WHAT IS A FLIP?

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My main theme is that a flip arises automatically out of the geometric invariant theory (GIT) of a $\mathbb{C}^*$ action. Thus the question in the title is answered in parallel with "What is a projective space (or projective variety)?" and "What is a blow-up?".

Up to now, flips and flops have arisen in Mori theory as an empirical phenomenon: a birational map $f\colon X \dashrightarrow X'$ between projective varieties which is an isomorphism on the complement of closed sets of codimension ≥ 2. (That is, restricting to open sets of X and X' where f and f^{-1} are regular defines an isomorphism $f\colon U \cong U'$ between opens $U \subset X$ and $U' \subset X'$ whose complements $X \setminus U$ and $X' \setminus U'$ have codimension ≥ 2. In jargon: *an isomorphism in codimension* 1, meaning that f and f^{-1} are isomorphisms in a neighbourhood of a sufficiently general point of any codimension 1 subvariety of X and X'.) My sermon today puts flips in a more general context, but still does not solve the problem of giving a convincing definition. I return briefly to more concrete questions on Mori flips in §3.

Acknowledgements. I am very grateful for the opportunity to make my third visit to the Tabernacle of 3-folds at Utah[1], which has been extremely pleasant and stimulating, and I hope that the Lord continues to bless Utah as a world centre of the church of algebraic geometry for a long time to come.

I have benefited from several conversations with Michael Thaddeus, and the material described in §2 is due entirely to him. Together with other workers in Mori theory, I have worried about the question in the title since about 1980. In particular, the idea of §3 that many flip singularities can be described as toric hypersurfaces goes back to the early 1980s. However, the revelation of the connection with $\mathbb{C}^*$ actions was vouchsafed to me in Mar–Apr 1992 at the Univ. of Valladolid while meditating on the scriptures [Mori] and [Kollár and Mori], in connection with a lecture course on Mori theory.

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[1] joke due to Greg Sankaran

§1. $\mathbb{C}^*$ ACTIONS IN GENERAL

1.1. Notation. A diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^n$ is given by

$$x_1,\dots,x_n \mapsto \lambda^{a_1}x_1,\dots,\lambda^{a_n}x_n \qquad \text{for } \lambda\in\mathbb{C}^*$$

where $a_i\in\mathbb{Z}$ are certain characters of $\mathbb{C}^*$. For my purposes, the most important thing about the a_i is their sign, and I reorder the coordinates on $\mathbb{C}^n$ so that the action is

$$\begin{aligned} x_1,\dots,x_l &\mapsto \lambda^{a_1}x_1,\dots,\lambda^{a_l}x_l \\ y_1,\dots,y_m &\mapsto \bar\lambda^{b_1}y_1,\dots,\bar\lambda^{b_m}y_m \qquad \text{with } a_i,b_j>0. \\ z_1,\dots,z_k &\mapsto z_1,\dots,z_k \end{aligned}$$

Here $l+m+k=n$. If $k>0$ then since the z_i are invariant, for most purposes

$$(\text{whole theory}) = \mathbb{C}^k\times(\text{case } k=0),$$

so for simplicity I mostly assume $k=0$, that is $l+m=n$.

The automorphism $\lambda\mapsto\lambda^{-1}$ interchanges the roles of the x_i and y_j, and l and m, so I usually assume that $l\ge m$. The cases $m\le 1$ are in a sense degenerate. As I'm about to explain, $m=0$ corresponds to the ordinary construction of (weighted) projective space, and $m=1$ to a (weighted) blowup (see the Exercise at the end of 1.4). A flip will only be visible if $l,m\ge 2$.

1.2. First case, $\mathbb{P}^{n-1}$. Consider the case $m=0$, $a_1=\dots=a_n=1$, that is, the usual action of $\mathbb{C}^*$ on $\mathbb{C}^n$ defining projective space:

$$x_1,\dots,x_n\mapsto\lambda x_1,\dots,\lambda x_n.$$

It's crucial to understand why this action is bad. In algebra, there are no $\mathbb{C}^*$-invariant polynomials except constants: the λ^d eigenspaces (character spaces) of $\mathbb{C}^*$ are just the homogeneous polynomials of degree d (because $h(\lambda\underline{x})=\lambda^d h(\underline{x})$), and for $d=0$ you get just the constants. In geometry, the origin 0 is in the closure of every orbit, because $\lambda\cdot x\to 0$ as $\lambda\to 0$. Thus the only continuous $\mathbb{C}^*$-invariant morphism from $\mathbb{C}^n$ is a constant.

(Figure 1.2.1)

As everyone knows, the solution to this problem is not very hard to find: we throw away the origin, then the action on $\mathbb{C}^n\setminus 0$ is good, and the quotient is $\mathbb{P}^{n-1}$. Functions on $\mathbb{P}^{n-1}$ are defined as follows: if h is a homogeneous polynomial of degree d, the open subset $U_h:(h\ne 0)\subset\mathbb{C}^n$ is $\mathbb{C}^*$ invariant, and an invariant regular function on U_h is given by g/h^c, where g is homogeneous of degree cd. Thus textbooks on algebraic geometry define the topology of projective space in terms of the open sets $V_h=U_h/\mathbb{C}^*\subset\mathbb{P}^{n-1}$ and the regular functions on V_h to be g/h^c, where g and h^c are homogeneous of the same degree, so that g/h^c is $\mathbb{C}^*$-invariant. (It is most common to take $d=1$ and $h=x_i$ to get the "standard" linear affine pieces of projective space.)

Note that just replacing $\mathbb{C}^n$ by $\mathbb{C}^n\setminus 0$ makes a dramatic difference to the group action and the behaviour of the quotient.

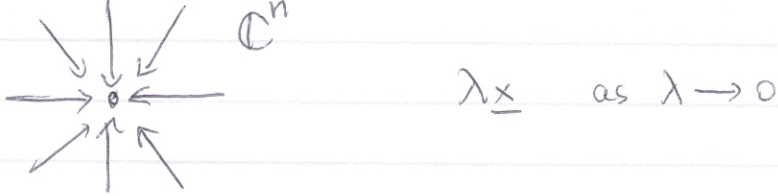


Figure 1.2.1. The standard $\mathbb{C}^*$ action on $\mathbb{C}^n$ defining $\mathbb{P}^{n-1}$

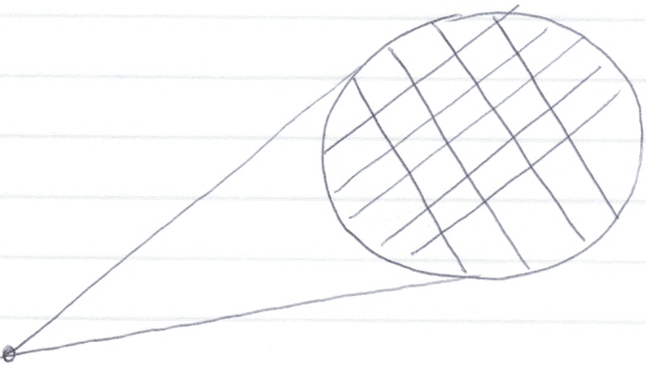


Figure 1.3.1. Cone over $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$

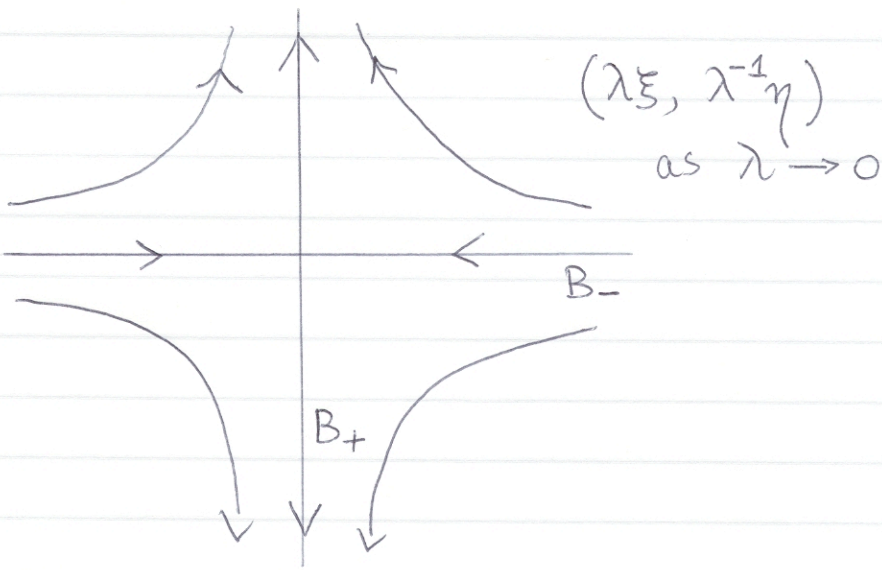


Figure 1.3.2. The action $(\xi, \eta) \mapsto (\lambda \xi, \lambda^{-1} \eta)$

Variation 1: Weighted projective space. The case when $\mathbb{C}^*$ acts on $\mathbb{C}^n$ by

$$x_1, \dots, x_n \mapsto \lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n$$

with all $a_i > 0$ is basically very similar to the case that all the $a_i = 1$. Just replace the degree of polynomials by their weights (where, for example, $x_1^2 x_2$ has weight $2a_1 + a_2$. The resulting projective variety $(\mathbb{C}^n \setminus 0)/\mathbb{C}^*$ is called a *weighted projective space*, and denoted by $\mathbb{P}(a_1, \dots, a_n)$ or $\mathbb{P}(\underline{a})$ or $\mathbb{P}^{(n-1)}(\underline{a})$. Open sets of $\mathbb{P}(\underline{a})$ and regular functions on them are defined in terms of weighted homogeneous polynomials. In general, $\mathbb{P}(\underline{a})$ has cyclic quotient singularities (orbifold points), corresponding to points at which the stabiliser of the $\mathbb{C}^*$ action jumps. For example, the weighted projective space $\mathbb{P}(1,1,2)$ is isomorphic to the ordinary quadratic cone in $\mathbb{P}^3$, with the singular point $(0,0,1)$ corresponding to the line of points of $\mathbb{C}^3$ having stabiliser $\{\pm 1\} \subset \mathbb{C}^*$.

Variation 2: Projective subvarieties of $\mathbb{P}^{n-1}$. Instead of just taking the quotient of $\mathbb{C}^n$, I could start with some affine variety $A \subset \mathbb{C}^n$ invariant under $\mathbb{C}^*$ (strictly bigger than 0), and consider the quotient $(A \setminus 0)/\mathbb{C}^* = X \subset \mathbb{P}^{n-1}$. This is the standard definition of a projective subvariety of $\mathbb{P}^n$ in algebraic geometry.

In order for $A \subset \mathbb{C}^n$ to be $\mathbb{C}^*$-invariant, it is necessary and sufficient for its defining ideal $I_A \subset k[x_1, \dots, x_n]$ to be $\mathbb{C}^*$-invariant. In this case the projective subvariety is naturally associated with the graded ring $k[x_1, \dots, x_n]/I_A$.

Variation 3: Proj R. Putting together Variations 1 and 2 gives the general construction of Proj R for a graded ring R. In more detail, let $k[x_1, \dots, x_n]$ be the polynomial ring with the weights $\mathrm{wt}(x_i) = a_i > 0$, and let I be a *homogeneous* prime ideal (that is, I is invariant under the $\mathbb{C}^*$ action

$$x_1, \dots, x_n \mapsto \lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n;$$

or equivalently, I is generated by a number of weighted homogeneous polynomials g_j). Then $R = k[x_1, \dots, x_n]/I$ is a graded finitely generated integral domain with $R_0 = k$ and $R_i = 0$ for $i < 0$, and every such R is of this form. Then $A \subset \mathbb{C}^n$ is a variety with a $\mathbb{C}^*$ action, and $\operatorname{Proj} R = X = (A \setminus 0)/\mathbb{C}^* \subset \mathbb{P}(a_1, \dots, a_n)$ is a projective subvariety of the weighted projective space.

R is determined by X together with the extra data of the sheaves $\mathcal{O}_X(k)$ for $k \geq 0$: namely, $R = \bigoplus_{k \geq 0} H^0(X, \mathcal{O}_X(k))$. Under extra conditions the $\mathcal{O}_X(k)$ are invertible sheaves (line bundles) with $\mathcal{O}_X(k) = \mathcal{O}_X(1)^{\otimes k}$.

1.3. The case of flips. Now let $l, m \geq 2$, and suppose $a_1 = \dots = a_l = b_1 = \dots = b_m = 1$ for simplicity, so that the $\mathbb{C}^*$ action is given by

$$x_1, \dots, x_l, y_1, \dots, y_m \mapsto \lambda x_1, \dots, \lambda x_l, \lambda^{-1} y_1, \dots, \lambda^{-1} y_m.$$

It's almost obvious that the ring of invariant polynomials $k[x_1, \dots, y_l]^{\mathbb{C}^*}$ is generated by the products $u_{ij} = x_i y_j$ for $i = 1, \dots, l$, $j = 1, \dots, m$, and related by the determinantal equations

$$\operatorname{rank} \begin{pmatrix} u_{11} & \dots & u_{1m} \\ \vdots & & \vdots \\ u_{l1} & \dots & u_{lm} \end{pmatrix} \leq 1.$$

(That is, the ideal of relations between the u_{ij} is generated by the 2×2 minors of the $l \times m$ matrix (u_{ij}).) The affine algebraic variety $X \subset \mathbb{C}^{lm}$ defined by these equations is well known in algebraic geometry. It is the affine cone over the Segre embedding of $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$. In other words, embed $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^{lm-1}$ by the bilinear forms $x_i y_j$, and take the affine cone over it.

(Figure 1.3.1)

I write $X = \mathbb{C}^n \mathbin{/\!/} \mathbb{C}^*$, and call it the GIT quotient of $\mathbb{C}^n$. Its affine coordinate ring is the ring of invariants $k[x_1, \dots, y_l]^{\mathbb{C}^*}$. Now consider the extent to which X is the orbit space of the $\mathbb{C}^*$ action. The action of $\mathbb{C}^*$ on $\mathbb{C}^n$ looks like this:

(Figure 1.3.2)

In other words, it has closed orbits $\mathbb{C}^* \cdot (\xi, \eta)$ with $\xi = (x_1, \dots, x_l) \neq (0, \dots, 0)$ and $\eta = (y_1, \dots, y_m) \neq (0, \dots, 0)$ (the orbit is isomorphic to $\mathbb{C}^*$), but it also has two bad locuses B_- and B_+, defined by $x_1 = \dots = x_l = 0$ and $y_1 = \dots = y_m = 0$ (intersecting in the origin). These are nonclosed orbits, since $\lambda(\xi, 0) \to 0$ as $\lambda \to 0$ and $\lambda(0, \eta) \to 0$ as $\lambda \to \infty$.

It is easy to check that the morphism $\mathbb{C}^n \to X = \mathbb{C}^n \mathbin{/\!/} \mathbb{C}^*$ defined by $x_i, y_j \mapsto x_i y_j$ has the following property: the inverse image of a point $P \in X$ with $P \neq 0$ is a closed orbit of $\mathbb{C}^*$ in $\mathbb{C}^n$, whereas the inverse image of $0 \in X$ is $B_+ \cup B_-$, which is a union of closures of many orbits. Thus X is a good quotient as far as the closed orbits are concerned, since there are enough $\mathbb{C}^*$-invariant functions to separate them, but the quotient map $\mathbb{C}^n \to X$ treats the nonclosed orbits fairly indiscriminately. Is there a better quotient? I say that there are several different ones, and that's where flips come in.

