The complete intersection of two or more quadrics

Miles Reid
Trinity College, Cambridge

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Except where otherwise stated, the results of this dissertation are original. No part of this dissertation has been submitted for a degree at any other university.

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Chapter 0

Introduction. Cohomology theories and algebraic cycles

This work is on the subject of the cohomology of algebraic varieties. In it I show that rather elementary geometric arguments suffice to compute the cohomology of some special varieties, namely the complete intersection of two quadrics; this enables me to check directly that certain well known conjectures hold in the case of these varieties.

This introduction is devoted to a brief sketch of what is known, and what the problems are, in the cohomology theory of algebraic varieties. I outline the methods that I will be using, their scope and their limitation; I will also touch upon what can be done using more sophisticated techniques. In the last section of the introduction, I will carry out a number of calculations of Betti numbers that belong to the various sections of the sequel, but that are conveniently treated together.

0.1 Topological cohomology theory

Let V be an algebraic variety[1] over the complex numbers $\mathbb{C}$, and let $V_{\mathbb{C}}$ denote the underlying topological space, with the complex topology. The most obvious way to associate cohomological invariants to V is to attach to V the cohomology groups $H^i(V_{\mathbb{C}}, \mathbb{Z})$ – together with the topological cup product

[1] For the purpose of §0, all varieties will be *connected and* proper, or even projective. (Italics in the footnotes indicates handwritten corrections that were inserted immediately following the Ph.D. oral exam.)

that these groups come equipped with. Suppose that V is nonsingular, of dimension n; then $V_{\mathbb{C}}$ is a real $2n$-dimensional manifold, so that the groups $H^i(V_{\mathbb{C}}, \mathbb{Z})$ have a number of reasonable properties, among which

(i) $H^i(V_{\mathbb{C}}, \mathbb{Z})$ is a finitely generated Abelian group for each i, and is zero if $i > 2n$.

(ii) $H^{2n}(V_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$, and the cup product

$$H^{2n-i}(V_{\mathbb{C}}, \mathbb{Z}) \otimes H^i(V_{\mathbb{C}}, \mathbb{Z}) \to H^{2n}(V_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$$

is nondegenerate, i.e., $a \cup b = 0$ for all $b \in H^i(V_{\mathbb{C}}, \mathbb{Z})$ implies that $a \in H^{2n-i}(V_{\mathbb{C}}, \mathbb{Z})$ is a torsion element. This property of a cohomology theory is called Poincaré duality[2].

(iii) If $\sigma \colon V \to V$ is an endomorphism, then by functoriality, σ induces a map

$$\sigma_i^* \colon H^i(V_{\mathbb{C}}, \mathbb{Z}) \to H^i(V_{\mathbb{C}}, \mathbb{Z})$$

on the cohomology. If V^σ is the locus of fixed points under σ, then the Euler–Poincaré characteristic $\chi(V^\sigma)$ is given by the Lefschetz fixed point formula[3]:

$$\chi(V^\sigma) = \sum_{i=0}^{2n} (-1)^i \operatorname{Tr}(\sigma_i^*), \tag{1}$$

where $\operatorname{Tr}(\sigma_i^*)$ is the trace of σ_i^*, calculated on a basis of the torsion free part of $H^i(V_{\mathbb{C}}, \mathbb{Z})$.

[2] *The statement of Poincaré duality given here is slightly weaker than the assertion which actually holds, namely that the pairing induces an isomorphism of H^{2n-i} with H^i.*

[3] *This statement of the Lefschetz fixed point formula only holds under the following hypotheses on σ:*

(i) The fixed point locus V^σ has multiplicity one, i.e., the graph Γ^σ intersects the diagonal $\Delta_V \subset V \times V$ transversally.

(ii) Either

(a) σ has finite order, i.e., $\sigma^n = \mathrm{id}$; or

(b) σ has only isolated fixed points, so that V^σ is a finite set, and $\chi(V^\sigma) = \mathrm{card}(V^\sigma)$.

These two footnotes were kindly pointed out by Professor Deligne.

0.2 The Weil conjectures and ℓ-adic cohomology

Let V be an algebraic variety defined over the finite field $\mathbb{F}_q$ with q elements; for any extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$, let $V(\mathbb{F}_{q^n})$ denote the set of points of V with values in $\mathbb{F}_{q^n}$. Then for each n, $V(\mathbb{F}_{q^n})$ is a finite set, and I will denote by $\#V(\mathbb{F}_{q^n})$ the number of elements in it. For example, if $V = \mathbb{P}^s$ is s-dimensional space, then

$$\#\mathbb{P}^s(\mathbb{F}_{q^n}) = 1 + q^n + q^{2n} + \cdots + q^{sn}.$$

Again, if $V = E$ is an elliptic curve over $\mathbb{F}_q$, then there is the famous formula, due to Hasse,

$$\#E(\mathbb{F}_{q^n}) = 1 + \alpha^n + \beta^n + q^n,$$

with α and β (conjugate) algebraic integers, and $\alpha\beta = q$. In fact, we also have that $|\alpha| = |\beta| = q^{1/2}$.

The Weil conjectures assert firstly that there exists a formula for $\#V(\mathbb{F}_{q^n})$ of the form

$$\#V(\mathbb{F}_{q^n}) = \sum_{i=0}^{2n} \sum_{j=1}^{b_i} (-1)^i \alpha_{ij}^n \tag{2}$$

with the α_{ij} algebraic integers; for each i, j, there exists j' such that

$$\alpha_{ij}\alpha_{2n-i\ j'} = q^n. \tag{3}$$

The second part of the Weil conjectures is the famous Riemann–Weil hypothesis, and asserts that the absolute value of the α_{ij} are given by

$$|\alpha_{ij}| = q^{i/2}. \tag{4}$$

The first step backwards in proving either part of these conjectures, which was taken by Weil himself, is to realize that the formula (2) is an immediate consequence of the existence of a cohomology theory, taking its values in the category of finite dimensional vector spaces over a field k of characteristic zero, and such that the condition (iii) above is satisfied.

To see this, note that $V(\mathbb{F}_q)$, as a subset of V, is just the fixed point set V^σ of the Frobenius endomorphism

$$\sigma\colon V \to V,$$

defined on the coordinates x of V by $x \mapsto x^q$. Similarly, $V(\mathbb{F}_{q^n})$ is just the fixed point set of σ^n, so that applying the Lefschetz fixed point formula (1), we obtain

$$\#V(\mathbb{F}_{q^n}) = \sum_{i=0}^{2n} (-1)^i \operatorname{Tr}((\sigma_i^*)^n).$$

This gives (2) immediately, on taking α_{ij} (for $j = 1, \dots, b_i$) to be the characteristic roots of the linear endomorphism σ_i^*. And the b_i are identified as being the ith Betti numbers of V (w.r.t. this cohomology theory).

A formula of type (3) then takes on the character of a duality theorem, expressing the characteristic roots of σ_i^* in terms of those of σ_{2n-i}^*. Before giving the statement of Poincaré duality that will imply (3), let me make a few general remarks.

Firstly, it is important to say that we are interested in the cohomology groups $H^i(V)$ not just as finite dimensional vector spaces over k, but as *Galois modules*, that is to say as finite dimensional vector spaces on which the Galois group $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts. Since this group is profinite cyclic, with generator the Frobenius σ, the action of $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $H^i(V)$ is determined by the action of σ. The following Galois module, the *Tate module* $k(1)$ will be indispensable in the sequel[4]:

(a) as a vector space, $k(1)$ is one dimensional over k;

(b) the action of σ on $k(1)$ is given by

$$\sigma(x) = qx \quad \text{for all } x \in k(1).$$

We can then define $k(n)$ as the n-fold tensor produce of $k(1)$ with itself, and $k(-n)$ as the Galois module dual to $k(n)$. For any Galois module H, $H(n)$ is the tensor product $H \otimes k(n)$.

The statement of Poincaré duality for a cohomology theory that will imply that the characteristic roots of σ_i^* satisfy (3) is

[4]Some people define $k(1)$ to be what I call $k(-1)$. *σ is the "geometric" Frobenius, i.e., the inverse in* $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ *of* $x \mapsto x^q$.

(a)

$$H^{2n}(V) = k(n),$$

the equality sign being taken as a canonical isomorphism; and

(b) the cup product

$$H^i(V) \otimes H^{2n-i}(V)(-n) \to k$$

is a perfect pairing.

A cohomology theory having the properties required, i.e., taking its values in the category of finite dimensional vector spaces over a field of characteristic zero, and satisfying the Lefschetz fixed point theorem and Poincaré duality, was constructed about 1962 by Artin and Grothendieck [1]; this then provides a proof of the first part of the Weil conjectures.

In fact, the cohomology constructed by Artin and Grothendieck has some other properties: it associates to a variety V the groups $H^i(V, \mathbb{Z}_\ell)$, which are finitely generated modules over the ℓ-adic numbers $\mathbb{Z}_\ell$. In this context, the Tate module $\mathbb{Z}_\ell(-1)$ has a natural interpretation as the $\mathbb{Z}_\ell$-module of ℓ^nth roots of unity in $\overline{\mathbb{F}}_q$.

The construction of ℓ-adic cohomology does not, of course, imply the Riemann–Weil hypothesis. We shall discuss the use of ℓ-adic cohomology in this context a little later.

0.3 Hodge theory and algebraic cycles

Let me return to the case that V is a smooth algebraic variety over the complex numbers $\mathbb{C}$; then Hodge discovered in the 1930s that there is a remarkable connection between the cohomology groups $H^i(V_\mathbb{C}, \mathbb{Z})$, which are defined by topological means, and the purely algebraic coherent cohomology[5]

$$H^{qp} = H^p(V, \Omega^q).$$

[5] Ω^q_V is the sheaf (for the Zariski topology) of algebraic q-forms on V. One could also consider the analogous cohomology of V (in the classical topology), with coefficients in the sheaf $\Omega^q_{V_{\mathrm{an}}}$ of holomorphic q-forms. These groups in fact coincide if V is complete, by the theorems of Serre (GAGA [21]).

In fact, there is a canonical decomposition

$$H^i(V_{\mathbb{C}}, \mathbb{C}) = H^i(V_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=i} H^{pq}(V). \tag{5}$$

The point of view that is commonly adopted nowadays is that the decomposition (5) is an additional structure given on the groups $H^i(V_{\mathbb{C}}, \mathbb{Z})$; in this sense it is analogous to the Galois module structure on the ℓ-adic groups $H^i(V, \mathbb{Z}_\ell)$. The groups $H^i(V_{\mathbb{C}}, \mathbb{Z})$, together with the cup product and the decomposition (5), will be referred to as the *Hodge structure* of V; when I write $H^i(V, \mathbb{Z})$ in the sequel, I always mean the groups $H^i(V_{\mathbb{C}}, \mathbb{Z})$ together with all this extra structure.

Rather than define explicitly the notion of Hodge structure in general (see [7]), let me just point out that the decomposition (5) is known to satisfy certain conditions; firstly $H^{qp} = \overline{H^{pq}}$, the bar denoting complex conjugation of $H^i(V_{\mathbb{C}}, \mathbb{C})$ w.r.t. the lattice $H^i(V_{\mathbb{C}}, \mathbb{Z})$. Then there are two conditions on the relation between the decomposition (5) and the cup product, the first asserting that

$$H^i(V, \mathbb{Z}) \otimes H^j(V, \mathbb{Z}) \to H^{i+j}(V, \mathbb{Z})$$

is a morphism of Hodge structures, i.e., the cup product of a pq by $p'q'$ class is a $p + p'$ $q + q'$ class; the second condition is a positivity assertion, which I will only state below in a more restricted context.

There is a natural analogue in Hodge theory of the "twisting by the Tate module" that has been mentioned above in the context of ℓ-adic cohomology. Although this appears rather baroque at first sight, it appears naturally on considering the cohomology class associated to an algebraic cycle, and it is convenient to have similar formalisms in the two cases of ℓ-adic or Hodge cohomology.

For simplicity in the following discussion, either assume that $H^i(V_{\mathbb{C}}, \mathbb{Z})$ is torsion free, or let us abuse the notation by writing $H^i(V_{\mathbb{C}}, \mathbb{Z})$ for this group modulo its torsion subgroup; we then have an inclusion of $H^i(V_{\mathbb{C}}, \mathbb{Z})$ into $H^i(V_{\mathbb{C}}, \mathbb{C})$.

Suppose then that $Y \subset V$ is an algebraic cycle; there are two different definitions possible of the cohomology class of Y: the first is purely topological, giving a class $\gamma'_V(Y) \in H^{2r}(V_{\mathbb{C}}, \mathbb{Z})$. The second gives us a class $\gamma_V(Y)$ in the coherent group $H^r(V, \Omega^r_V)$, and the definition of $\gamma_V(Y)$ is purely algebraic. Considering both these classes as elements of $H^{2r}(V_{\mathbb{C}}, \mathbb{C})$, it makes sense to ask if they are equal, or proportional. It turns out that they are proportional, with scaling factor $(2\pi i)^r$:

$$\gamma_V(Y) = (2\pi i)^r \gamma'_V(Y).$$

It is then obvious that the isomorphism of $H^{2n}(V_{\mathbb{C}}, \mathbb{C})$ with $\mathbb{C}$ which takes the *algebraic* cohomology class of a point into $1 \in \mathbb{C}$ takes the lattice $H^{2n}(V_{\mathbb{C}}, \mathbb{Z})$ into $(2\pi i)^{-n} \cdot \mathbb{Z} \subset \mathbb{C}$.

Definition 0.1 The *Hodge–Tate structure* $\mathbb{Z}(1)$ is the Hodge structure[6] that is purely of type $(1,1)$, one dimensional, and whose integral lattice is $(2\pi i)^{-1} \cdot \mathbb{Z} \subset \mathbb{C}$.

As before, there is the possibility of defining $\mathbb{Z}(n)$ for any integer n, and for any Hodge structure H, of defining $H(n)$. This gives meaning to the following assertions, which are the form that Poincaré duality takes in Hodge theory:

(a) There is a canonical isomorphism $H^{2n}(V, \mathbb{Z}) \cong \mathbb{Z}(n)$, given as indicated above by taking the algebraically defined class $\gamma_V(\text{point}) \in H^{2n}(V_{\mathbb{C}}, \mathbb{Z})$ into $1 \in \mathbb{C}$.

(b) The cup product $H^{2n-i}(V, \mathbb{Z}) \otimes H^i(V, \mathbb{Z}) \to \mathbb{Z}(n)$ induces an isomorphism of Hodge structures

$$\mathrm{Hom}(H^{2n-i}(V, \mathbb{Z}), \mathbb{Z}(n)) \to H^i(V, \mathbb{Z}).$$

Suppose now that H is a Hodge structure of weight r, i.e., so that $H^{pq} = 0$ unless $p + q = r$; let $H_{\mathbb{Z}}$ and $H_{\mathbb{C}}$ denote the lattice and the vector space underlying H, so that $H_{\mathbb{Z}} \subset H_{\mathbb{Z}} \otimes \mathbb{C} = H_{\mathbb{C}}$. Then we have the following concrete description of the Hodge structure $H(-n)$, which is of weight $r - 2n$:

> $H(-n)$ and H have the same underlying vector space $H_{\mathbb{C}}$. The lattice $H(-n)_{\mathbb{Z}} \subset H_{\mathbb{C}}$ is the lattice $(2\pi i) \cdot H_{\mathbb{Z}}$. Finally, the decomposition of $H(-n)_{\mathbb{C}}$ into H^{pq} is done in the simplest possible way, i.e.,
>
> $$H^{pq}(-n) = H(-n)^{pq} = H^{p+n\ q+n},$$
>
> and it is quite obvious that $H(-n)^{pq} = 0$ unless $p + q + 2n = r$, so that $H(-n)$ is indeed of weight $r - 2n$.

[6] It would be wrong of me to miss this opportunity to point out that half the world is out of step, i.e., call $\mathbb{Z}(-1)$ what I call $\mathbb{Z}(1)$.

We asserted above that the cohomology class $\gamma_V(Y)$ associated to a cycle $Y \subset V$ of codimension r is an element of $H^r(V,\Omega_V^r) = H^{rr}(V)$. On the other hand, the class $\gamma'_V(Y) = (2\pi i)^{-r} \cdot \gamma_V(Y)$ is an element of $H^{2r}(V_{\mathbb{C}},\mathbb{Z})$. Putting the two together, we have the assertion that $\gamma_V(Y)$ is an element of $H^{2r}(V,\mathbb{Z})(-r)$, that is, both integral and of weight $(0,0)$.

The *Hodge conjecture* is the assertion that every element of $H^{2r}(V,\mathbb{Z})(-r)$ that is integral and of weight $(0,0)$ is the cohomology class $\gamma_V(Y)$ for some algebraic cycle Y of codimension r in V. As a particular case, if V is a variety such that $H^{pq} = 0$ for $p \neq q$, then the Hodge conjecture is the assertion that the cohomology of V can be entirely spanned by algebraic cycles. An example of such a variety is the $2n$-dimensional intersection of two quadrics, which is studied in §3 of this work, and for which I prove that the classes of the n-planes in V span the cohomology $H^{2n}(V,\mathbb{Z})(-n)$; the other groups are already spanned by powers of the class of the hyperplane section, as explained a little later.

The following generalization of the Hodge conjecture is one of the principal sources of inspiration behind this work; it suggests that, at least for certain special varieties V, there is a way of understanding the cohomology of V in terms of simpler objects by constructing families of algebraic cycles on V.

Let S be a nonsingular algebraic variety of dimension m, and suppose that $Y \subset S \times V$ is an algebraic cycle of codimension $m+r$; when all the intersections $Y_s = Y \cap (\{s\} \times V)$ are transverse, then Y can be thought of as a family of cycles Y_s of V, of codimension $m+r$, parametrized by S. Let me indicate the way in which such a cycle gives rise to a morphism of Hodge structures

$$f_Y\colon H^i(S,\mathbb{Z}) \to H^{i+2r}(V,\mathbb{Z})(-r),$$

and thus to a sub-Hodge structure $\mathrm{Im}(f_Y)$ of $H^{i+2r}(V,\mathbb{Z})(-r)$.

Let $\gamma(Y) = \gamma_{S\times V}(Y)$ be the cohomology class of Y, which is an element of $H^{2(r+m)}(S\times V,\mathbb{Z})(-r-m)$, which is integral and of weight $(0,0)$, so that the cup product with $\gamma(Y)$ is a morphism of Hodge structure

$$m_Y\colon H^i(S\times V,\mathbb{Z}) \to H^{i+2(r+m)}(S\times V,\mathbb{Z})(-r-m).$$

On the other hand, we dispose of morphisms

$$p_1^*\colon H^i(S,\mathbb{Z}) \to H^i(S\times V,\mathbb{Z}),$$

by functoriality of H^i, and

$$p_{2*}\colon H^{i+2m}(S \times V, \mathbb{Z}) \to H^i(V, \mathbb{Z})(m),$$

which is the transpose of p_2^*, p_1 and p_2 being the two projections of $S \times V$ onto its factors; the fact that p_{2*} is a morphism of Hodge structures is an immediate consequence of the form in which we have stated Poincaré duality.

f_Y is then just the composite:

$$f_Y = p_{2*} \circ m_Y \circ p_1^*,$$

and it is easy to check that is has the desired effect on the dimensions.

The sub-Hodge structure $\mathrm{Im}(f_Y) \subset H^{i+2r}(V, \mathbb{Z})(-r)$ satisfies two special properties:

(a) It is integral, in the sense that it has an underlying integer lattice $\mathrm{Im}(f_Y)_{\mathbb{Z}}$ which is a sublattice of the integer lattice underlying $H^{i+2r}(V, \mathbb{Z})(-r)$.

(b) Since it is the image of $H^i(S, \mathbb{Z})$, there are restrictions on its Hodge decomposition: it only has nonzero Hodge groups H^{pq} if p and q are positive.

The *generalized Hodge conjecture* is the assertion that for any sub-Hodge structure H of $H^{2r+i}(V, \mathbb{Z})(-r)$ satisfying the two conditions (a) and (b), there should exist a variety S of dimension i, and a family of cycles Y_s of V of codimension $i + r$ parametrized by s, i.e., a cycle Y of codimension $i + r$ in $S \times V$, such that H is the image of $H^i(S, \mathbb{Z})$ by f_Y.

The most common example, and the only case of interest to me, is that in which H is in fact the whole of $H^{i+2r}(V, \mathbb{Z})(-r)$. By (b), this is satisfied if and only if the first r groups $H^{i+2r\,0}, \ldots, H^{i+r\,r}$ of the Hodge decomposition are trivial. In this case, the Hodge conjectures suggest at least that one can "account for" the cohomology of V in terms of that of a variety S of smaller dimension, i.

As an example, if V is an Abelian variety, $H^1(V, \mathbb{Z})$ is of weight $(1, 0) + (0, 1)$. We might hope to find a curve C, and a cycle on $C \times V$ inducing a surjective map

$$H^1(C, \mathbb{Z}) \to H^1(V, \mathbb{Z}).$$

It is known that this is in fact the case.

0.4 Complete intersections and Hodge level

There are a number of varieties which have only one interesting cohomology group. Firstly, to deal with the "uninteresting" ones: let V be a projective variety, that is, a closed subvariety of projective space. Then V has a number of nonzero cohomology classes: if V_H is a hyperplane section, then the class of V_H is an element of $H^2(V,\mathbb{Z})(-1)$, which I will denote by ξ. If V has degree d in projective space and $\dim V = n$, then the cup product ξ^n of ξ with itself n times is d times a generator of $H^{2n}(V,\mathbb{Z})(-n) \cong \mathbb{Z}$. In particular, the classes

$$\xi^r \in H^{2r}(V,\mathbb{Z})(-r)$$

are nonzero for $r = 0, 1, \dots, n$. Since these are algebraic cycles, they are of type $(0,0)$, and the conclusion is that for $r = 0, 1, \dots, n$, the Hodge group $H^{rr}(V)$ has dimension at least 1.

Definition 0.2 V will be said to have *purely n-dimensional cohomology* if the only nonzero Hodge groups are

$$H^{pq}(V) \quad \text{for } p+q=n \qquad \text{and} \quad H^{rr}(V) \quad \text{for } r = 0, \dots, n;$$

furthermore, the groups $H^{rr}(V)$ for $2r \neq n$, are required to be one dimensional.

If the cohomology of V is purely of dimension n, then V will be said to be of *Hodge level* k if $H^{pq} = 0$ for $|p-q| \geq k$. For instance, if V is a surface, then V has cohomology purely of dimension 2 if and only if the irregularity q vanishes; its Hodge level is 2 if $H^{2\,0} \neq 0$, and is zero if $H^{2\,0} = 0$. This is the case in which the Hodge conjecture in known, so that if V is a surface of Hodge level zero, its cohomology is spanned by algebraic cycles.

It is known that if $V \subset \mathbb{P}^N$ is the complete intersection of $N-n$ hypersurfaces, then the cohomology of V is purely of dimension n. I will give some examples later of varieties for which the Hodge level is small; the complete list of all the cases of complete intersections for which the Hodge level is ≤ 2 is given in [20].

Suppose that V has dimension $2n+1$, and Hodge level one, so that its only nonzero Hodge groups are $H^{n+1\,n}(V)$ and $H^{n\,n+1}(V)$. According to the generalized Hodge conjecture, one would hope to be able to construct a curve C, and an algebraic cycle Y on $C \times V$, such that the morphism defined above

$$f_Y \colon H^1(C,\mathbb{Z}) \to H^{2n+1}(V,\mathbb{Z})(-n)$$

is surjective.

For the special case that V is the intersection of two quadrics, this question is answered in the affirmative in §4; the relevant algebraic cycles are just the n-planes of V. In fact this result can also be obtained by the methods of Grothendieck, since the n-planes contained in any hyperplane section V_H of V span the cohomology of V_H.

There is a much more general construction that provides an Abelian variety $J(V)$, the *intermediate Jacobian* of V, such that

$$H^1(J(V), \mathbb{Z}) \xrightarrow{\cong} H^{2n+1}(V, \mathbb{Z})(-n),$$

that is valid for any variety V of dimension $2n+1$ and Hodge level 1. Unfortunately, there is no general way of constructing families of cycles of V, parametrized by $J(V)$, that would induce the above isomorphism. In §4, I show that the intermediate Jacobian $J(V)$ for the particular case of V the intersection of two quadrics, can naturally be regarded as parametrizing the n-planes of V.

The construction of $J(V)$ goes as follows. From V, we have the Hodge structure $H = H^{2n+1}(V, \mathbb{Z})(-n)$; H is purely of type $(1,0)+(0,1)$, and is *polarized*[7], i.e., there is given a nondegenerate form $\varphi\colon H \otimes H \to \mathbb{Z}(1)$ (induced by the cup product in $H^{2n+1}(V, \mathbb{Z})$); on p. 10 I mentioned without stating a certain positivity condition that φ is known to satisfy; in this restricted set-up, the condition is that the Hermitian form $\psi(x, y) = -i\varphi(x, \overline{y})$ is positive definite on H^{01}.

