Anyone know these guys?

Gavin Brown and Miles Reid

We observe that some of our diptych varieties have a beautiful description in terms of key 5-folds $V(k) \subset \mathbb{A}^{k+5}$ that are *almost homogeneous spaces.* By poetic licence, we could call them *quantum rational normal curves.* We write them out here, in the hope that someone can inform us that they occur elsewhere in math or sci-fi.

From the point of view of equations, $V(k)$ are serial unprojections or non-general crazy-Pfaffians. In geometry, they are almost homogeneous spaces for a central extension group G of $\mathrm{GL}(2)$ – in other words, closed orbits for a highest weight vector (in a slightly nonfamiliar representation of a nonsimple group).

The variety $V(k)$ in equations

We define 5-folds $V(k) \subset \mathbb{A}^{k+5}\left\langle x_{0\dots k}, a, b, c, z\right\rangle$ for each $k \geq 3$. First set up $2 \times k$ and $k \times (k-2)$ matrixes

$$M = \begin{pmatrix} x_0 & \dots & x_{i-1} & \dots & x_{k-1} \\ x_1 & \dots & x_i & \dots & x_k \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} a & & & & & \\ b & a & & & & \\ c & b & a & & & \\ & \vdots & & & \vdots & \\ & & & c & b & a \\ & & & & c & b \\ & & & & & c \end{pmatrix}$$

Our variety $V(k)$ is defined by two sets of equations (see below)

$$\text{(I)} \quad MN = 0 \quad \text{and} \qquad \text{(II)} \quad \bigwedge^2 M = z \cdot \bigwedge^{k-2} N. \tag{1}$$

(I) is a recurrence relation

$$ax_{i-1} + bx_i + cx_{i+1} = 0 \quad \text{for } i = 1, \dots, k-1.$$

(II) is a $(k-2)\times k$ adaptation of Cramer's rule giving the Plücker coordinates of the space of solutions of (I) up to a scalar factor z.

The ordering of minors in (II) is best understood in terms of the guiding cases

$$x_{i-1}x_{i+1} - x_i^2 = a^{i-1}c^{k-i-1}z \quad \text{and} \quad x_{i-1}x_{i+2} - x_i x_{i+1} = a^{i-1}bc^{k-i-2}z. \quad (2)$$

Note that the maximal $(k-2)\times(k-2)$ minors of N include a^{k-2} (delete the last two row) and c^{k-2} (delete the first two). More generally, deleting two adjacent rows $i-1, i$ gives $a^{i-1}c^{k-i-1}$ as a minor (only the diagonal contributes), whereas deleting two rows $i-1, i+1$ gives the minor $a^{i-1}bc^{k-i-2}$.

Thus our second set of equations is

$$x_{i-1}x_{j+1} - x_i x_j = zN(i-1, j).$$

Relations for $x_i x_j - x_k x_l$ for all $i + j = k + l$ can be obtained as combinations of these; for example

$$\begin{aligned} x_{i-1}x_{j+2} - x_{i+1}x_j &= x_{i-1}x_{j+2} - x_i x_{j+1} + x_i x_{j+1} - x_{i+1}x_j \\ &= zN(i-1, j+1) + zN(i, j). \end{aligned}$$

Theorem 1 *For $k \geq 3$, (I) and (II) define a reduced irreducible Gorenstein 5-fold*

$$V(k) \subset \mathbb{A}^{k+5}\langle x_{0\dots k}, a, b, c, z\rangle.$$

Also for $k = 2$, with (II) involving the 0×0 minors interpreted as the single equation $1 \cdot z = x_0x_2 - x_1^2$.

Lemma 2 *(i) z is a regular element for $V(k)$.*

(ii) The section $z = 0$ of $V(k)$ is the quotient of the hypersurface

$$\widetilde{W} : (g := au^2 + buv + cv^2 = 0) \subset \mathbb{A}^5\langle a, b, c, u, v\rangle$$

by the μ_k action $\frac{1}{k}(0,0,0,1,1)$. It is Gorenstein because

$$\frac{\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c \wedge \mathrm{d}u \wedge \mathrm{d}v}{g} \in \omega_{\mathbb{A}^5}(\widetilde{W}).$$

is μ_k invariant.