It's easiest to describe the geometric picture first. Since X is the cone on $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$, a union of generating lines over points of the base $(\xi, \eta) \in \mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$, there are 3 obvious ways of modifying it by taking account of the values of ξ and η.

The blowup $\widetilde{X}$. The blowup $\widetilde{X}$ consists of replacing the "cone" by a "cylinder", that is, the disjoint union of the same generators, in other words, the graph of the incidence relation (cone) $\times$ (base of cone) consisting of pairs

$$P \in \text{cone}, Q \in \text{base of cone} \quad \text{such that} \quad P \in \text{generator thro' } Q.$$

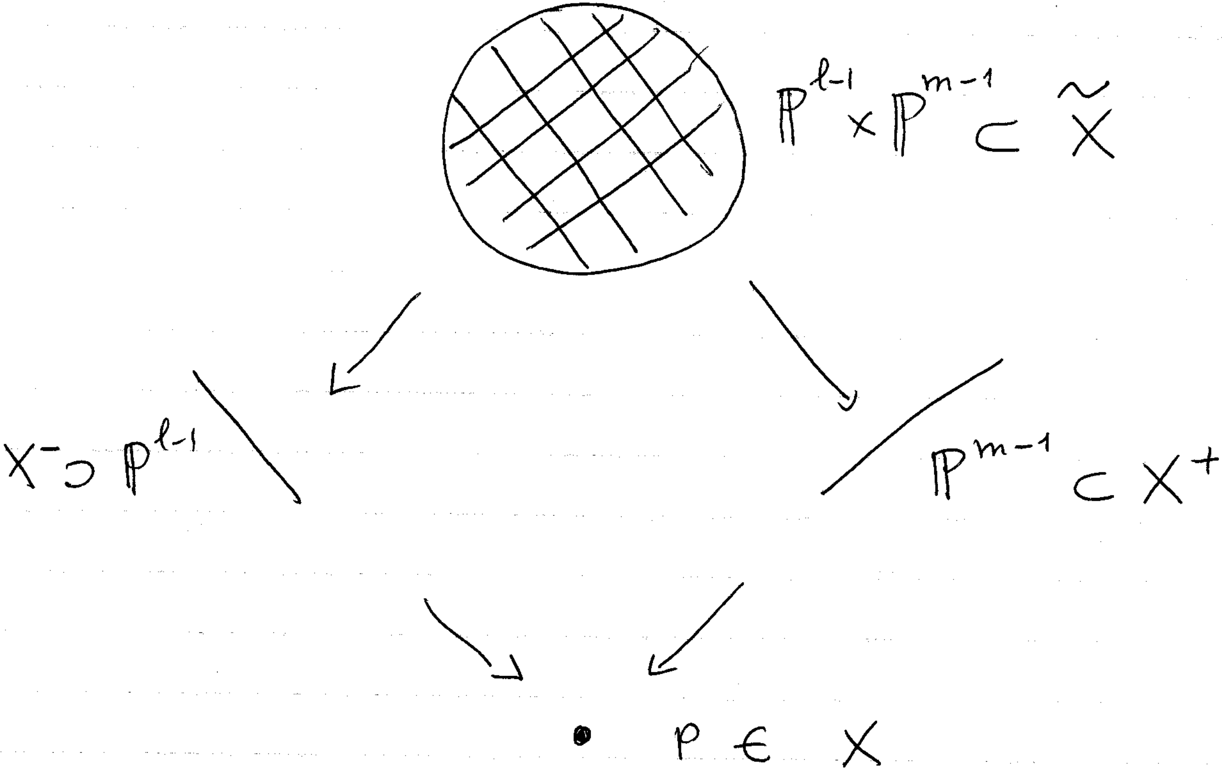
Thus $\widetilde{X}$ has a morphism $\sigma \colon \widetilde{X} \to X$ such that $\sigma^{-1}(0) = \mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$.

Before and after the flip $X^{\pm}$. In a similar way, by fixing the ratio ξ between the l rows of the matrix (u_{ij}), I can view X as a union of copies of $\mathbb{C}^m$ parametrised by points of $\mathbb{P}^{l-1}$. By taking the disjoint union of these generating m-planes, or equivalently the incidence relation between points of X and points of $\mathbb{P}^{l-1}$ given by the ratio of rows of (u_{ij}), I get a variety X^- having a morphism $f_- \colon X^- \to X$ with fibre over the origin $f_-^{-1}(0) = \mathbb{P}^{l-1}$. Interchanging the roles of the x_i and y_j gives $f_+ \colon X^+ \to X$ with ${f_+}^{-1}(0) = \mathbb{P}^{m-1}$.

The 4 varieties X, $X^{\pm}$ and $\widetilde{X}$ fit together in the following way:

(Figure 1.3.3)

In Mori theory, we sometimes call the rational map $X^- \dashrightarrow X^+$ a flip. It replaces $\mathbb{P}^{l-1} \subset X^-$ (with normal bundle $\mathcal{O}(-1)^m$) with $\mathbb{P}^{m-1} \subset X^+$ (with normal bundle $\mathcal{O}(-1)^l$).



$$\begin{tikzcd}
 & \tilde{X} \arrow[dl] \arrow[dr] & \\
X^- \arrow[dr] & & X^+ \arrow[dl] \\
 & X &
\end{tikzcd}$$

Figure 1.3.3

The standard bilinear flip arising from the cone over Segre emb. of $\mathbb{P}^{l-1}\times\mathbb{P}^{m-1}$.

The case $l = m = 2$ is a celebrated example of [Atiyah]: X is the ordinary double point $(xt = yz) \subset \mathbb{C}^4$ (where $x = u_{11}$, $y = u_{12}$, $z = u_{21}$, $t = u_{22}$), and the two modifications $X^\pm$ correspond to blowing up the ratios $x/y = z/t$ and $x/z = y/t$. The distinction between flips and flops is mainly of interest to the people who already know it, so I don't discuss it. However, the Atiyah case is called a *flop*, because $X^- \dashrightarrow X^+$ is a symmetric operation; the only flop of his career.

Theorem (GIT interpretation). *The varieties X and $X^\pm$ are all natural GIT quotients. Indeed, we've already seen $X = \mathbb{C}^n /\!/ \mathbb{C}^*$, and I claim that $X^\pm = (\mathbb{C}^n \setminus B_\pm)/\mathbb{C}^*$.*

Proof. $\mathbb{C}^n \setminus B_-$ is covered by $\mathbb{C}^*$-invariant open pieces $U_-^{(i)} = (x_i \neq 0)$. It's easy to see that the ring of $\mathbb{C}^*$-invariant regular functions on $U_-^{(1)}$ (say) is exactly the polynomial ring on x_i/x_1 for $i = 2, \dots, l$ and $y_i x_1$, and that these are parameters on an affine piece of X_-. Amen

Exercise. In case $m = 1$, everything works in exactly the same way, but $X^+ = X = \mathbb{C}^l$, and $f_- \colon X^- \to X$ is the blowup of the origin $0 \in \mathbb{C}^l$.

1.4. What is a flip? I claim that the flip diagram

$$\begin{array}{ccc} X^- & & X^+ \\ & \searrow \quad \swarrow & \\ & X & \end{array} \tag{1.4.1}$$

is a precise analogue of the construction of projective space for the $\mathbb{Z}$-graded polynomial ring $k[x_1, \dots, y_m]$. Indeed, the topology on X^- (say) is defined by open sets $(h \neq 0)$, where $h \in k[x_1, \dots, y_m]$ is a homogeneous polynomial of degree $d < 0$ (for example $h = x_i$), and the regular functions on $(h \neq 0)$ are given by g/h^c, where g is homogeneous of degree cd.

All 3 variations of 1.2 apply here also, so that a flip diagram (1.4.1) can be constructed from any $\mathbb{C}^*$ action on an (affine) variety A, or equivalently, any $\mathbb{Z}$-graded ring R. (I assume for safety that R is a normal integral domain, and contains homogeneous elements of both positive and negative degree, say at least one element of each degree ± 1.)

Geometrically, the action on $A = \operatorname{Spec} R$ has two bad locuses $B_\pm$, and the three varieties in the flip diagram are $X = A /\!/ \mathbb{C}^*$ and $X^\pm = (A \setminus B_\pm)/\mathbb{C}^*$. In terms of algebraic geometry, $X = \operatorname{Spec}(R^{\mathbb{C}^*})$, and $X^\pm$ are exact analogues of Grothendieck's construction of $\operatorname{Proj} R$: for example, X^- has an open set $U_h : (h \neq 0)$ for any homogeneous element $h \in R$ of negative degree d, and $U_h = \operatorname{Spec}(R[1/h]^{\mathbb{C}^*})$. The flip diagram (1.4.1) is the homogeneous spectrum of R: the scheme theoretic points of X^- are the $\mathbb{C}^*$-invariant (that is, homogeneous) prime ideals of R not containing the whole of R^-, etc.

The eigenspaces (character spaces) of $\mathbb{C}^*$ acting on R define a finitely generated $\mathbb{Z}$-graded sheaf of $\mathcal{O}_X$-algebras $\mathcal{R} = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}(k)$. The two sides of the flip are relative Projs:

$$X^\pm = \operatorname{Proj}_X(\mathcal{R}^\pm) \qquad \text{where } \mathcal{R}^- = \bigoplus_{k \leq 0} \mathcal{O}(k) \text{ and } \mathcal{R}^+ = \bigoplus_{k \geq 0} \mathcal{O}(k).$$

Remark. Another aspect of this Proj-like character of the flip diagram 1.4.1 is Grothendieck's Lefschetz principle: all the properties of the Proj are already implicit in the local ring of the vertex of the affine cone. To make sense of this in the present context, suppose that $B_0 \cap B_\infty$ (the subscheme defined by the sum of the two ideal sheaves $\mathcal{R}_{<0}$ and $\mathcal{R}_{>0}$) is a single point $0 \in A = \operatorname{Spec} R$; this is the analogue of the condition $R_0 = k$ for $\mathbb{N}$-graded rings. Then taking the direct sum of the eigenspaces of the $\mathbb{C}^*$-action on the local ring $\mathcal{O}_{0\in A}$ recovers R together with its grading.

1.5. Genuine flips. The construction of 1.3–4 is more general than the generally recognised phenomenon of a flip that is an isomorphism in codimension 1. Indeed, in 1.4, the reason for imposing the condition that R contains nonzero homogeneous elements of both positive and negative degrees was to avoid the case $X_- = \emptyset$, $X = \text{pt.}$, and X_+ an arbitrary projective variety; in the birational context, I suppose we don't want to consider spontaneous creation from the void as an elementary operation (however, compare Proposition 2.6, (2)). Next, as in the exercise at the end of 1.3, if $R_{>0}$ is a principal ideal then $X_- \to X$ is a blowup and $X_+ = X$. From this point of view, a flip and a blowup can be viewed as different aspects of the same construction.

Now let $X = A \mathbin{/\!/} \mathbb{C}^*$ be as in 1.3–4, and assume in addition that $B_\pm \subset A$ each has codimension ≥ 2 (to avoid the case of a blowup). It is not hard to prove the following.

Proposition. *(1) Each $\mathcal{O}(k)$ is a divisorial sheaf on X, that is, $\mathcal{O}(k) = \mathcal{O}_X(kD)$ for some Weil divisor $D \in \operatorname{WDiv} X$.*

(2) If k_1, k_2 are both positive (or both negative) and sufficiently divisible, then

$$\mathcal{O}(k_1) \otimes \mathcal{O}(k_2) \twoheadrightarrow \mathcal{O}(k_1 + k_2).$$

(3) For k sufficiently divisible (of either sign),

$$\mathcal{O}(-k) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(k), \mathcal{O}_X). \quad \square$$

Here (3) gives the "opposite" property familiar from Mori flips: the Weil divisor classes on X corresponding to ample divisors on $X^\pm$ are negative rational multiples of one another.

In the same way that a positively graded ring R can be recovered from $\operatorname{Proj} R$ and the data of its sheaves $\mathcal{O}(k)$, it is clear that R and its $\mathbb{C}^*$ action can be recovered from the flip diagram (1.4.1) and the sheaves $\mathcal{O}(k)$ on it. (See §3 for a particular case.)

* §2. Applications to GIT by M. Thaddeus

2.1. GIT, its aims. The general setup of geometric invariant theory is the following: V is an algebraic variety, say projective, and G an algebraic group acting on V. We want to construct the orbit space V/G as an algebraic variety. The most important applications are moduli problems. When we study some geometric objects, we may have a construction in coordinates depending on some parameters in a variety V, and changes of coordinates form a group G that acts on the parameters. For example, a plane curve $C_d \subset \mathbb{P}^2$ of degree d is determined in a given coordinate system by the coefficients of its defining equation, which form a projective space V of dimension $\binom{d+2}{2} - 1$, and projective coordinate changes in $\mathbb{P}^2$ define an action of $\mathrm{PGL}(3)$ on V.

*see also Yi Hu, Geometry and topology of quotient varieties of torus actions, Duke Math J. 68:1 (1992) 151–184

2.2. What the layman is usually told about GIT. Unfortunately, the orbit space V/G is almost always bad, for reasons we have already seen in §1: in geometry, because nonclosed orbits mean that the orbit space is non-Hausdorff; in algebra, because there are not enough G-invariant functions to separate the orbits. The solution (just as in the very simple case (1.2)) is to throw away some bad orbits.

Roughly, the main result is as follows: under reasonable assumptions on V and G, there exist dense open subvarieties $V^{\mathrm{s}} \subset V^{\mathrm{ss}} \subset V$ (of *stable* and *semistable* orbits) and a quotient algebraic variety $V \mathbin{/\!/} G$ with maps

$$\begin{array}{ccc} V & & \\ \cup & & \\ V^{\mathrm{ss}} & \xrightarrow{\pi^{\mathrm{ss}}} & V \mathbin{/\!/} G = X \\ \cup & & \cup \\ V^{\mathrm{s}} & \xrightarrow{\pi^{\mathrm{s}}} & V^{\mathrm{s}}/G \end{array} \tag{2.2.1}$$

with the properties: (1) $X = V \mathbin{/\!/} G$ is the best quotient algebraic variety (it is made up by taking Spec's of all G-invariant functions on G-invariant open sets). (2) V^{ss} is the domain of definition of the rational quotient map $\pi\colon V \dashrightarrow X$. (3) Restricted to V^{s}, the quotient map $\pi\colon V^{\mathrm{s}} \to V^{\mathrm{s}}/G \subset X$ is a good quotient, in the sense that the fibres of π are the G-orbits.

2.3. What the layman is not told. The description in 2.2 is incomplete because it omits one ingredient, on which the subvarieties $V^{\mathrm{s}} \subset V^{\mathrm{ss}} \subset V$, the quotient variety $X \mathbin{/\!/} G$ and the morphisms π^{s} and π^{ss} in (2.2.1) depend essentially. Namely to make the constructions of (2.2.1), I need to choose an embedding $V \subset \mathbb{P}^N$, and an action of G on the coordinates of $\mathbb{P}^N$ inducing the given action on X. The technical term for these choices is a *G-linearised very ample line bundle L*, but for brevity, and to avoid technicalities, I call it a *linearisation* of the action. I write V^{s}_L, $V \mathbin{/\!/}_L G = X_L$ etc. for the objects in (2.2.1) constructed according to a given L.

Important warning. The GIT software[2] is automatic and for many purposes transparent to the user. However, it works by taking G-invariants of a projective coordinate ring (e.g. $k[x_1, \dots, x_n]$ if $V = \mathbb{P}^{n-1}$), so it cannot start working if a linearisation is not installed. Moreover, it's easy to construct examples (see 2.6 below) of a variety V with an action of a group G, and two different linearisations L and M such that, say, $V^{\mathrm{ss}}_L = \emptyset$ and $V^{\mathrm{ss}}_M \subset V$ is dense.