The intermediate Jacobian $J(V)$ is now defined in terms of the Hodge structure H as follows:

> as a complex torus, $J(V) = H_{\mathbb{Z}} \backslash H_{\mathbb{C}}/H^{10} = H^{01}/H_{\mathbb{Z}}$; in the second of these expressions, $H_{\mathbb{Z}}$ is considered as embedded in H^{01} by the composite $H_{\mathbb{Z}} \hookrightarrow H_{\mathbb{C}} \xrightarrow{p} H^{01}$, p being the projection of $H_{\mathbb{C}}$ onto H^{01};
>
> the polarization of $J(V)$ is the positive definite Hermitian form ψ; this satisfies the requirement that $\mathrm{Im}(\psi)$ be integral on $H_{\mathbb{Z}}$, as can be seen on noticing that $\psi_{|H_{\mathbb{Z}}}$ is just the original form φ.

The above argument may be summarized by saying that the functor

$$H^1\colon A \mapsto H^1(A, \mathbb{Z})$$

[7] *Actually, principally polarized Hodge structures:* φ *induces* $H \xrightarrow{\cong} \big(H(-1)\big)^*$.

is an equivalence of categories between principally polarized Abelian varieties on the one hand, and polarized Hodge structures purely of type $(0,1)+(1,0)$ on the other.[8]

0.5 Algebraic cycles and the Riemann–Weil hypothesis

In the context of ℓ-adic cohomology, there is also a cycle map

$$\gamma_V\colon \begin{Bmatrix}\text{cycles on } V \text{ of}\\ \text{codimension } r\end{Bmatrix} \to H^{2r}(V,\mathbb{Z}_\ell)(-r).$$

The argument used above associates to a cycle $Y \subset S \times V$ a morphism on the cohomology

$$f_Y\colon H^i(S,\mathbb{Z}_\ell) \to H^{i+2r}(V,\mathbb{Z}_\ell)(-r).$$

For the same reason that f_Y preserved the Hodge structure, the new f_Y now commutes with the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. More precisely, suppose that V and S are smooth and of dimension n and m respectively, and defined over the field $\mathbb{F}_q$. And suppose that Y is an algebraic cycle, defined over $\mathbb{F}_q$, of codimension $m+r$ on $S \times X$.

Then there is a map of cohomology groups

$$f_Y\colon H^i(S,\mathbb{Z}_\ell) \to H^{i+2r}(V,\mathbb{Z}_\ell)(-r),$$

which can be defined as a composite of $p_2^*\colon H^i(S,\mathbb{Z}_\ell) \to H^i(S\times V,\mathbb{Z}_\ell)$, the multiplication in $H^i(S\times V,\mathbb{Z}_\ell)$ by the class

$$\gamma_{V\times S}(Y) \in H^{2(m+r)}(V\times S,\mathbb{Z}_\ell)(-m-r),$$

and p_{1*}. These operations are all morphisms of Galois modules, i.e., commute with the action of Galois; for p_2^* this is just functoriality, and for p_{1*} it is by functoriality and the care that was taken to express Poincaré duality in terms of morphisms of Galois modules; as for the middle step, it is the multiplication with an element that is fixed under Galois, since the cycle Y is defined over $\mathbb{F}_q$.

[8] *See previous footnote.*

The application of this to proving the Riemann–Weil hypothesis in particular cases is the following; suppose that we are interested in the characteristic roots of Frobenius acting on a certain vector space H. Then if we have, say, a surjection of another vector space L onto H, $f\colon L \to H$, and L is a Galois module in such a way that the arrow f commutes with the Galois action, then the roots of Frobenius on H are a subset of those on L.

In particular, suppose that we have a variety V, and that we can find a curve C and a cycle Y on $C \times V$ such that

$$f_Y\colon H^1(C, \mathbb{Z}_\ell) \to H^{2n+1}(V, \mathbb{Z}_\ell)(-n)$$

is surjective; we can suppose that all the varieties concerned are defined over some field $\mathbb{F}_q$. Then the Riemann–Weil hypothesis for V is a consequence of that for C, at least if V has cohomology purely of dimension $2n+1$.

The arguments of §4 of this work show that, in the particular case that V is the $(2n+1)$-dimensional intersection of two quadrics, there exists a curve C, and a cycle Y on $C \times V$, such that the map f_Y is an isomorphism; this proves a particular case of the Hodge conjectures in characteristic zero, and over a finite field, it reduces the Riemann–Weil hypothesis for V to that for the curve C, which is of course well known.

0.6 Deligne's method

I have so far been using the analogy between characteristic zero and characteristic p in a purely informal manner, the only relation between the two being that there is a common point of attack, namely that of trying to construct families of algebraic cycles, that will work either for verifying the Hodge conjecture, or for reducing the Riemann–Weil hypothesis to the case of curves.

On the other hand, in characteristic zero, there is a direct transcendental construction, sketched above, which to any variety of Hodge level one associates an Abelian variety $J(V)$, and an isomorphism of Hodge structures

$$H^{2n+1}(V, \mathbb{Z})(-n) = H^1(J(V), \mathbb{Z}).$$

If there were an algebraic way of producing, in characteristic p, such an Abelian variety $J(V)$, and an isomorphism of Galois modules

$$H^{2n+1}(V, \mathbb{Z})(-n) = H^1(J(V), \mathbb{Z}),$$

we would of course have reduced the Riemann–Weil hypothesis for V to that of Abelian varieties, which is a well known result of Weil's.

Deligne constructs such an Abelian variety in a number of cases, by using a reduction modulo p of the intermediate Jacobian, which was defined above transcendentally. The steps in the construction are:

(i) The construction of $J(V)$ sketched above is in fact algebraic, i.e., invariant under the automorphisms of $\mathbb{C}$; hence, if V is defined over a given number field K, then so is $J(V)$.

(ii) If V^0 is defined over $\mathbb{F}_q$, V a lifting of it to characteristic zero, then $J(V)$ has good reduction $J(V)^0$ at p.

(iii) The isomorphism of the cohomology of V^0 with that of $J(V)^0$ is a consequence of the isomorphism of the cohomology of V with that of $J(V)$.

The properties of V^0 that are essential for this construction to go through are, firstly the very reasonable one that V^0 must have a lifting V to characteristic zero which has Hodge level one. The second property, that is more technical, is that V^0 and V should both belong to large enough moduli spaces.

These conditions are satisfied for the complete intersections of Hodge level one, listed in [20]; in particular, this method proves the Riemann–Weil hypothesis for the cubic 3 or 5-fold, and for the odd dimensional intersection of two or three quadrics.

On the negative side, Deligne has shown that the methods used in this work, as well as his own methods, are incapable of proving the Riemann–Weil hypothesis for "general" varieties; thus, for instance, there can be no reduction of the cohomology of the generic surface of degree 5 in $\mathbb{P}^3$ to that of Abelian varieties.

0.7 The Hirzebruch formula and some Betti numbers

In this final section of the introduction, I want to describe a recipe for calculating the Betti numbers of a complete intersection, and to apply the recipe to the cases that I will be interested in later. The first step is to reduce to characteristic zero:

Fact 0.3 *Suppose that V is a smooth scheme proper over* $\mathrm{Spec}(R)$, *where R is a complete valuation ring, and let V_g and V_s be the general and special fibres of V; these are then complete nonsingular varieties over the quotient field K and the residue field k respectively. Then the following vector spaces all have the same dimension $b_i(V)$:*

$$H^i_{\mathrm{et}}(V_g, \mathbb{Q}_\ell), \quad H^i_{\mathrm{et}}(V_s, \mathbb{Q}_\ell), \quad H^i(V_g, \mathbb{Q}),$$

where ℓ is any prime other than the characteristic of k, and the final group is the ordinary rational cohomology of K if K has characteristic zero[9].

This is just the comparison theorem and the base change theorem in étale cohomology.

Let $V \subset \mathbb{P}^{n+r}$ be a nonsingular n-dimensional complete intersection of r hypersurfaces H_i of degrees $a_1, \dots, a_r$; the dimension h^{pq} of the Hodge group $H^{pq}(V)$ depends only on the a_i and n, so that it can be written $h^{pq}(a_1, \dots, a_r; n)$. Since V is a complete intersection, its cohomology is purely of dimension n, and the only nonzero h^{pq} are for $p = q$ or $p+q = n$; to dispose of the first of these, define h_0^{pq} to be the dimension of the primitive part of $H^{pq}(V)$, i.e., the orthogonal in $H^{pq}(V)$ to all the classes ξ^r of p. 14. Then $h_0^{pq} = h^{pq}$ if $p \neq q$, and $h_0^{pp} = h^{pp} - 1$.

Let $H(a_1, \dots, a_r) \in \mathbb{Z}[\![x, y]\!]$ be the formal power series in y and z defined by

$$H(a_1, \dots, a_r) = \sum_{p,q=0} h_0^{hq}(a_1, \dots, a_r; p+q) y^p z^q.$$

There is the following recipe for calculating the power series $H(a_1, \dots, a_r)$, which in fact turn out to be rational functions of y and z: firstly, if $r = 1$,

$$\begin{aligned} H(a) &= \frac{-(1+z)^a + (1+y)^a}{(1+z)^{a+1}y - (1+y)^{a+1}z} \\ &= \frac{a - 1 + \binom{a-1}{2}(y+z) + \binom{a-1}{3}(y^2 + yz + z^2) + \cdots}{1 - \binom{a}{2}yz - \binom{a}{3}yz(y+z) - \cdots}. \end{aligned}$$

Then $H(a_1, \dots, a_r)$ can be expressed as follows in terms of the $H(a_i)$:

$$H(a_1, \dots, a_r) = \sum_{P \subset [1,r]} \big((1+y)(1+z)\big)^{|P|-1} \prod_{i \in P} H(a_i),$$

[9] *And an embedding $K \subset \mathbb{C}$ is chosen.*

the sum being over all nonempty subsets of $[1, r]$ and $|P|$ denoting the number of elements of P.

We thus have

$$H(2) = \frac{1}{1 - yz} = 1 + yz + y^2z^2 + \cdots ,$$

so that the n-dimensional quadric has Betti numbers

$$b_{2r+1} = 0; \quad b_{2r} = \begin{cases} 1 & \text{if } 2r \neq n, \\ 2 & \text{if } 2r = n. \end{cases}$$

Next,

$$\begin{aligned} H(2,2) &= 2H(2) + (1+y)(1+z)(H(2))^2 \\ &= 2(1 + yz + y^2z^2 + \cdots) \\ &\qquad + (1 + y + z + yz)(1 + 2yz + 3y^2z^2 + \cdots). \end{aligned}$$

$h_0^{pq}(2,2)$, which is the coefficient of $y^p z^q$ in this series is then:

$$\begin{aligned} h_0^{pp} &= 2 + (p+1) + p = 2p + 3, \\ \text{and} \quad h_0^{p\,p+1} &= p + 1; \quad h_0^{pq} = 0 \text{ if } |p - q| > 1; \end{aligned}$$

thus an intersection of two quadrics of even dimension $2n$ has $2n$th Betti number $2n + 4$; if the dimension is $2n + 1$, then the $(2n + 1)$st Betti number is $2(n + 1)$, and the variety has Hodge level one.

Similarly, it is easy to see that the odd dimensional intersection of three quadrics is of Hodge level one, the relevant Hodge numbers here being

$$h^{n\,n+1} = (n+1)(2n+5)$$

For the proof of the Hirzebruch formula, see [10], p. 159, as well as [6, XI].

Chapter 1

Quadrics and their generators

In this paragraph I collect together for later convenience a number of results on quadrics. My special interest will be the case of quadrics that are either nonsingular, or are ordinary cones, and eventually the case of a pencil of quadrics whose general member is nonsingular, but with special members degenerating into ordinary cones.

If Q is a nonsingular quadric of dimension n, then a *generator* of Q is defined as being a linear subspace contained in Q and having the maximal dimension $[\frac{n}{2}]$. The generators of Q form a closed subvariety of the relevant Grassmannian, which is either connected or has two connected components according as n is odd or even. In either case, these varieties are rational and nonsingular. The notion of generator also extends to singular quadrics, which are just cones on nonsingular ones. It turns out to be convenient to define the generators of a singular quadric as being the $[\frac{n}{2}]$-planes that it contains, even if these are not the maximal linear subspaces. With this convention, a nonsingular quadric and an ordinary cone of the same dimension n have families of generators of the same dimension – so that it makes sense to ask how they vary in a family. The answer is contained in Theorem 1.10, where it is seen that if the quadrics have even dimension, the two families of generators of the nonsingular quadric join together to form a single family (that has to be counted "with multiplicity 2"), whereas the opposite happens for odd dimensional quadrics: the irreducible family of generators of the nonsingular quadric splits up into two components, which meet along a singular locus.

Notation k will denote an algebraically closed field of characteristic $\neq 2$, V a finite dimensional vector space over k, and $\mathbb{P} = \mathbb{P}(V)$ projective space.

Since $\mathrm{char}\, k \neq 2$, there is no distinction between the notions of quadratic form and symmetric bilinear form, and I will use the same letter, usually φ, to denote either. In particular, $\mathrm{Ker}\,\varphi$ will always mean the kernel of the bilinear form φ, that is, the collection of vectors orthogonal to all of V with respect to φ. φ determines a quadric $Q \subset \mathbb{P}$, and conversely, Q determines φ up to scalar multiplication

An r-plane of $\mathbb{P}$, s say, is of the form $s = \mathbb{P}(E)$, for a unique $(r+1)$-plane $E \subset V$; r-planes of $\mathbb{P}$ are parametrized by the Grassmannian variety $\mathrm{Gr}(r+1, V)$. The r-plane s is contained in Q if and only if the form φ vanishes identically on E, i.e., if and only if E is *isotropic* for φ. The r-planes contained in Q then form a closed subvariety of the Grassmannian $\mathrm{Gr}(r+1, V)$.

It is easy to see, for instance by using Lemma 1.3, that if Q is nonsingular, and of dimension n, then Q contains $[\frac{n}{2}]$-planes, and no bigger linear subspaces. Hence the following definition:

Definition 1.1 Let Q be a nonsingular quadric, of dimension n. Then a *generator* g of Q is an $[\frac{n}{2}]$-plane $g \subset Q$.

The generators of Q obviously form a closed subvariety $\mathrm{Gen}(Q)$ of the Grassmannian $\mathrm{Gr}(r+1, V)$.

Before dealing specifically with the generators of Q, I want to sketch the following argument for the dimension of the variety of r-planes of Q; we have already pointed out that this variety is empty if $2r > n$. If $2r \leq n$ then it is nonempty, and has dimension $\frac{(r+1)(2n-3r)}{2}$.

This can be seen by remarking that the Grassmannian $\mathrm{Gr}(r+1, V)$ has dimension $(r+1)(n+1-r)$, since V has dimension $n+2$; and imposing that φ vanishes identically on an r-plane is $\binom{r+2}{2}$ conditions – hence, the r-planes of Q form a variety of at least the indicated dimension. There is an easy infinitesimal calculation to show that these $\binom{r+2}{2}$ conditions are in fact independent, and this gives the required dimension.

Theorem 1.2 *Let Q be a nonsingular quadric, of dimension n, and let $G = \mathrm{Gen}(Q)$ be the variety of generators of Q. Then:*

(a) if $n = 2m+1$ then G is irreducible, nonsingular, and of dimension $\binom{m+2}{2}$. Further, G has affine coverings by affine space such that any two points g and $g' \in G$ belong to a common affine open.

(b) *If $n = 2m$, then $G = A \cup B$, A and B being disjoint irreducible varieties, which are nonsingular, and of dimension $\binom{m+1}{2}$. Further, A and B have affine covers by affine space such that any pair of points g and $g' \in A$ (resp. B) belong to a common affine open of A (resp. B). Finally, g and $g' \in G$ belong to the same component if and only if $\dim(g \cap g')$ is congruent modulo 2 to $\dim(g) = m$.*

The next few pages will be devoted to a proof of this theorem, and to obtaining a number of other results that will be useful later.

Lemma 1.3 *Let φ be a nondegenerate quadratic form on V. Then:*

(a) *If $\dim V = 2r + 1$, then there exist a basis $e_0, e_1, \ldots, e_r, f_1, \ldots, f_r$ of V such that $\varphi(\sum_i x_i e_i + \sum_i y_i f_i) = x_0^2 + \sum_i x_i y_i$; in other words, with respect to this basis, φ has the matrix form*

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix},$$

where the blocks are 1×1, $1 \times r$, $r \times 1$ and $r \times r$.

(b) *If $\dim V = 2r$, then there is a basis $e_1, \ldots, e_r, f_1, \ldots, f_r$ of V such that $\varphi(\sum_i x_i e_i + \sum_i y_i f_i) = \sum_i x_i y_i$, or in matrix form*

$$\varphi = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Furthermore, one can impose that $e_1, \ldots, e_s$ form a basis, or even a given *basis, of any given s-dimensional isotropic plane E of V.*

Remarks 1.4 (i) From this lemma, it is easy to deduce that a nonsingular quadric Q of dimension n has $[\frac{n}{2}]$-planes, but no bigger linear subspaces.

(ii) Let $E^\perp$ denote the orthogonal in V of E; since E is isotropic, $E \subset E^\perp$, and we can consider the quotient $E^\perp/E$. The form φ induces a form $\widehat{\varphi}$ on $E^\perp/E$, and the images in $E^\perp/E$ of (e_0), $e_{s+1}, \ldots, e_r, f_{s+1}, \ldots, f_r$ form a basis for $E^\perp/E$ with respect to which $\widehat{\varphi}$ has one of the indicated matrix forms; in particular $\widehat{\varphi}$ is nondegenerate. Isotropic planes of V containing E correspond naturally to isotropic planes (of dimension s smaller) of $E^\perp/E$. From this it is easy to see that the generators of Q containing a given $([\frac{n}{2}] - 1)$-plane $p \subset Q$ form naturally a plane conic (respectively, a pair of points of $\mathbb{P}^1$).

Proof of Lemma 1.3 By induction on $\dim V$. Let e_1 satisfy $\varphi(e_1) = 0$, e_1 being in E, or one of a given basis of E if this has been stipulated in advance; otherwise e_1 exists because k is algebraically closed and $\dim V > 1$. Then, since φ is nondegenerate, there exists an f_1' such that $\varphi(e_1, f_1') = 1$; then $f_1 = f_1' - e_1\varphi(f_1')$ satisfies $\varphi(e_1, f_1) = 1$, and $\varphi(f_1) = 0$. Now $\varphi_{|\{e_1,f_1\}}$ is nondegenerate, so that

$$V = \{e_1, f_1\} \oplus \{e_1, f_1\}^\perp$$

is a direct sum decomposition for φ. Induction completes the proof, since $\dim\{e_1, f_1\}^\perp = \dim V - 2$, and the result is trivial if $\dim V = 0$ or 1.

The following is a convenient way of introducing coordinate charts on Grassmannians: let $\mathrm{Gr} = \mathrm{Gr}(k, V)$ be the Grassmannian of k-dimensional vector subspaces of V, and let $\dim V = k + m$. Let $E \in \mathrm{Gr}$, and let F be a direct complement of E in V; the coordinate neighbourhood $A(F)$ is going to consist of all $E' \in \mathrm{Gr}$ with $E \cap F = 0$ – this is clearly a neighbourhood of E. The coordinate charts $x(E, F)$ are defined as follows: any direct complement E' of F in V is the graph of a unique mapping x, where $-x$ is the composite

$$E \xrightarrow{i_1} E \oplus F = V = E' \oplus F \xrightarrow{p_2} F.$$

The map $X\colon A(F) \to \mathrm{Hom}(E, F)$ given by $E' \mapsto x$ is the required coordinate chart. It is convenient to put the same thing into matrix form: let $e_1, \dots, e_k$ and $f_1, \dots, f_m$ base E and F. If, w.r.t. this basis, x has the matrix $X = (x_{ij})$ then E', the graph of x, is based by the k vectors $e_i + \sum_j x_{ij} f_j$. I intend to write this in the suggestive form $E' = (IX)$, so that in particular $E = (I0)$.

Using the above argument and Lemma 1.3, it is easy to describe a covering of $G = \mathrm{Gen}(Q)$ by affine open sets. To prove the remaining assertions of Theorem 1.2, it will only remain to check that these overlap as required.

Let $\dim Q = 2n$, and suppose that $g = \mathbb{P}(E) \in \mathrm{Gen}(Q)$. E is an isotropic $n+1$ plane of V, so that I can apply Lemma 1.3 to put φ in the form $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Let $E' = (IX)$ be an $(n+1)$-plane near to E. Then $\mathbb{P}(E')$ is a generator of Q if and only if $\varphi(e') = 0$, for all $e' \in E'$. But it is trivial to see that this is if and only if

$$\varphi(IX) = (IX)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I \\ {}^{\mathrm{t}}X \end{pmatrix} = 0, \quad \text{i.e.,} \quad X + {}^{\mathrm{t}}X = 0.$$

There is an intrinsic way of saying the same thing, without the use of a basis: let F be any isotropic direct complement of E in V (these exist, by Lemma 1.3); then φ identifies E^* with F, so that if $E' = \mathrm{Graph}(x)$, $x\colon E \to F$, then the quadratic form φ' defined on E by $\varphi'(e) = \varphi(e + x(e))$ is identified with the quadratic form associated to the bilinear form $x \in \mathrm{Hom}(E, F) = \mathrm{Hom}(E, E^*)$. Obviously, E' is isotropic if and only if φ' is identically zero, which takes place if and only if x is a skewsymmetric form.

Either of these arguments give a covering of $\mathrm{Gen}(Q)$ by affine coordinate patches of dimension $\binom{n+1}{2}$.

For the case of $\dim Q = 2n+1$, the argument is analogous: let $g = \mathbb{P}(E)$ be a given generator, $e_1, \dots, e_{n+1}, f_1, \dots, f_{n+1}, e_0$ the basis of Lemma 1.3, w.r.t. which φ has the matrix $\left(\begin{smallmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Then $F = \{f_1, \dots, f_{n_1}, e_0\}$ is a direct complement of E in V – and an $(n+1)$-plane E' disjoint from F can be written $E' = (IXY)$. Then E' is isotropic for φ if and only if

$$\varphi(IXY) = (IXY) \begin{pmatrix} 0 & I & 0 \\ I & 0 & \\ & & \end{pmatrix} \begin{pmatrix} I \\ {}^{\mathrm{t}}X \\ {}^{\mathrm{t}}Y \end{pmatrix} = 0, \quad \text{so that} \quad X + {}^{\mathrm{t}}X + Y\,{}^{\mathrm{t}}Y = 0.$$

These equations may be rewritten $x_{ii} = \frac{1}{2}y_i^2$, $x_{ij} = y_i y_j - x_{ji}$, so that y_i, x_{ij} $(i > j)$ form a system of affine coordinates in a neighbourhood of E.

The following lemma completes the proof of Theorem 1.2:

Lemma 1.5 *Let $g = \mathbb{P}(E)$ and $g' = \mathbb{P}(E')$ be two generators of the nonsingular quadric Q. Then:*

(a) if $\dim Q = 2n+1$, there exists a third generator $g'' = \mathbb{P}(F)$ of Q disjoint from g and g'.

(b) if $\dim Q = 2n$, then there exists such a third generator if and only if $\dim(g \cap g') \equiv \dim g \mod 2$.

Remark 1.6 It is clear that g and g' belong to a common affine neighbourhood of the above type if and only if there exists a third generator of Q disjoint from both of them.

Proof of Lemma 1.5 I show first of all that the condition on the parity of $\dim(g \cap g')$ in (b) is necessary. This is so since if $E' = \mathrm{Graph}(x)$, with $x\colon E \to F$, then $E \cap E' = \mathrm{Ker}(x)$; it is well known that the rank of a skewsymmetric matrix is necessarily even, and this implies the result.

Now let E and E' be two $(n+1)$-dimensional isotropic planes of V, and suppose either that $\dim V$ is odd, or that $\dim(E/E \cap E')$ is even. I must construct a third isotropic plane F of V, of dimension $n+1$, and meeting E and E' in 0.

Let C (resp. C') be a complement of $E \cap E'$ in E (resp. in E'), so that $E + E' = E \cap E' \oplus C \oplus C'$. Then $\varphi_{|C \oplus C'}$ is nondegenerate, since otherwise E or E' would be contained in a strictly larger isotropic subspace. Hence φ puts C and C' in duality, so that I can choose a basis of C, and this specifies the dual basis of C'. $(C \oplus C')^{\perp}$ is also nondegenerate for φ, and $E \cap E' \subset (C \oplus C')^{\perp}$ is a maximal isotropic subspace. I can thus choose a basis $e_1, \dots, e_{n+1}, f_1, \dots, f_{n+1}, (e_0)$, such that $e_1, \dots, e_r$ span C, $e_{r+1}, \dots, e_{n+1}$ span $E \cap E'$, $f_1, \dots, f_r$ span C', and φ has the matrix form

$$\varphi = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}.$$

I now choose F to be spanned by

$$f_{r+1}, \dots, f_{n+1}, e_1 + f_2, e_2 - f_1, e_3 + f_4, \dots, e_{2[\frac{r}{2}]-1} + f_{2[\frac{r}{2}]}, e_{2[\frac{r}{2}]} + f_{2[\frac{r}{2}]-1},$$

and finally, if r is odd, $e_r - \frac{1}{2} f_r + e_0$. It is clear that F is isotropic, of dimension $n+1$, and does not meet either E or E', so that the lemma is proved.