(iii) *Also z, a, c is a regular sequence, and the section $z = a = c = 0$ of $V(k)$ is the tent consisting of $\frac{1}{k}(1,1)$ with coordinates $x_0, \dots, x_k$ and two copies of $\mathbb{A}^2$ with coordinates x_0, b and x_k, b.*

Proof First, if $c \neq 0$ then a, b, c, x_0, x_1 are free parameters, and the recurrence relation (I) gives $x_2, \dots, x_k$ as rational function of these. One checks that the first equation in (II) gives $z = -\frac{ax_0^2 + bx_0x_1 + cx_1^2}{c^{k-1}}$ and the remainder follow. Similarly if $a \neq 0$.

If $a = c = 0$ and $b \neq 0$ then one checks that x_0, x_k, b are free parameters, $x_i = 0$ for $i = 1, \dots, k-1$ and $z = \frac{x_0 x_k}{b^{k-2}}$. Finally, if $a = b = c = 0$ then $x_0, \dots, x_k$ and z obviously parametrise $\frac{1}{k}(1,1) \times \mathbb{A}^1$.

Therefore, no component of $V(k)$ is contained in $z = 0$, which proves (i).

After we set $z = 0$, the equations (II) become $\bigwedge^2 M = 0$, and define the cyclic quotient singularity $\frac{1}{k}(1,1)$ (the cone over the rational normal curve). Introducing u, v as the roots of $x_0, \dots, x_k$, with $x_i = u^{k-i}v^i$, boils the equations $MN = 0$ down to the single equation $g := au^2 + buv + cv^2 = 0$. This proves (ii). (iii) is easy.

The equations as Pfaffians

The equations of $V(k)$ fit together as 4×4 crazy Pfaffians of a skew matrix. For this, edit M and N to get two new matrixes,

$$M' = \begin{pmatrix} x_0 & \dots & x_{i-1} & x_i & \dots & x_{k-2} \\ x_1 & \dots & x_i & x_{i+1} & \dots & x_{k-1} \\ x_2 & \dots & x_{i+1} & x_{i+2} & \dots & x_k \end{pmatrix}$$

which is $3 \times (k-1)$ and N', the $(k-1) \times (k-3)$ matrix with the same display as N (that is, delete the first (or last) row and column of N). Equations (I) can be rewritten $(a, b, c)M' = 0$.

Now the equations (1) can be written as the Pfaffians of the $(k+2) \times (k+2)$ skew matrix

$$\begin{pmatrix} cz & -bz & \\ & az & M' \\ & & \bigwedge^{k-3} N' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} c & -b & \\ & a & M' \\ & & z \bigwedge^{k-3} N' \end{pmatrix}$$

the factor z in the first 3×3 block floats over to the final 3×3 block, allowing us to cancel z in $\mathrm{Pf}_{12.3(i-4)}$ for the recurrence relation $ax_{i-1}+bx_i+cx_{i+1}=0$. The remaining Pfaffians give (II).

Remark 3 This is a mild form of crazy Pfaffian (by analogy with Riemenschneider's quasi-determinantal): there is a multiplier z between the $(3,3)$ and $(4,4)$ entries, and when evaluating crazy Pfaffians you include z as a factor whenever you cross it.

Written out in more detail, the big matrix is

$$\begin{pmatrix} cz & -bz & x_0 & \dots & x_{i-1} & x_i & \dots & x_{k-2} \\ & az & x_1 & \dots & x_i & x_{i+1} & \dots & x_{k-1} \\ & & x_2 & \dots & x_{i+1} & x_{i+2} & \dots & x_k \\ & & & c^{k-3} & \dots & & \dots & \dots \\ & & & & c^{k-i-1}a^{i-2} & -bc^{k-i-2}a^{i-2} & \dots & \dots \\ & & & & & c^{k-i-2}a^{i-1} & \dots & \dots \\ & & & & & & \dots & \dots \\ & & & & & & & a^{k-3} \end{pmatrix}$$

with bottom right $(k-1)\times(k-1)$ block equal the $(k-3)$rd wedge of N'.

Sanity check

Our family starts with $k\geq 3$; the case $k=2$ would give the hypersurface $ax_0+bx_1+cx_2=0$, with $z:=x_0x_2-x_1^2$. The first regular case is $k=3$, which gives the 5×5 skew determinantal

$$\begin{pmatrix} c & -b & x_0 & x_1 \\ & a & x_1 & x_2 \\ & & x_2 & x_3 \\ & & & z \end{pmatrix}$$

a regular section of the affine Grassmannian $\mathrm{aGr}(2,5)$. The case $k=4$ is

$$\begin{pmatrix} c & -b & x_0 & x_1 & x_2 \\ & a & x_1 & x_2 & x_3 \\ & & x_2 & x_3 & x_4 \\ & & & zc & -zb \\ & & & & za \end{pmatrix},$$

the standard extra symmetric 6×6 determinantal of [Dicks] and [Reid1].