2.4. Thaddeus' principle. *Suppose that V and G are as usual, and that L and M are linearisations such that V^{s}_L and V^{s}_M are both dense in V. Then X_L and X_M are birational, and $X_L \dashrightarrow X_M$ is a composite of flips in the sense of §1.*

Of course, it comes as no surprise that X_L and X_M are birational, since there is a G-invariant open dense set $V^{\mathrm{s}}_L \cap V^{\mathrm{s}}_M \subset V$ on which either quotient is simply the orbit space of the G action.

[2] GIT ©1965 is a trademark of David Mumford, Inc.

Remarks. (1) The listener versed in the doctrine of cones in algebraic geometry (Hironaka, Kleiman, Mori and co.) will guess at once that the linearisations L form a convex cone in a finite dimensional real vector space, and that the different biregular models X_L partition this cone into chambers, with a flip $X_{L_1} \dashrightarrow X_{L_2}$ as you cross a wall; in a different context, compare [Reid1], Theorem 8.2.

(2) Thaddeus' proof at present uses a reduction to the case of $\mathbb{C}^*$ acting on $\mathbb{P}^n$. Although very simple and pleasant, it does not give any information about the chain of flips. It would be very interesting to have an argument more directly involving the GIT of the action of G on V, L and V, M. One could hope that flips correspond to parabolic subgroups and so forth, as in Kirwan's work on stratification of the semistable locus [Kirwan].

In sketch form, Thaddeus' proof of the principle is in two steps:

2.5. Step 1. Reduction to $\mathbb{C}^*$ action on $\mathbb{P}^n$. The idea is the following. We're given two G-linear embeddings $\varphi_L \colon V \hookrightarrow \mathbb{P}^l$ and $\varphi_M \colon V \hookrightarrow \mathbb{P}^m$. A classical construction of projective geometry gives a $\mathbb{P}^1$-bundle $\mathbb{P}(L \oplus M) \to V$: put the two projective spaces as disjoint linear subspaces of a common projective space $\mathbb{P}^{l+m+1}$, and join up corresponding points of $\varphi_L(V)$ and $\varphi_M(V)$.

(Figure 2.5.1)

$\mathbb{P}(L \oplus M) \to V$ contains two sections $\varphi_L(V)$ and $\varphi_M(V)$ given by the inclusions φ_L and φ_M. The construction of $\mathbb{P}(L \oplus M) \to V$ is equivariant: G has a linear action on $\mathbb{P}(L \oplus M) \subset \mathbb{P}^{l+m+1}$ by just taking the diagonal action on the direct sum of $\mathbb{C}^{l+1}$ and $\mathbb{C}^{m+1}$. In addition, there is a natural geometric action of $\mathbb{C}^*$ on $\mathbb{P}(L \oplus M)$ that pushes towards the two ends as $\lambda \to 0$ and $\lambda \to \infty$. The point is that this action has two different linearisations, namely

$$\begin{aligned} &\lambda \colon \xi, \eta \mapsto \lambda\xi, \eta \\ \text{and} \qquad &\lambda \colon \xi, \eta \mapsto \xi, \lambda^{-1}\eta \end{aligned}$$

Now there are several ways of viewing the quotient $\mathbb{P}(L \oplus M) \mathbin{/\!/} G \times \mathbb{C}^*$. Dividing first by either of the $\mathbb{C}^*$ actions and then by G gives $\mathbb{P}(L \oplus M) \mathbin{/\!/} G \times \mathbb{C}^* = X_L$ and X_M. On the other hand, dividing first by the G-action gives the quotient $Y = \mathbb{P}(L \oplus M) \mathbin{/\!/} G$ as a projective variety with a given embedding $Y \hookrightarrow \mathbb{P}^N$ (defined by G-invariant forms) having a $\mathbb{C}^*$ action with two different linearisations, such that $Y \mathbin{/\!/} \mathbb{C}^* = X_L$ and X_M respectively. Thus X_L and X_M are contained as subvarieties in quotients of $\mathbb{P}^N$ by a $\mathbb{C}^*$ action with two different linearisations.

2.6. Step 2. Study of $\mathbb{C}^*$ action on $\mathbb{P}^n$. A linearised $\mathbb{C}^*$ action on $\mathbb{P}^n$ is given by

$$x_0, \dots, x_n \mapsto \lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n \qquad \text{for } \lambda \in \mathbb{C}^*$$

where $a_i \in \mathbb{Z}$. However, since in $\mathbb{P}^n$ it is only the ratios of the homogeneous coordinates x_i that have any meaning, the same action on $\mathbb{P}^n$ is also given by

$$x_0, \dots, x_n \mapsto \lambda^{a_0 - d} x_0, \dots, \lambda^{a_n - d} x_n$$

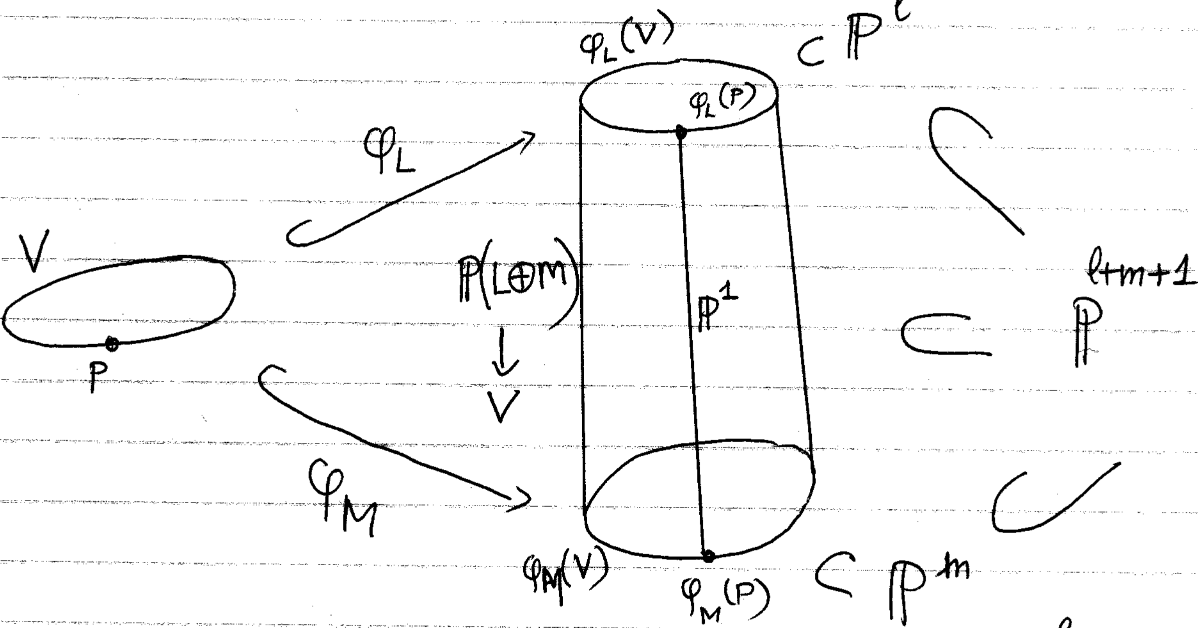


Figure 2.5.1 Construction of $\mathbb{P}(L \oplus M) \subset \mathbb{P}^{l+m+1}$

for any $d \in \mathbb{Z}$; I write L_d for this linearisation of the $\mathbb{C}^*$ action, abbreviate $(\mathbb{P}^n)^{\mathrm{s}}_{L_d}$ to $(\mathbb{P}^n)^{\mathrm{s}}_d$ and $/\!/_{L_d}$ to $/\!/_d$, etc. I assume that the a_i are all distinct and well spaced-out, say

$$a_1 \geq a_0 + 2, \quad a_2 \geq a_1 + 2, \quad \ldots, \quad a_n \geq a_{n-1} + 2.$$

This is purely for definiteness, and to get a result that is convenient to describe. If all the a_i are distinct one reduces easily to this by taking a cyclic cover of the $\mathbb{C}^*$ action (passing to a root of λ), or equivalently, working with a Veronese embedding. The effect of coincidences among the a_i is straightforward, and is discussed in the remark after the proposition.

Proposition. *(1) If $d \notin [a_0, a_n]$ then $(\mathbb{P}^n)^{\mathrm{ss}}_d = \emptyset$; if $d = a_0$ or a_n then $(\mathbb{P}^n)^{\mathrm{s}}_d = \emptyset$, and $(\mathbb{P}^n)^{\mathrm{ss}}_d$ is the affine space $(x_0 \neq 0)$ (respectively $(x_n \neq 0)$). The quotient $\mathbb{P}^n /\!/_d \mathbb{C}^*$ is a single point (as with the bad action at the start of (1.2)).*

(2) If $d \in (a_0, a_1)$ then $(\mathbb{P}^n)^{\mathrm{ss}}_d = \mathbb{P}^n \setminus (1, 0, \ldots, 0)$, $(\mathbb{P}^n)^{\mathrm{s}}_d = (\mathbb{P}^n)^{\mathrm{ss}} \setminus (x_0 = 0)$ (that is, $\mathbb{P}^n$ with a point and a complementary hyperplane deleted), and

$$\mathbb{P}^n /\!/_d \mathbb{C}^* = \mathbb{P}^{(n-1)}(a_1 - a_0, \ldots, a_n - a_0) \quad \textit{(weighted projective space).}$$

The case $d \in (a_{n-1}, a_n)$ is similar. Thus the passage from $d < a_0$ to $d > a_0$ creates a $\mathbb{P}^{(n-1)}$ from the void, and the passage from $d < a_n$ to $d > a_n$ annihilates a $\mathbb{P}^{(n-1)}$.

(3) If $d = a_k$ for some $k = 1, \ldots, n-1$ then $(\mathbb{P}^n)^{\mathrm{s}} = (\mathbb{P}^n)^{\mathrm{ss}}$ is the complement of the $\mathbb{P}^{k-1}$ with coordinates $x_0, \ldots, x_{k-1}$ and the $\mathbb{P}^{n-k-1}$ with coordinates $x_{k+1}, \ldots, x_n$. The quotient $X_k = \mathbb{P}^n /\!/_{L_d} \mathbb{C}^*$ is a weighted projective cone over a product of weighted projective spaces

$$\mathbb{P}^{(k-1)}(a_k - a_0, \ldots, a_k - a_{k-1}) \times \mathbb{P}^{(n-k-1)}(a_{k+1} - a_k, \ldots, a_n - a_k). \tag{2.6.1}$$

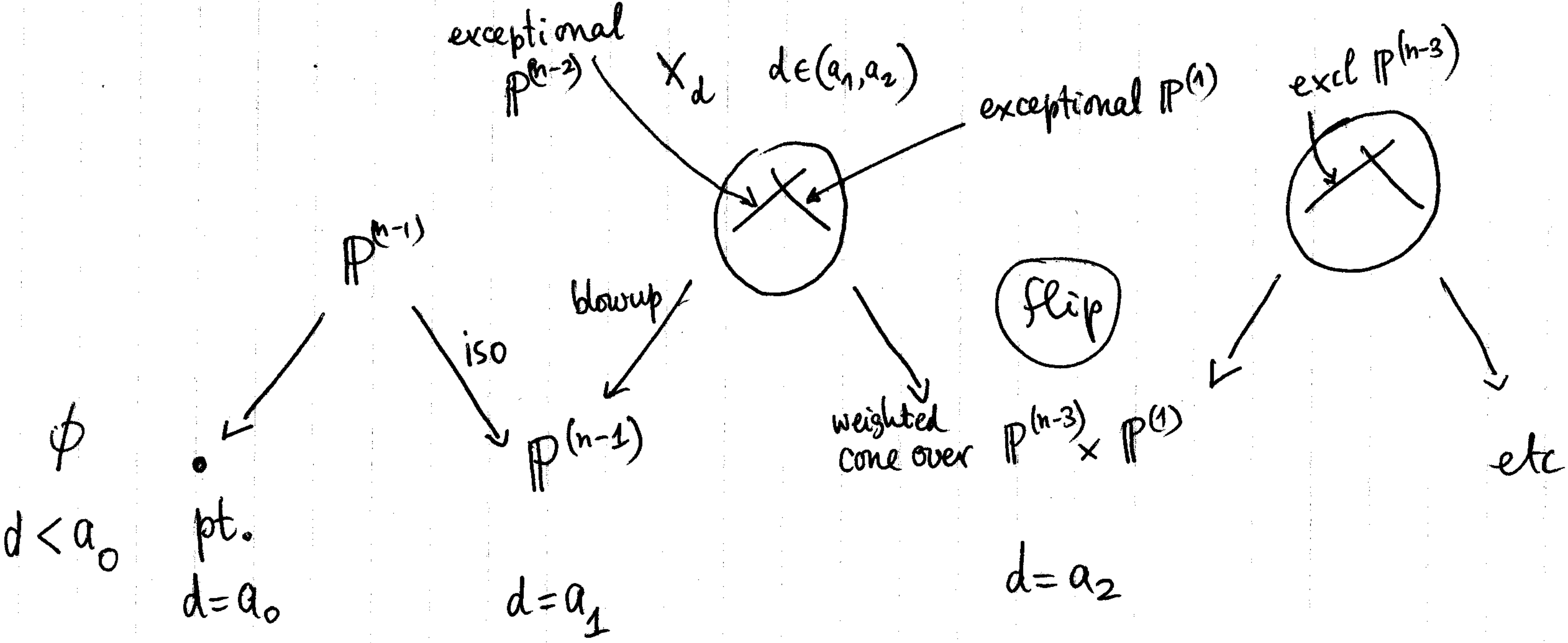
(4) If $d \in (a_k, a_{k+1})$ then $(\mathbb{P}^n)^{\mathrm{s}} = (\mathbb{P}^n)^{\mathrm{ss}}$ is the complement of the $\mathbb{P}^k$ with coordinates $x_0, \ldots, x_k$ and the $\mathbb{P}^{n-k-1}$ with coordinates $x_{k+1}, \ldots, x_n$. The quotient variety X_d is independent of $d \in (a_k, a_{k+1})$ (but its natural embedding to projective space defined by $\mathbb{C}^*$ invariant rational monomials varies with d). X_d has two birational morphisms $X_d \to X_k$ and $X_d \to X_{k+1}$ which contract to a point weighted projective spaces $\mathbb{P}^{(n-k-1)}$ and $\mathbb{P}^{(k)}$ respectively. Over the cone point of X_k, the passage from $d < a_k$ to $d > a_k$ is the flip of 1.3 (with the weights of (2.6.1)).

Remark. Coincidences between the a_i, just lead to cones in (3) with positive dimensional vertexes; the flip takes place along the whole vertex in a locally trivial way.

(Figure 2.6.2)

Proof. This is all an easy exercise in the style of 1.2–3. [Hint: Consider homogeneous monomials $x^m = \prod x_i^{m_i}$ on $\mathbb{P}^n$ that are invariant under the linearised $\mathbb{C}^*$ action, that is, $\sum (a_i - d) m_i = 0$. For (3), $(x_k \neq 0)$ is a $\mathbb{C}^*$ invariant open set, and the ratios x_i/x_k for $i = 0, \ldots, k-1$ and x_j/x_k for $j = k+1, \ldots, n$ play exactly the roles of the x_i and y_j in 1.3.]