1.1 Cones

It will turn out to be important to have a result analogous to Theorem 1.2 for the case of quadrics having the simplest possible singularities, i.e., ordinary cones.

Definition 1.7 A variety $X \subset \mathbb{P}$ is said to be a *cone* from $O \subset X$ if for all $o \in O$, and all $x \in X$, the line ox joining the two is contained in X.

Let Q be a quadric in $\mathbb{P}(V)$, corresponding to the quadratic form φ on V. Then it is clear that Q is a cone with vertex O if and only if $P^{-1}(O) \subset \operatorname{Ker}(\varphi) \subset V$.

Let $K = \operatorname{Ker}(\varphi)$, and let V' be any direct complement of K in V, so that $V = K \oplus V'$, and $\varphi_{|V'}$ is nondegenerate. Obviously, the pair $(V', \varphi_{|V'}) \cong (V/K, \widehat{\varphi})$, so that the choice of V' is not important.

Now let $\mathbb{P}' = \mathbb{P}(V') \subset \mathbb{P}$, and $Q_0 = Q \cap \mathbb{P}'$. Then clearly, Q is a cone with vertex $O = \mathbb{P}(K)$, and with base Q_0, in the sense that Q is the join $O * Q_0$ of O and Q_0; since Q_0 is nonsingular, every quadric is a cone, with a well-defined vertex, and a nonsingular base determined up to projective equivalence.

Let $\dim K = k$; then $k = 0$ if and only if Q is nonsingular. k is called the *degeneracy* of Q; if $k = 1$, the Q is said to be *simply degenerate*, or to be an *ordinary cone.*

If Q is k-fold degenerate, then it can contain linear subspaces of dimension bigger than $[\frac{n}{2}]$: the maximal linear subspaces of Q are obviously the cones $g * O$, with g a generator of Q_0, so that they have dimension $[\frac{n-k}{2}] + k$. However, I still want to define the *generators* of Q to be the $[\frac{n}{2}]$-planes, reserving the name of *degenerators* for bigger linear subspaces. Theorem 1.8 will give a description of the family of generators on a simple cone. Before stating it, I would like to give the following plausibility argument for the dimension of the family of linear subspaces of any quadric.

Let Q be a quadric of dimension n and degeneracy k. Let Q be a cone with vertex O and base Q_0; Q_0 has dimension $n-k$, and O dimension $k-1$. A generic r-plane g of Q will meet O as little as possible, so that the join $g*O$ will be as big as possible. However, $g_0 = g * O \cap Q_0$ can only have dimension $[\frac{n-k}{2}]$, since Q_0 is nonsingular. Let $s = \min(r, [\frac{n-k}{2}])$, so that for generic g, g_0 has dimension s. To determine g, one must choose g_0, an s-plane of Q_0, and then choose any r-plane of the join $g_0 * O$ – so that the family of r-planes g has dimension[1]

$$\frac{(s+1)(2n-2k-3s)}{2} + (r+1)(k+s-r).$$

Theorem 1.8 *Suppose Q a simple cone. Then:*

(a) *If $\dim Q = 2n+1$, then Q has a family $G = \mathrm{Gen}(Q)$ of generators. $G = A \cup B$, where A and B are irreducible and of dimension $\binom{n+2}{2}$, and $A \cap B$, consisting of all the generators of Q passing through the vertex O (which is now just a point), is nonsingular and of codimension one in both A and B.*

(b) *If $\dim Q = 2n$, there is a single irreducible family G of generators, which is naturally identified with the family of generators of a base Q_0 of Q. Hence $\dim G = \binom{n+1}{2}$.*

[1] As before, this variety is empty if $r > [\frac{n+k-1}{2}]$.

In both (a) and (b), the families considered have affine open coverings by affine space, such that any two points belong to a common affine open.

Proof (b) is clear: indeed, let Q_0 be a base of Q, which is then a nonsingular quadric of dimension $2n-1$. An n-plane of Q necessarily intersects Q_0 in an $(n-1)$-plane, which is then a generator of Q_0.

(a) Let Q_0 again denote a base of Q, and let A_0 and B_0 be the two families of generators of Q_0. Let $A \subset G$ and $B \subset G$ be the subsets consisting of generators g such that the intersection $g_0 = g * O \cap Q_0$ is contained in a generator of A_0, resp. of B_0; since g_0 is either an n-plane or an $(n-1)$-plane, this generator is uniquely determined (by Remark 1.4 after Lemma 1.3, together with the remark that the two generators containing a given $(n-1)$-plane are necessarily in different families). The morphisms

$$a\colon A \to A_0 \quad \text{and} \quad b\colon B \to B_0$$

defined by $g \mapsto$ (the unique generator containing g_0) have as fibres the variety of n-planes of $g_0 * O$, which is obviously just $\mathbb{P}^{n+1}$; hence A and B are smooth and of dimension $\binom{n+1}{2} + n + 1 = \binom{n+2}{2}$. It is fairly obvious that a (resp. b) also fibres $A \cap B$ in $\mathbb{P}^n$s, namely in the subvariety of $\mathbb{P}^{n+1}$ of n-planes through O. This completes the proof.

1.2 Pencils of quadrics

To study the intersection $X = Q_1 \cap Q_2$ of two quadrics, it is necessary to study the pencil Φ of quadrics through X. If Q_1 and Q_2 correspond to the quadratic forms φ_1 and φ_2, then Φ consists of the quadrics of $\mathbb{P}$ with equations $\lambda\varphi_1 + \mu\varphi_2$, with $(\lambda, \mu) \in \mathbb{P}^1$.

Systems of quadrics have been studied in [6, XII] under the additional assumption that all the quadrics of the system are nonsingular. I want to give here the results that I need in §3 and §4 is as elementary a context as possible.

Recall that the quadratic forms on a given vector space V form a vector space, which in characteristic $\neq 2$ can be denoted S^2V^*. So far, quadratic forms have been considered implicitly only up to multiplication by nonzero scalars, i.e., as points of the projective space $\mathbb{P}(S^2V^*)$.

Definition 1.9 A *pencil of quadratic forms* Φ on V is a projective line $\mathbb{P}^1_\Phi \subset \mathbb{P}(S^2V^*)$.

The *pencil of quadrics* associated to Φ, usually denoted by the same letter, is the pencil of quadric hypersurfaces of $\mathbb{P}$ with equations φ_λ, $\lambda \in \mathbb{P}^1_\Phi$.

If Φ is a pencil of quadrics, consider the subvariety

$$Z_\Phi \subset \mathbb{P}^1 \times \mathbb{P} \quad \text{defined by} \quad Z_\Phi = \bigl\{(\lambda, x) \bigm| x \in Q_\lambda\bigr\}.$$

The first projection $Z_\Phi \to \mathbb{P}^1$ is proper and flat, with fibres the quadrics Q_λ.

From now on, I will assume that the pencil Φ is *nonsingular* in the following strong sense: the quadrics Q_λ, $\lambda \in \mathbb{P}^1$ are either nonsingular or simple cones; and for the values λ such that Q_λ is degenerate, if e_λ is a basis for $\operatorname{Ker}(\varphi_\lambda)$, $\varphi_\mu(e_\lambda) \neq 0$ for some, and hence for all, values of $\mu \neq \lambda$. This condition will be discussed in §2; it implies in particular that the generic Q_λ is nonsingular (Proposition 2.1).

Let $\dim Q_\lambda = n$. Then each of the Q_λ contains $[\frac{n}{2}]$-planes. These families of generators fit together in the following way: let $\operatorname{Gen}(\Phi) \subset \mathbb{P}^1_\Phi \times \mathrm{Gr}$, where $\mathrm{Gr} = \mathrm{Gr}([\frac{n}{2}] + 1, V)$ is the usual Grassmannian, be defined by

$$\operatorname{Gen}(\Phi) = \bigl\{(\lambda, E) \bigm| \varphi_\lambda \equiv 0\bigr\}.$$

It is obvious that the first projection $\operatorname{Gen}(\Phi) \to \mathbb{P}^1$ has as (set theoretic) fibres the families of generators of Q_λ, i.e., the families of generators discussed in Theorem 1.2 or Theorem 1.8 for good or bad values of λ.

Suppose that the pencil Φ is nonsingular in the above strong sense. Then there are just a finite number of values of $\lambda \in \mathbb{P}^1$, say $\lambda_0, \dots, \lambda_N$ for which Q_λ is singular. Let $\operatorname{Sing}(\Phi) = \{\lambda_0, \dots, \lambda_N\} \subset \mathbb{P}^1$.

Theorem 1.10 $\operatorname{Gen}(\Phi)$ *is nonsingular, and:*

(a) If $\dim Q_\lambda$ *is odd, the fibres of* $p_1\colon \operatorname{Gen}(\Phi) \to \mathbb{P}^1$ *are reduced, and are the families of generators described in Theorems 1.2 and 1.8, (a).*

(b) If $\dim Q_\lambda$ *is even, then* p_1 *has the factorization*

$$\begin{tikzcd} \operatorname{Gen}(\Phi) \arrow[d, "p_1"'] \arrow[dr, "p"] & \\ \mathbb{P}^1 & C \arrow[l, "q"] \end{tikzcd} \qquad (*)$$

where C is nonsingular, q is a double covering ramified precisely in $\mathrm{Sing}(\Phi)$*, and p is smooth. In particular the fibres of p_1 are the pairs of families of generators of Theorem 1.2 above $\lambda \notin \mathrm{Sing}(\Phi)$, and is the family of generators of Theorem 1.8, (b) counted with multiplicity two above points of* $\mathrm{Sing}(\Phi)$.

Proof The only difficult assertion is the existence of the factorization $(*)$, and fortunately this is general nonsense. The remaining assertions can be checked by infinitesimal calculations at points of $\mathrm{Gen}(\Phi)$. I will only carry out one of these, that giving the ramification of C at points of $\mathrm{Sing}(\Phi)$; the others are analogous, and easier.

Suppose then that $\dim Q_\lambda$ is even. Then the general fibre of p_1 is the family of generators of Theorem 1.2, (b), which has two connected components. There is a canonical way of factorising any map as a composite $q \circ p$, with q finite, and the fibres of p (geometrically) connected, namely its Stein factorization ([EGA], III, 4.3). The only other fact I need about the Stein factorization is that $p_*(\mathcal{O}_{\mathrm{Gen}(\Phi)}) = \mathcal{O}_C$ (as $\mathcal{O}_C$-algebra), which is more or less the definition of C.

Clearly the fibres of q must be a pair of points over a generic point of $\mathbb{P}^1$, and a single point (with some subscheme structure) over a point of $\mathrm{Sing}(\Phi)$. Hence $q\colon C \to \mathbb{P}^1$ is a double covering ramified in $\mathrm{Sing}(\Phi)$. I still need to prove that $\mathrm{Gen}(\Phi)$ and C are nonsingular, that the differential of the map p is nonzero (and hence of maximal rank), and that q has ordinary ramification at points of $\mathrm{Sing}(\Phi)$.

Now all these can be checked by an infinitesimal calculation: for once $\mathrm{Gen}(\Phi)$ is nonsingular, it is normal, so that C is also normal (the affine rings $\Gamma(U, \mathcal{O}_C)$ of C are just $\Gamma(V, \mathcal{O}_{\mathrm{Gen}(\Phi)})$, which are integrally closed). And if we check that the second derivative of the map p_1 is nonzero, then it's not possible for both p and q to fail to be smooth, nor for q to be worse than ordinarily ramified. Since q is certainly not smooth, this will prove the theorem.

Lemma 1.11 $\mathrm{Gen}(\Phi)$ *is nonsingular, and the map p_1 has nonzero second derivative.*

Proof Let $x \in \mathrm{Sing}(\Phi)$, $g \in \mathrm{Gen}(\Phi)$ over x, and let $\mathcal{O}_x, m_X$ and $\mathcal{O}_g, m_g$ be the local rings of x in $\mathbb{P}^1$ and g in $\mathrm{Gen}(\Phi)$ respectively, together with their maximal ideals. Then p_1 induces a map

$$p_1\colon \mathcal{O}_x \to \mathcal{O}_g$$

taking m_x into m_g. The second derivative of p_1 is (the dual of) the map $m_x/m_x^3 \to m_g/m_g^3$ induced by p_1.

Now to carry out the infinitesimal calculation: Q_x is a singular quadric of the pencil, so let Q_x be associated to the form φ_0 on V, and let e_0 generate $\operatorname{Ker}(\varphi_0)$, so that $\varphi_1(e_0) \neq 0$ for some other φ_1 in the pencil. g corresponds to a maximal isotropic subspace E of V, and we must have $e_0 \in E$. It is convenient to choose as complement of e_0 in V the orthogonal $e_0^\perp$ of e_0 w.r.t. φ_1. Then $\varphi_0{}_{|e_0^\perp}$ is nondegenerate, and $E \cap e_0^\perp$ is a maximal isotropic subspace of $e_0^\perp$ w.r.t. φ_0, so that Lemma 1.3 can be used to provide a basis $e_0, \ldots, e_n, f_0, \ldots, f_n$ such that the matrix of φ_0 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 1 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \quad \text{with } 1\times 1,\ n \times 1 \text{ and } n\times n \text{ blocks.}$$

In other words, the only nonzero products are $\varphi_0(f_0) = 1$, $\varphi_0(e_i, f_i) = 1$ for $i > 0$. Recall also that e_0 is orthogonal to the rest of the basis w.r.t. φ_1; we can take it that $\varphi_1(e_0) = 1$.

A neighbourhood of (g, x) in $\mathrm{Gr} \times \mathbb{P}^1$ is given by $\{(E', \lambda)\}$, where E' is the $(n+1)$-plane of V spanned by the vectors $(e_i + \sum x_{ij} f_j)$, and $(E', \lambda) \in \operatorname{Gen}(\Phi)$ if and only if

$$(\varphi_0 + \lambda\varphi_1)\Big(e_i + \sum x_{ij} f_j, e_{i'} + \sum x_{i'j} f_j\Big) = 0 \quad \text{for all } i, i'.$$

These can be divided up as follows:

$$\lambda + (x_{00})^2 + \lambda\varphi_1\Big(\sum x_{0k} f_k\Big) = 0 \tag{00}$$

$$x_{0j} + x_{00}x_{j0} + \lambda\varphi_1\Big(\sum x_{0k} f_k, e_j + \sum x_{jk} f_k\Big) = 0, \quad j \neq 0 \tag{0j}$$

$$x_{ij} + x_{ji} + x_{i0}x_{j0} + \lambda\varphi_1\Big(e_i + \sum x_{ik} f_k, e_j + \sum x_{jk} f_k\Big) = 0, \quad i \geq j \neq 0. \tag{ij}$$

It's easy to read off from these relations that $\lambda \not\equiv 0 \mod m_x^3$, which proves what was required.

Appendix to §1. The cohomology of quadrics

In this appendix I wish to give an elementary calculation of the cohomology of the nonsingular quadrics studied in §1. The method is as follows: one can write down some obvious cohomology classes, namely those of r-plane sections of X, or those of r-planes lying in X. It turns out to be quite easy to see that these generate the cohomology of X – indeed, they certainly span a $\mathbb{Z}_\ell$-submodule, and it is not too hard to see that this has the right dimension, and that the cup product restricted to this submodule satisfies Poincaré duality.

The Hirzebruch formula (see §0, p. 18 and the references given there) reads:

$$H(2) = \frac{1}{1-yz} = 1 + yz + y^2z^2 + \cdots .$$

Hence the Betti numbers of the quadric X are as follows:

$$\dim X = 2n, \quad b_i(X) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ 1 & \text{if } i \text{ is even, } 0 \le i \le 4n \text{ and } i \neq 2n, \\ 2 & \text{if } i = 2n. \end{cases}$$

$$\dim X = 2n+1, \quad b_i(X) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ 1 & \text{if } i \text{ is even, } 0 \le i \le 4n+2. \end{cases}$$

Let $\eta \in H^2(X, \mathbb{Z}_\ell)(-1)$ be the class of the hyperplane section. Then $\eta^i \in H^{2i}(X, \mathbb{Z}_\ell)(-i)$ is nonzero, since the cup product $\eta^{\dim X} = 2$, considered as an element of $\mathbb{Z}_\ell \cong H^{2\dim X}(X, \mathbb{Z}_\ell)(-\dim X)$.

The case of $\dim X$ odd is now easy to polish off: let $a \in H^{2n}(X, \mathbb{Z}_\ell)(-n)$ be the class of a generator – all generators are rationally equivalent by Theorem 1.2, (a), so that this is uniquely determined. Now $\eta^n = 2a$, as can easily be seen by considering any generator A of X, and an $(n+1)$-plane L containing A: for $L \cdot X = A \cup A'$. Hence:

Theorem 1.12 *The classes*

$$1, \eta, \dots, \eta^{n-1}, a, \eta a, \dots, \eta^{n+1} a$$

generate $H^{2i}(X, \mathbb{Z}_\ell)(-i)$ *for* $i = 0, 1, \dots, 2n+1$ *as* $\mathbb{Z}_\ell$*-module.*

Consider now the case of even dimensional X, say $\dim X = 2n$. Let a and b be the classes of the generators from the two families of Theorem 1.2, (b). Firstly note that $\eta a = \eta b$, since both are the cohomology class of generators of the hyperplane section of X, and all such generators are rationally equivalent even in the hyperplane section; further, $\eta^n a = 1$.

Theorem 1.13 *The classes*

$$1, \eta, \dots, \eta^{n-1}, a \textit{ and } b, \eta a, \dots, \eta^n a$$

generate $H^{2i}(X, \mathbb{Z}_\ell)(-i)$ *for* $i = 0, 1, \dots, 2n$ *as* $\mathbb{Z}_\ell$*-module.*

Proof It remains to establish that a and b are linearly independent, and that they span $H^{2n}(X, \mathbb{Z}_\ell)(-n)$; but both of these assertions follow once it is established that the cup product on $\mathbb{Z}a \oplus \mathbb{Z}b$ is nondegenerate and unimodular; in fact, the following are the only two possibilities for the cup product:

$$\begin{aligned} a^2 = b^2 = 1, &\quad ab = 0 \quad \text{if } n \text{ is even, or} \\ a^2 = b^2 = 0, &\quad ab = 1 \quad \text{if } n \text{ is odd.} \end{aligned}$$

For in the proof of Theorem 1.2, (b), it was seen that one can find generators from either family meeting a given generator in either one point or not at all, and the above intersection numbers come at once on chasing the parity.

Chapter 2

Generalities on the intersection of two quadrics

In this paragraph I will study those properties of the intersection of two quadrics $X = Q_1 \cap Q_2$ which are common to the cases of even and odd dimensions. There are two principal results: Proposition 2.1, which discusses the nonsingularity of X and that of the pencil of quadrics through X, and constructs an essentially unique basis of V w.r.t. which the two quadratic forms defining X are both diagonalized; and Theorem 2.6, which proves that the families of linear subspaces of X are nonsingular.

The proof of Theorem 2.6 is just a computation of the tangent space at a point of the family; and this involves some linear algebra that is messy at the best of times, and which can best be done by choosing a particular basis. However, it is not as easy to choose bases to bring two quadratic forms to a standard shape as it was for just one; the fact that there is a unique basis bringing both forms to diagonal form is an example of this.

The orthogonal reflections in these basic vectors turns out to provide a way of picking good bases (Lemma 2.2); they will also turn out to be important in §3.

I will preserve the notation of the end of §1, so that if $X = Q_1 \cap Q_2$, Φ will denote the pencil of quadrics through X.

If $e \in V$, let $e^\perp$ denote its orthogonal w.r.t. Φ, that is

$$e^\perp = \bigcap_{\lambda \in \mathbb{P}^1} e^{\perp_{\varphi_\lambda}} = e^{\perp_{\varphi_0}} \cap e^{\perp_{\varphi_1}}.$$

For most $e \in V$, this will be a subspace of V of codimension 2. It is clear that if the codimension of $e^\perp$ is less than 2, then $e \in \operatorname{Ker}(\varphi_\lambda)$ for some $\lambda \in \mathbb{P}^1$.

If $x \in X$, then the tangent space $T_{x,X}$ to X at x is the intersection of the tangent spaces to Q_1 and Q_2 at x, so that if $x \in \mathbb{P}$ is represented by $e \in V$, then $T_{x,X} = \mathbb{P}(e^\perp)$. Hence X is nonsingular and of codimension 2 in $\mathbb{P}$ if and only if for all $x \in X$, $e \in V$ representing x, $e \notin \operatorname{Ker}(\varphi_\lambda)$ for any $\lambda \in \mathbb{P}^1$; or put the other way round, $e \in \operatorname{Ker}(\varphi_\lambda) \implies \varphi_\mu(e) \neq 0$ for some (and hence for all) other points $\mu \in \mathbb{P}^1$.

Proposition 2.1 *The following four conditions are equivalent:*

(a) X *is nonsingular and of codimension* 2 *in* $\mathbb{P}$.

(b) The pencil Φ *is nonsingular in the sense of §1, Definition 1.9.*

(c) $\det(\varphi_\lambda)$, *which is a polynomial in* λ *of degree* $\dim V$, *is not identically zero and has* $n+1$ *distinct roots.*

(d) There exists a basis of V, *orthogonal for all the* φ_λ, *such that*

$$\varphi_1\left(\sum x_i e_i\right) = \sum x_i^2,$$
$$\varphi_2\left(\sum x_i e_i\right) = \sum \lambda_i x_i^2,$$

and $\lambda_i \neq \lambda_j$ *for* $i \neq j$.

Furthermore, the basis $\{e_i\}$ *in (d) is unique up to change of signs in front of the* e_i.

Proof It is clear that the condition $e \in \operatorname{Ker}(\varphi_\lambda) \implies \varphi_\mu(e) \neq 0$ implies that $\operatorname{Ker}(\varphi_\lambda)$ is at most one dimensional, so that (a) $\implies$ (b) by the argument above.

(b) $\implies$ (c). Let $\lambda = 0$ be a root of $\det(\varphi_\lambda)$. I will prove that $\det(\varphi_\lambda) = \lambda \cdot f(\lambda)$ with $f(\lambda)$ a polynomial and $f(0) \neq 0$. Let $\operatorname{Ker}(\varphi_0)$ be spanned by e_0, and let $\varphi_1(e_0) \neq 0$, so that $e_0^\perp$ is a hyperplane of V. Then the decomposition $V = \{e_0\} \oplus e_0^\perp$ is orthogonal for all φ_λ, $\lambda \in \mathbb{P}^1$, and hence I can calculate the determinant $\det(\varphi, V) = \det(\varphi_{|\{e_0\}}) \cdot \det(\varphi_{|e_0^\perp})$, and $\varphi_{|e_0^\perp}$ is nondegenerate by hypothesis.

(c) $\implies$ (d). Since $\det(\varphi_\lambda)$ is not identically zero, I can take φ_1 to be some nondegenerate quadratic form of the pencil. The required basis is going to consist of the $e_i \in \operatorname{Ker}(\varphi_{\lambda_i})$, for λ_i a singular value of the pencil. If $i \neq j$

then e_i and e_j are orthogonal for φ_{λ_i} and φ_{λ_j}, and hence for all forms of the pencil. Clearly, there is a unique (up to sign) way of normalizing the e_i such that $\varphi_1(e_i) = 1$. The e_i then form a basis, since there are $\dim V$ of them and they are orthogonal w.r.t. a nondegenerate quadratic form, and hence linearly independent.

(d) $\implies$ (a) is an easy computation.

Finally, the uniqueness of the basis in (d) is more or less obvious: if there is such a basis, then $e_i \in \operatorname{Ker}(\lambda_i\varphi_1 - \varphi_2)$, so that the basis coincides with the one constructed above.

From now on, let $\operatorname{Sing}(\Phi) = \{\lambda_1, \dots, \lambda_{\dim V}\}$, and e_i span $\operatorname{Ker}(\lambda_i\varphi_1 - \varphi_2)$.

The pair $\{V, \Phi\}$ has a number of symmetries: the orthogonal reflections in the base vectors e_i. In fact these are the only symmetries that are k-linear: for any symmetry must take e_i into an element of $\operatorname{Ker}(\lambda_i\varphi_1 - \varphi_2)$, with a given value of $\varphi_1(e_i)$, hence into $\pm e_i$. For any value of i, let σ_i denote the reflection in e_i (taking e_i to $-e_i$, and fixing the other e_j), and let G' denote the group of linear transformations of V generated by the σ_i, and $G = G'/\pm 1$ the corresponding group of transformations of $\mathbb{P}$, and of $X \subset \mathbb{P}$.

