On the Hasse principle for bielliptic surfaces

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To Sir Peter Swinnerton-Dyer

From the geometer's point of view, bielliptic surfaces can be described as quotients of abelian surfaces by freely acting finite groups, that are not abelian surfaces themselves. Together with abelian, K3 and Enriques surfaces they exhaust the class of smooth and projective minimal surfaces of Kodaira dimension 0. Because of their close relation to abelian surfaces, bielliptic surfaces are particularly amenable to computation. At the same time they display phenomena not encountered for rational, abelian or K3 surfaces, for example, torsion in the Néron–Severi group, finite geometric Brauer group, non-abelian fundamental group. This curious geometry is reflected in amusing arithmetical properties of these surfaces over number fields.

The behaviour of rational points on bielliptic surfaces was first studied by Colliot-Thélène, Swinnerton-Dyer and the second author [CSS] in relation with Mazur's conjectures on the connected components of the real closure of $\mathbb{Q}$-points. The second author then constructed a bielliptic surface over $\mathbb{Q}$ that has points everywhere locally but not globally; moreover, this counterexample to the Hasse principle cannot be explained by the Manin obstruction [S1] (see also [S2], Ch. 8). D. Harari [H] showed that bielliptic surfaces give examples of varieties with a Zariski dense set of rational points that do not satisfy weak approximation; moreover this failure cannot be explained by the Brauer–Manin obstruction.

A discrete invariant of a bielliptic surface is the order n of the canonical class in the Picard group. The possible values of n are 2, 3, 4 and 6. The surface contructed in [S1] has $n = 2$. Until now this was the only known counterexample to the Hasse principle that cannot be explained by the Manin obstruction. In this note we construct a similar example in the case $n = 3$. The difference is that we now need to consider elliptic curves with complex multiplication. The actual construction turns out to be somewhat simpler than in [S1]. In contrast, for the bielliptic surfaces with $n = 6$ we prove that the Manin obstruction to the Hasse principle is the only one (under the assumption that the Tate–Shafarevich group of its Albanese variety is finite).

1 Bielliptic surfaces

Let k be a field of $\mathrm{char}\, k = 0$, and $\overline{k}$ be an algebraic closure of k. For a k-variety X we write $\overline{X} = X \times_k \overline{k}$.

Definition 1 A *bielliptic surface* X over k is a smooth projective surface such that $\overline{X}$ is a minimal surface of Kodaira dimension 0, and is not a K3, abelian or Enriques surface.

Bielliptic surfaces over $\overline{k}$ were classified by Bagnera and de Franchis (see [B], VI.20). Their theorem says that any bielliptic surface over $\overline{k}$ can be obtained as the quotient of the product of two elliptic curves $E \times F$ by a freely acting finite abelian group. The geometric genus of any bielliptic surface is 0. For a bielliptic surface X let n be the order of $K_{\overline{X}}$ in $\mathrm{Pic}\, \overline{X}$. It follows from the Bagnera–de Franchis classification that n can be $2, 3, 4$ or 6 (*loc.cit.*).

Proposition 1 *Let X be a bielliptic surface over k. There exists an abelian surface A, a principal homogeneous space Y of A, and a finite étale morphism $f\colon Y \to X$ of degree n, that is a torsor under the group scheme μ_n.*