The exercise can be understood most naturally in the language of toric geometry. Monomials of degree c in $x_0, \ldots, x_n$ are obviously the lattice points of the simplex Δ obtained



($\mathbb{P}^{(m)}$ stands for <u>weighted</u> projective spaces)

<u>Figure 2.6.2.</u> Quotients of a $\mathbb{C}^*$ action on $\mathbb{P}^n$.

by intersecting the positive quadrant in $\mathbb{R}^{n+1}$ with the affine hyperplane $\sum m_i = c$. Write Γ_d for the convex polygon given as the intersection of Δ with the variable affine hyperplane $\sum(a_i - d)m_i = 0$. Then X_d is the toric variety associated with Γ_d. In Thaddeus's expression, this amounts to putting the simplex Δ obliquely through a meat-slicer. Amen

§3. Mori extremal divisorial contractions and extremal flips

3.1. Mori theory. Mori's theory of minimal models of 3-folds works with projective 3-folds having at worst *Q-factorial terminal singularities*, a class of singularities that is now quite well understood (see [YPG]). The aim is to pass from a given projective 3-fold X to a birational variety X', where (1) the final model X' is either a minimal model or a Mori fibre space (the analogue of a ruled surface), and (2) the passage from X to X' is by a succession of highly restricted "elementary" steps, called *extremal divisorial contractions* or *extremal flips* (the 3-fold analogues of contracting a -1-curve on a surface). In this sermon, I don't have time to discuss either the virtues of the final model, nor the way in which the whole approach is driven by the canonical class.

I preach instead on some of the remaining problems concerning extremal divisorial contractions and extremal flips. The picture is as follows (I omit a reasonable number of technical assumptions).

(Figure 3.1.1)

The main outstanding problem in minimal models of 3-folds is that, although the extremal divisorial contractions and extremal flips have strong categorical properties, and we know how to prove that they exist, we don't at present have lists of them comparable to the lists of terminal singularities ([YPG], Theorem 6.1). My main point here is that the doctrine of $\mathbb{C}^*$ actions discussed in §1 reduces contractions and flips in a purely formal way to the local study of a $\mathbb{C}^*$ action on a Gorenstein 4-fold point $0 \in A$.

It seems likely that the case when $0 \in A$ is a hypersurface singularity is at least a very significant component of the lists. The singularities of A are isolated modulo the $\mathbb{C}^*$ action, and therefore finitely determined, so that all the methods of singularity theory are available (see for example [Montaldi and van Straten] for the de Rham complex and the equivariant Milnor fibre).

3.2. What we know about flips. I summarise very briefly the results of the two giant foundational papers on flips, referring to Figure 3.1.1, (b). [Mori] contains a detailed analysis of the the flipping contraction $X^- \to X$ (the left-hand side of the diagram); his main result is that given the left-hand side, the right-hand side also exists. [Kollár and Mori] give a classification of the flip singularity X in terms of its general hyperplane section, and show how to construct and to some extent classify the right-hand side. The disavantage with the knowledge provided by the sum of these two papers is that in general they do not provide a useable description of both sides of the flip—in other words, with present technology, you can't see both $X^- \to X$ and $X^+ \to X$ on the same screen. (Also, the technology is not extremely user-friendly as it stands. It's fair to say that at present Kollár and Mori are the only two people in the world to have understood the two papers.) In particular, it seems difficult in most cases to determine the $\mathbb{Z}$-graded algebra $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(nK_X)$.

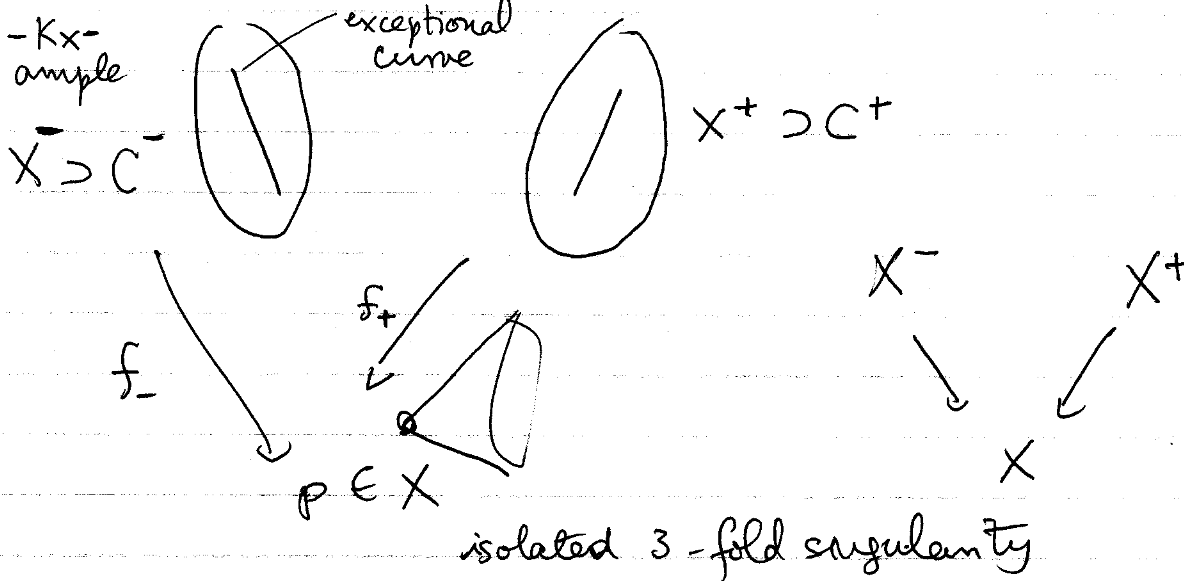


Figure 3.1.1. (a) Flip

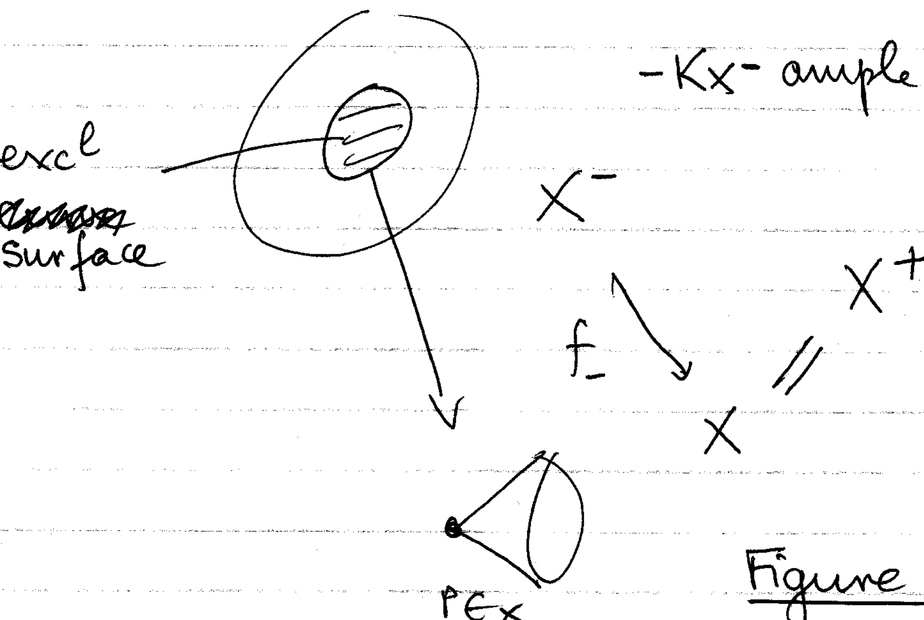


Figure 3.1.1, (b) Divisorial Contraction

Incidentally, flips gained notoriety as a difficult subject because their existence was the main stumbling-block in the theory for most of the 1980s, but most of the potential applications of Mori theory involve a concrete description of varieties and birational transformations between them, and classifying the extremal divisorial contractions is every bit as difficult and as important as the flips; of course, we know a lot more about the flips thanks to the work of Mori and Kollár.

3.3. The reduction to a local $\mathbb{C}^*$ action. Let $X^- \to X$ be a contraction of an extremal ray that is either a flipping or divisorial contraction to a point (I'm ignoring divisorial contractions to a curve), and write

$$\begin{array}{ccc} X^- & & X^+ \\ & \searrow \quad \swarrow & \\ & X & \end{array} \tag{3.3.1}$$

for the flip. In the case of a divisorial contraction, set $X^+ = X$; in the flipping case, the existence of the flip $X^+ \to X$ is proved in [Mori]. In what follows I consider $P \in X$ as a variety germ in the Zariski topology (the scheme $\operatorname{Spec} \mathcal{O}_{P\in X}$).

In either case, by definition of extremal ray, the relative Weil divisor class group $\mathrm{WCl} = \mathrm{WCl}(X^-/X)$ is finitely generated and of rank 1, that is

$$\mathrm{WCl} \cong \mathbb{Z} \oplus \mathrm{Tors}.$$

Moreover, $-K_{X^-}$ is an ample generator. The presence of a finite torsion group just means that at the same time as the $\mathbb{C}^*$ action, providing the GIT interest, there is also a fairly harmless finite Abelian cover going on.

The following result is very similar to the cyclic covering trick (see e.g. [YPG], (3.6)).

Theorem 3.3.1. *There exists a local variety $0 \in A$ and an action of the dual group $\mathrm{WCl}^{\mathrm{dual}} = \mathbb{C}^* \oplus \mathrm{Tors}$ on A such that (3.3.1) is the GIT quotient in the sense discussed in Theorem 1.3.*

The $\mathbb{C}^*$ action on $0 \in A$ has the two bad locuses $B_\pm$ such that $B_+ \cap B_- = 0$ (set-theoretically), and $B_\pm$ have dimension $2+2$ in the flip case and $3+1$ in the divisorial contraction case.

Revelation 3.3.2. *$0 \in A$ is a Gorenstein rational singularity.*

Construction and proof. To prove the theorem, I just construct a more-or-less tautological WCl-graded sheaf of $\mathcal{O}_X$ algebras $\mathcal{R}$ such that the graded piece of degree $0 \in \mathrm{WCl}$ is $\mathcal{O}_X$, and such that the positive and negative subalgebras $\mathcal{R}^\pm$ have $X^\pm$ as their Proj. (I say that $D \in \mathrm{WCl}$ is *negative* if it is ample on X^-, for example, $-K_{X^-}$.)

In the flipping case, WCl equals the local divisor class group $\mathrm{WCl}_P X$, so just set $\mathcal{R} = \bigoplus_{D\in\mathrm{WCl}} \mathcal{O}_X(D)$. The multiplication $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \to \mathcal{O}_X(D_1 + D_2)$ is the natural one. In degree $D = 0$ I have $\mathcal{R}_0 = \mathcal{O}_X$ by choice, in negative degree I have $\mathcal{R}_D = f_{-*}\mathcal{O}_{X^-}(D)$, and in positive degree $\mathcal{R}_D = f_{+*}\mathcal{O}_{X^+}(D)$ (where I write D for the birational transform on D). Thus $\operatorname{Proj}_X(\mathcal{R}^\pm) = X^\pm$ just follows because $\mathcal{O}_X(D)$ is ample on $X^\pm$ for D positive or negative.

In the divisorial contraction case, I set $\mathcal{R}_D = \mathcal{O}_X$ in nonnegative degree D and $\mathcal{R}_D = f_{-*}\mathcal{O}_{X^-}(D)$ in negative degree (this is a subsheaf of finite colength in $\mathcal{O}_X(f_{-*}(D))$, which is a torsion element of $\mathrm{WCl}_P X$). Then, as in the flip case, $\mathcal{R}_0 = \mathcal{O}_X$ by choice and $\mathrm{Proj}_X(\mathcal{R}^-) = X^-$ because $\mathcal{O}_X(D)$ is ample on X^- for D negative, and $\mathrm{Proj}_X(\mathcal{R}^+) = X^- = X$.

A is simply $\mathrm{Spec}_X \mathcal{R}$. The bad locuses B_0 and B_∞ of the $\mathbb{C}^*$ action are defined by the ideals $\mathcal{R}_{<0}$ and $\mathcal{R}_{>0}$. Since all the sheaves are locally free on $X \setminus P$ it follows that for every D_1 and D_2, the multiplication $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \to \mathcal{O}_X(D_1 + D_2)$ maps onto a subsheaf of finite colength. It's easy to deduce from this that the ideal $\mathcal{R}_{<0} + \mathcal{R}_{>0} \subset \mathcal{R}$ also has finite colength, so that $B_0 \cap B_\infty = 0$ is set-theoretically a single point.

I've never worked out the proof of 3.3.2, but it shouldn't be too hard. A Gorenstein should be a tautological consequence of $K_X \in \mathrm{WCl}$. The revelation that A has rational singularities should follow from $R^i f_{\pm *}\mathcal{O}_{X^\pm} = 0$. Amen

3.4. Problems. The ideas of §1 and Theorem 3.3.1 provide a formal language in which one can eventually understand flips and extremal contractions. I still hope that the answer will be not much more complicated than for terminal singularities. The main difficulty is that I do not know too much about the Gorenstein 4-fold rational singularity $0 \in A$. My revelation doesn't say if $0 \in A$ is terminal, but it seems likely. In principle the results of [Mori] and [Kollár and Mori] tell us a lot about the flipping case.

If $0 \in A$ is nonsingular, the flip or divisorial contraction is toric ([Reid2]), and this case can be easily understood. Gavin Brown has some calculations under the assumption that $0 \in A$ is a hypersurface singularity. In this case, the condition that $X^- = (A \setminus B_-)/\mathbb{C}^*$ has only terminal singularities has a toric interpretation similar to that of [YPG], Chap. II. It is at least possible that in this case $0 \in A$ is always a cDV 4-fold point.

Example. [Kollár and Mori], 13.7. Let $f_{m-1}(\xi_1, \xi_2)$ be a homogeneous polynomial of degree $m-1$ with coefficients in k. Consider $\mathbb{C}^5$ with coordinates x_1, x_2, x_4, y_1, y_2 (the x_4 is not a misprint) and the $\mathbb{C}^*$ action

$$x_1, x_2, x_4, y_1, y_2 \mapsto \lambda x_1, \lambda x_2, \lambda^m x_4, \lambda^{-1} y_1, \lambda^{-1} y_2.$$

Consider the $\mathbb{C}^*$ invariant 4-fold cA_{m-2} singularity $0 \in A \subset \mathbb{C}^5$ given by

$$x_4 y_1 = f_{m-1}(x_1, x_2).$$

It's easy to see that the $\mathbb{C}^*$ invariant functions are

$$\begin{aligned} u_1 &= x_1 y_1, \quad v_1 = x_2 y_1, \\ u_2 &= x_1 y_2, \quad v_2 = x_2 y_2, \quad v_3 = x_4 y_2^m, \end{aligned}$$

and that $X = A \,/\!/\, \mathbb{C}^*$ is the triple point in $A^5(u_1, u_2, v_1, v_2, v_3)$ defined by

$$\operatorname{rank} \begin{pmatrix} u_1 & v_1 & f_{m-1}(u_2, v_2) \\ u_2 & v_2 & v_3 \end{pmatrix} \leq 1.$$

Then the elements

$$x_1, x_2, x_3 = x_4 y_2^{m-1} \in \mathcal{O}_X(-K_X) \quad \text{and} \quad x_4 \in \mathcal{O}_X(-mK_X)$$

generate $\mathcal{R}_-$, and $y_1, y_2 \in \mathcal{O}_X(K_X)$ generate $\mathcal{R}_+$ as $\mathcal{O}_X$ algebras. Thus X_- is the graph of the rational map $X \dashrightarrow \mathbb{P}(1^3, m)$ given by the weighted ratios of x_1, x_2, x_3, x_4, and X^+ the graph of $X \dashrightarrow \mathbb{P}^1$ given by y_1, y_2. The whole flip picture is as follows:

(Figure 3.4.1)

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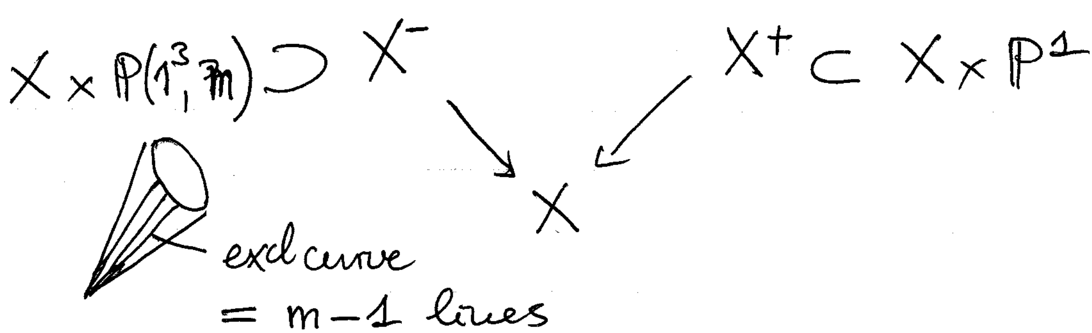


Dear Miles,

The following is the simplest case of my example, where the ring A is a complete intersection of codimension 2.