The use of the symmetries $\sigma \in G'$ for finding good bases of V is based on the following simple idea: let $x \in X$ be represented by $f \in V$, so that $\varphi_1(f) = \varphi_2(f) = 0$, and let e_i be such that $e_i \in \operatorname{Ker}(\lambda_i\varphi_1 - \varphi_2)$, and $\varphi_1(e_i, f) = 1$ (say). Then e_i and f span a 2-plane on which φ_1 and φ_2 are proportional: $\varphi_2 = \lambda_i\varphi_1$. Choosing the basis f, $\sigma_i(f) = f - e_i/\sqrt{2}$ of this 2-plane, these take the matrix forms $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\lambda_i \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ respectively.

Lemma 2.2 *Let X be a nonsingular intersection of two quadrics, and let $g \subset X$ be an r-plane, so that $g = \mathbb{P}(E)$ with E an $(r+1)$-plane isotropic for both φ_1 and φ_2. Let* $\dim V = N$.

Then there exists a basis $f_1, \dots, f_N$ of V such that $f_1, \dots, f_{r+1}$ span E, and such that w.r.t. this basis, φ_1 and φ_2 have the matrix forms

$$\varphi_1 = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 & L & * \\ L & 0 & * \\ * & * & * \end{pmatrix}, \quad L = \operatorname{diag}(\lambda_1, \dots, \lambda_{r+1}), \qquad (*)$$

with $\lambda_i \neq \lambda_j$.

Remark 2.3 The result proved is in fact stronger, but awkward to state in a more useful form: I prove that there exists a subset $\{\lambda_1, \dots, \lambda_{r+1}\} \subset \operatorname{Sing}(\Phi)$, and a basis $f_1, \dots, f_{r+1}$ of E such that the two-planes f_i, e_i are of the above form, and such that f_j is orthogonal to e_i (for $i \neq j$), so that $\sigma(f_i) = \sigma_i(f_i)$, where $\sigma = \sigma_1 \circ \dots \circ \sigma_{r+1}$.

Proof Let φ_1 be chosen as a nonsingular form of the pencil. Then the mapping $\widehat{\varphi}_1 \colon V \to E^*$ given by $v \mapsto \varphi_1(v, \ )$ is surjective; and in particular the images $\widehat{e}_1, \dots, \widehat{e}_N$ of the basis $e_1, \dots, e_N$ spans E^*. Hence I can pick a maximal linearly independent set $\widehat{e}_1, \dots, \widehat{e}_{r+1}$ of them to base E^*. Let $f_1, \dots, f_{r+1}$ be the dual basis of E, and let $\sigma = \sigma_1 \circ \dots \circ \sigma_{r+1}$. Then clearly, $\sigma(f_i) = \sigma_i(f_i)$, and letting $\sigma(f_i) = f_{r+i+1}$, we see that $f_1, \dots, f_{2r+2}$ span a subspace of V for which φ_1 has the matrix form $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, and is hence nondegenerate. Let $f_{2r+3}, \dots, f_N$ be any orthonormal base of the orthogonal complement of this space w.r.t. φ_1. It's easy to see that w.r.t. this base, φ_1 and φ_2 have the stated matrix form.

Corollary 2.4 *$2r \leq \dim X$.*

Proof It is clear from the indicated shape of the matrices that $2r - 1 \leq \dim X$. But if $\dim X = 2r - 1$, it is easy to see that the matrices of φ_1 and φ_2 are $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & L \\ L & 0 \end{smallmatrix}\right)$, which contradicts the nonsingularity of X, since the roots of the characteristic polynomial $\det(\lambda\varphi_1 - \varphi_2)$ would then be double.

Addendum 2.5 *Suppose that X has equations φ_1 and φ_2, having the matrix form of Lemma 2.2:*

$$\varphi_1 = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 & L & A \\ L & 0 & B \\ {}^{\mathrm{t}}A & {}^{\mathrm{t}}B & C \end{pmatrix}.$$

Then for X to be nonsingular, it is necessary that no complete row of A or B vanish.

Proof For suppose, say, that the ith row of A were to vanish. Then f_i is in the kernel of $\lambda_i\varphi_1 - \varphi_2$, and $\varphi_1(f_i) = \varphi_2(f_i) = 0$, so that X is singular by the discussion preceding Proposition 2.1.

Theorem 2.6 *Let X be a nonsingular intersection of two quadrics, and let $\dim X = 2r+k$. Let* Gr *be the Grassmannian of r-planes of $\mathbb{P} = \mathbb{P}^{2r+k+2}$, and let $S \subset \mathrm{Gr}$ be the subvariety of planes lying in X. Then S is nonsingular, reduced, and*[1] *of dimension $k(r+1)$.*

Proof As we have already seen, Gr is of dimension $(r+1)(r+k+2)$, and of course nonsingular. I must show that S is given locally by $(r+1)(r+2)$ equations, and that the tangent space to S at any point, given by the linear parts of these equations, is of dimension $k(r+1)$.

Let $s = \mathbb{P}(E) \in S$, so that s is an r-plane of X. Applying Lemma 2.2, I can take φ_1 and φ_2 in standard form. If $s' = \mathbb{P}(E')$ is a neighbouring r-plane of $\mathbb{P}$, let E' be spanned by the $r+1$ vectors

$$f_i + \sum_{j=r+2}^{2r+2} x_{ij} f_j + \sum_{j=2r+3}^{2r+k+3} y_{ij} f_j.$$

I will as usual abbreviate this to $E' = (IXY)$, with X an $(r+1)$ square matrix, and Y and $(r+1)\times(k+1)$ matrix. s lies in X if and only if φ_1 and φ_2 both vanish identically on E', which can be written (using the matrix forms of Addendum 2.5 to Lemma 2.2)

$$\varphi_1(IXY) = X + {}^{\mathrm{t}}X + Y{}^{\mathrm{t}}Y = 0 \quad \text{and}$$
$$\varphi_2(IXY) = XL + L{}^{\mathrm{t}}X + Y{}^{\mathrm{t}}A + A{}^{\mathrm{t}}Y + Y{}^{\mathrm{t}}B{}^{\mathrm{t}}X + XB{}^{\mathrm{t}}Y + YC{}^{\mathrm{t}}Y = 0.$$

It is easy to see that there are here at most $(r+1)(r+2)$ equations (essentially, two for each pair ij, $1 \le i \le j \le r+1$). And the tangent space to S at s is given by the linear terms, i.e.,

$$X + {}^{\mathrm{t}}X = 0 \tag{1}$$
$$XL + L{}^{\mathrm{t}}X + Y{}^{\mathrm{t}}A + A{}^{\mathrm{t}}Y = 0. \tag{2}$$

$(1)_{ij}$ is $x_{ij} = -x_{ji}$, and $(1)_{ii}$ is $x_{ii} = 0$. Recalling that $L = \mathrm{diag}(\lambda_1, \dots, \lambda_{r+1})$ with $\lambda_i \neq \lambda_j$, $(2)_{ij}$ is just

$$x_{ij} = (\lambda_i - \lambda_j)^{-1} \sum y_{ik} a_{jk} + y_{jk} a_{ik};$$

[1] *S is purely of dimension $k(r+1)$ – I do not assert that S is nonempty.*

and $(2)_{ii}$ is just

$$\sum_j y_{ij} a_{ji} = 0.$$

Thus $(2)_{ij}$ determines x_{ij} in terms of the y_{ij}, and $(2)_{ii}$ imposes one linear condition on each row of Y – a linear condition that is not trivial, on account of Addendum 2.5 to Lemma 2.2. Hence the tangent space to S at s has dimension precisely $k(r+1)$, i.e., the number of y_{ij} less the number of relations $(2)_{ii}$. This completes the proof of the theorem.

Chapter 3

Two quadrics – the even dimensional case

Throughout this paragraph, X will denote a nonsingular intersection of two quadrics, and $\dim X = 2n$. I will be interested in the set Σ of n-planes on X, and in particular I want to prove that there are enough n-planes to span the cohomology. The results are as follows:

Theorem 3.8 gives a complete description of the set Σ: X has 2^{2n+2} n-planes, and their intersection properties are established.

Theorem 3.14 asserts that the n-planes of X span the cohomology. It is proved by means of a calculation of the intersection form on the sublattice of the cohomology groups spanned by the classes of the n-planes. This form turns out to be unimodular and of rank $2n+4$, which proves the theorem, since the Hirzebruch formula for the nth Betti number of X implies that $H^n(X)$ is $(2n+4)$-dimensional.

In Theorem 3.19, I discuss the symmetry properties of Σ. The group of symmetries of Σ is easily seen to be the Coxeter group D^{2n+3}; this is also the (orthogonal) symmetry group of the cohomology lattice. Theorem 3.19 will provide an essential step in the proof of the results of the next paragraph.

The work of the proof of Theorem 3.8 is contained in two lemmas: the first shows, by means of a messy matrix computation, that X contains at least one n-plane. The other discusses the r-planes of a $2r$-dimensional vector space U which are isotropic under two quadratic forms ψ_1 and ψ_2, whose matrices are $\psi_1 = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ and $\psi_2 = \left(\begin{smallmatrix} 0 & L \\ L & 0 \end{smallmatrix}\right)$, where L is diagonal with distinct entries.

I preserve the notation of §2. In particular,

$$\operatorname{Sing}(\Phi) = \bigl\{\lambda_i \bigm| \det(\lambda_i\varphi_1 - \varphi_2) = 0\bigr\}, \quad \text{and } e_i \in \operatorname{Ker}(\lambda_i\varphi_1 - \varphi_2).$$

Lemma 3.1 *Suppose that X and X' are two intersections of two quadrics, and suppose that* $\operatorname{Sing}(\Phi)$ *and* $\operatorname{Sing}(\Phi')$ *are projectively equivalent as subsets of $\mathbb{P}^1$, so that up to possible permutation and changes of basis in the pencil Φ' we may take*

$$\lambda_i = \lambda_i'.$$

Then there exists 2^{2n+2} (projective) isomorphisms of X with X'.

Proof This is completely obvious in view of Proposition 2.1. In the particular case $X = X'$, the group of automorphisms of X contains the group G introduced in §2 (and will only be bigger than G if $\operatorname{Sing}(\Phi)$ has nontrivial projective automorphisms).

Lemma 3.2 *Suppose that $f(X)$ is a given polynomial of degree $2n+3$ with distinct roots $\lambda_1, \dots, \lambda_{2n+3}$, and suppose that $n+1$ of these roots, say $\lambda_1, \dots, \lambda_{n+1}$ have been selected out; say*

$$f(X) = \prod_{i=1}^{2n+3}(X - \lambda_i), \quad g(X) = \prod_{i=1}^{n+1}(X - \lambda_i).$$

Then there exist $a, b_1, \dots, b_{n+1}, c_1, \dots, c_{n+1}$ such that, letting A and B be the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}, \quad B = \begin{pmatrix} a & b & c \\ {}^{t}b & 0 & L \\ {}^{t}c & L & 0 \end{pmatrix},$$

where $L = \operatorname{diag}(\lambda_1, \dots, \lambda_{n+1})$, $b = (b_1, \dots, b_{n+1})$ and $c = (c_1, \dots, c_{n+1})$, we have

$$\det(AX - B) = f(X).$$

Proof Let $h(X)$ be the quotient $f(X)/g(X)$, so that $f(X) = g(X)h(X)$; then $h(X)$ is a monic polynomial of degree $n+2$. Now expand out the determinant $\det(XA-B)$ by the first row, using the fact that $\det(XI-L) = g(X)$:

$$\pm\det(XA-B) = g(X)\left\{(X-A)g(X) + \sum_{i=1}^{n+1} 2b_ic_ig_i(X)\right\},$$

where I've put

$$g_i(X) = \frac{g(X)}{(X-\lambda_i)} = \prod_{\substack{j=1\\ j\neq i}}^{n+1}(X-\lambda_j).$$

The lemma will be proved if I can show that a suitable choice of the a and $2b_ic_i$ will make the polynomial in curly brackets equal to any monic polynomial $h(X)$ of degree $n+2$. This is quite easy: divide $h(X)$ by $g(X)$ with remainder, giving

$$h(X) = (X-a)g(X) + r(X), \quad \text{with } \deg(r(X)) \leq n.$$

I now claim that the polynomials $g_i(X)$ span the vector space of polynomials of degree $\leq n$. For there are $n+1$ of them, and they are linearly independent, since if $\sum a_ig_i(X) = 0$, then in particular $\sum a_ig_i(\lambda_j) = 0$, so that $a_j = 0$.

Corollary 3.3 *X contains n-planes.*

For by Lemma 3.1, X can be given by $\varphi_1 = \varphi_2 = 0$, where φ_1 and φ_2 are given by the matrices of Lemma 3.2.

I will need to discuss the following situation: s and s' are two n-planes of X, and $\dim(s\cap s') = n-r$; let me fix the following notation $s = \mathbb{P}(E)$, $s' = \mathbb{P}(E')$, and C, resp. C' a complement of $E\cap E'$ in E, resp. in E'. Suppose that a basis of V is chosen such that $f_{n-r+3},\dots,f_{n+2}$ span C, $f_{n+3},\dots,f_{n+r+2}$ span C' and $f_{n+r+3},\dots,f_{2n+3}$ span $E\cap E'$. Then w.r.t this basis, φ_1 and φ_2 both have the matrix form

$$\begin{pmatrix} * & * & * & * \\ * & 0 & * & 0 \\ * & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.$$

For $i = 1, \dots, r$, let $c_i = f_{n-r+2+i}$, and $c'_i = f_{n+2+i}$.

Let me now proceed to a discussion of the following standard situation: C and C' are two r-dimensional vector spaces, $U = C \oplus C'$, and ψ_1 and ψ_2 quadratic forms on U given w.r.t. some bases c_i and c'_i of C and C' by the matrices

$$\psi_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} 0 & L \\ L & 0 \end{pmatrix},$$

where, as usual, L is a diagonal matrix with distinct entries $\lambda_1, \dots, \lambda_r$.

Lemma 3.4 *The only r planes of U which are isotropic for both ψ_1 and ψ_2 are the obvious ones, i.e.,*

$$C_I = \mathrm{Span}\{c_i \text{ for } i \in I, \text{ and } c'_i \text{ for } i \notin I\}, \quad \textit{for } I \subset [1, r].$$

In particular, U has only 2^r such r-planes.

Proof It is obvious by inspection that the forms $\lambda\psi_1 - \psi_2$ have rank $\geq 2r - 2$ for all λ, and that $\{c_i\} \oplus \{c'_i\} \subset \mathrm{Ker}(\lambda_i\psi_1 - \psi_2)$. I claim that if E is an r-plane of U isotropic for $\lambda_i\psi_1 - \psi_2$, then E must meet $\{c_i\} \oplus \{c'_i\}$, which obviously implies the result. This is true since if not, $E \oplus \{c_i\} \oplus \{c'_i\}$ is isotropic for $\lambda_i\psi_1 - \psi_2$; and this is an $r + 2$-plane, which contradicts that the rank is at least $2r - 2$.

Corollary 3.5 *Let $U' = U \oplus E$, and $\psi'_i = (\psi_i, 0)$, where E is a vector space of dimension s. Then the only $(r + s)$-planes of U' isotropic for both the ψ'_i are the $C_I \oplus E$.*

The proof is obvious.

Since I wish to apply this result to n-planes on X, I had better prove that the relevant matrices can be simultaneously put in diagonal form. For this I use the trivial remark that the two quadratic forms $\left(\begin{smallmatrix} 0 & A \\ {}^tA & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & B \\ {}^tB & 0 \end{smallmatrix}\right)$ can simultaneously be put in the form of Lemma 3.4 if and only if $\det(XA - B)$ has distinct roots; together with the corollary to the following proposition.

Proposition 3.6 *Let m_{ij} and a_{ij} be a large collection of indeterminates, and form the matrix*

$$M = \begin{array}{cc} \begin{pmatrix} * & * & * & * \\ * & 0 & A & 0 \\ * & {}^tA & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} n-r+2 \\ r \\ r \\ n-r+1 \end{array} \end{array}$$

where the blocks have the size indicated (the diagonal boxes being square), A is filled with a_{ij}s and the asterisked boxes with m_{ij}s.

Then there is a polynomial identity in $\mathbb{Z}[a_{ij}, m_{ij}]$ asserting that $\det(A)$ *divides* $\det(M)$*, i.e., there exists a polynomial $q(a_{ij}, m_{ij})$ such that*

$$\det(A) \cdot q(a_{ij}, m_{ij}) = \det(M).$$

Proof On substituting values (from any ring), $\det(A) = 0 \implies \det(M) = 0$. Hence by the Nullstellensatz, $\det(A)$ divides some power of $\det(M)$, say $(\det(M))^s$. But $\mathbb{Z}[a_{ij}, m_{ij}]$ is a unique factorization domain, and $\det(A)$ is irreducible, and hence the result.

Corollary 3.7 *Let*

$$M_i = \begin{pmatrix} * & * & * & * \\ * & 0 & A_i & 0 \\ * & {}^{\mathrm{t}}A_i & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix} \quad \textit{for } i = 1, 2;$$

then $\det(A_1 X - A_2) \;\big|\; \det(M_1 X - M_2)$*. Hence, if* $\det(M_1 X - M_2)$ *has distinct roots $\lambda_1, \dots, \lambda_{2n+3}$,* $\det(A_1 X - A_2)$ *also has distinct roots, and all the roots of* $\det(A_1 X - A_2)$ *are among the λ_i.*

I am now in a position to state Theorem 3.8. Let Σ denote the set of n-planes on X, and let G be the group of automorphisms of X discussed in §2; recall that G is generated by the reflections σ_i in the e_i. Since $\sigma_1 \circ \cdots \circ \sigma_{2n+3} = 1$ in G, every element g of G has a unique expression $g = \sigma_{i_1} \circ \cdots \circ \sigma_{i_{m(g)}}$, with $m(g) \le n + 1$. Define the *length* of g to be $m(g)$.

Finally for three n-planes s, s' and t, let us say that t is *between* s and s' if $t \subset \operatorname{Span}(s, s')$.

Theorem 3.8 *Let s and s' be any two elements of Σ, with* $\dim(s \cap s') = n - r$*. Then there are 2^r n-planes of X between s and s', of which $\binom{r}{r'}$ satisfy*

$$\dim(s \cap t) = n - r + r' \quad \textit{and} \quad \dim(s' \cap t) = n - r'.$$

The group G acts simply transitively on Σ, so that in particular Σ has 2^{2n+2} elements. For any $g \in G$, $s \in \Sigma$,

$$\dim(s \cap g(s)) = n - m(g),$$

so that for $s \in \Sigma$ fixed, there are $\binom{2n+3}{r}$ elements t such that $\dim(s \cap t) = n - r$*.*

Proof Let $s = \mathbb{P}(E)$, $s' = \mathbb{P}(E')$; then Proposition 3.6 and the remarks preceding it show that I can choose bases c_i and c'_i of C and C' so that φ_1 and φ_2 have the matrix forms

$$\varphi_1 = \begin{pmatrix} * & * & * & * \\ * & 0 & I & 0 \\ * & I & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} * & * & * & * \\ * & 0 & L & 0 \\ * & L & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.$$

Corollary 3.5 to Lemma 3.4 is just the first assertion of the theorem. I can suppose without loss that $L = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$. To show that G acts transitively on Σ, it will suffice to show that $g = \sigma_1 \circ \cdots \circ \sigma_r$ takes s into s'; by induction, it will be enough to show that $\sigma_1(s)$ is between s and s', and meets s' in an $(n-r+1)$-plane. And to see this, it is enough to show that $e_1 \in E + E'$. In fact, $e_1 \in E + \{c'_1\}$, for otherwise $E + \{c_1, e_1\}$ would be an $(n+3)$-plane isotropic for $\lambda_1\varphi_1 - \varphi_2$, which is impossible since $\lambda_1\varphi_1 - \varphi_2$ is simply degenerate only.

It is easy to see that G acts transitively on Σ by the following argument; let $g = \sigma_1 \circ \cdots \circ \sigma_r$, with $r \leq n+1$. Then E has some nonzero x which is orthogonal to $e_2, \ldots, e_r$, so that $g(x) = \sigma_1(X)$, but not orthogonal to e_1, so that e_1 is a linear combination of x and $\sigma_1(x) = g(x)$. Hence $e_1 \in E + g(E)$, and since e_1 is certainly not in E, $E \neq g(E)$, so that $g(s) \neq s$.

From the uniqueness of the g taking s into s', we deduce that

$$\dim(s \cap g(s)) = n - m(g),$$

and thus the last assertion of the theorem.

The following is a labelling for G, and thus, once chosen a base point $s \in \Sigma$, a labelling for Σ:

Corollary 3.9 *For $I \subset [1, 2n+3]$, let $g_I = \prod_{i \in I} \sigma_i$, and $s_I = g_I(s)$. Then the only ambiguity in the labelling is that $g_{C(I)} = g_I$, and $s_{C(I)} = s_I$, where $C(I)$ is the complement of I in $[1, 2n+3]$.*

I want now to calculate the intersection number of two elements of Σ. Recall that the intersection number of two n-cycles of S is defined, and depends purely on the cohomology class of the cycles. For $s \in \Sigma$, denote the cohomology class $\gamma_X(s) \in H^{2n}(X, \mathbb{Z}_\ell)(-n)$ by the corresponding capital S, and that of s_I by S_I.

Let $\eta \in H^{2n}(X, \mathbb{Z}_\ell)(-n)$ denote the cohomology class of a linear section of X of dimension n; this does not depend on the particular linear section, so that for any $(n+2)$-plane $H \subset \mathbb{P}$, $\gamma_X(X \cap H) = \eta$.

Let i and $j \in [1, 2n+3]$, and let $s \in \Sigma$ be a chosen base point; then let H be the $(n+2)$-plane spanned by s and s_{ij}. $X \cap H$ clearly contains s, s_i, s_j and s_{ij}, and it is easy to see that it cannot be any bigger, so that

$$X \cap H = s \cup s_i \cup s_j \cup s_{ij},$$

and so $S + S_i + S_j + S_{ij} = \eta$.

Lemma 3.10 *Let* $\dim(s \cap s') = r$*. Then* $S \cdot S' = (-1)^r([\frac{r}{2}] + 1)$.

Proof The particular cases $r = -1$ and $r = 0$ are easy: if s and s' are disjoint, then obviously $S \cdot S' = 0$, and if they meet transversally at one point, then $S \cdot S' = 1$. The other obvious intersection number is

$$S \cdot \eta = 1,$$

since for any s, a general $(n+2)$-plane of P meets it in just one point.

I prove the lemma by induction on r. Suppose that

$$s' = s_{i_1, \ldots, i_{n-r}},$$

and that i and j are two indices distinct from $i_1, \ldots, i_{n-r}$; then s meets s'_i and s'_j in an $(r-1)$-plane, and s'_{ij} in an $(r-2)$ plane. Since

$$\eta = S' + S'_i + S'_j + S'_{ij},$$

using the inductive values for $S \cdot S'_i$, $S \cdot S'_j$ and $S \cdot S'_{ij}$, we have

$$S \cdot S' = 1 - 2(-1)^{r-1}\left(\left[\frac{r-1}{2}\right] + 1\right) - (-1)^{r-2}\left(\left[\frac{r-2}{2}\right] + 1\right),$$

and since $[\frac{r-2}{2}] = [\frac{r}{2}] - 1$ and $[\frac{r-1}{2}] = [\frac{r}{2}] - \frac{1+(-1)^r}{2}$, it follows at once that

$$S \cdot S' = (-1)^r \left(\left[\frac{r}{2}\right] + 1\right).$$

Corollary 3.11 *$S^2 = (-1)^n([\frac{n}{2}]+1)$; if s and t are two n-planes such that* $\dim(s \cap t) = n-2$, *then* $(S-T)^2 = (-1)^n \cdot 2$.

The verification is immediate.

Let $F(\Sigma, \eta)$ denote the free Abelian group on the elements of Σ and η, and let $R(\Sigma, \eta)$ be the subgroup of $F(\Sigma, \eta)$ generated by the "relations"

$$\eta - s - s_i - s_j - s_{ij},$$

for all $s \in \Sigma$ and $i \neq j = 1, \dots, 2n+3$. Let $A(X)$ denote the quotient:

$$A(X) = F(\Sigma, \eta)/R(\Sigma, \eta).$$

It is obvious that the map $\Sigma \cup \{\eta\} \to H^{2n}(X)(-n)$ defined by

$$s \mapsto S, \quad \eta \mapsto \eta$$

extends to a unique map $A(X) \to H^{2n}(X)(-n)$, which defines the cup product of two elements of $A(X)$.

I want first to show that $A(X)$ is spanned by $2n+4$ elements, so that its rank (as an Abelian group) is at most $2n+4$.