The first really new case is $k=5$, with equations

$$\begin{pmatrix} c & -b & x_0 & x_1 & x_2 & x_3 \\ & a & x_1 & x_2 & x_3 & x_4 \\ & & x_2 & x_3 & x_4 & x_5 \\ & & & zc^2 & -zbc & z(b^2-ac) \\ & & & & zac & -zab \\ & & & & & za^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} zc & -zb & x_0 & x_1 & x_2 & x_3 \\ & za & x_1 & x_2 & x_3 & x_4 \\ & & x_2 & x_3 & x_4 & x_5 \\ & & & c^2 & -bc & b^2-ac \\ & & & & ac & -ab \\ & & & & & a^2 \end{pmatrix}$$

We first arrived at this matrix by guesswork, determining the superdiagonal entries c^2, ac, a^2 and those immediately above $-bc, -ac$ by eliminating variables to smaller cases; the entry b^2-ac is then fixed so that the bottom 4×4 Pfaffian vanishes identically.

Alternative Proof of Theorem 1 A by-now standard application of serial unprojection [PR] and [Reid2]. We can start with any of the codimension 2 complete intersections

$$\begin{pmatrix} x_{i-1}x_{i+1} = x_i^2 + a^{i-1}c^{k-i-1}z \\ ax_{i-1} + bx_i + cx_{i+1} = 0 \end{pmatrix} \subset \mathbb{A}^7 \left\langle x_{i-1}, x_i, x_{i+1}, a, b, c, z \right\rangle$$

and add the remaining variables one at a time by unprojection.

The variety $V(k)$ by apolarity

We can treat $V(k)$ as an almost homogeneous space under $\mathrm{GL}(2)\times\mathbb{G}$. For this, view $x_0,\dots,x_k$ as coefficients of a binary form and a,b,c as coefficients of a binary quadratic form in dual variables, so that the equations $MN = 0$ or $(a, b, c)M' = 0$ are the apolarity relations.

More formally, write U for the given representation of GL(2) and write

$$q = au'^2 + 2bu'v' + cv'^2 \in S^2U^\vee$$

and

$$f = x_0u^k + kx_1u^{k-1}v + \cdots + x_kv^k \in S^kU.$$

One includes a binomial coefficient $\binom{k}{i}$ as multiplier in the coefficient of u^iv^{k-i}, a standard move in this game.

The second polar of f is the polynomial

$$\Phi(u, v, u', v') = \frac{1}{k(k-1)}\left(\frac{\partial^2 f}{\partial u^2}u'^2 + 2\frac{\partial^2 f}{\partial u\partial v}u'v' + \frac{\partial^2 f}{\partial v^2}v'^2\right)$$

$$\begin{aligned}
&= \sum_{i=0}^{k-2}\binom{k-2}{i}x_iu^{k-i-2}v^iu'^2 \\
&\quad + 2\sum_{i=1}^{k-1}\binom{k-2}{i-1}x_iu^{k-i-1}v^{i-1}u'v' + \sum_{i=2}^{k}\binom{k-2}{i-2}x_iu^{k-i}v^{i-2}v'^2 \\
&= \sum_{i=0}^{k-2}\binom{k-2}{i}x_iu^{k-2-i}v^iu'^2 \\
&\quad + 2\sum_{i=0}^{k-2}\binom{k-2}{i}x_{i+1}u^{k-2-i}v^iu'v' + \sum_{i=0}^{k-2}\binom{k-2}{i}x_{i+2}u^{k-2-i}v^iv'^2
\end{aligned}$$

Substituting $u'^2 \mapsto a$, $u'v' \mapsto \frac{1}{2}b$, and $v'^2 \mapsto c$ in this and equating to zero gives our recurrence relation $(a, b, c)M = 0$.

Moreover, the second set of equation follow from the first by substitution, provided (say) that $c \neq 0$ and we fix the value of $x_0x_2 - x_1^2$; for example, in

$$x_ix_{i+2} - x_{i+1}^2$$

substituting $x_{i+2} = -\frac{a}{c}x_i - \frac{b}{c}x_{i+1}$ gives

$$x_i(-\frac{a}{c}x_i - \frac{b}{c}x_{i+1}) - x_i^2 = -\frac{a}{c}x_i^2 - (\frac{b}{c}x_i + x_{i+1})x_{i+1},$$

and we can substitute $-\frac{a}{c}x_{i-1}$ for the bracketed expression, to deduce that

$$x_i x_{i+2} - x_{i+1}^2 = \frac{a}{c}(x_{i-1}x_{i+1} - x_i^2).$$

etc.