Let $m_1, m_2, \sigma_1, \sigma_2, \alpha_1, \alpha_2, \delta$ be positive integers such that

$$(m_1, m_2) = (m_1 m_2, \delta) = 1, \sigma_2 \delta m_2 < m_1$$

.

Let $x_1, \cdots, x_4, z, u$ be the variables with weights

$$m_1, m_2, \sigma_2 \delta m_2 - m_1 (< 0), -m_2, \delta, 0$$

.

Let $A \subset \mathbb{C}^6$ be defined by

$$\begin{aligned} x_1 x_3 &= x_2^{\sigma_2 \delta} u^{\alpha_2} - z^{\sigma_2 m_2} \\ x_2 x_4 &= u^{\alpha_1} - x_3^{\sigma_1 \delta} z^{\sigma_1 (m_1 - \sigma_2 \delta m_2)} \end{aligned}$$

with the G_m-action induced by the above weights. Let

$$\begin{aligned} B^- &= \{x_1 = x_2 = z = 0\} = \{x_1 = x_2 = 0\} \\ B^+ &= \{x_3 = x_4 = 0\} \end{aligned}$$

following your notation. Then $X^- = (A - B^-)/G_m$ is covered by two open sets $U_1 = \{x_1 \neq 0\}$ and $U_2 = \{x_2 \neq 0\}$. On U_1, we have coordinates

$$(\bar{x}_2, \bar{x}_4, \bar{z}, u) = \frac{1}{m_1}(m_2, -m_2, \delta, 0)$$

with equation

$$\bar{x}_2 \bar{x}_4 = u^{\alpha_1} - \bar{z}^{\sigma_1 (m_1 - \sigma_2 \delta m_2)} \cdot \{\bar{x}_2^{\sigma_2 \delta} u^{\alpha_2} - \bar{z}^{\sigma_2 m_2}\}^{\sigma_1 \delta},$$

and on U_2, we also have coordinates

$$(\bar{x}_1, \bar{x}_3, \bar{z}, u) = \frac{1}{m_2}(m_1, -m_1, \delta, 0)$$

with equation

$$\bar{x}_1 \bar{x}_3 = u^{\alpha_2} - \bar{z}^{\sigma_2 m_2}.$$

It is easy to see that X^- has only terminal singularities and $-K_X$ is positive. If we put a further condition that

$$(\alpha_1, \sigma_1) = (\alpha_2, \sigma_2) = 1,$$

then X^- is analytically Q-factorial, whence $Cl(A//G_m) = \mathbb{Z}$. $X^+ = (A - B^+)/G_m$ has a similar description. It can also be checked that

$$A \simeq \oplus_{D:\ \text{Weil Divisor on } X^-} H^0(X^-, \mathcal{O}(D)).$$

Yours,

Mori

What is a flip? Part II — Bonn 18th Mar 1993

§1. Intro and aim
§2. $\mathbb{C}^*$ cover
§3. Flips in moduli
§4. Flips in Mori theory, "how to ride an elephant"

§1. Intro and aim. A constr. of n-fold terminal singularities

Let $P\in S$ be canonical Gorenstein $(n-1)$ dim^l sing., $T\to S$ a $\mathbb{Z}/m$ cyclic cover, etale in codim 1. $T\hookrightarrow Y$ 1-para deform of T, equivariant wrt $\mathbb{Z}/m$, $z\mapsto\varepsilon^a z$ hcf$(a,m)=1$, $\varepsilon^m=1$. Y "terminating" T.

data: $(n-1)$-dim^l ~~cyclic cover~~ of canonical Gor. sing.

$$\left\{\begin{array}{ccc} T & \hookrightarrow & Y \quad \text{1-parameter terminating} \\ \downarrow & & \downarrow \\ P\in S & \hookrightarrow & X=Y/(\mathbb{Z}/m) \end{array}\right.$$

Theorem. This is a list of all n-fold terminal sings for which $\exists\, S\in|-K_X|$ canonical.

Example $n=3$. ① Always $\exists$ canonical $S\in|-K_X|$
② $T\to S$ is $\mathbb{Z}/m$ cover of Du Val sings = Klein qt sings
③ 1-para smoothing is easy
$\Rightarrow$ [YPG, Theorem 6.1] Mori's classfn of terminal sings.

e.g.
$$\left\{\begin{array}{ll} T: & xy=u^k \\ S: & (xy=u^k)/\tfrac{1}{m}(1,-1,0) \\ & vw=u^{km} \quad v=x^m,\ w=y^m \\ Y: & xy=f(u,z^m)=u^k+\cdots+z^{\nu m} \\ X: & Y/\tfrac{1}{m}(1,-1,0,a) \end{array}\right.$$

$\varepsilon=\exp 2\pi i/m$ acts by $x\mapsto\varepsilon x,\ y\mapsto\varepsilon^{-1}y,\ u\mapsto u$

equivariant $z\mapsto\varepsilon^a z$

hcf$(a,m)=1$

main family. Remember $(xy=f(u,z^m))/\tfrac{1}{m}(1,-1,0,a)$.

Aim. If $\exists$ canonical elephant, and we know what $\mathbb{C}^*$ cover is, then $\exists$ similar classification of flips (and contractions) of n-folds in terms of $(n-1)$-dim^l data.

§2. $\mathbb{C}^*$ cover. $\mathbb{C}^*$ action on (local) variety $\longleftrightarrow$ $\mathbb{Z}$-grading

$$R = \bigoplus_{n\in\mathbb{Z}} R_n \qquad \mathbb{C}^* \text{ acts } f \mapsto \lambda^n f \text{ for } f \in R_n$$

$\operatorname{Proj} R$ = diagram
$$\left\{\begin{tikzcd} X^- \arrow[dr] & & X^+ \arrow[dl] \\ & X & \end{tikzcd}\right\}$$
$X = \operatorname{Spec} R_0 = \operatorname{Spec} R^{\mathbb{C}^*}$

$X^- = \operatorname{Proj} R^-$

$X^+ = \operatorname{Proj} R^+$

where $R = R^- \oplus R^+$ $\quad R^- = \bigoplus_{n\leq 0} R_n \quad R^+ = \bigoplus_{n\geq 0} R_n$. rewrite EGA II

This is $\mathbb{C}^*$ quotient. $A = \operatorname{Spec} R$, $X = A/\!/\mathbb{C}^*$ = categorical quotient $= \operatorname{Spec} R^{\mathbb{C}^*}$

$$X^- = \left(A \setminus \text{variety } V\Big(\bigoplus_{n<0} R_n\Big)\right) \Big/ \mathbb{C}^*$$
$$= \left\{x\in A \;\middle|\; \begin{array}{l}\exists f \text{ homog of deg } n<0 \\ \text{s.t. } f(x)\neq 0\end{array}\right\} \Big|\ \text{set } \left(\sqrt[n]{f}\right) = 1$$

$\mathbb{C}^*$ quotient is "homog. coords."

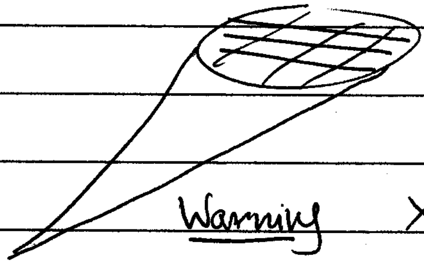
<u>Example</u> $A = k[\underbrace{x_1,\dots x_\ell}_{x_i\mapsto\lambda x_i}, \underbrace{y_1,\dots y_m}_{y_j \mapsto \lambda^{-1} y_j}]$ $\quad \ell, m \geq 1$

$A^{\mathbb{C}^*} = k[(x_i y_j)] \Rightarrow \therefore X$ = cone over Segre embd $\mathbb{P}^{\ell-1}\times\mathbb{P}^{m-1}\subset\mathbb{P}^{\ell m-1}$

$X^\pm$ = graph of one ruling

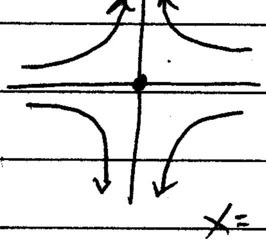
$$\begin{tikzcd} X^- \arrow[dr] & & X^+ \arrow[dl] \\ & X & \end{tikzcd}$$

ratio $(x_1 : \dots : x_\ell)$ defined



<u>Warning</u>
$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr] & \\ X^- \arrow[dr] & & X^+ \arrow[dl] \\ & X & \end{tikzcd}$$

[usually $\tilde{X}$ is useless.]



$\mathbb{C}^*$ action on A

$\mathcal{B}^\pm$ = nonclosed orbits

$X = A/\!/\mathbb{C}^*$ contracts them

$(A\setminus\mathcal{B}^-)/\mathbb{C}^* = X^-$ . .

$$\begin{tikzcd} X^- \arrow[rr, dashed] \arrow[dr, "\text{contract first}"'] & & X^+ \arrow[dl, "\text{resolve opposite}"] \\ & X & \end{tikzcd}$$

[forget simult resoln

"local" Lefschetz principle.

§3. Flips in moduli

Principle (Thaddeus, Dolgachev and Hu, Gullemin and Sternberg, ... Archimedes)
practical cases many others

Quotient $V/\!/G$ depends on choices. $V^{ss}(L)/G$
Two different choices differ by sequence of flips:

$$V/\!/^{L_1} G \dashrightarrow \cdots \dashrightarrow V/\!/^{L_2} G$$

§4. Flips in Mori theory

Def. canonically directed n-fold flip

$$\begin{tikzcd} X^- \arrow[dr] & & X^+ \arrow[dl] \\ & X & \end{tikzcd}$$

$X^\pm$ are n-folds with terminal sings (X local)

$f_\pm : X^\pm \longrightarrow X$ birational morphisms $\pm K_{X^\pm}$ rel. ample.

Not symmetric $\begin{cases} X^- \longrightarrow X & \text{Given by cone + freedom} \\ X^+ & \text{~~prove exists (conjecture in general)~~} \end{cases}$

$\begin{cases} X^+ = \operatorname{Proj} \bigoplus_{n\geq 0} \mathcal{O}_X(nK_X) & \text{is well-defined, rigid, } X^+ \to X \text{ small} \\ X^- \text{ is "anticanonical model", not unique.} \end{cases}$

[Mori] and [Kollár-Mori] assume extremal. $\operatorname{Rk} \operatorname{Pic} X^-/X = 1$.
I allow e.g. contract divisors, and isolated curves.
Case $X^+ \xrightarrow{\cong} X$ allowed "divisorial contraction".

$$\mathcal{R} = \bigoplus_{n\leq 0} f_* \mathcal{O}_{X^-}(-nK_{X^-}) \oplus \bigoplus_{\geq 0} f^+_* \mathcal{O}_{X^+}(nK_{X^+})$$

$$\begin{tikzcd} & A = \operatorname{Spec} \mathcal{R} \arrow[dl] \arrow[d] \arrow[dr] & \\ X^- \arrow[dr] & & X^+ \arrow[dl] \\ & X & \end{tikzcd}$$

diagram is a $\mathbb{C}^*$ quotient.

Elephant: $\forall\, S_1 \subset |-K_{X^-}|$ gives rise to

$$\begin{tikzcd} S_1 \arrow[dr, "f_1"'] & & S_2 \arrow[dl, "f_2"] \\ & S & \end{tikzcd} \subset \begin{tikzcd} X^- \arrow[dr] & & X^+ \arrow[dl] \\ & X & \end{tikzcd}$$

S, S_1, S_2 Gorenstein

$f_1^* \omega_S = \omega_{S_1}$ crepant.

$s \in \mathrm{Hom}(\omega_{X^-}, \mathcal{O}_{X^-}) \subset \mathrm{Hom}(\omega_X, \mathcal{O}_X) = \mathrm{Hom}(\omega_{X^+}, \mathcal{O}_{X^+})$

and if $X^- \to X$ is small, also $=$

<u>Proof</u>. for $x \in \omega$ $s(x) \in \mathcal{O}_X$ is regular in codim 1, $\therefore$ regular

$$0 \to \omega_{X^-} \xrightarrow{s} \mathcal{O}_{X^-} \to \mathcal{O}_{S_1} \to 0$$

$$\omega_{S_1} = \mathrm{Ext}^1_{\mathcal{O}_{X^-}}(\mathcal{O}_{S_1}, \omega_{X^-})$$

$$0 \to \underset{\underset{0}{\parallel}}{\mathrm{Hom}(\mathcal{O}_{S_1}, \omega_{X^-})} \to \underset{\underset{\omega_{X^-}}{\parallel}}{\mathrm{Hom}(\mathcal{O}_{X^-}, \omega_{X^-})} \xrightarrow{s} \underset{\underset{\mathcal{O}_{X^-}}{\parallel}}{\mathrm{Hom}(\omega_{X^-}, \omega_{X^-})} \to$$

$$\to \underset{\underset{\omega_{S_1}}{\parallel}}{\mathrm{Ext}^1(\ldots)} \to 0 \qquad \therefore \mathcal{O}_{S_1} = \omega_{S_1} \qquad \left(K_{S^-} = (\underbrace{K_{X^-} + S^-}_{0})\big|_{S^-}\right)$$

<u>Canonical elephant conjecture</u>

for general $s \in \mathrm{Hom}(\omega_{X^-}, \mathcal{O}_{X^-})$ $S_1 \in |-K_{X^-}|$ has canonical sings.

$$S_1 \searrow S \swarrow S_2$$ ~~crepant maps between Gor. canonical sings.~~

Status ① known in some cases

② I believe it in 3 dims. ($\exists$ start of proof).

$$\begin{array}{ccc} & A & \\ \swarrow & \downarrow & \searrow \\ X^- & & X^+ \\ \searrow & & \swarrow \\ & X & \end{array} \supset \begin{array}{ccc} & B & \\ \swarrow & \downarrow & \searrow \\ S_1 & & S_2 \\ \searrow & & \swarrow \\ & S & \end{array}$$

$B \subset A$ is Cartier divisor. ($s=0$)

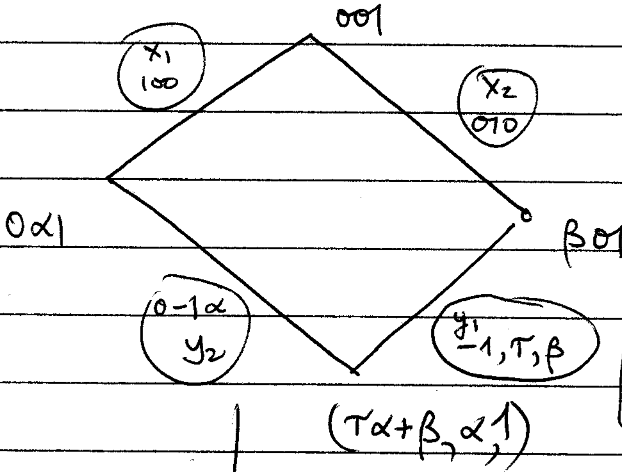
$A = \bigoplus R_n$

$s \in R_{-1}$

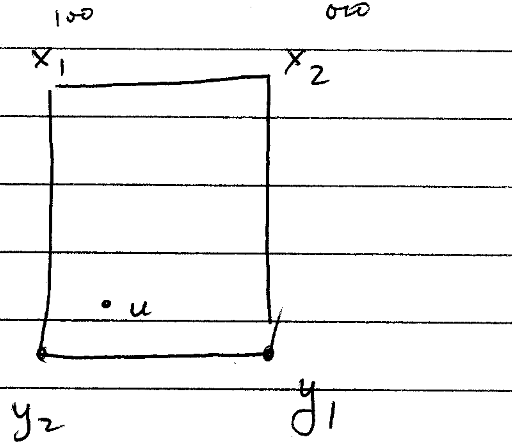
$\therefore$ flip is data in $(n-1)$ dimensions + 1-para smoothing.