Lemma 3.12 *Let $s \in \Sigma$ be a base point. Then the elements*

$$\eta, s, \quad \textit{and} \quad s_i \quad (\textit{for } i = 1, \dots, 2n+3)$$

generate $A(X)$. Indeed, $(-1)^r s_{i_1 \dots i_r} = [\frac{r}{2}]\eta - (r-1)s - \sum_{j=1}^r s_{i_j}$ for any distinct $i_1, \dots, i_r$; note that this holds for $r \geq n+1$, provided that the i_j are distinct.

Proof The proof is similar to that of Lemma 3.10, by induction on r, beginning trivially if $r = 0$ or 1. Suppose the formula proved for $r-1$ and $r-2$. Then

$$\begin{aligned}(-1)^r s_{i_1 \dots i_r} &= (-1)^r \eta + (-1)^{r-1} s_{i_1 \dots i_{r-1}} + (-1)^{r-1} s_{i_2 \dots i_r} \\ &\qquad - (-1)^{r-2} s_{i_2 \dots i_{r-1}}, \\ &= \left\{(-1)^r + 2\left[\frac{r-1}{2}\right] - \left[\frac{r-2}{2}\right]\right\}\eta \\ &\qquad - \{2(r-2) - (r-3)\}s - \sum_{j=1}^r s_{i_j},\end{aligned}$$

which gives the required formula for r after simplification.

Recall that in G we have the relation $\sigma_1 \circ \cdots \circ \sigma_{2n+3} = 1$, since the composite of all the reflections is -1 on V, and therefore acts trivially on $\mathbb{P}$. Hence there is an ambiguity in the labelling of $t \in \Sigma$ as $s_{i_1 \ldots i_r}$. In fact $s_{1\ldots n+1} = s_{n+2\ldots 2n+3}$, which allows me to write down a relation between the generating set of Lemma 3.12:

Lemma 3.13 *$A(X)$ is spanned by the $2n+4$ elements*

$$s_i \quad \text{for } i = 1, \ldots, 2n+3 \quad \text{and} \quad \eta - 2s.$$

We have $s = \sum_{i=1}^{2n+3} s_i - (n+1)(\eta - 2s)$, and there is obviously a similar formula for η.

Proof $s = s_{1\ldots 2n+3}$, so that by Lemma 3.12,

$$-s = (-1)^{2n+3} s_{1\ldots 2n+3} = -(n+1)\eta + (2n+2)s + \sum_{i=1}^{2n+3} s_i$$

as claimed.

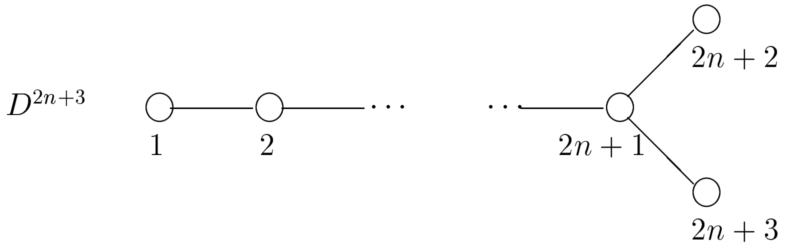
When I have proved that the cup product matrix of the above spanning set of $A(X)$ is nondegenerate, this will prove at one go that the vectors s_i, $\eta - 2s$ are linearly independent, and further that the map $A(X) \to H^{2n}(X)(-n)$ is injective. Before doing this, let me introduce a further piece of notation: let $A^{\perp}$ denote the orthogonal in $A(X)$, w.r.t. the cup product, of η. We then have that $A^{\perp} \oplus \{\eta\}$ is a sublattice of finite index in $A(X)$.

Theorem 3.14 *The cup product on $A(X)$ makes it into an unimodular lattice of rank $2n+4$. Indeed, $A^{\perp} \oplus \{\eta\} \subset A(X)$ is of index 4, and $A^{\perp}$ is isomorphic to the Witt lattice D^{2n+3} (unfortunately, with the cup product being $(-1)^n$ times the usual one on D^{2n+3}).*

Proof Let me first show that the second sentence implies the first. The discriminant of the form D^{2n+3} is well known to be 4, and that of the lattice $\{\eta\}$ is obviously 4. Hence the sublattice $A^{\perp} \oplus \{\eta\}$ has discriminant 16, and hence $A(X)$, together with the cup product, is unimodular.

Now for a definition of the Witt lattice D^{2n+3}; for further details, see Bourbaki [3] or Conway [5].

As a group, D^{2n+3} is the free Abelian group on $2n+3$ elements $r_1, \ldots, r_{2n+3}$. The scalar products $r_i \cdot r_j$ are given as follows: for all i, $r_i^2 = 2$; and for $i \neq j$, $r_i \cdot r_j = -1$ or 0, according as the ith and jth vertex of the following Dynkin diagram are joined by a branch or not:



To see that this lattice has discriminant 4, it suffices to calculate the determinant of the matrix $(r_i \cdot r_j)$.

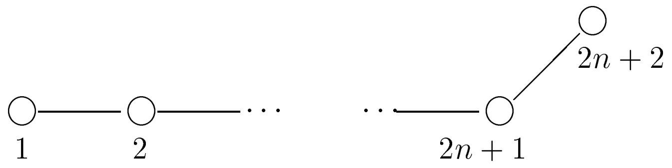
Corollary 3.11 to Lemma 3.10 shows us that $A(X)$, and even $A^{\perp}$, contains a whole host of vectors with squared length 2: for each pair s, t of n-planes of X which meet in an $n-2$-plane, $(s-t)^2 = (-1)^n \cdot 2$. In particular, if $s \in \Sigma$ is chosen, for any $i \neq j$,

$$(s_i - s_j)^2 = (s - s_{ij})^2 = (-1)^n \cdot 2.$$

It is easy to see, with the help of Lemma 3.10, that for i, j, k, l distinct,

$$(s_i - s_j)(s_j - s_k) = -(-1)^n, \quad \text{and} \quad (s_i - s_j)(s_k - s_l) = 0.$$

Hence, if I set $r_i = (s_i - s_{i+1})$ for $i = 1, \ldots, 2n+2$, it is obvious that the multiplication table of the r_i is given by the subdiagram



Finally, to complete the diagram, I set

$$r_{2n+3} = s - s_{2n+2\,2n+3} = s_{2n+2} + s_{2n+3} - (\eta - 2s).$$

The product I must check are

$$r_i \cdot r_{2n+3} = 0 \quad \text{for} \quad i \neq 2n+1, 2n+3,$$

which are obvious for reasons of symmetry; and

$$\begin{aligned} r_{2n+1} \cdot r_{2n+3} &= (s_i - s_j)(s - s_{jk}), \quad \text{with } i, j, k \text{ distinct}, \\ &= -(s_i - s_j) \cdot s_{jk}; \end{aligned}$$

and since s_j and s_{jk} meet in an $(n-1)$-plane, and s_i and s_{jk} in an $(n-3)$-plane,

$$\begin{aligned} &= -(-1)^{n-3}\left(\left[\frac{n-3}{2}\right]+1\right) + (-1)^{n-1}\left(\left[\frac{n-1}{2}\right]+1\right) \\ &= -(-1)^n, \end{aligned}$$

as desired.

This shows that $A^\perp$ contains as a sublattice of finite index the Witt lattice D^{2n+3} (up to $(-1)^n$). When we know that the index of $A^\perp \oplus \{\eta\}$ in $A(X)$ is at least 4, this will prove that $A^\perp$ is actually equal to D^{2n+3}, and will prove the theorem.

To see this, note that $s \in A(X)$, that $4s = \eta + (s - s_i) + (s - s_j) + (s - s_{ij})$ and is hence in $A^\perp \oplus \{\eta\}$, but that neither s nor $2s$ are, for the intersection of η with any element of $A^\perp \oplus \{\eta\}$ is a multiple of 4.

The theorem is proved.

Corollary 3.15 *The algebraic cycles S for $s \in \Sigma$ span the cohomology of X, $H^{2n}(X)(-n)$.*

For they span an unimodular sublattice isomorphic to $A(X)$, hence of rank $2n+4$. But this is the $2n$th Betti number of X, by the Hirzebruch formula.

Addendum 3.16 *The map $\Sigma \mapsto H^{2n}(X)(-n)$ given by $s \mapsto S$ is injective.*

Proof For if $s \neq t$, then $\dim(s \cap t) \neq n$, and hence

$$(S-T)^2 = 2(-1)^n\left(\left[\frac{n}{2}\right]+1\right) - 2(-1)^r\left(\left[\frac{r}{2}\right]+1\right) \neq 0.$$

Addendum 3.17 *Let us define the* roots *of $A^\perp$ to be the vectors of length $(-1)^n \cdot 2$. Then the previous list*

$$(s_i - s_j) \quad \textit{and} \quad (s - s_{ij}) \quad \textit{for all } i \neq j$$

is complete (and unambiguous). Furthermore,

$$\sigma_i(s_j - s_k) = (s_{ij} - s_{ik}) = -(s_j - s_k) \quad \textit{for all distinct } i, j, k. \qquad (*)$$

Proof It is easy to see that no two roots of the above list are equal, by looking at their scalar product with each other. They form a complete list for no better reason than that D^{2n+3} has precisely $2(2n+3)(2n+2)$ roots, as is well known.

For the last assertion $(*)$, it is enough to write down

$$\begin{aligned} s + s_i + s_j + s_{ij} &= \eta \\ s + s_i + s_k + s_{ik} &= \eta, \end{aligned}$$

and to subtract.

To conclude this paragraph, I will discuss the symmetry group of the set Σ, and compare it with that of the cohomology lattice. Recall that the Weyl group $W(A^\perp)$ is by definition the group generated by the reflections in the roots of $A^\perp$. $W(A^\perp)$ acts on $A^\perp$, and also on $A(X)$, respecting the scalar product, and leaving η fixed. It is not difficult to see that $W(A^\perp)$ is, in fact, the full group of automorphisms of the triple $(A, \cdot\,, \eta)$. For any such automorphism must induce one of $A^\perp$ – and the automorphisms of $A^\perp$ are well known to be an extension of $W(A^\perp)$ by (± 1); and it is obvious that there is no automorphism of $A(X)$ fixing η and being -1 on $A^\perp$; for call such a thing τ, and observe that $s + \tau s = \frac{1}{2}\eta \notin A(X)$.

I need to know the following two facts about $W(A^\perp)$, which are well known; see, for details, Bourbaki [3].

(a) $W(A^\perp)$ is in fact generated by the reflections in the $2n+3$ roots r_i.

(b) The order of $W(A^\perp)$ is $2^{2n+2} \cdot (2n+3)!$ – in fact $W(A^\perp)$ is the semidirect product of $(\mathbb{Z}/2\mathbb{Z})^{2n+2}$ with the symmetric group S_{2n+3}.

Now to discuss the symmetry properties of Σ. Let us say that two elements $s, t \in \Sigma$ are *very incident* if $\dim(S \cap t) = n - 1$. It is clear that I can recover all the incidence properties of Σ from a knowledge of which elements are very incident: for $\dim(s \cap t) \geq n - r$ if and only if there exists a chain $s = s_0, s_1, \ldots, s_r = t$, with s_i and s_{i+1} very incident. The automorphisms of Σ are thus just the automorphisms of an incidence geometry. Let me denote the group of them by $\mathrm{Aut}(\Sigma)$.

We have already seen that the group G, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2n+2}$, acts simply transitively on Σ. Hence $\mathrm{Aut}(\Sigma)$ is the semidirect product of G with the stabilizer in $\mathrm{Aut}(\Sigma)$ of any element s of Σ. Let $s_1, \ldots, s_{2n+3}$ be the $2n+3$ elements of Σ very incident to s. It is obvious that any automorphism of Σ permutes the s_i; and from the system of labelling the elements of Σ that I have been using throughout this paragraph, it is obvious that any such permutation is induced by a unique automorphism of Σ. Hence I have proved the following proposition.

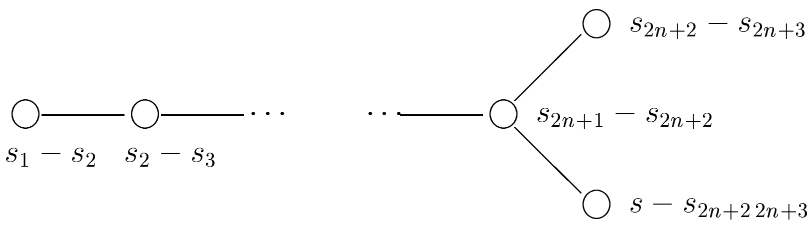
Proposition 3.18 $\mathrm{Aut}(\Sigma)$ *is the semidirect product of* G *and* S_{2n+3}, *this last group acting by permuting the* s_i, *for some chosen* $s \in \Sigma$.

Recall that $A(X)$ was defined as the quotient of $F(\Sigma, \eta)$ by the relations $R(\Sigma, \eta)$; so that any automorphism of Σ can be persuaded to act on $A(X)$ is a natural way, fixing η. And since the scalar product of elements of Σ depends only on their incidence properties, which are preserved by $\mathrm{Aut}(\Sigma)$, this action is orthogonal.

Theorem 3.19 $\mathrm{Aut}(\Sigma) \cong W(A^{\perp}) = \mathrm{Aut}(A(X), \,\cdot\,, \eta)$.

Proof The second equality is just a restatement of the above discussion. For the first, I will prove that $\mathrm{Aut}(\Sigma)$ maps onto the standard generators of $W(A^{\perp})$, so that the map is surjective; the isomorphism of the two groups will then follow, because the poor things both have the same order $2^{2n+2} \cdot (2n+3)!$.

Let me redraw the roots generating $A^{\perp}$ of Theorem 3.14.



It is easy to see that the permutation $(i\,i+1)$ has the following effect on the above basis (for $i=1,\dots,2n+2$)

$$\begin{aligned}
&(s_j - s_{j+1}) \text{ stays fixed if } j \neq i-1, i, i+1;\\
&(s_{i-1} - s_i) \mapsto (s_{i-1} - s_{i+1}) = (s_{i-1} - s_i) + (s_i - s_{i+1});\\
&(s_i - s_{i+1}) \mapsto -(s_i - s_{i+1});\\
&(s_{i+1} - s_{i+2}) \mapsto (s_i - s_{i+1}) + (s_{i+1} - s_{i+2});
\end{aligned}$$

and $(s - s_{2n+2\,2n+3})$ is left fixed for $i \neq 2n+1$, and for $i = 2n+1$,

$$(s - s_{2n+2\,2n+3}) \mapsto (s - s_{2n+1\,2n+3}),$$

which is equal to $(s - s_{2n+2\,2n+3}) + (s_{2n+1} - s_{2n+2})$ on account of $(*)$ of Addendum 3.17.

This is precisely the effect that the reflection in the root $r_i = (s_i - s_{i+1})$ has on the basis.

Finally, to capture the last generator of $W(A^\perp)$, namely the reflection in $r_{2n+3} = s - s_{2n+2\,2n+3}$, I have to be a little cleverer: let

$$\tau = (2n+2\;2n+3) \circ \sigma_{2n+2} \circ \sigma_{2n+3} \in \operatorname{Aut}(\Sigma).$$

It is obvious that this leaves fixed the roots $(s_i - s_{i+1})$ for $i = 1,\dots,2n$. We must also check up that τ has the desired effect on the last three roots:

$$\begin{aligned}
(s_{2n+2} - s_{2n+3}) &\mapsto (s_{2n+2\,2n+3} - s) \mapsto (s_{2n+3} - s_{2n+2})\\
&\qquad\qquad\qquad\qquad\;\; \mapsto (s_{2n+2} - s_{2n+3});\\
(s_{2n+1} - s_{2n+2}) &\mapsto -(s_{2n+1} - s_{2n+2}) \mapsto (s - s_{2n+1\,2n+2})\\
&\mapsto (s - s_{2n+1\,2n+3}) = (s - s_{2n+2\,2n+3}) + (s_{2n+1} - s_{2n+2}),
\end{aligned}$$

where all the mysterious steps are by application of $(*)$; finally

$$(s - s_{2n+2\,2n+3}) \mapsto (s_{2n+3\,2n+2} - s),$$

which completes the verification that τ acts on $A^\perp$ as the reflection in $(s - s_{2n+2\,2n+3})$, and the proof of Theorem 3.19.

Chapter 4

Two quadrics – the odd dimensional case

In this paragraph, X will denote a nonsingular intersection of two quadrics, and $\dim X = 2n+1$. I will be interested in much the same problem as in §3, namely in giving a description of the variety S on n-planes of X, and in showing that in some precise sense these n-planes span the cohomology of X. Note that Theorem 2.6 assures us that S is nonsingular and of dimension $n+1$; it is obvious that S is nonempty, since the hyperplane sections X_H of X have been seen to contain n-planes.

Recall the construction of the curve C given in Theorem 1.10: to X we associate the pencil Φ of quadrics through X. By Proposition 2.1, this has only the singularities considered in the hypotheses of Theorem 1.10, so that this theorem constructs for us the diagram

$$\begin{tikzcd} \mathrm{Gen}(\Phi) \arrow[d, "p_1"'] \arrow[dr, "p"] & \\ \mathbb{P}^1 & C \arrow[l, "q"] \end{tikzcd}$$

C is a double cover of $\mathbb{P}^1$ with ordinary ramification at the $2n+4$ points of $\mathrm{Sing}(\Phi)$, so that it is a hyperelliptic curve of genus $n+1$, and is canonically attached to X. Note that if X given by the equations in diagonal form

$$X: \quad \sum x_i^2 = \sum \lambda_i x_i^2 = 0,$$

then C can be written

$$C: \quad z^2 = \prod (x - \lambda_i y).$$

For $c \in C$, denote $G(c)$ the fibre $p^{-1}(c)$ of p over c. This is a rational variety with the affine opens discussed in Theorem 1.2.

The principal geometric result is Theorem 4.8, which asserts that S is (noncanonically) isomorphic to the Jacobian $J(C)$ of the curve C. Theorem 4.14 discusses the cohomology of X: the only interesting cohomology group of X, $H^{2n+1}(X)(-n)$, is isomorphic to $H^1(C) \cong H^1(S)$, in either Hodge or ℓ-adic cohomology. In fact, what I prove is a little stronger: the motive $h(X)$ contains as a direct summand the motive $h^1(C)(n)$; this of course implies that $H^{2n+1}(X)(-n)$ is isomorphic to $H^1(C)$ for any cohomology theory H having the "right" Betti numbers. The assertion in terms of Hodge theory is of course equivalent with the assertion that S is (again noncanonically) isomorphic to the intermediate Jacobian $J(X)$ of X.

The proof of Theorem 4.8 proceeds in two steps: firstly, I show that S is birationally equivalent to the $(n+1)$st symmetric power $C^{(n+1)}$ of the curve C, which is well known to be birational to the Jacobian $J(C)$; secondly, it follows from a remark of Deligne's that to show that such an S is isomorphic to $J(C)$, it suffices to prove that the degree of its canonical divisor K_S is zero. It turns out to be possible to compute the degree of this divisor, by a Chern class argument in the Grassmannian containing S.

The proof of Theorem 4.14 is analogous to that of Theorem 3.14, in that I have a map from the cohomology group $H^{2n+1}(S)$ to $H^{2n+1}(X)$; to show that this map in injective, and thus isomorphic, I only have to show that the cup product on $H^{2n+1}(X)$ induces a nondegenerate pairing on $H^{2n+1}(S)$, or equivalently, by Poincaré duality, an isomorphism from $H^{2n+1}(S)$ to $H^1(S)$.

It turns out to be possible to put this into a more geometric form: let $D \subset S \times S$ be the "incidence divisor"

$$D = \bigl\{(s,t) \bigm| s \text{ and } t \text{ meet}\bigr\}.$$

Then D induces a map $d\colon S \to {}^{\mathrm{t}}S$ of S to its dual Abelian variety; this map in turn induces a map d^* from $H^{2n+1}(S)$ to $H^1(S)$, which is of course the same as the map induced by the cup product in $H^{2n+1}(S)$, and which I have to show is an isomorphism.

This leads me to write down diagrams of the form

$$\begin{tikzcd} C \arrow[r] & S \arrow[dd] \arrow[dr] & \\ & & X(-n) \arrow[dl] \\ C & S(-n) \arrow[l] & \end{tikzcd}$$

in the *category of correspondences*; these diagram are of course inspired by the corresponding part of Clemens and Griffiths' paper [4], although they only write down such a diagram on the level of cohomology groups. I give a very much simplified proof of one of their main results in Appendix 4.3.

A slightly unsatisfactory feature of my proof of Theorem 4.8 is that, although I prove that a certain nonempty subset open of S, defined by a rather generic-looking condition, is birational to $C^{(n+1)}$, I have no direct way of seeing that this open is in fact dense – which would at once imply that S is irreducible. A way round this difficulty is found in Appendix 4.2, in which I show that Theorem 3.19, together with the main results of Lefschetz theory, proves directly that S is connected.

The last section of this paragraph is devoted to a brief sketch of the relation of the geometric parts of my work with the classical theory of Kummer varieties; the first and last sections are more or less independent of the rest of the paragraph, and could probably be read by any 19th century geometer.

Let $s \subset X$ be an n-plane; then s is in particular an n-plane of each of the quadrics Q_λ of the pencil Φ:

Lemma 4.1 *Let $\widetilde{C}_s \subset \mathrm{Gen}(\Phi)$ be defined by the conditions*

$$\widetilde{C}_s = \big\{(\lambda, g) \bigm| g \supset s\big\}.$$

Then $\widetilde{C}_s$ is a nonsingular curve, carried isomorphically onto C by p.

Proof Let $s = \mathbb{P}(E)$, E an $(n+1)$-plane such that $E \cap \mathrm{Ker}(\varphi_\lambda) = 0$ for all $\lambda \in \mathbb{P}^1$. φ_λ induces a quadratic form on the two dimensional vector space $E^\perp/E$, and this is no more degenerate than φ_λ. For any nonsingular value of λ, there are thus just two generators of Q_λ which contain s, one from each family if Q_λ is an ordinary cone, then there is a unique such generator.

Hence $\widetilde{C}_s$ is a curve (which is easily seen to be reduced, on writing down the equations), and p takes it into C in a one-to-one fashion. Hence $\widetilde{C}_s$ is nonsingular and isomorphic to C.

There is a natural map $r\colon \widetilde{C}_s \to S$ defined as follows: if $g \in \widetilde{C}_s$, then g is a generator of some quadric Q_λ. X is the intersection $Q_\lambda \cap Q_\mu$ for any other $\mu \in \mathbb{P}^1$, and hence $X \cap g = Q_\mu \cap g = s \cup s_g$, for some n-plane $s_g \subset X$. Set $r(g) = s_g$.

Let C_s denote the image of $\widetilde{C}_s$ in S. It is obvious that it is the closure in S of the subvariety of those s' such that $\dim(s \cap s') = n-1$, i.e., of those s' *very incident* to s; and further that $r\colon \widetilde{C}_s \to C_s$ is one-to-one, and therefore birational. For if s and s' are very incident, then s and s' span an $(n+1)$-plane L of $\mathbb{P}$, such that $L \cap X = s \cup s'$; and such an L is clearly a generator of a unique quadric Q_λ of the pencil Φ.

I have thus proved the following proposition.

Proposition 4.2 *Let $C_s = C_s^1$ be the closure in S of the subvariety ${C_s^1}'$ defined by the incidence condition*

$$ {C_s^1}' = \bigl\{ t \bigm| \dim(s \cap t) = n-1 \bigr\}. $$

Then C_s is a curve, and is birationally equivalent to C.

The extension of this result to more general incidence subvarieties of S presents only one difficulty: let $s \in S$ be chosen, and suppose that $t \in S$ satisfies $\dim(s \cap t) = r$. Then I can follow the reasoning much used in §3, letting $s = \mathbb{P}(E), t = \mathbb{P}(E')$, C and C' being the usual complements of $E \cap E'$ in E and E', and write down the matrices

$$ \varphi_1 = \begin{pmatrix} * & * & * & * \\ * & 0 & A & 0 \\ * & A & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} * & * & * & * \\ * & 0 & B & 0 \\ * & B & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}. $$

Now I want to use Lemma 3.4 to associate to t the $n-r$ n-planes of X, meeting s in an $(n-1)$-plane; and I can do this if and only if I can put the matrices A and B simultaneously in diagonal form, i.e., if and only if $\det(AX - B)$ has distinct roots.

Definition 4.3 Let $C_s^i \subset S$ be the closure in S of the subvariety ${C_s^i}'$ defined by the following two conditions:

$$ {C_s^i}' = \left\{ t \;\middle|\; \begin{array}{l} \dim(s \cap t) = n - i \text{ and} \\ \det(XA - B) \text{ has distinct roots} \end{array} \right\}. $$

It is fairly obvious that the second condition does not depend on the choice of the bases of E, E', C, C' concerned. $C_s^{i\prime}$ is nonempty, as follows at once from Lemma2.2.