Proof The natural map $\mathrm{Pic}\, X \to \mathrm{Pic}\, \overline{X}$ is injective, hence nK_X is a principal divisor. We write $nK_X = (\phi)$, where $\phi \in k(X)^*$. Let Y be the normalization of the covering of X given by $t^n = \phi$. Then the natural map $f\colon Y \to X$ is unramified, and is a torsor under μ_n (cf. [CS], 2.3.1, 2.4.1). This implies that $K_Y = f^* K_X = 0$. By the classification of surfaces, $\overline{Y}$ is an abelian surface. (It is not K3 as the only unramified quotients of K3 surfaces are Enriques surfaces.) Let A be the Albanese variety of Y, defined over k (see [L], II.3). Then $\overline{A}$ is the Albanese variety of $\overline{Y}$. The choice of a base point makes $\overline{Y}$ an abelian variety isomorphic to $\overline{A}$, so that $\overline{Y}$ is naturally a principal homogeneous space of $\overline{A}$. Choose $\overline{y}_0 \in Y(\overline{k})$, then we have an isomorphism $\overline{Y} \to \overline{A}$ that sends $\overline{y}$ to $\overline{y} - \overline{y}_0$. Then $\rho(g) = {}^g\overline{y}_0 - \overline{y}_0$ is a continuous 1-cocycle of $\mathrm{Gal}(\overline{k}/k)$ with coefficients in $A(\overline{k})$. Let A^ρ be the principal homogeneous space of A defined by ρ; it corresponds to the twisted Galois action $(g, \overline{a}) \mapsto {}^g\overline{a} + \rho(g)$, where $\overline{a} \in A(\overline{k})$ (see [S], III.1, or [S2], 2.1). Then the above $\overline{k}$-isomorphism $\overline{Y} \to \overline{A}$ descends to a k-isomorphism $Y \to A^\rho$. $\square$

Note that the analogue of the proposition fails in higher dimension because there are many more possibilities for Y.

2 Group action on principal homogeneous spaces of abelian varieties

Let A be an abelian variety over k, and Z a principal homogeneous space of A. Suppose that a k-group scheme Γ acts on Z. This gives rise to a Galoisequivariant action of the group $\Gamma(\overline{k})$ on the set $Z(\overline{k})$. The action of Γ on Z naturally defines an action of Γ on A, the Albanese variety of Z. Then the action of the group $A(\overline{k})$ on the set $Z(\overline{k})$ is both Galois and Γ-equivariant. Let A^Γ be the Γ-invariant group subscheme of A. Similarly, let $Z^\Gamma \subset Z$ be the closed subscheme consisting of points fixed by Γ.

Proposition 2 *Suppose that a k-group scheme Γ acts on Z in such a way that Z^Γ is a nonempty scheme (i.e., some $\overline{k}$-point of Z is fixed by $\Gamma(\overline{k})$). Then* $[Z] \in \mathrm{Im}[H^1(k, A^\Gamma) \to H^1(k, A)]$.

Proof Take $\overline{x} \in Z(\overline{k})$, fixed by $\Gamma(\overline{k})$. Then a 1-cocycle of $\mathrm{Gal}(\overline{k}/k)$ sending $g \in \mathrm{Gal}(\overline{k}/k)$ to ${}^g\overline{x} - \overline{x} \in A(\overline{k})$ represents the class $[Z] \in H^1(k, A)$. For any $\gamma \in \Gamma(\overline{k})$ we have

$$\gamma({}^g\overline{x} - \overline{x}) = \gamma \cdot {}^g\overline{x} - \gamma \cdot \overline{x} = {}^g({}^{g^{-1}}\gamma \cdot \overline{x}) - \overline{x} = {}^g\overline{x} - \overline{x}.$$

Therefore, ${}^g\overline{x} - \overline{x} \in A^\Gamma(\overline{k})$. □

It is easy to see that Z^Γ is a principal homogeneous space of A^Γ. The A-torsor Z is the push-forward of the A^Γ-torsor Z^Γ with respect to the natural injection of group schemes $A^\Gamma \to A$. This gives an alternative proof of the proposition.

Corollary 1 *Let $A_1 = A/A^\Gamma$, and $\alpha\colon A \to A_1$ the natural surjection. Then* $[Z] \in H^1(k, A)[\alpha_*]$, *where*

$$H^1(k, A)[\alpha_*] = \ker[\alpha_*\colon H^1(k, A) \to H^1(k, A_1)].$$

We now consider the case when $Z = C$ is a curve of genus 1 equipped with a faithful action of Γ that has a fixed point. Then $A = E$ is the Jacobian of C. We shall write $\mathrm{Aut}_0(\overline{E})$ for the automorphism group of $\overline{E}$ as an elliptic curve. Now $\Gamma(\overline{k}) \subset \mathrm{Aut}_0(\overline{E})$, hence $\Gamma(\overline{k})$ is a cyclic group of order n, where n can be 1, 2, 3, 4 or 6. A straightforward calculation shows that, excluding the trivial case $n = 1$, we have one of the following possibilities:

n	$\#E^\Gamma(\overline{k})$
6	1
4	2
3	3
2	4

By Corollary 1 the first line of this table shows that if a cyclic group scheme of order 6 acts on a curve of genus 1, then this curve has a k-point. As a consequence of this fact we obtain in the next section a simple description of bielliptic surfaces with $n = 6$.