A normal form for a quadric under GL(2) is uv, so that a typical solution to the equations is

$$(a, b, c) = (0, 1, 0), \quad (x_{0\dots k}) = (1, 0, \dots, 1).$$

This is a "highest weight vector", and $V(k)$ is its closed orbit.

Application to diptych varieties

The diptych varieties for d, e with $de = 4$ are unprojections of pullbacks of $V(k)$.

Case $[2, 2]$

The diptych variety has variables the $x_{0\dots k}$, $y_{0\dots 2}$ of Figure 1, together with

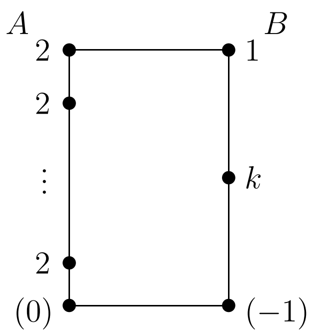

Figure 1: Case $[2, 2]$

A, B, L, M. The two bottom equations are

$$x_1 y_0 = A^{k-1}B^k + x_0^2 L \quad \text{and} \quad x_0 y_1 = ABx_1 + y_0 M$$

The pentagram y_1, y_0, x_0, x_1, x_2 adjoins x_2, then the long rally of *flat* pentagrams $y_1, x_{i-1}, x_i, x_{i+1}, x_{i+2}$ adjoin $x_3, \dots, x_k$, with matrixes

$$\begin{pmatrix} y_1 & x_1 & M & x_2 \\ & y_0 & AB & x_0L \\ & & x_0 & A^{k-2}B^{k-1} \\ & & & x_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 & x_{i+1} & LM & x_{i+2} \\ & x_{i-1} & AB & x_i \\ & & x_i & (AB)^{k-i-2}(LM)^{i-1}BM \\ & & & x_{i+1} \end{pmatrix}$$

and Pfaffian equations

$$y_1x_i = ABx_{i+1} + LMx_{i-1}, \quad x_{i-1}x_{i+1} = x_i^2 + (AB)^{k-i-1}(LM)^{i-1}BM$$
$$\text{and} \quad x_{i-1}x_{i+2} = x_ix_{i+1} + (AB)^{k-i-2}(LM)^{i-1}BMy_1.$$

These are the equations of $V(k)$ after the substitution

$$(a, b, c, z) \mapsto (LM, y_1, AB, BM).$$

Thus to make our diptych variety, pull back $V(k) \subset \mathbb{A}^{k+5}$ by that substitution, then adjoin y_0, y_2 as unprojection variables. [1]

Case $[4, 1]$ with even $l = 2k$

Omit the odd numbered x_i, giving Figure 2. The diptych variety has variables

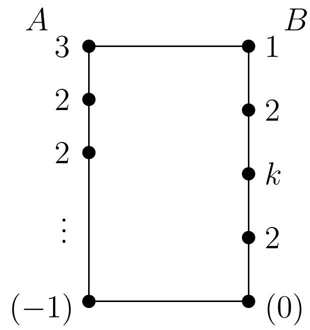

Figure 2: Case $[4, 1]$ with even $l = 2k$

$x_{0\dots k}$, $y_{0\dots 4}$, A, B, L, M with the two bottom equations

$$x_1y_0 = A^{k-1}B^{2k-1}y_1 + x_0^3L \quad \text{and} \quad x_0y_1 = A^kB^{2k+1} + y_0M$$

[1] We still have to deal with the unprojection, here and below.

We adjoin y_2, then $x_2, \dots, x_k$ by a game of pentagrams centred on a long rally of flat pentagrams, with y_2 against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and Pfaffian equations

$$y_2 x_i = AB^2 x_{i+1} + LM^2 x_{i-1}, \quad x_{i-1}x_{i+1} = x_i^2 + (AB^2)^{k-i-1}(LM^2)^{i-1}BM$$
$$\text{and} \quad x_{i-1}x_{i+2} = x_i x_{i+1} + (AB^2)^{k-i-2}(LM^2)^{i-1}BMy_2$$