It is perhaps worth remarking that if s and t belong to a common nonsingular hyperplane section X_H of X, then the second condition is automatically fulfilled, by what we have seen in §3.

Proposition 4.4 *The C_s^i defined above are birationally equivalent to the ith symmetric power $C^{(i)}$ of the curve C, or equivalently, to that of C_s, $C_s^{(i)}$.*

Corollary 4.5 *One component of S is birationally equivalent to $C^{(n+1)}$, and hence to $J(C)$.*

Proof By definition, $C_s^{i\prime}$ is open and dense in C_s^i. Let me first construct the map $C_s^{i\prime} \to C_s^{(i)}$: let $t \in C_s^{i\prime}$, and suppose that matrices of φ_1 and φ_2 written down as on the previous page, with $A = I$ and B diagonal with distinct entries. Then by Lemma 3.4, there exist precisely i n-planes $s_1, \ldots, s_i$ of X such that

$$\dim(s \cap s_j) = n - 1 \quad \text{and} \quad \dim(t \cap s_j) = n - i + 1.$$

The first of these conditions is just that $s_j \in C_s$, so that the required map $C_s^{i\prime} \to C_s^{(i)}$ is given by $t \mapsto (s_1, \ldots, s_i)$.

To show that this map is birational, it suffices to see that for some open dense set of $C_s^{(i)}$, the i-tuple $s_1, \ldots, s_i$ defines a unique t such that

$$\dim(s_j \cap t) = n - i + 1.$$

Note that this is obvious if there is some nonsingular hyperplane section X_H of X containing s and all the s_j, from what has been seen in §3. But the existence of such a hyperplane section is an open condition on the s_j, and $C_s^{(i)}$ is irreducible. So that to prove the proposition it suffices to prove the following.

Lemma 4.6 (Bertini's theorem) *Let $s \subset X$ be any n-plane. Then there exists a nonsingular hyperplane section X_H of X containing s.*

Proof Let $\mathbb{P}^*$ denote the dual projective space, and $\mathbb{P}_s \subset \mathbb{P}^*$ the subspace of hyperplanes H containing s. $\mathbb{P}_s$ is clearly an $(n+2)$-dimensional projective space. The proof is going to consist in showing that the subvariety of $\mathbb{P}_s$ consisting of hyperplanes meeting X singularly has dimension at most $n+1$. Notice that X_H is singular at x if and only if H meets X nontransversally at x, i.e., if and only if $H \supset T_{x,X}$. But now $T_{x,X}$ is a hyperplane in H, so that $s \cap T_{x,X}$ has dimension at least $n-1$.

Consider the following diagram

$$\begin{array}{ccccc} & Z & \subset & X \times \mathbb{P}_s & \\ p_1 \swarrow & & \searrow p_2 & & \\ X & & \mathbb{P}_s & & \end{array}$$

where Z is defined by $Z = \{(x, H) | T_{x,X} \subset H\}$.

The lemma will be proved if I can show that $\dim Z \leq n+1$. For this, consider the projection map p_1: the fibres of p_1 are one dimensional over the subvariety X_1, and zero dimensional over X_2, where

$$\begin{aligned} & X_1 = \{x \mid s \subset T_{x,X}\}, \\ \text{and} \quad & X_2 = \{x \mid \dim(s \cap T_{x,X}) = n-1\}. \end{aligned}$$

Hence I will be finished if I show that X_1 has dimension at most n, and X_2 has dimension at most $n+1$. The first of these is easy: I claim that $X_1 = s$; for if $x \in X$, and $T_{x,X} \supset s$, then either $x \in s$ or x and s together span an $(n+1)$-plane of X, which contradicts the nonsingularity of X.

For the second assertion, note that X_2 is given in X by the condition that, if x is represented in V by e, that the two linear forms $\varphi_1(e, \ )$ and $\varphi_2(e, \ )$ on E (where $s = \mathbb{P}(E)$) are linearly dependent. But this means that for some $\lambda \in \mathbb{P}^1$, $\varphi_\lambda(e, \ )$ is zero on E, i.e., that for some λ, e is in the orthogonal w.r.t. φ_λ of E. But for all λ, $E \cap \mathrm{Ker}(\varphi_\lambda) = 0$, so that this orthogonal has codimension $n+1$ in V. It is clear that the intersection of X with any $(n+2)$-plane has dimension exactly n, so that X_2 has dimension $n+1$.

This complete the proof of the lemma, as well as that of Proposition 4.4 and Corollary 4.5.

From now on I will assume the result of Appendix 4.2, namely that S is irreducible. The Corollary 4.5 now asserts that S is birational equivalent to the Jacobian $J(C)$ of the curve C.

Lemma 4.7 (Deligne) *Let S be a nonsingular variety, and α a rational map of S to an Abelian variety A, of dimension n,*

$$\alpha\colon S \to A.$$

Assume that

(i) α is a birational equivalence;

(ii) the canonical class K_S is zero.

Then α is an isomorphism.

Furthermore, if S is projective, then (ii) can be weakened to

(ii′) the degree of K_S is zero.

Proof Firstly, recall that A is an absolutely minimal model, so that α is in fact a morphism. This is an easy consequence of the general nonsense of the existence of the Picard scheme: for there is a morphism

$$S \to (\operatorname{Pic}^0 S)^* = \operatorname{Alb}(S).$$

α then induces a rational map $\operatorname{Alb}(S) \to A$, which is a morphism on account of the miraculous properties enjoyed by Abelian varieties.

Since α is a morphism between two nonsingular varieties, it induces a map

$$\mathrm{d}\alpha\colon T_S \to \alpha^*(T_A)$$

given locally by the Jacobian matrices $\left(\frac{\partial \alpha_i}{\partial s_j}\right)$. The determinant $\det(\mathrm{d}\alpha)$ is the dual of a mapping $\omega\colon \alpha^*(\Omega^n_A) \to \Omega^n_S$, and since these bundles are trivial, ω is either an isomorphism or the zero map. But ω vanishes precisely at the locus of S where α fails to be étale – and so cannot vanish identically, since α is an isomorphism on an open of S. Hence, ω is an isomorphism, and hence α is everywhere étale, and therefore an isomorphism.

If we do not assume that Ω^n_S is trivial, then the conclusion is that it has a section, the image of 1 under ω; hence $\Omega^n_S = \mathcal{O}_S(D)$, where D, the locus of zeros of this sections, is an effective divisor. The degree of $\mathcal{O}_S(D)$ is just the degree of the divisor D, and this can be zero only if D itself is the zero divisor.

Theorem 4.8 *S is isomorphic as a variety to the Jacobian $J(C)$ of the curve C, and is thus a principal homogeneous space under an Abelian variety.*

Proof Lemma 4.7 and the previous work has reduced the proof of this theorem to the computation of a single integer, namely the degree of the canonical divisor K_S of S, in some suitable projective embedding.

This calculation, which relies on the fact that the degree of a bundle can be calculated in terms of its first Chern class, can be summarized as follows: let $\mathrm{Gr} = \mathrm{Gr}(n+1, 2n+4)$ be the Grassmannian of n-planes of $\mathbb{P}$, so that $S \subset \mathrm{Gr}$. We can write down the exact sequence of bundles on S

$$0 \to T_S \xrightarrow{\mathrm{d}i} i^* T_{\mathrm{Gr}} \longrightarrow \nu(S, \mathrm{Gr}) \to 0,$$

where $i\colon S \to \mathrm{Gr}$ is the inclusion map, and $\nu = \nu(S, \mathrm{Gr})$ is the normal bundle to S in Gr. The Chern classes of T_{Gr} and ν are easy to write down, so that those of T_S can be calculated by the product rule. And Ω_S^n is easily expressed in terms of T_S, so that its degree can be calculated.

Recall that over Gr there is a canonically defined $(n+1)$-vector bundle E, the *tautological bundle*. This is the vector bundle whose fibre over the point $g \in G$ is the space of linear forms on g. The global sections $\Gamma(E)$ form[1] the vector space V^* dual to V; any linear form α on V induces a section of E, by being the linear from $\alpha_{|g}$ in the fibre over g.

Gr is projectively embedded by the line bundle $\mathcal{O}(1) = \bigwedge^{n+1}(E)$. It is with respect to this embedding that the degree of K_S will be calculated.

Let $c(E) = 1 + c_1 + \cdots + c_{n+1}$ be the total Chern class of E. For my purpose, it will be sufficient to consider these as cohomology classes, $c_i \in H^{2i}(\mathrm{Gr})(-i)$, although they also turn up naturally as algebraic cycles – the Schubert cycles in the Chow ring of Gr.

The point is now that both T_{Gr} and $\nu(S, \mathrm{Gr})$ have a simple description in terms of the bundle E. Firstly, for T_{Gr} there is the Euler exact sequence:

$$0 \to \mathrm{End}(E) \to E \otimes v \to T_{\mathrm{Gr}} \to 0,$$

where V denotes abusively the trivial bundle $V \times \mathrm{Gr}$ over Gr; see for example [12]. The middle terms is isomorphic to the sum of $2n+4 = \dim V$ copies of E, and the first to $E \otimes E^*$. Hence

$$c(T_{\mathrm{Gr}}) = (c(E))^{2n+4} \cdot (E \otimes E^*)^{-1}. \tag{0}$$

[1] Note that this is only true of E as an algebraic vector bundle – as a topological vector bundle, it has of course many more sections.

Now for $\nu(S, \mathrm{Gr})$. Let $S^2(E)$ be the second symmetric power of E; this is the vector bundle whose fibre over $g \in \mathrm{Gr}$ is the *space of all quadratic forms* on g. As earlier, the global sections of $S^2(E)$ is the space $S^2(V^*)$ of quadratic forms on V. The two forms φ_1 and φ_2 are thus in a natural way sections of $S^2(E)$, and it is clear that $S \subset \mathrm{Gr}$ is just the locus defined by the vanishing of these sections. Indeed, $g \in S$ if and only if φ_1 and φ_2 both vanish identically on g, in other words, if and only if φ_1 and φ_2, as sections of $S^2(E)$, both vanish at g.

I can consider S to be defined by the vanishing of the single section $\varphi = (\varphi_1, \varphi_2)$ of $S^2(E) \oplus S^2(E)$. S is nonsingular and of codimension $2\binom{n+1}{2}$ in Gr, which is the rank of this bundle, so that φ has a nondegenerate zero along S. It is then not difficult to see that the normal bundle $\nu(S, \mathrm{Gr})$ is naturally isomorphic to the restriction $i^*(S^2(E) \oplus S^2(E))$ of $S^2(E) \oplus S^2(E)$ to S. Hence I can rewrite the above mentioned exact sequence

$$0 \to T_S \to i^*(T_{\mathrm{Gr}}) \to i^*(S^2(E) \oplus S^2(E)) \to 0,$$

so that $c(T_S) = i^*(c(T_{\mathrm{Gr}}) \cdot c(S^2(E))^{-2})$, where I am using the fact that i^* commutes with the product, as well as the usual properties of Chern classes. The final result is

$$c(T_S) = i^* \left(c(E)^{2n+4} \cdot c(E \otimes E^*)^{-1} \cdot c(S^2(E))^{-2} \right).$$

Now I claim that the first Chern class of T_S, $c_1(T_S)$ is zero, and this suffices to prove that the degree of Ω_S^{n+1} is zero. In fact, I will be finished if I show that $c_1(\Omega_S^{n+1}) = 0$, since the degree of Ω_S^{n+1} is a product of this class.

To see this is an easy exercise in the formulae given by Hirzebruch [10], p. 159. Indeed, let

$$c(E) = 1 + c_1 + \cdots + c_{n+1} = \prod_{i=1}^{n+1} (1 + \alpha_i)$$

be a (purely formal) decomposition of the total Chern class of E in some extension of $H^*(\mathrm{Gr})$. Then we have

$$c(E \otimes E^*) = \prod_{i,j} (1 + \alpha_i - \alpha_j), \quad \text{product taken over all } i \text{ and } j;$$

$$c(S^2(E)) = \prod_{i,j} (1 + \alpha_i + \alpha_j), \quad \text{taken over all } i \leq j.$$

Notice that these expressions are symmetric polynomials in the α_i, so that they define genuine elements of $H^*(\mathrm{Gr})$. Now

$$c_1(E \otimes E^*) = 0,$$

since all the α_i occur as often with plus signs as with minus; and

$$c_1(S^2(E)) = \sum_{i \leq j}(\alpha_i + \alpha_j) = (n+2)\sum_i \alpha_i = (n+2)c_1,$$

as can be seen by observing that α_1 turns up twice in $(\alpha_1 + \alpha_1)$, and n times in $(\alpha_1 + \alpha_j)$. Hence

$$c_1(T_S) = i^*((2n+4)c_1 - 0 - 2(n+2)c_1) = 0.$$

Finally, Ω_S^{n+1} is the dual of the $(n+1)$st alternating power of T_S, $\bigwedge^{n+1}(T_S)$, so that $c_1(\Omega_S^{n+1}) = -c_1\bigl(\bigl(\bigwedge^{n+1} T_S\bigr) = -c_1(T_S) = 0$.

The theorem is proved.

Let $T \subset S \times X$ be the obvious incidence subvariety:

$$T = \bigl\{(s,x) \bigm| x \in s\bigr\};$$

then I have the two projection maps

$$\begin{array}{ccc} & T & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ S & & X \end{array}$$

and by construction p_1 fibres T in n-planes. I have the following result for p_2:

Lemma 4.9 *p_2 is onto, and the generic fibre of p_2 has 2^{2n} points; in other words, through a generic point of X pass 2^{2n} n-planes.*

Proof The n-planes through $x \in X$ are obvious contained in the tangent plane $T_{x,X}$ to X at x, and hence in the intersection $T_{x,X} \cap X$; this is clearly an intersection of two quadrics in $T_{s,X}$, and it is also clear that it is a cone with vertex x – for it is the intersection of two conical varieties $T_{x,X} \cap Q_1$ and $T_{x,X} \cap Q_2$. Now I claim that it is sufficient to prove that $T_{x,X} \cap X$ is in fact a cone on a nonsingular intersection of two quadrics for generic $x \in X$ – for then the n-planes through x are just the cones on the $(n-1)$-planes on the base of this cone, and the result follows Theorem 3.8.

To prove this condition, it is sufficient to produce a single point x of X for which $T_{x,X} \cap X$ is such a cone on a nonsingular base; for the condition on x is clearly open, and X is irreducible. For this, let X be given by the equations in diagonal form

$$\sum x_i^2 = \sum \lambda_i x_i^2 = 0,$$

and let x be a point $(x_0, x_1, x_2, 0, \ldots, 0)$, whose only nonzero coordinates are the first three; then it can be seen that $T_{x,X} \cap X$ is the cone with vertex x and base X_0 given by the equations

$$\sum_{i \geq 3} x_i^2 = \sum_{i \geq 3} \lambda_i x_i^2 = 0.$$

(To see this, one can actually carry out the computation, or reason as follows: let $x, x_3, \ldots, x_{2n+4}$ be chosen as coordinates in $T_{x,X}$, so that $x_0 = \alpha_0 x, x_1 = \alpha_1 x$ and $x_2 = \alpha_3 x$; then $T_{x,X} \cap X$ is given in $T_{x,X}$ by two equations

$$ax^2 + \sum x_i^2 = bx^2 + \sum \lambda_i x_i^2 = 0,$$

for some a and b; and the only way for the variety these define to be conical is for a and b to vanish.) It is clear that X_0 is nonsingular, by the criterion of Proposition 2.1, and this proves the lemma.

To state Theorem 4.14 in its most general form, it is convenient to use the language of the category of correspondences; I need to use a number of well-known facts about algebraic cycles – for the detailed definition, see the surveys of Kleiman [14] and Manin [17], and the references given there.

For any smooth variety V over the (given) ground field k, $C^*(V)$ will denote the Chow ring of V, so that $C^r(V)$ is the group of cycles of *codimension* r on V modulo rational equivalence.

The *category of correspondences* is the category whose objects are smooth varieties V over k, and in which for any two objects V and V', $\mathrm{Hom}^*(V, V')$ is the graded group

$$\mathrm{Hom}^*(V, V') = C^{n'-*}(V \times V'),$$

where $n' = \dim V'$. Thus an *r-correspondence* from V to V', written $\alpha\colon V \to V'(-r)$ is an $(n+r)$-cycle of $V \times V'$. It is to be thought of as making a point of V correspond to an r-cycle of V'.

The definition of composition of two correspondences is as follows: if $\alpha\colon V \to V'(-r)$ and $\beta\colon V' \to V''(-r')$, then $\alpha \in C^{n'-r}(V \times V')$, $\beta \in C^{n''-r'}(V' \times V'')$, and $\beta \circ \alpha$ is given by

$$\beta \circ \alpha = p_{13*}\bigl\{(\alpha \times 1_{V''}) \cdot (1_V \times \beta)\bigr\},$$

where p_{13} is the projection of $V \times V' \times V''$ onto its outer factors, and the lower star and the multiplication inside the curly brackets have the usual meaning in Chow theory. $\beta \circ \alpha$ has codimension $(n' - r) + (n'' - r') - n'$, so that it is an $(r + r')$-correspondence from V to V''.

If $f\colon V \to V'$ is an ordinary morphism of varieties, then the graph of f, $\Gamma_f \subset V \times V'$ is an n-cycle, so that it defines a 0-correspondence from V to V', noted f_*.

It is not difficult to check that what I have just defined is a category, and that the natural map that is the identity on the objects, and takes the morphism f into the 0-correspondence f_* is a covariant functor from the category of varieties to the category of correspondences.

Since $C^*(V \times V')$ is the same thing as $C^*(V' \times V)$, the correspondence category has the following symmetry: for any correspondence $\alpha\colon V \to V'(-r)$, we can define the *transpose* ${}^t\alpha$ of α as being the self-same cycle α, considered as a cycle on $V' \times V$; thus ${}^t\alpha$ is a $(n - n' + r)$ correspondence from V' to V. In particular, to any morphism $f\colon V \to V'$, we have the transpose $f^* = {}^tf_*$ of the 0-correspondence f_*; f^* is an $(n - n')$-correspondence from V' to V, and $f \mapsto f^*$ is a contravariant functor[2].

If $\alpha\colon V \to V'(-r)$ is a correspondence, we can define homomorphisms

$$\begin{aligned} &\alpha_*\colon C^*(V) \to C^{*-(n-n'+r)}(V') \\ \text{and}\quad &\alpha^*\colon C^*(V) \leftarrow C^{*+r}(V') \end{aligned}$$

in the following way: $\alpha \in C^{n'-r}(V \times V')$, so that cup-producting with α induces maps m_α,

$$m_\alpha\colon C^*(V \times V') \to C^{*+n'-r}(V \times V').$$

[2]I have actually defined the dual of Manin's correspondence category; this makes f_* and f^* seem more natural, but it means that the cohomology functors will vary contrarily.

Then the required maps are the composites

$$\alpha_* = p_{2*} \circ m_\alpha \circ p_1^*, \quad \alpha^* = p_{1*} \circ m_\alpha \circ p_2^*.$$

It is easy to see that these maps do have the indicated effect on the superscripts, and that there is no ambiguity in continuing to denote f_* and f^* the maps induced by the morphism f on the cycle groups.

Most of the interest in the category of correspondences comes from the fact that the maps α_* and α^* can also be defined on the cohomology groups, so that any cohomology theory can be considered as a functor on this bigger category.

I will not recall the definition of a Weil cohomology H; for this, see [14]. One can either consider H to be a cohomology theory, with coefficients in a field of characteristic zero, and implicitly satisfying a number of axioms; alternatively, one can assume from the start that H is either Hodge theory (if $k = \mathbb{C}$), or ℓ-adic cohomology.

H comes equipped with a cycle map

$$\gamma_V \colon C^*(V) \to H^{2*}(V)(-*),$$

which is functorial, and commutes with the cup product defined on both sides. Using γ_V, we can define maps α_* and α^* on the cohomology for any correspondence $\alpha \colon V \to V'(-r)$.

Indeed, let m_α be the cup product with the class $\gamma_V(\alpha)$:

$$m_\alpha \colon H^*(V \times V') \to H^{*+2(n'-r)}(V \times V')(-n' + r).$$

We can then define maps

$$\begin{aligned} &\alpha_* = p_{2*} \circ m_\alpha \circ p_1^* \colon H^*(V) \to H^{*-2(n-n'+r)}(V')(n - n' + r), \\ \text{and} \quad &\alpha^* = p_{1*} \circ m_\alpha \circ p_2^* \colon H^*(V) \leftarrow H^{*+2}(V')(-r). \end{aligned}$$

Needless to say, to be complete, one should check that everything is functorial and is compatible with the cup product structures; in particular, I will be using implicitly obvious little facts such as that the identity correspondence $V \to V$, defined by the diagonal $\Delta_V \in C^n(V \times V)$, induces the identity of the cohomology.

I now wish to discuss a special case of correspondences that will turn up in the proof of Theorem 4.14, namely that of correspondences between curves.

Let C and C' be two nonsingular curves, and α a 0-correspondence between them, so that $\alpha \in C^1(C \times C')$. We have maps

$$\alpha\colon \begin{cases} C^1(C) \to C^1(C') \\ J(C) \to J(C') \\ H^1(C) \to H^1(C') \end{cases}$$

and I would like to compare them.

Recall that the *degree map* $\deg\colon C^1(C) \to \mathbb{Z}$ is the map that takes the 0-cycle $\sum n_i c_i$ into the integer $\sum n_i$.

Lemma 4.10 *Let $\alpha\colon C \to C'$ be a given correspondence, and suppose that for some fixed cycle $y \in C^1(C')$, α_* is given on the 0-cycles by*

$$\begin{array}{rccc} \alpha_*\colon & C^1(C) & \to & C^1(C') \\ & x & \mapsto & \deg(x)y, \end{array}$$

or equivalently, that α_* is the zero map on the cycles of degree zero.

Then

$$\alpha_*\colon J(C) \to J(C') \quad \textit{and} \quad \alpha_*\colon H^1(C) \to H^1(C')$$

are both zero.

Furthermore, α is then rationally equivalent in $C^1(C \times C')$ to a sum of horizontal and vertical components, i.e.,

$$\alpha \sim \sum n_i(\{c_i\} \times C') + \sum m_i(C \times \{c'_i\}).$$

Remark 4.11 (Equivalent hypothesis) The hypothesis of the lemma is equivalent with the following condition on α:

> $\alpha \in C^1(C \times C')$ is represented by the cycle $\sum n_i Z_i$, with the Z_i reduced and irreducible. There exists some dense open $U \subset C$, and some cycle $y \in C^1(C')$, such that, for all $c \in U$, the intersection $(\{c\} \times C') \cap Z_i$ is transverse for all i, and such that the cycle $\sum n_i\bigl((\{c\} \times C') \cap Z_i\bigr)$ is rational equivalent to y.

Proof of the equivalence In any case, for some open of C, the intersection $(\{c\} \times C') \cap Z_i$ is transverse; for the map $p_1 \colon Z_i \to C$ is either a constant (if Z_i is a vertical component), or is generically étale. Hence

$$\alpha_*(c) = \sum n_i\big((\{c\} \times C') \cap Z_i\big)$$

whenever the right hand side is defined, and it is defined for generic $c \in C$.

The equivalence of the two hypotheses is now clear; for if $\alpha_* = \deg(\ )y$, then $\sum n_i\big((\{c\} \times C') \cap Z_i\big)$ is rationally equivalent to y whenever it is defined. And conversely, if $\sum n_i\big((\{c\} \times C') \cap Z_i\big)$ is rationally equivalent to y for some open U, for some open U, then any cycle x on C can be pushed about under rational equivalence until it is supported inside U, and $\alpha_*(X) = \deg(x)y$.

Proof of Lemma 4.10 The assertion on the Jacobians is more or less obvious, since the (geometric) points of $J(C)$ are just the 0-cycles of C modulo rational equivalence, so that α_*, being zero on the geometric points of $J(C)$, is the zero map.

C and C' are supposed nonsingular, so that $C \times C'$ is a nonsingular surface, and the notion of 1-cycle, Weil divisor, and Cartier divisor coincide. Hence it will suffice to prove that the Cartier divisor α on $C \times C'$ is linearly equivalent to a divisor of the form $p_1^*(x) + p_2^*(y)$ for some divisor x and y on C and C' respectively.

This is easy modulo the existence and the principal properties of the Picard variety of C'. For the divisor α on $C \times C'$ induces a morphism $C \to \mathrm{Pic}(C')$, whose effect on the geometric points of C is given by $c \mapsto i_c^*(\alpha)$, where i_c is the inclusion map of $\{c\} \times C'$ into $C \times C'$. It is obvious that $i_c^*(\alpha) = \alpha_*(c)$, at least for c in the dense open of C where the intersection of $\{c\} \times C'$ with the components Z_i of α are transverse. Hence $\alpha_* \colon C \to \mathrm{Pic}(C')$ takes a dense open of C to the point $y \in \mathrm{Pic}(C')$, and hence it is the constant map.