3 A case when the Manin obstruction to the Hasse principle is the only one

Proposition 3 *Let X be a bielliptic surface over k such that the order of $K_{\overline{X}}$ in* $\operatorname{Pic}\overline{X}$ *is 6. There exist an elliptic curve E and a curve D of genus 1 such that the group scheme μ_6 acts on E by automorphisms of an elliptic curve (in particular, preserving the origin), and acts on D by translations, in such a way that $X=(E\times D)/\mu_6$.*

Proof The Bagnera–de Franchis classification ([B], VI.20) says that for any bielliptic surface $\overline{X}$ with $K_{\overline{X}}$ of order 6 in $\operatorname{Pic}\overline{X}$ there exist elliptic curves C_1 and C_2 over $\overline{k}$ such that:

(1) μ_6 acts on C_1 by automorphisms of an elliptic curve (in particular, preserving the origin);

(2) the group scheme μ_6 is a subgroup of C_2;

(3) $X=(C_1\times C_2)/\mu_6$.

The free action of μ_6 on $C_1\times C_2$ makes the finite étale map $C_1\times C_2\to\overline{X}$ a torsor under μ_6. Let us compare it with the torsor $\overline{Y}\to\overline{X}$ constructed in Proposition 1.

Recall that the type of a Z-torsor under a group of multiplicative type S is a certain functorial map $\widehat{S}\to\operatorname{Pic}\overline{Z}$, where $\widehat{S}$ is the module of characters of S (see [S2], Definition 2.3.2). A torsor under a group of multiplicative type over an integral projective $\overline{k}$-variety is uniquely determined up to isomorphism by its type (this follows from the fundamental exact sequence of Colliot-Thélène and Sansuc, see [CS], [S2], (2.22)). Therefore it is enough to compare the respective types. There is an exact sequence

$$0\to\operatorname{Hom}(\mu_6,\overline{k}^*)=\mathbb{Z}/6\to\operatorname{Pic}\overline{X}\to\operatorname{Pic}\overline{Y},$$

where the second arrow is the type of the torsor $\overline{Y}\to\overline{X}$, and a similar sequence for $C_1\times C_2\to\overline{X}$ ([S2], (2.4) and Lemma 2.3.1). Since the canonical class of an abelian surface is trivial, $K_{\overline{X}}$ is in the image of $\mathbb{Z}/6$ in $\operatorname{Pic}\overline{X}$, and hence it is a generator of that image. Thus the types of both torsors are the same (up to sign). Hence the pair ($\overline{Y}$, the action of μ_6) can be identified with the pair ($C_1\times C_2$, the action of μ_6).

Let A be the Albanese variety of Y. This is an abelian surface defined over k. Let s be the k-endomorphism of A given by $s=\sum_{\sigma\in\mu_6}\sigma$. Let A_1 (respectively A_2) be the connected component of 0 in $\ker(s)$ (respectively in

A^{μ_6}). Note that s acts as 0 on $J_1 = \mathrm{Jac}(\overline{C}_1) \subset \overline{A}$, and as multiplication by 6 on $J_2 = \mathrm{Jac}(\overline{C}_2) \subset \overline{A}$. Therefore, $\overline{A}_1 = J_1$, $\overline{A}_2 = J_2$. Now the map

$$A_1 \times A_2 \to A, \quad (x, y) \mapsto x + y,$$

is an isomorphism, since over $\overline{k}$ it is the natural isomorphism $J_1 \times J_2 \to \overline{A}$. This proves that A is a product of two elliptic curves over k. Hence Y, which is a principal homogeneous space of A, is a product of two curves of genus 1 over k: $Y = E \times D$, where $C_1 \simeq \overline{E}$, $C_2 \simeq \overline{D}$.