Choose $S_1 \searrow S \swarrow S_2$ choose H_1, H_2 Weil $\mathbb{Q}$-Cartier ~~polarisations~~ s.t.

$f_{1*}H_1 + f_{2*}H_2 \overset{\text{lin}}{\sim} 0$ on S. (gives B). Choose 1-para equivt. deform



$\boxed{\ker = (m_1\, m_2\, 0)}$

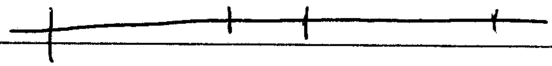


$\mathbb{C}^*$ action

$$\begin{pmatrix} m_2 & -m_2 \\ -m_1 & m_1 \\ m_1\alpha & m_2\beta \end{pmatrix}$$

$$x_1 y_1 = x_2^\tau u^\beta + z^N$$

$$x_2 y_2 = u^\alpha + x_1^? z^?$$



$0m$ $(m_1\alpha, m_2\beta)$ $m0$

$$\left(m_2(\tau\alpha+\beta), \alpha(m_1 - \tau m_2)\right)$$

$\delta N = \tau m_2$

wts	x_1	x_2	y_1	y_2	u
	m_1	m_2		$-m_2$	0
			$-(m_1-\tau m_2)$		
					z

$\sigma \mid \tau$

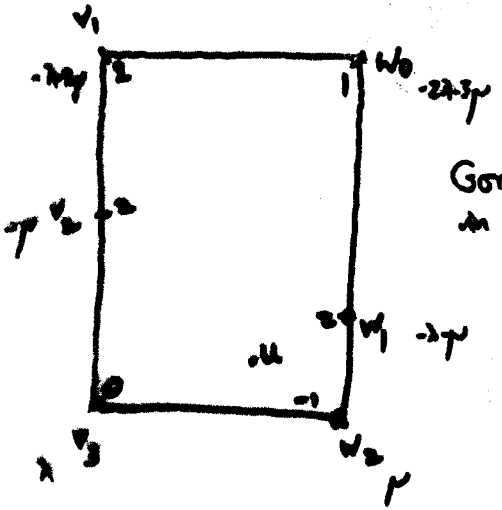
$\boxed{\text{“}x_2 = 1\text{”}}$ $\left(x_1 y_1 = u^\beta + z^N\right) / \mathbb{Z}/m_2\,(-1, 1, 0)$

$\boxed{\text{“}x_1 = 1\text{”}}$ $\left(x_2 y_2 = u^\alpha + y_1^? z^?\right) / \mathbb{Z}/m_1\,(m_2, -m_2, 0)$

after subst $y_1 = x_2^\tau u^\beta + z^N$

get $x_2 y_2 = u^\alpha + \ldots z^M$

Santa Cruz 1995



Gorenstein cone in $M = \mathbb{Z}^3$
(origin behind screen, not coplanar)

$$B: \begin{cases} w_1 v_1 = w_0 u^{\beta} \\ w_0 v_2 = v_1^2 u^{\alpha} \\ v_1 v_3 = v_2^2 \\ v_2 w_2 = u^{\alpha+2\beta} \\ w_1 v_3 = v_2 u^{\alpha+\beta} \\ \qquad = w_2^{-1} u^{2\alpha+3\beta} \\ w_0 w_2 = w_1^2 \end{cases}$$

+ other equations

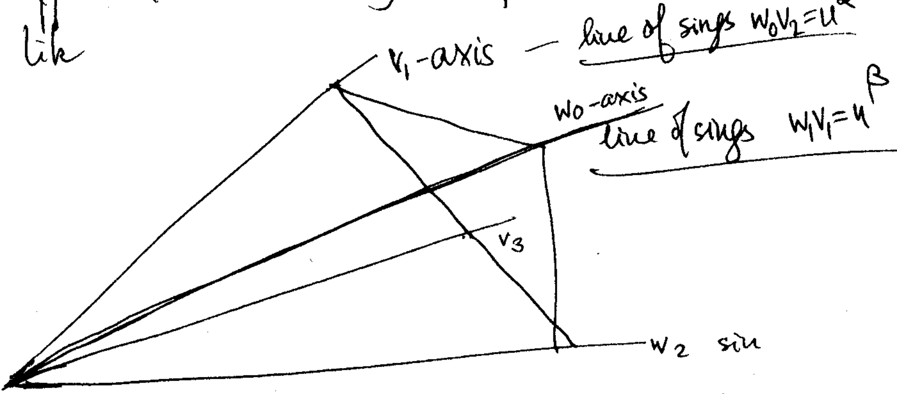
Slide 1.

Example

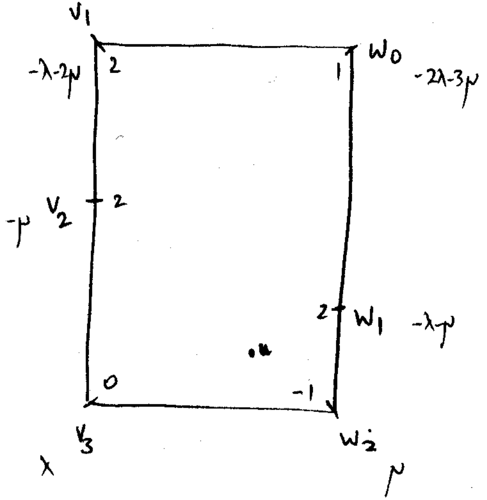
Slide 1 shows a quadrilateral cone σ in $\mathbb{Z}^3 = M$ whose interior ideal is generated by 1 elt. u.

$\mathbb{C}[M \cap \sigma]$ is generated by marked monomials $v_1, v_2, v_3, w_0, w_1, w_2, u$ related by equations shown, and

$B = \operatorname{Spec} \mathbb{C}[M \cap \sigma]$ is a Gorenstein affine toric ~~variety~~ 3-fold. It looks like



Main problem deform B to $B \subset A$ to a 4-fold affine variety to smooth singular lines [equivariant wrt. action of $\mathbb{C}^*$]



+other eqns

B

$$\left\{\begin{aligned} w_1v_1 &= w_0u^{\beta} \\ w_0v_2 &= v_1^2u^{\alpha} \\ v_1v_3 &= v_2^2 \\ v_2w_2 &= u^{\alpha+2\beta} \\ w_1v_3 &= v_2u^{\alpha+\beta} \\ &= w_2^{-1}u^{2\alpha+3\beta} \\ w_0w_2 &= w_1^2 \end{aligned}\right.$$

[all eqns]

$$\operatorname{rank}\begin{pmatrix} w_2 & w_1 & u^{\beta} & \boxed{*} \\ w_1 & w_0 & v_1 & v_2 \\ u^{\alpha+\beta} & v_1u^{\alpha} & v_2 & v_3 \end{pmatrix} \leqslant 1$$

Answer

A:

$$v_1w_1 = w_0u^{\beta} + v_2t^{2\lambda+2\mu}$$

@v_1 $$w_0v_2 = v_1^2u^{\alpha} + t^{2\lambda+4\mu}$$

$\Rightarrow$ $$w_0w_1v_1 = w_0^2u^{\beta} + v_1^2u^{\alpha}t^{2\lambda+2\mu} + t^{4\lambda+6\mu}$$ @w_0

explain how this gives the answer along v_1-axis & w_0-axis

$$v_1v_3 = v_2^2 + u^{\beta}t^{2\mu}$$

@v_3 $$w_2v_2 = u^{\alpha+2\beta} + v_3^2t^{2\lambda}$$

$$v_3w_1 = v_2u^{\alpha+\beta} + w_2t^{2\mu}$$

$\Rightarrow$ (after the flip) $$w_2v_3v_1 = u^{2\alpha+3\beta} + v_3^2u^{\alpha+\beta}u^{2\mu} + w_2^2t^{2\mu}$$ @w_2

<u>Answer</u> A defined by

$$v_1 w_1 = w_0 u^{\beta} + v_2 t^{2\lambda+2\mu}$$

$$w_0 v_2 = v_1^2 u^{\alpha} + t^{2\lambda+4\mu}$$

$$v_1 v_3 = v_2^2 + u^{\beta} t^{2\mu}$$

$$w_2 v_2 = u^{\alpha+2\beta} + v_3^2 t^{2\lambda}$$

$$v_3 w_1 = v_2 u^{\alpha+\beta} + w_2 t^{2\mu}$$

$$w_0 w_2 = w_1^2 + v_2^2 u^{\alpha} t^{2\lambda}$$

+ other equations

(1) (2) $\Longrightarrow w_0 v_1 w_1 = w_0^2 u^{\beta} + \cdots + t^{4\lambda+6\mu}$

is locally $v_1 w_1 = u^{\beta} + \cdots + t^{4\lambda+6\mu}$
$w_0 = 1$

<u>isolated</u>

$$v_1 w_1 = w_0 u^{\beta} + v_2\, t^{2\lambda+2\mu}$$

$$v_1 v_3 = v_2^2 + u^{\beta} t^{2\mu}$$

$$w_1 v_2 = v_1 u^{\alpha+\beta} + v_3 t^{2\lambda+2\mu}$$

$$w_0 v_2 = v_1^2 u^{\alpha} + t^{2\lambda+4\mu}$$

$$w_0 v_3 = v_1 v_2 u^{\alpha} + w_1 t^{2\mu}$$

$$w_2 v_2 = u^{\alpha+2\beta} + v_3^2 t^{2\lambda}$$

$$v_3 w_1 = u^{\alpha+\beta} v_2 + w_2 t^{2\mu}$$

$$v_1 w_2 = u^{\beta} w_1 + v_2 v_3 t^{2\lambda}$$

$$w_0 w_2 = w_1^2 + v_2^2 u^{\alpha} t^{2\lambda}$$

§1. What is a flip? (I)

Mori category: 3-folds with mild sings.

$$\begin{tikzcd} X^- \arrow[dr, "\text{small}"] & & X^+ \arrow[dl] \\ & X & \end{tikzcd}$$

$-K_{X^-}$ ample $\qquad$ K_{X^+} ample

flips $\Longleftrightarrow$ certain $\mathbb{Z}$-graded alg.
(X affine)

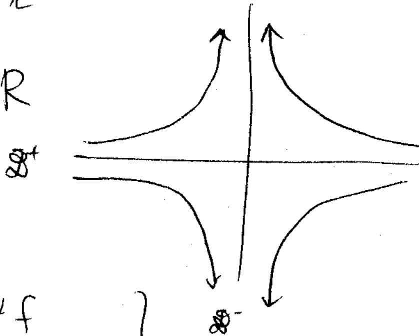
$$R(X, K_X) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathcal{O}_X(nK_X))$$

$$= \bigoplus_{n \le 0} H^0(X^-, nK_{X^-}) + \bigoplus_{n \ge 0} H^0(X^+, nK_{X^+})$$

$$X^{\pm} = \operatorname{Proj} R^{\pm}, \quad X = \operatorname{Spec} R^0$$

$\lambda \in \mathbb{C}^*$ acts on $R = \bigoplus_{n \in \mathbb{Z}} R_n$ by λ^n on R_n

$\mathbb{C}^*$ acts on $A = \operatorname{Spec} R$



$\mathfrak{B}^\pm$ = bad locuses

$$= \left\{ x \in A \;\middle|\; \begin{array}{l} f(x) = 0 \ \forall f \\ \text{homog of deg } \begin{array}{l} > 0 \\ (\text{resp } < 0) \end{array} \end{array} \right\}$$

$$\begin{tikzcd}
A \setminus \mathfrak{B}^- \arrow[d, "\mathbb{C}^*"] & & A \setminus \mathfrak{B}^+ \arrow[d] \\
X^- \arrow[dr] & & X^+ \arrow[dl] \\
& X &
\end{tikzcd}$$

$\operatorname{Proj} R^- = X^-$

$X = \operatorname{Spec} R_0$

$A = \operatorname{Spec} R$ is Gorenst. <u>canonical sings.</u>

$\because \quad A \setminus \mathfrak{D}^+ \cap \mathfrak{D}^-$

$$\begin{array}{c} A^- = \operatorname{total}(K_{A^-}) \\ \downarrow \\ X^- \end{array} \qquad \begin{array}{c} A^- = \operatorname{tot}(K_{X^-}) \\ \downarrow \\ X^- \end{array}$$

So K_{A^-} is canonically trivial line bdle

$$\det \begin{pmatrix} \frac{\partial x_i}{\partial y_i} & 0 \\ 0 & \left(\det \frac{\partial x_i}{\partial y_i}\right)^{-1} \end{pmatrix} = 1$$

$$\begin{tikzcd}
 & A & & \tilde{A} \arrow[ll] \arrow[dd] \\
A^- \arrow[ur, hook] \arrow[d] & & A^+ \arrow[ul, hook'] \arrow[d] & \\
X^- \arrow[dr] & & X^+ \arrow[dl] & \\
 & X & & \tilde{X} \arrow[ll]
\end{tikzcd}$$

$K_{\tilde{A}}$

nonsing

§2.