It is then clear that the divisor $\alpha - p_2^*(y)$ induces the zero map from C to $\mathrm{Pic}(C')$, and so $\alpha - p_2^*(y)$ is linearly equivalent to a divisor of the form $p_1^*(x)$, for some divisor x on C.

From the fact that α is rationally equivalent to a sum of horizontal and vertical components, it is easy to deduce that α_* is zero on the first cohomology; indeed, by linearity, we may assume that α is either $\{c\} \times C'$ or $C \times \{c'\}$, so that the cohomology class $\gamma_V(\alpha)$ is $\gamma_C(c) \otimes 1_{C'}$ or $1_C \otimes \gamma_{C'}(c')$ in $H^2(C \times C')(-1)$.

Using the Künneth decomposition

$$H^n(C \times C') = \bigoplus_i H^i(C) \otimes H^{n-i}(C'),$$

it is obvious that the map $m_\alpha \colon H^1(C \times C') \to H^3(C \times C')(-1)$ is the sum of the pieces

$$\begin{aligned} 1 \otimes m_{\gamma_{C'}(c')} \colon H^1(C) \otimes H^0(C') &\to H^1(C) \otimes H^2(C')(-1) \\ m_{\gamma_C(c)} \otimes 1 \colon H^0(C) \otimes H^1(C') &\to H^2(C)(-1) \otimes H^1(C'), \end{aligned}$$

neither of which induces anything nontrivial from $H^1(C)$ to $H^1(C')$.

The lemma is proved.

Corollary 4.12 *Suppose that $\alpha \colon C \to C$ is a correspondence such that $\alpha_* \colon C^1(C) \to C^1(C)$ is given by*

$$x \mapsto (\pm 1)x + \deg(x)y,$$

for some constant cycle $y \in C^1(C)$; or equivalently, by the argument developed above, suppose that all points c of an open dense subset of C "correspond" in the sense of the equivalent hypothesis of Lemma 4.10 under α to a cycle rationally equivalent to $(\pm 1)c + y$, for some fixed fixed 0-cycle $y \in C^1(C)$.

Then α is rationally equivalent to a cycle of the form

$$\alpha \sim (\pm 1)\Delta_C + \sum n_i(\{c_i\} \times C) + \sum m_i(C \times \{c_i'\});$$

and the maps $\alpha_* \colon H^1(C) \to H^1(C)$ and $\alpha_* \colon J(C) \to J(C)$ are ± 1 times the identity.

Proof The identity correspondence $\Delta_C \subset C \times C$ induces the identity in the cycle groups, as well as in the cohomology groups and the Jacobians, in view of the functoriality that is implicit in everything that I have said. Hence by adding or subtracting Δ_C from the given correspondence α, I can bring myself back to the case dealt with in Lemma 4.10.

Corollary 4.13 [3] *α_* is also the identity on the 1-motive $h^1(C)$.*

[3] This corollary can be omitted at a first reading. For a definition of the category of motives, see [17]. It is perhaps worth pointing out that I am still on respectable, i.e., nonconjectural, grounds.

“Proof” We have the well known decomposition

$$h(C) = 1 \oplus h^1(C) \oplus L;$$

(see [17]). The identity correspondence Δ_C is the identity on all of $h(C)$; horizontal and vertical components can only add a nontrivial morphism from 1 to 1 and from L to L.

Let me now introduce the correspondences that are going to figure in Theorem 4.14. I return to the previous notations, i.e., X is a $(2n+1)$-dimensional nonsingular intersection of two quadrics, S is its variety of n-planes. $T \subset S \times X$ is the incidence cycle mentioned in Lemma 4.9: $T = \{(s,x) | x \in s\}$. T is a $(2n+1)$-cycle of $S \times X$, so that it defines correspondences

$$\alpha\colon S \to X(-n) \quad \text{and} \quad {}^{\mathrm{t}}\alpha\colon X \to S.$$

In $S \times S$ we have the cycle D defined by

$$D = \bigl\{(s,t) \bigm| s \text{ and } t \text{ meet}\bigr\},$$

which is a divisor, since it is easy to see from Lemma 4.9 that it has dimension at least $2n+1$, and it can't be any bigger without being the whole of $S \times S$, which contradicts for instance the known fact that X has disjoint n-planes.

D defines a correspondence $\beta\colon S \to S(-n)$. It is fairly obvious that β is nothing other than the composite ${}^{\mathrm{t}}\alpha \circ \alpha$.

We have already seen that if $s \in S$ is any chosen base point, there is a map $r_s\colon C \to S$, which is birational onto its image C_s (Proposition 4.2). Banishing any further mention of the base point s, we have another pair of correspondences

$$r_*\colon C \to S \quad \text{and} \quad r^*\colon S \to C(n).$$

Recall that for any variety V, with base point $v \in V$ chosen, there is defined a mapping $V \to \mathrm{Alb}(V)$, which is universal among maps of V into Abelian varieties, taking v to 0. In particular, for S a principal homogeneous space under an Abelian variety, and $t \in S$ fixed, $S \cong \mathrm{Alb}(S)$, the isomorphism taking t into 0.

If D is a divisor on $S \times S$, then D induces a morphism from S to $\mathrm{Pic}(S)$, and if we pick base points t and $i_t^*(D)$ in S and the relevant component of $\mathrm{Pic}(S)$, we obtain a morphism

$$D_*\colon\ \mathrm{Alb}(S) \to \mathrm{Pic}^0(S).$$

I will say that D *induces a principal polarization* of S if D_* is an isomorphism. Recall also that Alb is a covariant functor (on pointed varieties); and finally,[4] that for a curve, $\mathrm{Alb}(C) = J(C) = \mathrm{Pic}^0(C)$.

Theorem 4.14 *I have the following diagrams:*

(a) in the category of correspondences

$$\begin{tikzcd}
C \arrow[r, "r_*"] & S \arrow[dd, "\beta"'] \arrow[dr, "\alpha"] & \\
 & & X(-n) \arrow[dl, "{}^t\alpha"] \\
C & S(-n) \arrow[l, "r^*"'] &
\end{tikzcd}$$

(b) for any Weil cohomology H

$$\begin{tikzcd}
H^1(C) \arrow[r, "r_*"] & H^{2n+1}(S)(-n) \arrow[dd, "\beta_*"'] \arrow[dr, "\alpha_*"] & \\
 & & H^{2n+1}(X)(-n) \arrow[dl, "\alpha^*"] \\
H^1(C) & H^1(S) \arrow[l, "r^*"'] &
\end{tikzcd}$$

(c) in the category of Abelian varieties

$$\begin{tikzcd}
J(C) \arrow[r, "r_*"] & \mathrm{Alb}(S) \arrow[d, "\beta"'] \\
J(C) & \mathrm{Pic}^0(S) \arrow[l, "r^*"']
\end{tikzcd}$$

(c′) if $k = \mathbb{C}$,

$$\begin{tikzcd}
J(C) \arrow[r, "r_*"] & \mathrm{Alb}(S) \arrow[dd, "\beta"'] \arrow[dr, "\alpha_*"] & \\
 & & J(X) \arrow[dl, "\alpha^*"] \\
J(C) & \mathrm{Pic}^0(S) \arrow[l, "r^*"'] &
\end{tikzcd}$$

[4] See Lang's book [16] for details on Abelian varieties.

where $J(X)$ is the intermediate Jacobian of X, defined purely from the Hodge structure of $H^*(X)$.

(d) Finally, in the category of motives,[5]

$$\begin{array}{ccccccc} h^1(C) & \xrightarrow{i} & h(C) & \xrightarrow{r_*} & h(S) & & \\ & & & & & \searrow^{\alpha} & \\ & & & & \beta\downarrow & & h(X)(-n) \\ & & & & & \swarrow^{{}^t\alpha} & \\ h^1(C) & \xleftarrow{p} & h(C) & \xleftarrow{r^*} & h(S)(-n) & & \end{array}$$

i and p being the inclusion and the projection of the first factor of $h(C) = 1 \oplus h^1(C) \oplus L$.

Let $\varphi\colon C \to C$ denote the composite correspondence

$$\varphi = r^* \circ \beta \circ r_*.$$

Then for all c in an open dense subset $U \subset C$,

$$\varphi_*(c) = (-1)^n c + y \in C^1(C), \tag{A}$$

where $y \in C^1(C)$ is some constant, so that φ satisfies the hypotheses of Corollary 4.5. Hence:

(a) $\varphi\colon C \to C$ differs from $(-1)^n$ times the identity correspondence only by horizontal and vertical components.

(b) $\varphi_*\colon H^1(C) \to H^1(C)$ is an isomorphism, so that $H^1(C)$ is a direct summand of $H^1(S)$, $H^{2n+1}(X)(-n)$ and $H^{2n+1}(S)(-n)$. In fact, the Betti numbers $b_1(C) = b_1(S) = b_{2n+1}(S) = b_{2n+1}(X) = n+1$ for the classical or the ℓ-adic cohomologies; for any Weil cohomology with these Betti numbers, we thus have that all the arrows in diagram (b) are isomorphisms.

(c) $\varphi_*\colon J(C) \to J(C)$ is an isomorphism, so that all the arrows in diagram (c) are isomorphisms, and the divisor D on $S \times S$ induces a principal polarization of S.

[5] See footnote 3 on page 70.

(c′) *If $k = \mathbb{C}$ and H = Hodge theory, then it follows from (b) that all the arrows in (c′) are isomorphisms; this also follows from (c), in view of the fact that $J(X)$ has dimension $b_{2n+1}(X) = n+1$.*

(d) *Finally, the motive $h^1(C)(n)$ is a direct summand of $h(X)$.*

Of the above assertions, only (A) has to be proved, since the others then follow from Lemma 4.10 and its corollaries.

Proof of (A) I will write C to denote also an open subset of C, which will shrink at several stages in the proof; since r_s maps C birationally to $C_s \subset S$, I will consider C as embedded in S. Points of S or of C will be denoted s, t, etc.; the same letters will be used to denote the corresponding n-planes of X.

Let me describe the effect of the various maps

$$\begin{array}{ccccc} C^1(C) & \xrightarrow{r_*} & C^{n+1}(S) & & \\ & & & \searrow^{\alpha_*} & \\ & & \beta_* \downarrow & & C^{n+1}(X) \\ & & & \swarrow_{\alpha^*} & \\ C^1(C) & \xleftarrow{r^*} & C^1(S) & & \end{array}$$

on the various cycles of interest.

$$r_* \text{ takes a point } t \in C \text{ into the point } t \text{ of } S.$$
$$\alpha_* \text{ takes a point } t \in S \text{ into the } n\text{-cycle } t \text{ of } X.$$

To describe β_* and α^*, I need to introduce the notation: for $t \in S$, $D_t \subset S$ is the sum of the components of dimension n (divisors) of the subvariety $\{t' \mid t' \text{ meets } t\}$. This subvariety does have components of dimension n, for instance by Lemma 4.9. Then

$$\beta_* \text{ takes the point } t \text{ into the divisor } D_t;$$
$$\text{whereas } \alpha^* \text{ takes the } n\text{-cycle } t \text{ into the divisor } D_t.$$

Finally, r^* is defined as follows: $r^*(D) = D' \cap C$, where D' is a divisor of S rationally equivalent to D and in general position w.r.t. C, that is, the intersection of D' with C_s is everywhere transverse, and only takes place inside the dense open of C that we are interested in.

Then (A) can be restated as: for t belonging to a dense open of C, $(D_t)' \cap C$ is rationally equivalent, as a 0-cycle of C, to $(-1)^n t + \text{constant}$.

Recall that C is birationally the subvariety of S consisting of n-planes of X very incident to s (Proposition 4.2). By Lemma 4.6, for generic $t \in C$, t and s belong to a common nonsingular hyperplane section X_H of X.

D_t consists of those n-planes of X that meet t; unfortunately, $C \subset D_t$, since an n-plane that is very incident to s, which is itself very incident to t, meets t automatically (if $n \geq 2$). I wish to construct explicitly a $(D_t)'$ that is rationally equivalent to D_t, and meets C generally. For this, I will use the properties of X_H developed in §3.

The idea is as follows: I find an n-cycle of X_H that is rationally equivalent to t, and is a sum of n-planes disjoint from s; then I will have $\alpha^*(t) = D_t$ rationally equivalent to a sum of $D_{t'}$, with t' an n-plane of X_H disjoint from s. These $D_{t'}$ meet C generally, and the intersection can also be reckoned using the methods of §3.

I will now be using the notation of §3, with X_H replacing X: s is a given n-plane, so that the n-planes very incident to s can be labelled $s_1 = t$, $s_2, \ldots, s_{2n+3}$. If $\eta \in C^{n+1}(X)$ is the class of the $n+2$-plane section, then whenever four n-planes a, b, c, d are such that

$$X \cap (n+2\text{-plane}) = a + b + c + d,$$

then $a+b+c+d$ is rationally equivalent to η; the calculations of Lemma 3.12 are therefore valid in $C^{n+1}(X)$ – I am using here the obvious fact that if two cycles in X_H are rationally equivalent in X_H then they are a fortiori so in X.

Hence $u_i = t_{2\ldots\hat{i}\ldots n+2}$, for $i = 2, \ldots, n+2$, are $n+1$ n-planes of X_H that are disjoint from t and s. Let $u = t_{2\ldots n+2} = s_{1\ldots n+2} = s_{n+3\ldots 2n+3}$. Then applying Lemma 3.12, I obtain

$$(-1)^{n+1} t = \left[\frac{n+1}{2}\right] \eta - nu - \sum_{i=2}^{n+2} u_i \in C^{n+1}(X).$$

Applying α^* to both sides,

$$(-1)^n D_t = \text{constant} - nD_u + \sum D_{u_i} \in C^1(S).$$

u and the u_i are disjoint from s, and contained in the nonsingular hyperplane section X_H. The intersections $C_s \cap D_u$, resp. $C_s \cap D_{u_i}$ consist of the n-planes of X that are very incident to s, and incident to u, resp. u_i; any such n-plane must of course be contained in X_H, so that they are given by Theorem 3.14 and Lemmas 3.10–3.13: in fact

$$\begin{aligned} C_s \cap D_u &= \{s_i, i = n+3, \dots, 2n+3\}, \\ \text{and} \quad C_s \cap D_{u_i} &= \{s_j, i \neq j = 1, \dots, n+2\}. \end{aligned} \tag{0}$$

The following argument can now be used to show that the intersections in (0) are transverse; which is equivalent to saying that the intersection of the three subvarieties of the relevant Grassmannian

$$\begin{aligned} S &= \{t \mid t \subset X\} \\ U_1 &= \{t \mid t \text{ is very incident to } s\} \\ U_2 &= \{t \mid t \text{ meets } u\}, \end{aligned}$$

which is certainly supported on the subsets given in (0), is in fact reduced. It is easy to see that $U_1 \cap U_2$ is a sub-*scheme* of U_3, with

$$U_3 = \{t \mid t \subset H\};$$

indeed, a point of $U_1 \cap U_2$ with values in a ring R is the span of an $(n-1)$-plane of s (i.e., a point of s^*) and a point of u, with values in the same ring R; (recall that I am supposing u and s disjoint). This span is obviously a point of U_3, since it is an n-plane, and is contained in the span of s and u, and hence a fortiori in H.

U_3 is obviously just the Grassmannian of n-planes of H, so that $S \cap U_3$ is the subscheme of U_3 consisting of n-planes of X_H, and is hence reduced by Theorem 2.6.

Hence $r^*(D_t) = \varphi_*(t)$ can be calculated as

$$\begin{aligned} r^*(D_t) &= \text{constant} + n(-1)^n r^*(D_u) + (-1)^n r^* \left(\sum D_{u_i}\right) \\ &= \text{constant} + n(-1)^n \sum s_j + (-1)^n \sum_i \sum s_j, \end{aligned}$$

the summation being over s_j in the sets indicated in (0), i.e.,

$$\begin{aligned} &= \text{constant} + n(-1)^n \sum_{n+2}^{2n+3} s_j + (-1)^n \sum_{i=2}^{n+2} \sum_{i \neq j=1}^{n+2} s_j \\ &= \text{constant} + n(-1)^n \sum_{1}^{2n+3} s_j + (-1)^n s_1; \end{aligned}$$

this can be seen on observing that all the s_i except for s_1 turn up n times, and s_1 turns up $n+1$ times. Recall that $s_1 = t$, so that to show

$$r^*(D_t) = \text{constant} + t,$$

it suffices to see that $\sum_1^{2n+3} s_j$ is a constant in $C^1(C)$. This is just the sum of the $2n+3$ n-planes of X_H that are very incident to s, so that it can only depend on H; but it is then a constant, since H varies in a rational variety.

This completes the proof of (A), and hence of Theorem 4.14.

The assertion (c) of Theorem 4.14 has the following corollary

Corollary 4.15 *Let $t_i \in S$, and n_i be integers. Then*

$$\sum n_i t_i = 0 \text{ in } C^{n+1}(S) \iff \sum n_i t_i = 0 \text{ in } C^{n+1}(X).$$

Proof $\Longrightarrow$ is obvious in view of the fact that $\alpha_*\colon C^{n+1}(S) \to C^{n+1}(X)$ takes t_i into t_i.

$\Longleftarrow$: $\sum n_i t_i = 0$ in $C^1(X)$ implies that $\alpha^*(\sum n_i t_i) = 0$ in $C^1(S)$; but this is also equal to $\beta_*(\sum n_i t_i)$, and if β is an isomorphism, then $\sum n_i t_i = 0$ in $C^{n+1}(S)$.

This makes the C_s^i of Proposition 4.2 and Definition 4.3 have a natural meaning. For simplicity, suppose that a base point $t \in S$ is chosen in addition to the $s \in S$ that has been used to define[6] the C_s^i. Then $S = \mathrm{Alb}(S)$, and is an Abelian variety, so that we have in particular an involution $-1\colon S \to S$, and $s + (-1)s = 2t$ in $C^{n+1}(S)$.

Corollary 4.16 *Let $m\colon C_s^{(i)} \to S$ be the map given by*

$$(s_j) \mapsto \sum s_j.$$

Then $C_s^i \subset S$ is a translate of the image of the map $(-1)^{i+1}m$:

$$C_s^i = \text{constant} + \mathrm{Im}((-1)^{i+1}m).$$

[6]There is no reason for not choosing $t = s$ all over again.

Proof The idea is to show that for $u \in C_s^{i\prime}$ (see Proposition 4.2 and Definition 4.3) if $s_1, \dots, s_i$ are the i n-planes such that

$$\dim(s \cap s_j) = n - 1, \quad \text{and} \quad \dim(u \cap s_j) = n - i + 1,$$

as in Proposition 4.4, then

$$(-1)^{i+1} u = \text{constant} + \sum_{j=1}^{i} s_j \in C^{n+1}(X),$$

which implies the result by Corollary 4.5.

For this I need only restrict myself to the dense open of $C_s^{i\prime}$ consisting of u such that s and u are contained in a common nonsingular hyperplane section X_H of X. Then, by an argument I have already used in Theorem 4.14, the relation

$$(-1)^i u = \left[\frac{i}{2}\right] \eta - (i-1)s - \sum_{j=0}^{i} s_j,$$

obtained in Lemma 3.12, is valid in $C^{n+1}(X)$, where η is as usual the class in $C^{n+1}(X)$ of an $(n+2)$-plane section of X.

It is clear by continuity that this relation holds for all $u \in C_s^i$ once it does for a dense open subset, so that the corollary is proved.

4.1 Generalized Kummer varieties, and n-planes "of the second kind"

To conclude this paragraph, I would like to show the connection between the variety of n-planes of X and a generalization of the work of Kummer and Klein on quadratic complexes. This last section is independent of the cohomological arguments that have preceded.

Let $Q \subset \mathbb{P}^5$ be a nonsingular quadric; then it is well known that Q can be identified with the Grassmannian of lines in $\mathbb{P}^3$, $\mathrm{Gr}(2,4)$, in its Plücker embedding – and, with this identification in mind, we can refer to Q as the *Klein quadric* [12].

There is a beautiful geometric interpretation of the generators of the Klein quadric Q: for any point $p \in \mathbb{P}^3$, write

$$g_p = \bigl\{ t \in \mathrm{Gr}(2,4) \bigm| t \text{ passes through } p \bigr\},$$

and dually, for any hyperplane $p^* \in \mathbb{P}^{3*}$,

$$g_p^* = \{t \in \mathrm{Gr}(2,4) \mid t \subset p^*\}.$$

Then g_p and g_p^*, as subvarieties of Q, are 2-planes, so that we have two 3-dimensional varieties of generators of Q, in accordance with Theorem 1.2; what is new is that these varieties of generators are actually isomorphic to $\mathbb{P}^3$.

Definition 4.17 A *line complex* is the intersection of the Grassmannian $\mathrm{Gr}(2,4)$, in its Plücker embedding, with a hypersurface H^d of $\mathbb{P}^5$. The *degree* of the complex is the degree d of the hypersurface H^d. If $d = 1$, the complex is a linear complex, if $d = 2$, a quadratic complex, and so on.

A complex of degree d is thus obtained by imposing a condition "of degree d" on the lines of $\mathbb{P}^3$; if g_p is a generator of Q, $g_p \cap H^d$ is just the set of lines through p satisfying this condition, and just forms the generators of a cone, of degree d, with vertex p. The case $d = 2$, the quadratic complex, is the one that has attracted most attention in the literature, the case $d = 1$ being easy, and $d \geq 3$ being too difficult.

The intersections $g_p \cap H^d$ are conics – if we look at the point p such that g_p degenerates into a line pair, we obtain the classical point of view on Kummer surfaces; the connection with the variety of lines on the intersection $Q \cap H^2$, which I want to discuss and generalize, does not seem to have occurred to the ancients, although I am not the first to spot it either [19].

Suppose now that the intersection $Q \cap H^2$ is a nonsingular intersection of two quadrics, and that Q is still identified with the Klein quadric. Define the following subvarieties of $\mathbb{P}^3$, identified as one of the families of generators of Q as above:

$$K_0 \subset K_1 \subset K_2 = \mathbb{P}^3,$$

defined by

$$K_1 = \{p \in \mathbb{P}^3 \mid g_p \cap H^2 \text{ degenerates into a line pair}\}$$
$$K_0 = \{p \in \mathbb{P}^3 \mid g_p \cap H^2 \text{ is a double line}\}.$$

Then K_1 is a surface of degree 4 in $\mathbb{P}^3$, which is nonsingular outside the 16 points of K_0. The symmetry properties of K_1 and its 16 double points, as well as the relation between K_1 and the dually defined $K_1^* \subset \mathbb{P}^{3*}$, have been widely studied in the 19th century [13].

Before giving the connection with my own work, I should like to discuss briefly the widest possible generalization of these ideas. Let Q^1 be a quadric, and assume that Q^1 is

(a) nonsingular, and of dimension $2n+2$, or

(b) an ordinary cone of dimension $2n+2$, or

(c) nonsingular, and of dimension $2n+1$.

Let G be one component of the variety of generators of Q^1. I will continue to use the same letter g to denote a point of G or the $(n+1)$-plane (resp. n-plane) of $\mathbb{P}$ that corresponds to it. By Theorems 1.2 and 1.8, G is nonsingular of dimension $\binom{n+2}{2}$, and has a cell decomposition.

G is going to be the required generalization of $\mathbb{P}^3$ – in view of the fact that it has a cell decomposition, one should expect that its geometry is no more complicated than, say, that of Grassmannians.

Suppose now that Q^2 is a second quadric, and that $Q^1 \cap Q^2$ is nonsingular. The generalization of the classical ideas is to consider the subvarieties of G consisting of g such that the intersection $g \cap Q^2$ acquires a certain degeneracy.

For this, let φ be a quadratic form defining Q^2, and for $g \in G$ with $g = \mathbb{P}(E)$, let $\operatorname{rank}(g) = \operatorname{rank}(\varphi_{|E})$ – then $\operatorname{rank}(g) = n+2$ for general $g \in G$ in cases (a) and (b), and $n+1$ in case (c); and $\operatorname{rank}(g)$ drops as $g \cap Q^2$ becomes degenerate. Let

$$K_0 \subset K_1 \subset \cdots \subset K_n \subset K_{n+1} = G$$

be defined by

$$\begin{aligned} K_i &= \{ g \in G \mid \operatorname{rank}(g) \le i+1 \} \quad \text{in cases (a) or (b),} \\ &= \{ g \in G \mid \operatorname{rank}(g) \le i \} \quad \text{in case (c).} \end{aligned}$$

The K_i are called *over-generalized Kummer varieties.*

In both cases, we have that K_n is the subvariety of G where $g \cap Q^2$ is degenerate, so that K_n is a divisor in G.

In case (c), K_{-1} is empty by definition; in the other cases, K_{-1} is empty because if $g \in G$ and $\operatorname{rank}(g) = 0$, then φ vanishes identically on E, so that g is an $(n+1)$-plane of the intersection $Q^1 \cap Q^2$, which contradicts the nonsingularity of this intersection.