These are the equations of $V(k)$ after the substitution

$$(a, b, c, z) \mapsto (LM^2, y_2, AB^2, BM).$$

Case $[1, 4]$ with even $l = 2k$

Omit the even numbered x_i, giving Figure 3. The diptych variety has vari-

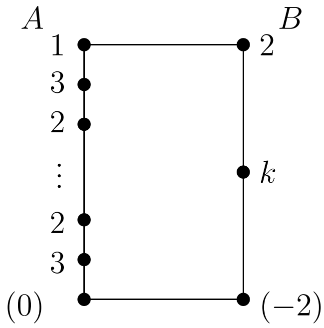

Figure 3: Case $[4, 1]$ with even $l = 2k$

ables $x_{0\dots k}$, $y_{0\dots 2}$, A, B, L, M with the two bottom equations

$$x_1 y_0 = A^{2k-1}B^k + x_0 L \quad \text{and} \quad x_0 y_1 = x_1^2 A^2 B + y_0^2 M$$

As before, adjoining $x_2, \dots, x_k$ features a long rally of flat pentagrams, with y_1 against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and Pfaffian equations

$$y_1 x_i = A^2 B x_{i+1} + L^2 M x_{i-1}, \quad x_{i-1}x_{i+1} = x_i^2 + (A^2B)^{k-i-1}(L^2M)^{i-1}AL$$
$$\text{and} \quad x_{i-1}x_{i+2} = x_i x_{i+1} + (A^2B)^{k-i-2}(L^2M)^{i-1}BMy_2$$

These are the equations of $V(k)$ after the substitution

$$(a, b, c, z) \mapsto (L^2M, y_1, A^2B, BM).$$

Case $[1, 4]$ with odd $l = 2k + 1$

This is $[1, 4]$ read from the top, but $[4, 1]$ read from the bottom, so is a mix of the two preceding cases. Omit the odd numbered x_i, giving Figure 4. The

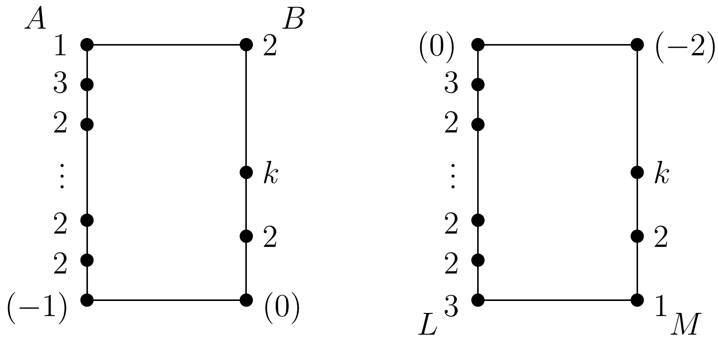

Figure 4: Case $[1, 4]$ with odd $l = 2k + 1$

diptych variety has variables $x_{0\dots k}$, $y_{0\dots 3}$, A, B, L, M with the two bottom equations

$$x_1 y_0 = y_1 A^{2k-3} B^{k-1} + x_0^3 L \quad \text{and} \quad x_0 y_1 = A^{2k-1} B^k + y_0 M$$

Adjoin y_2 then x_2 by

$$\begin{pmatrix} y_1 & A^2B & M & y_2 \\ & y_0 & A^{2k-3}B^{k-1} & x_0^2 L \\ & & x_0 & y_1 \\ & & & x_1 \end{pmatrix} \quad \text{then} \quad \begin{pmatrix} y_2 & x_1 & M & x_2 \\ & y_1 & A^2B & x_0 LM \\ & & x_0 & y_2 A^{2k-5} B^{k-2} \\ & & & x_1 \end{pmatrix}$$

After this, adjoining $x_3, \dots, x_{k-1}$ is the usual long rally of flat pentagrams, with y_2 against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and

$$\begin{pmatrix} y_2 & x_{i+1} & LM^2 & x_{i+2} \\ & x_{i-1} & A^2B & x_i \\ & & x_i & (A^2B)^{k-i-3}(LM^2)^{i-1} ABM y_2 \\ & & & x_{i+1} \end{pmatrix}$$

and the Pfaffian equations

$$y_2 x_i = A^2 B x_{i+1} + LM^2 x_{i-1}, \quad x_{i-1}x_{i+1} = x_i^2 + (A^2B)^{k-i-2}(LM^2)^{i-1}ABMy_2$$
$$\text{and} \quad x_{i-1}x_{i+2} = x_i x_{i+1} + (A^2B)^{k-i-3}(LM^2)^{i-1}ABMy_2^2$$

These are the equations of $V(k-1)$ after the substitution

$$(a,b,c,z) \mapsto (LM^2, y_2, A^2B, BM).$$