What is a flip? (II)

$$\begin{tikzcd}[column sep=small]
 & B \arrow[rrrr, hook, "t=0"'] \arrow[d, "\mathbb{C}^*"] & & & & A \arrow[d, "\mathbb{C}^*"] & \\
S^- \arrow[dr] & {} & S^+ \arrow[dl] & & X^- \arrow[dr] & {} & X^+ \arrow[dl] \\
 & S \arrow[rrrr, hook] & & & & X &
\end{tikzcd}$$

$t \in R_{-1}$ $\qquad 0 \to \omega_X \xrightarrow{t} \mathcal{O}_X \longrightarrow \mathcal{O}_S \to 0$

$S \subset X$ Gorenstein scheme

$S \in |-K_X|$

$\therefore (K_X + S)_{|S} = 0$

General elephant

S^-, S, S^+ surfaces

with Du Val sings.

$$\begin{tikzcd}
 & B \arrow[dd] \arrow[rr, hook, "(t=0)"] & & A \arrow[d, "\mathbb{C}^* \text{ quotient}"] \\
S_1 \arrow[dr] & & S_2 \arrow[dl] & {} \\
 & S & &
\end{tikzcd}$$

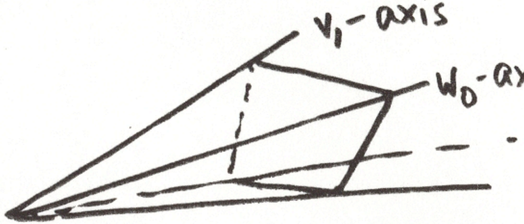
construction
involving Du Val singS
and their crepant
resoln
+ polarising divisors

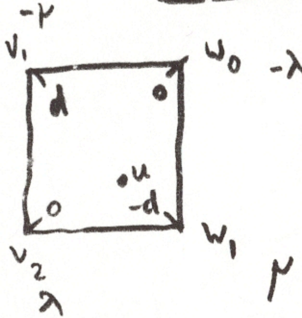
[A is equivariant
deformation of B]

Example

$$\left.\begin{aligned} v_1 w_1 &= u^{\alpha} \\ v_2 w_0 &= v_1^{d} u^{\beta} \end{aligned}\right\} \text{ in } \mathbb{C}^5$$

$v_1 \; v_2$
$w_0 \; w_1 \; u$





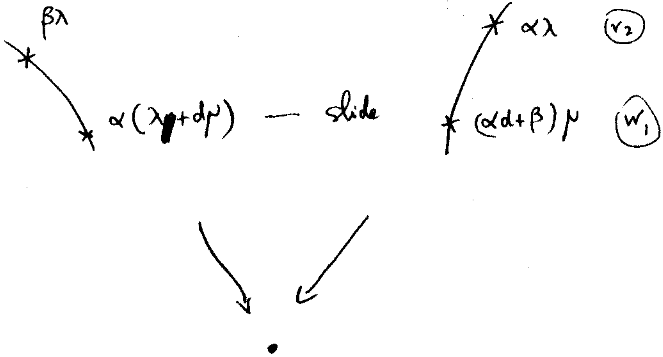
weights of $\mathbb{C}^*$ action
weight $u = 0$.

Affine piece $w_0 \neq 0$: $w_0 \mapsto 1$

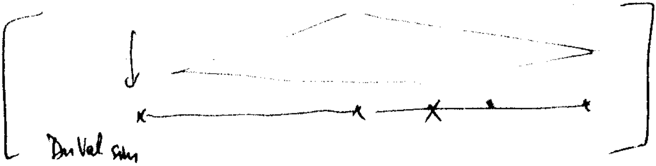
eliminate $v_2 = v_1^{d} u^{\beta}$

$$(v_1 w_1 = u^{\alpha}) / (\mathbb{Z}/\lambda+\mu d)$$

(V1) $\left(V_2W_0 = u^{\beta}\right) / \left(\mathbb{Z}/\mu\right) \quad \frac{1}{\mu}(-\lambda, \lambda, 0)$



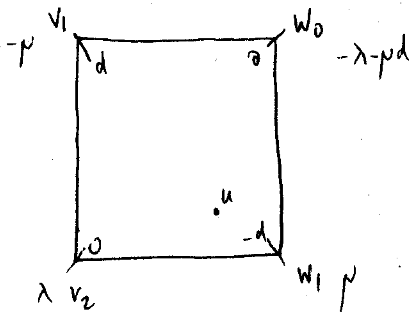
$[\alpha\lambda + \beta\lambda + \alpha\mu d]$

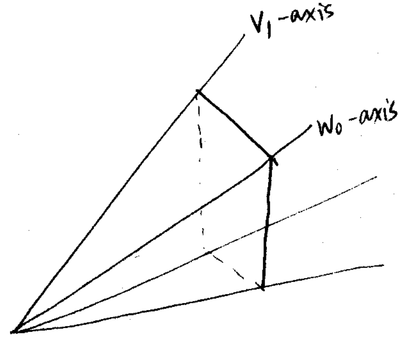


Example

$$V_1W_1 = u^\alpha$$

$$V_2W_0 = V_1^d u^\beta$$





S covered by 2 pieces $V_1 \neq 0$, $W_1 \neq 0$

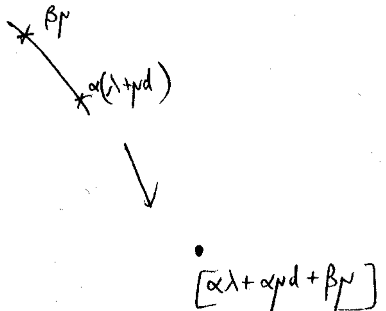
near $V_1 \neq 0$ coords V_2, W_0, W_1, u

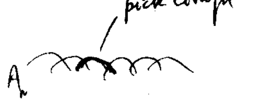
eqn $(V_2W_0 = u^\beta)/\mathbb{Z}/\mu$ $\frac{1}{\mu}(\lambda,-\lambda,0)$

$\rightsquigarrow$ Du Val sing cyclic gp $\frac{1}{\beta\mu}(1,-1)$

$W_1 \neq 0$ coords $V_1\ W_0\ W_1\ u$

eqn $(V_1W_1 = u^\alpha)/\mathbb{Z}/(\lambda+\mu d)$ $\frac{1}{\lambda+\mu d}(\mu,-\mu,0)$





$$V_1W_1 = u^\alpha + \left(t^\lambda v_2\right)^a \quad ②$$

$$V_2W_0 = V_1^d u^\beta + t^{d\mu} \quad ①$$

near V_1-axis local coordinates

$$V_2\, W_0\, u\, t$$

$$\left(V_2W_0 = u^\beta + t^{d\mu} \right) / \mathbb{Z}/\mu \quad \tfrac{1}{\mu}(\lambda, -\lambda, 0, 1)$$

near W_0-axis local coords

$$V_1\, W_0\, W_1\, u\, t$$

<u>must eliminate</u> V_2 from ②

$$W_0^a V_1W_1 = W_0^a u^\alpha + t^{\lambda a}(W_0V_2)^a$$
$$= W_0^a u^\alpha + \dots + t^{a(\lambda+d\mu)}$$

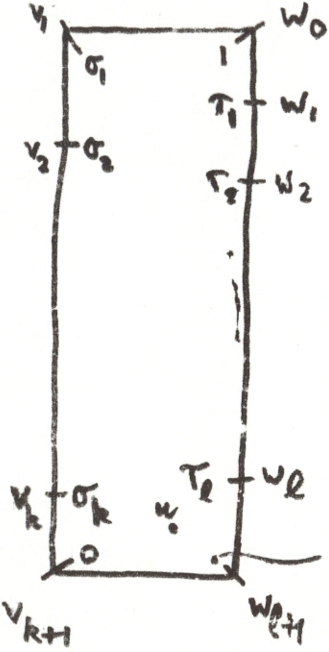
Gorenstein quadrilateral cone
(main case)

a, b coprime

$a > b \geqslant 2$

$\exists!\ i, j$ s.t.

$ai + bj + 1 = ab$



$$[1, \sigma_1, \dots, \sigma_k] = \frac{j}{a}$$

$$[1, \tau_1, \dots, \tau_\ell] = \frac{i}{b}$$

$\tau_{\ell+1} < 0$

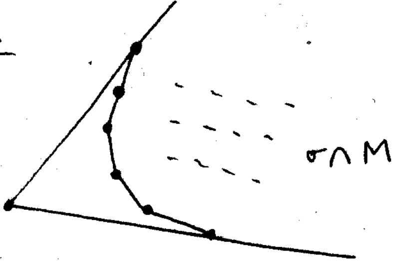
$$v_{i-1} v_{i+1} = v_i^{\sigma_i}$$

"tag equation"

$$-\frac{a}{b} = [\tau_{\ell+1}, \tau_\ell, \dots, \tau_1]$$

$$-\frac{b}{a} = [0, \sigma_k, \dots, \sigma_1]$$

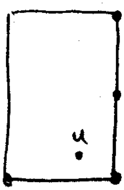
L.1.



minimal set of generators = vertices on Newton boundary

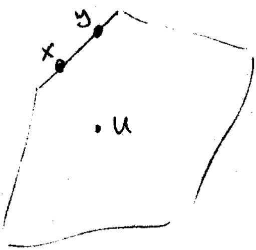
L.2 $B = \operatorname{Spec} k[\sigma\cap M]$ is Gorenstein

$\Longleftrightarrow \exists!\ u$ interior generator



$\because$ $u\,\dfrac{dx_1}{x_1}\wedge\cdots\wedge\dfrac{dx_n}{x_n}$ bases ω_B.

L.3.



3-dim^l case

$x\ y$ consecutive vertexes

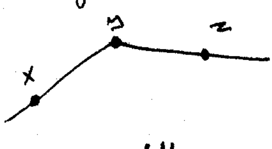
$\Rightarrow$ $x\ y\ u$ basis of M

$\because$ x, y base $M_0 \subset M$

$M_0 = \langle x, y\rangle$

u generates interior $\Rightarrow u, M_0$ base M

L.4.



$$\begin{pmatrix} y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & \sigma & \alpha \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ u \end{pmatrix}$$

$z = x^{-1} y^{\sigma} u^{\alpha}$ $\qquad$ $\begin{pmatrix} 0 & 1 \\ -1 & \sigma \end{pmatrix} \in SL(2,\mathbb{Z})$, affine extn

$$\begin{pmatrix} 0 & 1 \\ -1 & \sigma_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & \sigma_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & \sigma_i \end{pmatrix} = \begin{pmatrix} -q_{i-1} & n_{i-1} \\ -q_i & n_i \end{pmatrix}$$

$$\frac{q_i}{n_i} = [\sigma_i, \sigma_{i-1}, \dots \sigma_1]$$

$$= \sigma_i - \cfrac{1}{\sigma_{i-1} - \cfrac{1}{\ddots - \cfrac{1}{\sigma_1}}}$$

e.g. $\begin{pmatrix} 0 & 1 \\ -1 & \sigma_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & \sigma_2 \end{pmatrix} = \begin{pmatrix} -1 & \sigma_2 \\ -\sigma_1 & \sigma_1\sigma_2 - 1 \end{pmatrix}$

$$\frac{\sigma_1\sigma_2 - 1}{\sigma_1} = \sigma_2 - \frac{1}{\sigma_1}$$

$1 - \frac{1}{2} = \frac{i}{b}$ $\quad i = 1,\ b = 2$

$1 - \frac{1}{2 - \frac{1}{2}} = \frac{j}{a} = \frac{1}{3}$ $\quad j = ,\ a = 3$

$3 + 2 + 1 = 6$

$$\begin{pmatrix} w_{l+1} \\ v_{k+1} \\ u \end{pmatrix} = \begin{pmatrix} (b-i) & j & ? \\ b & a & ? \\ & & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ w_0 \\ u \end{pmatrix}$$

Mori. This corresponds to a flips with $\mathrm{Excl}(S^-) = \mathrm{Excl}(X^-)$

$$\Longleftrightarrow \quad \left.\begin{array}{l} \sigma_1 = \sigma_3 = \sigma_5 \cdots \\ \sigma_2 = \sigma_4 = \cdots \end{array}\right\}$$

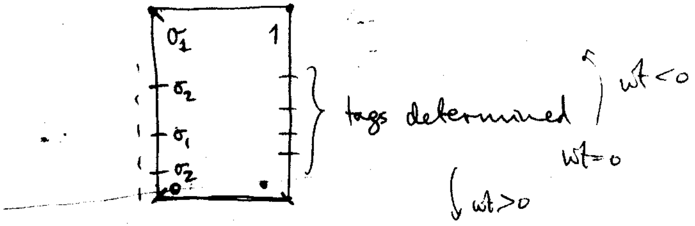
and flip is entirely determined by top corner equations

$$\begin{cases} w_0 v_2 = v_1^{\sigma_1} u^{\alpha} + \cdots + t^{\sigma_1} wtv_1 \\ v_1 w_1 = w_0 u^{\beta} + \cdots + v_2^{\sigma_2 - 1} t^{?} \end{cases}$$

$$\subset \mathbb{C}^6 \quad (v_1, w_0) \neq (0,0)$$

<u>Mori.</u> Correspond to flips [redacted] s.t. $S_1 \supset$ excl curve of X^- (1)

$\Longleftrightarrow$



$\dashrightarrow$ deformations of B explicitly determined by 2 top corner equations

<u>wt t = -1</u>

[redacted] $v_2 w_0 = v_1^{\sigma_1} u^{\alpha} + \cdots + t^{\sigma_1 \mathrm{wt}\, v_1}$ $\big/ \left(\mathbb{Z}/_{\mathrm{wt}\, v_1}\right)$

$v_1 w_1 = w_0 u^{\beta} + v_2$ [redacted] $^{\sigma_2 - 1}$

$v_1 w_1 w_0^{\sigma_2 - 1} = w_0^{\sigma_2} u^{\beta} + \cdots + t^{\sigma_2 \mathrm{wt}\, w_0}$ $\big/ \mathbb{Z}/_{\mathrm{wt}\, w_0}$

eqn of A @ v_1 and @ w_0

$$\begin{cases} w_0 v_2 = v_1^{\sigma_1} u^{\alpha} + \dots + t^{\sigma_1 \text{ wt } v_1} \\ v_1 w_1 = w_0 u^{\beta} + \dots + v_2^{\sigma_2 - 1} t^{\text{wt } w_0 - (\sigma_2 - 1)\text{ wt } v_2} \end{cases}$$

$[v_1 w_0 \neq 0]$

$\cancel{v_1 w}$

$w_0 \neq 0$ can subst for v_2

$$w_0^{\sigma_2 - 1} v_1 w_1 = w_0^{\sigma_2} u^{\beta} + \dots + t^{\sigma_2 \text{ wt } w_0}$$

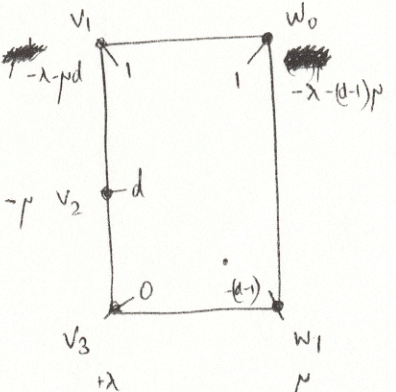
$A_{(v_1)}$ $\quad w_0 v_2 = v_1^{\sigma} u^{\alpha} + \dots + t^{\sigma_1 \text{ wt } v_1} \subset \mathbb{C}^5_{w_0 v_2 v_1 u t}$ $\quad [v_1 \neq 0]$

SS

$A_{(w_0)}$ $\quad w_0^{\sigma_2 - 1} v_1 w_1 = w_0^{\sigma_2} u^{\beta} + \dots + t^{\sigma_2 \text{ wt } w_0} \subset \mathbb{C}^5_{w_0 v_1 w_1 u t}$ $\quad [w_0 \neq 0]$

$$v_2 = \frac{v_1^{\sigma} u^{\alpha} + \dots + t^{\sigma_1 \text{ wt } v_1}}{w_0}$$

Pfaffian example



$$\left\{\begin{aligned} v_1v_3 &= v_2^d \\ v_2w_0 &= v_1u^\alpha \\ v_1w_1 &= w_0u^\beta \\ v_3w_0 &= v_2^{d-1}u^\alpha \\ v_2w_1 &= u^{\alpha+\beta} \end{aligned}\right.$$

B

to deform

$$\left.\begin{aligned} v_2w_0 &= v_1u^\alpha + t^{\lambda+\mu d} \\ v_1v_3 &= v_2^d + t^{d\mu}u^\beta \\ v_2w_1 &= u^{\alpha+\beta} + v_3t^\lambda \\ v_3w_0 &= v_2^{d-1}u^\alpha + t^{d\mu}w_1 \\ w_1v_1 &= w_0u^\beta + v_2^{d-1}t^\lambda \end{aligned}\right\}$$

A

Pfaffian form $\longrightarrow$ $\begin{pmatrix} t^{d\mu} & v_2^{d-1} & w_0 & v_1 \\ & v_3 & u^\alpha & v_2 \\ & & w_1 & u^\beta \\ -\mathrm{Sym} & & & t^\lambda \end{pmatrix}$

to deform & see behaviour @ v_1 w_0

@ v_1 local coords $v_1\ w_0\ v_2\ u\ t$

local equ $v_2 w_0 = v_1 u^\alpha + t^{\lambda+\mu d}$ } OK

@ w_0 local coords $v_1\ w_0\ w_1\ u\ t$

local equ ?