In case (c), K_0 consists just of the n-planes of $Q^1 \cap Q^2$, and we have seen in §3 that this is just 2^{2n+2} (reduced) points. In (a) and (b), K_0 consists of those $(n+1)$-planes of Q^1 that touch Q^2 along an n-plane s – so that the intersection $g \cap Q^2$ for $g \in K_0$ is s counted twice. I hope to show that there are also 2^{2n+2} of these, corresponding to certain n-planes "of the second kind" of $Q^1 \cap Q^2$.

I expect the following to hold:

Guess 4.18 *K_i is of dimension $\binom{n+2}{2} - \binom{n+1-i}{2}$ and is nonsingular outside K_{i-1}.*

I will prove this only in a much more restricted case that is related to the n-planes of $Q^1 \cap Q^2$, namely the case (a) or (b), with $i = 1$; these are the generators g of Q^1 such that $g \cap Q^2$ splits up into a pair $s \cup s'$ of n-planes of $Q^1 \cap Q^2$.

I should point out that the above guess is only the first in a long chain of questions about the geometry of the K_i; by analogy with the classical case, there is still an enormous amount to discover in the field.

Let me now return to the usual notation of this paragraph; suppose that a point $c \in C$ has been chosen (cf. p. 56), so that $G(c)$ is one component of the variety of generators of, say, Q^1. I do not assume that Q^1 is nonsingular, but it is, or course, at worst an ordinary cone, so that I am in case (a) or (b) above.

S as usual denotes the variety of n-planes of X. An argument of the type used in the paragraphs preceding Proposition 4.2 shows that there is a morphism

$$r\colon S \to G(C)$$

taking any point $s \in S$ into the unique generator g of the family $G(c)$ containing it. It is clear that the image of r is precisely K_1, for we have $\operatorname{rank}(g) = 2$ if and only if $g \cap Q^2$ degenerates into a pair of n-planes; and thus r is generically 2-to-1.

Definition 4.19 An n-plane $s \in S$ is said to be *of the second kind* for c if $r(s) \in K_0$.

A definition that is obviously equivalent is to say that there exists a generator $g \in G(c)$ of Q^1 that touches Q^2 along s, i.e., such that $g \cap Q^2$ is the n-plane s counted twice.

Proposition 4.20 *$r\colon S \to K_1$ is a double covering, étale outside the points of the second kind for c.*

Proof We have already seen that r is generically 2-to-1; to see that r is étale outside the n-planes of the second kind for c is going consist of a verification that the map

$$\mathrm{d}r\colon T_{s,S} \to T_{r(S),G}$$

has maximal rank, i.e., is injective. This is an infinitesimal computation, relying on the methods of §§1–2, particularly Lemmas 1.3–2.2. I will use the matrix notation of §2 without further commentary.

Firstly, I need an equivalent way of saying that an n-plane $s \in S$ is of the second kind for c. Suppose that $s = \mathbb{P}(E)$, $r(s) = g = \mathbb{P}(E')$; then I can choose a basis $e_1, \dots, e_{2n+4}$ of V such that $e_1, \dots, e_{n+1}$ base E, $e_1, \dots, e_{n+1}, e_{2n+3}$ base E', and such that the matrices of φ_1 and φ_2 with respect to these bases are

$$\varphi_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & J \end{pmatrix}, \quad \text{and} \quad \varphi_2 = \begin{pmatrix} 0 & L & A \\ L & 0 & B \\ {}^{\mathrm{t}}A & {}^{\mathrm{t}}B & C \end{pmatrix},$$

with L diagonal with distinct entries, and J either $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ if Q^1 is nonsingular, or $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ if Q^1 is an ordinary cone. As in the slightly different case of Addendum 2.5, the nonsingularity of X implies that no row of A or b vanishes completely.

By definition, s is of the second kind for c if and only if the rank of φ_2 restricted to E' is 1. But the matrix of φ_2 restricted to E' is

$$\begin{pmatrix} 0 & A_1 \\ {}^{\mathrm{t}}A_1 & c_{11} \end{pmatrix},$$

with A_1 the first row of A, and c_{11} the first entry of C; this has rank 1 if and only if $A_1 = 0$, the condition $c_{11} \neq 0$ then following from the nonsingularity of X.

Recall that the tangent space $T_{s,S}$ is given by (IXY), with X and Y infinitesimal matrices, and

$$X + {}^{\mathrm{t}}X = 0 \tag{4.1}$$

$$XL + L{}^{\mathrm{t}}X + Y{}^{\mathrm{t}}A + A{}^{\mathrm{t}}Y = 0. \tag{4.2}$$

The $(n+2)$-plane neighbouring E' in G and containing (IXY), can obviously be based as

$$\begin{pmatrix} I & X & Y \\ 0 & u & 1v \end{pmatrix},$$

i.e., can be spanned by (IXY) and another vector $(0\ u\ 1v)$ close to e_{2n+3}; in fact it has a unique basis of the form

$$\begin{pmatrix} I & X & 0\ Y_2 \\ 0 & u & 1\ v \end{pmatrix},$$

Y_2 being the second column of Y.

It is not difficult to see that u and v, a priori linear in X and Y, must vanish; so that $\mathrm{d}r\colon T_{s,S} \to T_{g,G}$ is given by

$$(X\ Y) \mapsto (X\ 0Y_2);$$

to show that $\mathrm{d}r$ is injective, I am thus down to checking that, provided some $a_{1i} \neq 0$ (so that s is not of the second kind for c), the equations (4.1) and (4.2) imply that

$$X = Y_2 = 0 \implies Y_1 = 0.$$

This is easy: firstly, the equation $(4.2)_{ii}$ reads

$$a_{1i}y_{i1} + a_{2i}y_{i2} = 0,$$

so that $y_{i1} = 0$; then $(4.2)_{ij}$, using the fact that all the x_{ij}, y_{j2} and $y_{i1} = 0$, and of course again using $a_{1i} \neq 0$, implies at once that $y_{j1} = 0$.

This completes the proof of the proposition.

Corollary 4.21 *K_1 is nonsingular outside K_0, and is of dimension $n+1$.*

Since $S \to K_1$ is a double covering, there is at least a rational involution $\sigma\colon S \to S$ interchanging the sheets; in fact, since S is an Abelian variety, σ is morphic.

Lemma 4.22 *σ is (up to the choice of base point) minus the identity; in particular, σ has 2^{2n+2} fixed points.*

Proof For $g \in G(c)$, $g \cap Q^2 = g \cap X$ is an n-cycle of X, and since $G(c)$ is a rational variety, this is a constant in $C^{n+1}(X)$. But for $g \in K_1$, $g \cap Q^2 = s \cup \sigma(s)$, so that

$$s + \sigma(s) = \text{constant} \in C^{n+1}(X).$$

Hence by Corollary 4.15,

$$s + \sigma(s) = \text{constant} \in C^{n+1}(S),$$

which is the assertion of the lemma.

It is easy to see that the fixed points of σ correspond just to the ramification points of $r\colon S \to G(c)$, and thus to the n-planes of the second kind for c.

Corollary 4.23 *X has 2^{2n+2} n-planes of the second kind for c. K_0 consists of just 2^{2n+2} points.*

Let $(1, \sigma)$ be the group of automorphisms of S consisting of σ and the identity, and let K denote the quotient variety $K = S/(1, \sigma)$. The map $r\colon S \to K_1$ is $(1, \sigma)$-invariant, so that there is a factorization

$$\begin{tikzcd} S \arrow[d, "r"'] \arrow[dr] & \\ K_1 & K \arrow[l, "r'"] \end{tikzcd}$$

What we have seen is that the map r' is biregular outside K_0; I expect that a careful infinitesimal study of the singularity of K_1 at a point of K_0 would reveal that r' is in fact an isomorphism – this is at any rate what happens in the case $n = 1$, where the singularities of K and K_1 are ordinary double points.

4.2 Appendix. The irreducibility of S

This appendix is devoted to showing that the irreducibility of the variety S is an easy consequence of some results of Lefschetz theory, together with the results of §3 on the nonsingular hyperplane sections of X. These pages owe a lot, in particular their existence, to several conversations with Professor Deligne.

X_H will denote the generic hyperplane section of X; it is a nonsingular intersection of two quadrics and has dimension $2n$. It is defined over the field K of rational functions on $\mathbb{P}^*$. As before, $\mathbb{P}^*$ denotes the projective space dual to $\mathbb{P}$, and $X^* \subset \mathbb{P}^*$ denotes the dual variety of X, and consists of those hyperplanes of $\mathbb{P}$ which intersect X nontransversally.

The fundamental group $\pi_1(\mathbb{P}^* \setminus X^*)$ acts on X_H; classically, this action is obtained by pushing X around loops. From a purely algebraic point of view, $\pi_1(\mathbb{P}^* \setminus X^*)$ is a subgroup of the Galois group $\mathrm{Gal}(\overline{K}/K)$, and acts on X_H by Galois action. I wish to show that the action of $\pi_1(\mathbb{P}^* \setminus X^*)$ on X is transitive on the n-planes of X_H, and to deduce the irreducibility of S from this.

I will need the following results from Lefschetz theory (see [6, XVIII], concerning the action of $\pi_1(\mathbb{P}^* \setminus X^*)$ on the cohomology group $H^{2n}(X_H, \mathbb{Z}_\ell)(-n)$; let π denote the image of $\pi_1(\mathbb{P}^* \setminus X^*)$ in the group $\mathrm{GL}(H^{2n}(X_H, \mathbb{Z}_\ell)(-n))$. Then

(i) π is a Coxeter group, that is, π is a finite group generated by orthogonal reflections s_i in vectors $r_i \in H^{2n}(X_H, \mathbb{Z}_\ell)(-n)$ of length $(-1)^n \cdot 2$.

(ii) Let H_0 be the orthogonal in $H^{2n}(X_H, \mathbb{Z}_\ell)(-n)$ of the vector η. The action of π fixes η, so that π acts on H_0. Then this action is an irreducible representation of π on H_0.

(iii) π preserves the lattice $A(X_H)$ generated by the cohomology classes of n-planes of X_H.

In view of (iii), and the fact that $A(X_H)$ spans $H^{2n}(X_H, \mathbb{Z}_\ell)(-n)$ as a vector space, the group π can be considered as a subgroup of

$$\mathrm{Aut}(A(X_H), \ . \ , \eta) = W(A^\perp).$$

Theorem 4.24 $\pi = W(A^\perp)$.

Proof (i) implies that there is a root system R^1 in $H^{2n}(X_H, \mathbb{Z}_\ell)(-n)$, all the roots of which have length $(-1)^n \cdot 2$, and such that π is the Weyl group $W(R^1)$ generated by reflections in the roots of R^1.

(ii) then implies that R^1 is an irreducible root system, and that the roots of R^1 span H_0 – so that R^1 has rank $2n+3$.

The same assertions also hold for $W(A^\perp)$, the corresponding root system R^2 being of type D^{2n+3}. The proof of the theorem is going to consist firstly in checking that $R^1 \subset R^2$, and then in using the classification of root systems to show that $R^1 = R^2$.

$R^1 \subset R^2$ is quite straightforward: if $r \in R^1$, then s_r is an element of π, and hence of $W(A^\perp)$, and s_r^2 is the identity. Then s_r has to be the reflection in a root r' of R^2, and then necessarily $r = r'$, and hence $r \in R^2$.

The theorem is now a consequence of the following lemma:

Lemma 4.25 *Let R^1 and R^2 be irreducible root systems, all of whose roots have length* 2, *and let* $\operatorname{rank}(R^1) = \operatorname{rank}(R^2) = n$. *Suppose that $R^1 \subset R^2$; then*

$$R^1 = R^2, \quad \textit{or} \quad R^1 = A^7 \subset E^7 = R^2, \quad \textit{or} \quad R^1 = D^8 \subset E^8 = R^2.$$

Proof By the classification of root systems, I only have to consider the system A^n, D^n for $n \geq 3$, and E^n for $8 \geq n \geq 5$.

If $R^1 \subset R^2$, the discriminants $d(R^1)$ and $d(R^2)$ are related by the formula $d(R^1) = a^2 \cdot d(R^2)$, where a is the index of the lattice spanned by R^1 in that spanned by R^2. The discriminants of the root systems under consideration are

$$d(A^n) = n+1; \quad d(D^n) = 4; \quad \text{and} \quad d(E^n) = 9-n,$$

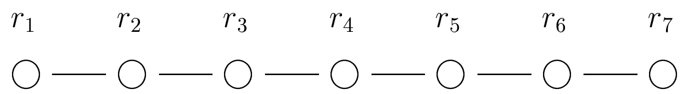
so that the only inclusions that are not immediately excluded are

$$A^n \subset D^n \quad (\text{if } n+1 \text{ is an even square}), \text{ or}$$
$$D^5 = E^5, \quad A^7 \subset E^7, \quad \text{and} \quad D^8 \subset E^8.$$

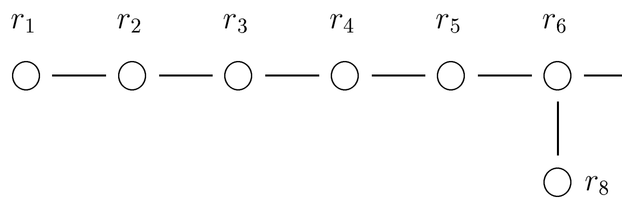
One can exclude $A^n \subset D^n$ if $n \geq 4$ by the following argument: if $A^n \subset D^n$, then we would have an inclusion $W(A^n) \subset W(D^n)$ between the Weyl groups. $W(A^n)$ is the symmetric group $\mathfrak{S}_{n+1}$, and $W(D^n)$ is the semidirect product of $\mathfrak{S}_n$ and $(\mathbb{Z}/2)^{n-1}$; if $\mathfrak{S}_{n+1} \subset W(D_n)$ then consider the restriction of the projection of $W(D_n)$ to $\mathfrak{S}_n$ to the simple subgroup $\mathfrak{A}_{n+1} \subset \mathfrak{S}_{n+1}$. this has to be the zero map, so that $\mathfrak{A}_{n+1}$ must be contained in $(\mathbb{Z}/2)^{n-1}$, which is absurd.

The exceptional cases are $A^3 = D^3$, $D^5 = E^5$, and the two inclusions in the statement of the lemma. To see that these do not take place, consider the Dynkin diagrams

A^7 :



D^8 :



In either case, the vector $\frac{1}{2}(r_1 + r_3 + r_5 + r_7)$ is of length 2, and has integral scalar product with all the roots r_i; hence A^7 and D^8 are contained in strictly bigger root systems of the same rank. The fact that these bigger root systems are necessarily E^7 and E^8 is an easy consequence of the fact that their discriminant is at most 2 and 1 in the two cases.

Theorem 4.24 has the following corollary:

Corollary 4.26 $\pi_1(\mathbb{P}^* \setminus X^*)$ *acts transitively on the* n*-planes of* X_H.

Proof By Addendum 3.16, and the fact that the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$ on the n-planes of X_H and its action on the cohomology $H^{2n}(X_H, \mathbb{Z}_\ell)(-n)$ are compatible with the cycle map, it suffices to show that π acts transitively on the cohomology classes of n-planes.

This is obvious from Theorem 3.19, since the group $\mathrm{Aut}(\Sigma(X_H))$ was seen to be transitive on the n-planes.

Corollary 4.27 S *is irreducible.*

Proof In $S \times \mathbb{P}^*$ consider the incidence subvariety

$$Z = \big\{(s, H) \bigm| s \subset H\big\}.$$

I have the two projections

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S & & \mathbb{P}^*. \end{tikzcd}$$

Let $Z_0 \subset Z$ be the inverse image of $\mathbb{P}^* \setminus X^*$:

$$\begin{tikzcd} & Z_0 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S & & \mathbb{P}^* \setminus X^*; \end{tikzcd}$$

then p_1 is surjective by Lemma 4.6 (the Bertini lemma); and so to show that S is irreducible, it will suffice to show that Z_0 is. On the other hand, p_2 is a finite étale covering. Thus to show that Z_0 is irreducible, it is enough to see that $\pi_1(\mathbb{P}^* \setminus X^*)$ acts transitively on the fibres of p_2, which is precisely the assertion of Corollary 4.26.

4.3 Appendix. The cubic threefold

In this appendix, I want to show that the methods developed in §4 are sufficient to give an elementary proof of one of the main results of the papers of on the one hand Bombieri and Swinnerton-Dyer [22] and on the other Clemens and Griffiths [2].

Let X be a nonsingular cubic threefold. Then it is easy to see that the family of lines on X is 2-dimensional, since the nonsingular hyperplane sections X_H of X contain lines. It can in fact be seen that there is a nonsingular surface S parametrizing the lines of X, and that the family of lines cover X, in the sense that at least one line passes through each point of X. Letting $T \subset S \times X$ be the incidence cycle,

$$T = \bigl\{(x,s) \bigm| x \in s\bigr\},$$

we obtain a 1-correspondence $\alpha\colon S \to X(-1)$; the transpose ${}^{\mathrm{t}}\alpha$ and the composite $\beta = {}^{\mathrm{t}}\alpha \circ \alpha$ fit into a diagram

$$\begin{tikzcd} S \arrow[d, "\beta"'] \arrow[dr, "\alpha"] & \\ S(-1) & X(-1) \arrow[l, "{}^{\mathrm{t}}\alpha"] \end{tikzcd}$$

analogous to that occurring in Theorem 4.14.

Let $s \in S$, so that s is a line[7] of X; following Bombieri and Swinnerton-Dyer, we can consider the curve $C_s \subset S$ defined by

$$C_s = \{t \mid t \text{ meets } s\}.$$

The inclusion $i\colon C_s \to S$ is going to define another pair of correspondences, to make up the diagram in Theorem 4.28.

The curve C_s is naturally equipped with an involution; for if t and s are two lines of X that meet, then t and s define a 2-plane H which meets X in s and t. The residual intersection is then a third line, $\sigma_s(t)$, say, and the involution

$$\sigma\colon C_s \to C_s$$

is defined by $t \mapsto \sigma_s(t)$.

Bombieri and Swinnerton-Dyer prove that the quotient of C_s by the group $(1, \sigma)$ is naturally a plane curve D_s of degree 5, and having at worst nodes at the stationary points of σ; the map $C_s \to D_s$ is thus a double covering, with ordinary ramification at the singular points of D_s, and étale outside the singularities of D_s.

σ is obviously an involution, i.e., $\sigma^2 = 1$. The decomposition of a vector space into eigenspaces under an involution has its analogue in the category of motives, or in that of Abelian varieties: the (rational) correspondences $\frac{1}{2}(1+\sigma)$ and $\frac{1}{2}(1-\sigma)$ are projectors in the category of correspondences, i.e., they satisfy $p^2 = p$; and since $\frac{1}{2}(1+\sigma) + \frac{1}{2}(1-\sigma) = 1$, and the two projectors are orthogonal (because $\sigma^2 = 1$), we have a decomposition

$$C_s = \left(C_s, \tfrac{1}{2}(1+\sigma)\right) \oplus \left(C_s, \tfrac{1}{2}(1-\sigma)\right).$$

Similarly, there is a decomposition (up to possible isogeny of degree 2^r, which will be unmentioned, but possibly present, throughout this appendix) of $J(C_s)$ into two parts, $J^+(C_s)$ which is isomorphic to $J(D_s)$ and $J^-(C_s)$, which is the Prym variety $\operatorname{Prym}(C_s/D_s)$ associated to the double covering $C_s \to D_s$) (for details, see [18]).

Before stating Theorem 4.28, let me calculate the dimension of the Prym variety $\operatorname{Prym}(C_s/D_s)$, which will prove an important corollary. The easiest case is that in which D_s is nonsingular, so that C_s is an étale double cover;

[7] The line $s \subset X$ will later be assumed to be "general enough".

since D_s is of degree 5 in $\mathbb{P}^2$, D_s has genus 6, and the genus g of C_s is given by the Hurwitz formula:

$$2 - 2g = 2(2 - 2 \cdot 6),$$

so that $g = 11$. Hence, the dimension of $\mathrm{Prym}(C_s/D_s)$ is $11 - 6$, i.e., 5.

If D_s starts acquiring nodes, but remains irreducible, the notion of Prym variety is a little more complicated; suffice it to say that the Prym of the covering C_s over D_s is the same thing as that of the normalization $\widetilde{C}_s$ over $\widetilde{D}_s$, the part of $\mathrm{Pic}(C_s)$ and $\mathrm{Pic}(D_s)$ that are not Abelian cancelling out. $\widetilde{D}_s$ has genus $6 - n$, and $\widetilde{C}_s$ is a double cover of $\widetilde{D}_s$, with ramification at precisely the $2n$ points of $\widetilde{C}_s$ lying over the notes of D_s; hence, using the Hurwitz formula, the genus of $\widetilde{C}_s$ is $11 - n$, so that again the Prym $\mathrm{Prym}(C_s/D_s)$ has dimension 5.

It is not difficult to check that $\mathrm{Prym}(C_s/D_s)$ still has dimension 5, even when D_s becomes reducible. For instance, if D_s become the union of 5 lines, C_s is the union of 5 elliptic curves, double covering the components of D_s with ramification at the four points where two components cross. $\mathrm{Prym}(C_s/D_s)$ is then just the product of these five elliptic curves. I do not know if this actually happens for a nonsingular cubic 3-fold, but I don't see why it shouldn't.

Theorem 4.28 *Consider the diagram in the category of correspondences:*

$$\begin{array}{ccccc} C_s & \xrightarrow{i_*} & S & & \\ & & & \searrow^{\alpha} & \\ & & \beta\downarrow & & X(-1) \\ & & & \swarrow^{{}^t\alpha} & \\ C_s & \xleftarrow{i^*} & S(-1) & & \end{array}$$

Then the composite $i^* \circ \beta \circ i_* \in C^1(C_s, C_s)$ is equal to $\sigma - 1_{C_s}$ plus a sum of horizontal and vertical components.

Corollaries 4.29 *For any Weil cohomology H, let $H^1(C_s)^-$ denote the subspace of $H^1(C_s)$ on which the involution σ^* acts as -1. Then $H^1(C_s)^-$ is a direct summand of $H^1(S)$, $H^3(S)(-1)$, and $H^3(X)(-1)$. In particular, for Hodge or ℓ-adic theory, where it is known that these are all 5-dimensional* *vector spaces, there is a diagram*

$$\begin{tikzcd}
H^1(C_s)^- \arrow[r, "i_*"] & H^3(S)(-1) \arrow[d, "\beta_*"'] \arrow[dr, "\alpha_*"] & \\
& & H^3(X)(-1) \arrow[dl, "\alpha^*"] \\
H^1(C_s)^- & H^1(S) \arrow[l, "i^*"'] &
\end{tikzcd}$$

in which all the arrows are isomorphism. The composite $i^* \circ \beta_* \circ i_*$ *is equal to* -2, *so that a little care must be taken to neglect* 2*-torsion if we want isomorphisms on the integral cohomology.*

The Prym variety $\mathrm{Prym}(C_s/D_s)$ *is isomorphic to the Abelian varieties* $\mathrm{Alb}(S)$, $\mathrm{Pic}(S)$ *and* $J(X)$ *if this last is defined.*[8] *Indeed, it is a direct summand, and all the varieties concerned have dimension* 5.

Finally, the 1*-motive* $(h^1(C), \frac{1}{2}(1-\sigma))$ *is a direct summand of the motives* S, $S(-1)$, *and* $X(-1)$.

Proof of the theorem As in the proof of Theorem 4.14, it is sufficient to see that for t in a dense open subset of C_s, the intersection of C_s and C_t is transverse, and the sum $\sum t'$, taken over all $t' \in C_s \cap C_t$, is rationally equivalent to $\sigma(t) - t + \text{constant}$ in $C^1(C_s)$.

We can assume that s is chosen not to pass through any of the Eckardt points of X, so that through any point $p \in s$ there pass only a finite number of lines, and through a general $p \in s$ there pass precisely 6.

It is not difficult to see then that if $t \in C_s$, so that t and s meet at a point p, and span a 2-plane H, the intersection $C_s \cap C_t$ is contained scheme theoretically in the intersection of S with the subvarieties of the Grassmannian variety consisting of lines that either pass through p, or are contained in H. It is then clear that this intersection is a reduced subscheme of S, and hence that the intersection is transverse, for generic $t \in C_s$.

Hence the relevant sum is $\sigma(t) + \sum t_i$, where the t_i are the four lines other than s and t that pass through p. And $t + \sum t_i$ is a constant in $C^1(C_s)$, for the usual reason that it can only depend on $p \in s$, and s is a rational variety.

The theorem is proved.

[8] Recall that I am ignoring 2-isogenies. Clemens and Griffiths in fact have a more precise statement, that identifies $J(X)$ with the Prym $\mathrm{Prym}(C_s/D_s)$ as defined by Mumford.

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