki].

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