By the Bagnera–de Franchis theorem the group scheme μ_6 acts on E with a fixed point. By the remark preceding the statement of the proposition, this point is unique, and hence is k-rational. Hence E is an elliptic curve (isomorphic to A_1). $\square$

See the beginning of the next section (or, in more generality, [S2], 5.2) for the definition of the Manin obstruction.

Corollary 2 *Let k be a number field. The Manin obstruction is the only obstruction to the Hasse principle on the bielliptic surfaces X over k such that the order of $K_{\overline{X}}$ in $\mathrm{Pic}\,\overline{X}$ is 6, and the Tate–Shafarevich group of the Albanese variety of X is finite.*

Proof By the previous proposition we have $X = (E \times D)/\mu_6$. Consider the curve $D' = D/\mu_6$ of genus 1, and let $p\colon X \to D'$ be the natural surjective map. Let J' be the Jacobian of D'. It is known ([B], VI) that the Albanese variety of any bielliptic surface has dimension 1. Using the universal property of the Albanese variety (see [L], II.3) and the connectedness of the fibres of p one easily checks that J' is the Albanese variety of X.

Let $\{Q_v\}$ be a collection of local points on X, for all places v of k, that satisfies the Brauer–Manin conditions. Then $\{p(Q_v)\}$ satisfies the Brauer–Manin conditions on D'. If $\text{Ш}(J')$ is finite, then D' has a k-point by a theorem of Manin (see [S2], Theorem 6.2.3). Call this point Q. The inverse image of Q in D defines a class $\rho \in H^1(k, \mu_6) = k^*/k^{*6}$. Consider the twisted torsor $E^\rho \times D^\rho \to X$. Now D^ρ has a k-point over Q. But the action of μ_6 on E preserves the origin, hence the twisted curve E^ρ has a k-point. Therefore, we obtain a k-point on $E^\rho \times D^\rho$, and hence on X. $\square$

Note that for the bielliptic surfaces of Corollary 2 the quotient of $\mathrm{Br}\,X$ by the image of $\mathrm{Br}\,k$ is infinite, but in the proof we only used the Brauer–Manin conditions given by the elements of the conjecturally finite group $\text{Ш}(J')$.

Corollary 2 is a particular case of a more general situation. Let Γ be an algebraic group acting on varieties V and W such that the action on W is free. Suppose that V has a k-point fixed by Γ. If the Manin obstruction to the Hasse principle is the only one on W/Γ, then the same is true for $(V \times W)/\Gamma$.

4 Main construction and example

Now assume $k = \mathbb{Q}$, and let $\mathbf{A}_\mathbb{Q}$ be the ring of adèles of $\mathbb{Q}$. For a projective variety X we have $X(\mathbf{A}_\mathbb{Q}) = \prod_v X(\mathbb{Q}_v)$, where v ranges over all places of $\mathbb{Q}$ including the real place. Let $X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$ be the subset of $X(\mathbf{A}_\mathbb{Q})$ consisting of the families of local points $\{P_v\}$ satisfying all the Brauer–Manin conditions. These conditions, one for each $A \in \mathrm{Br}\, X$, are

$$\sum_{\text{all } v} \mathrm{inv}_v\, A(P_v) = 0,$$

where inv_v is the local invariant at the place v, which is a canonical map $\mathrm{Br}\, \mathbb{Q}_v \to \mathbb{Q}/\mathbb{Z}$ provided by local class field theory. The Brauer–Manin conditions are satisfied for any $\mathbb{Q}$-point of X by the Hasse reciprocity law, so that we have $X(\mathbb{Q}) \subset X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$. If the last set is empty, this is an obstruction to the existence of a $\mathbb{Q}$-point on X; it is called the Manin obstruction.

We now give a construction of bielliptic surfaces X for which $X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}} \neq \emptyset$, but $X(\mathbb{Q}) = \emptyset$. Then X is a counterexample to the Hasse principle that is not explained by the Manin obstruction.