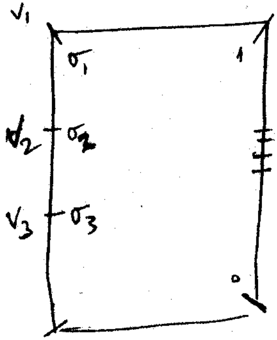
$v_1 w_1 = w_0 u^\beta + v_2^{d-1} t^\lambda$ involves v_2 not among local coords

$\therefore$ must eliminate

$$\boxed{v_1 (w_0^{d-1} w_1) = w_0^d u^\beta + (v_1 u^\alpha + t^{\lambda+\mu d})^{d-1} t^\lambda}$$

$N = d[\lambda + (d-1)\mu] = d \text{ wt } w_0$

subst. $w_0 = 1$ $\rightsquigarrow$ see $\underbrace{v_1 w_1 = u^\beta + \dots + t^N}_{\text{terminal.}}$ $/\mathbb{Z}/\text{wt } w_0$



$$w_0 v_1 w_1 = w_0^2 u^\beta + w_0 v_2 t^{2\lambda+4\mu}$$

$$w_0 v_2 = v_1^2 u^\alpha + t^{2\lambda+4\mu}$$

$$v_1 v_3 = v_2^2 + \cdots + u^\beta t^{2\mu}$$

$v_2 v_4 =$

$$w_0^2 v_1 v_3 = (w_0 v_2)^2 + w_0^2 u^\beta t^{2\mu}$$

$$(v_1^2 u^\alpha + t^{2\lambda+4\mu})^2 \qquad w_0 v_1 w_1 - \cdots$$

$$(w_0 v_2)^2 \qquad t^{2\lambda+4\mu}$$

[end terms must cancel]

$$x = a - \frac{1}{b-x}$$

$$a(b-x) - 1 = \cancel{bx} - x^2 \qquad \det\begin{pmatrix} x-a & a \\ b & x-b \end{pmatrix} = 1$$

$$x^2 - bx + ab - \cancel{a}x - \cancel{a}$$

$$\boxed{x^2 - (a+b)x + ab = 1}$$

$$\cancel{x^2 \quad a+b}$$

Computations for Example 1

tag eqn @ w_0 $\qquad w_1 v_1 w_0 = w_0^2 u^\beta + t^{4\lambda+6\mu}$ $\qquad$ NB [not generator of ideal]

@ w_1 $\qquad w_0 v_2 = v_1^2 u^\alpha + t^{2\lambda+4\mu}$

means: 2 varieties $\bar{A}^*_{w_0}$: $w^* v_1 = w_0^2 u^\beta + t^{4\lambda+6\mu}$ $\qquad \mathbb{C}^5(v_1\ w_0\ u\ t\ w^*)$

$w_0 v_2 = v_1^2 u^\alpha + t^{2\lambda+4\mu}$ $\qquad \mathbb{C}^5(v_1\ w_0\ u\ t\ v_2)$

A^- = glued together [redacted] $\bar{A}^*_{w_0} \setminus (w_0 = 0)$ $\qquad v_1\ w_0\ u\ t\ w^*$

$$\varphi \,\big|\wr \qquad\qquad \downarrow$$

$\bar{A}^*_{v_1} \setminus (v_1 = 0)$ $\qquad v_1\ w_0\ u\ t \quad v_2 = \dfrac{v_1^2 u^\alpha + t^{2\lambda+4\mu}}{w_0}$

$$\Gamma(A^-) = \Gamma(A^-_{w_0} \cup A^-_{v_1}) = \frac{k[v_1\ w_0^{\pm}\ u\ t\ w^*]}{(E_{w_0})} \cap \frac{k[v_1^{\pm}\ w_0\ u\ t\ v_2]}{(E_{v_1})}$$

Need to show $\underbrace{v_2, v_3, v_4, \dots v_\ell, w_k}, \underbrace{w_{k-1}, \dots, w_1} \in \Gamma$

[Computationally "easy" prove the existence of flip classify [in principle]] $\qquad$ computationally harder

NB v_2 not local coord at w_0

$$v_1 w_1 = w_0 u^{\beta} + \underline{\underline{v_2}}\, t^{2\lambda+2\mu}$$

$$w_0 v_2 = v_1^2 u^{\alpha} + t^{2\lambda+4\mu}$$

$$v_1 v_3 = v_2^2 + u^{\beta} t^{2\mu} \qquad v_3 \in \Gamma[v_1^{-1}]$$

$$w_0^2 v_1 v_3 = (w_0 v_2)^2 + w_0^2 u^{\beta} t^{2\mu}$$

$$\underline{v_1^4 u^{2\alpha}} + \underline{2 v_1^2 u^{\alpha} t^{2\lambda+4\mu}} + \underline{t^{4\lambda+8\mu}} + w_0 (v_1 w_1 - v_2 t^{2\lambda+2\mu}) t^{2\mu}$$

$$w_0 v_1 w_1 t^{2\mu} - (v_1^2 u^{\alpha} + \underline{t^{2\lambda+4\mu}}) t^{2\lambda+4\mu}$$

$$w_0^2 v_3 = v_1^3 u^{2\alpha} + v_1 u^{\alpha} t^{2\lambda+4\mu} + w_0 w_1 t^{2\mu} \qquad \underline{\therefore v_3 \in \Gamma[w_0^{-1}]}$$

$$w_2 v_2 = u^{\alpha+2\beta} + v_3^2 t^{2\lambda}$$

$$v_1^2 v_2 w_2 = (v_1^2 u^{\alpha}) u^{2\beta} + (v_1 v_3)^2 t^{2\lambda}$$

$$= (w_0 v_2 - t^{2\lambda+4\mu}) u^{2\beta} + (v_2^2 + u^{\beta} t^{2\mu})^2 t^{2\lambda}$$

$$= w_0 v_2 u^{2\beta} + v_2^4 t^{2\lambda} + 2 v_2^2 u^{\beta} t^{2\lambda+2\mu}$$

$$v_1^2 w_2 = w_0 u^{2\beta} + v_2^3 t^{2\lambda} + 2 v_2 u^{\beta} t^{2\lambda+2\mu}$$

$$V_3 W_1 = V_2 u^{\alpha+\beta} + \cancel{? \; w_0 \text{ or } V_2}? \quad w_2 t^{2\mu} \,?$$

@ W_2 $\qquad w_2 V_3 W_1 = (W_2 V_2) u^{\alpha+\beta} + \underline{w_2^2 ? \, t^{2\mu}}$ $\qquad W_2 = 1$

$$\boxed{u^{2\alpha+3\beta} + V_3^2 t^{2\mu}}$$

$$\boxed{V_3 W_1 = V_2 u^{\alpha+\beta} + w_2 t^{2\mu}}$$

eqn @ W_2

$$W_2 V_3 W_1 = \left(u^{2\alpha+3\beta} + V_3^2 u^{\alpha+\beta} u^{2\mu}\right) + W_2^2 t^{2\mu}$$

$$\boxed{W_2 = 1 \Rightarrow \text{OK}}$$

$\times V_2$ to get rid of W_2

$$V_2 V_3 W_1 = V_2^2 u^{\alpha+\beta} + (V_2 W_2) t^{2\mu}$$

$$= \left(V_1 V_3 - \cancel{u^{\beta} t^{2\mu}}\right) u^{\alpha+\beta} + \left(\cancel{u^{\alpha+2\beta}} + V_3^2 t^{2\lambda}\right) t^{2\mu}$$

$$\therefore \boxed{V_2 W_1 = V_1 u^{\alpha+\beta} + V_3^2 t^{2\lambda+2\mu}}$$

$$V_1^2 V_2 W_1 = \left(V_1^2 u^{\alpha}\right) V_1 u^{\beta} + (V_1 V_3)^2 t^{2\lambda+2\mu}$$

$$\left(W_0 V_2 - t^{2\lambda+4\mu}\right) V_1 u^{\beta} + \left(V_2^2 + u^{\beta} t^{2\mu}\right)^2 t^{2\lambda+2\mu}$$

$$= W_0 u^{\beta} V_1 V_2 - V_1 u^{\beta} t^{2\lambda+4\mu} + V_2^4 t^{2\lambda+2\mu} + 2 V_2^2 u^{\beta} t^{2\lambda+4\mu} + u^{2\beta} t^{2\lambda+6\mu}$$

$$\left(V_1 W_1 - V_2 t^{2\lambda+2\mu}\right) V_1 V_2$$

$$= V_1^2 W_1 V_2 - V_1 t^{2\lambda+2\mu} \left(V_1 V_3 - u^{\beta} t^{2\mu}\right)$$

$$v_1 v_2 w_1 = (v_1^2 u^\alpha) u^\beta + (v_1 v_3) v_3 t^{2\lambda+2\mu}$$

$$= (w_0 v_2 - t^{2\lambda+4\mu}) u^\beta + (v_2^2 + u^\beta t^{2\mu}) v_3 t^{2\lambda+2\mu}$$

$$\cancel{w_0 v_2^3} = \cancel{v_1^2 v_2 u^\alpha} + v_2 t^{2\lambda+4\mu}$$

$$\cancel{w_0 v_1 v_3} = \cancel{w_0 v_2^2} + w_0 u^\beta t^{2\mu}$$

$$v_1 (w_0 v_3 - v_1 v_2 u^\alpha) = v_2 t^{2\lambda+4\mu} + w_0 u^\beta t^{2\mu}$$

$$= t^{2\mu} (v_1 w_1)$$

$$\boxed{w_0 v_3 = v_1 v_2 u^\alpha + t^{2\mu} w_1}$$

$$\cancel{w_1 v_1 v_3 = w_1 v_2^2 + u^\beta w_1 t^{2\mu}}$$

$$\cancel{v_1 w_1 v_3 = v_1 v_2 u^{\alpha+\beta} + v_1 w_2 t^{2\mu}}$$

$$v_2 \left(\cancel{v_2 w_1 = v_1 u^{\alpha+\beta} + v_3^2 t^{2\lambda+2\mu}} \right)$$

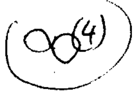
$$w_2 v_2$$

$$w_0 w_2 = w_1^2 + ?$$

$$v_1 w_0 w_2 = \boxed{v_1 w_1} w_1 + v_1 ? \qquad - u^{\alpha+\beta} t^{2\lambda+2\mu}$$

$$(w_0 u^\beta + v_2 t^{2\lambda+2\mu}) w_1 + v_1 ?$$

$$w_0 u^\beta w_1 + (v_1 u^{\alpha+\beta} + v_3^2 t^{2\lambda+2\mu}) t^{2\lambda+2\mu}$$



(4)

$$W_0 V_3 = V_1 V_2 u^{\alpha} + t^{2\mu} W_1 \quad (*)$$

$$V_3 W_1 = V_2 u^{\alpha+\beta} + W_2 t^{2\mu}$$

$$\times V_2 \quad V_2 V_3 W_1 = V_2^2 u^{\alpha+\beta} + V_2 W_2 t^{2\mu}$$

$$= \left(V_1 V_3 - \cancel{u^{\beta} t^{2\mu}}\right) u^{\alpha+\beta} + \left(\cancel{u^{\alpha+2\beta}} + V_3^2 t^{2\lambda}\right) t^{2\mu}$$

$$V_2 W_1 = V_1 u^{\alpha+\beta} + V_3 t^{2\lambda+2\mu} \quad (*)$$

deformation of original eqns

$$V_1 V_2 W_1 = \left(V_1^2 u^{\alpha}\right) u^{\beta} + V_1 V_3 t^{2\lambda+2\mu}$$

$$\left(W_0 V_2 - \cancel{t^{2\lambda+4\mu}}\right) u^{\beta} + \left(V_2^2 + \cancel{u^{\beta} t^{2\mu}}\right) t^{2\lambda+2\mu}$$

$$V_1 W_1 = W_0 u^{\beta} + V_2 t^{2\lambda+2\mu}$$

$$V_2 W_1^2 = V_1 W_1 u^{\alpha+\beta} + W_1 V_3 t^{2\lambda+2\mu}$$

$$= \left(W_0 u^{\beta} + V_2 t^{2\lambda+2\mu}\right) u^{\alpha+\beta} + \left(V_2 u^{\alpha+\beta} + W_2 t^{2\mu}\right) t^{2\lambda+2\mu}$$

$$= W_0 u^{\alpha+2\beta} + 2 V_2 u^{\alpha+\beta} t^{2\lambda+2\mu} + W_2 t^{2\lambda+4\mu}$$

$$u^{\alpha+2\beta} = W_2 V_2 - V_3^2 t^{2\lambda}$$

$$= W_0 W_2 V_2 - W_0 V_3^2 t^{2\lambda} + \cancel{W_2 t^{2\lambda+4\mu}} + \cancel{2} V_2 u^{\alpha+\beta} t^{2\lambda+2\mu}$$

$$- V_1 V_2 V_3 u^{\alpha} t^{2\lambda} - V_3 W_1 t^{2\lambda+2\mu}$$

$$\cancel{V_2 u^{\alpha+\beta} + W_2 t^{2\mu}}$$

$$W_1^2 = W_0 W_2 + u^{\alpha+\beta} t^{2\lambda+2\mu} - V_1 V_3 u^{\alpha} t^{2\lambda}$$

$$u^{\beta} w_1 v_2 = \cancel{v_1 u^{\alpha+2\beta} + u^{\beta} v_3 t^{2\lambda+2\mu}}$$

$$v_1 W_2 \cancel{W_2} \cancel{=} - v_1 v_3^2 t^{2\lambda} + u^{\beta} v_3 t^{2\lambda+2\mu}$$

$$v_3 (-v_2^2 t^{2\lambda})$$

$$u^{\beta} W_1 = v_1 W_2 - v_3 v_2 t^{2\lambda}$$

$$v_1 W_1^2 = W_0 W_1 u^{\beta} + v_2 W_1 t^{2\lambda+2\mu}$$

$$W_0 v_1 W_2 - W_0 v_2 v_3 t^{2\lambda} \qquad (v_1 u^{\alpha+\beta} + v_3 t^{2\lambda+2\mu}) \times t^{2\lambda+2\mu}$$

$$W_0 v_1 W_2 - W_0 v_2 v_3 t^{2\lambda} + v_1 u^{\alpha+\beta} t^{2\lambda+2\mu} \cancel{+ v_3 t^{4\lambda+4\mu}}$$

$$- v_1^2 u^{\alpha} v_3 t^{2\lambda} - \cancel{v_3 t^{4\lambda+4\mu}}$$

$$- v_1$$

$$W_1^2 = W_0 W_2 - \cancel{v_1 u^{\alpha} v_3 t^{2\lambda}} + \cancel{v_1 u^{\alpha+\beta} t^{2\lambda+2\mu}}$$

$$- v_2^2 u^{\alpha} t^{2\lambda}$$