Theorem 1 *Let E be an elliptic curve over $\mathbb{Q}$ with a nontrivial action of the group scheme μ_3. Let $\alpha\colon E \to E_1$ be the degree 3 isogeny with kernel E^{μ_3}. Let D be an elliptic curve with a group subscheme isomorphic to μ_3. Assume that:*

(i) $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts nontrivially on* E^{μ_3}*;*

(ii) $\#\text{Ш}(E)[\alpha_*] = 3$*;*

(iii) *C is a principal homogeneous space of E representing a nontrivial element of $\text{Ш}(E)[\alpha_*]$;*

(iv) $\mathrm{Sel}(D, \mu_3) = 0$*, that is, for any principal homogeneous space of D obtained from a nontrivial class in $H^1(\mathbb{Q}, \mu_3) = \mathbb{Q}^*/\mathbb{Q}^{*3}$, there exists a place v where it has no $\mathbb{Q}_v$-point.*

Then $X = (C \times D)/\mu_3$ is a counterexample to the Hasse principal not explained by the Manin obstruction.

Let us give an example of curves C and D satisfying the conditions of the theorem. Let ζ be a primitive cubic root of unity.

Let C be the plane cubic curve $x^3 + 11y^3 + 43z^3 = 0$, where the root of unity ζ acts by $(x : y : z) \mapsto (x : y : \zeta z)$. The Jacobian E of C is the plane curve $x^3 + y^3 + 473z^3 = 0$, with the action of μ_3 given by the same formula. One easily checks that Condition (i) is satisfied. Condition (ii) is verified in Example 4.3 of [F2]. The elements of $H^1(\mathbb{Q}, E)[\alpha_*]$ are given by the curves $mx^3 + m^2y^3 + 473z^3 = 0$ with m a cube-free integer. The curve C corresponds to $m = 11$. It has been known for some time [Se] that C has points everywhere locally but not globally. This gives Condition (iii). (See also [Ba], VI.)

Let D be the elliptic curve $u^3+v^3+w^3 = 0$, with $(1 : -1 : 0)$ as the origin. The group subscheme of D generated by $(1 : -\zeta : 0)$ is isomorphic to μ_3. The translation by this element is $(u, v, w) \mapsto (u : \zeta v : \zeta^2 w)$. The elements of the Selmer group $\mathrm{Sel}(D, \mu_3)$ are represented by the principal homogeneous spaces D_a defined by $u^3 + av^3 + a^2u^3 = 0$, where a is a cube free integer. Let p be a prime factor of a. Then D_a has no $\mathbb{Q}_p$-point. Therefore, the only curve D_a with points everywhere locally is D itself, so that $\mathrm{Sel}(D, \mu_3) = 0$, which is our Condition (iv).

Remark On changing some of the conditions of the theorem one obtains bielliptic surfaces for which the Manin obstruction to the Hasse principle is the only one. We replace Condition (ii) by the condition $\text{Ш}(E)[\alpha_*] = 0$, and instead of Condition (iii) we require that C is any principal homogeneous space of E whose class is in $H^1(\mathbb{Q}, E)[\alpha_*]$. We drop Condition (i) and keep Condition (iv). Then the Manin obstruction is the only obstruction to the Hasse principle for the surfaces $(C\times D)/\mu_3$. For the proof, consider the torsor $C \times D \to (C \times D)/\mu_3$ under μ_3. Under our assumptions the class of twists $C^\rho \times D^\rho$, $\rho \in \mathbb{Q}^*/\mathbb{Q}^{*3}$, satisfies the Hasse principle. By descent theory ([S2], Corollary 6.1.3 (2)) this implies our statement.

5 Proof of the theorem

Consider the alternating Cassels pairing $\text{Ш}(E) \times \text{Ш}(E) \to \mathbb{Q}/\mathbb{Z}$. Its restriction to $\text{Ш}(E)[\alpha_*]$ gives an alternating pairing

$$\text{Ш}(E)[\alpha_*] \times \text{Ш}(E)[\alpha_*] \to \mathbb{Q}/\mathbb{Z}. \tag{1}$$

The kernel of the last pairing is the image of $\alpha^t_*\colon \text{Ш}(E_1) \to \text{Ш}(E)$, where $\alpha^t\colon E_1 \to E$ is the dual isogeny. (This seems to be part of the folklore; see [F1] for a proof.) Since $\text{Ш}(E)[\alpha_*] \cong \mathbb{Z}/3\mathbb{Z}$ by Condition (ii), the pairing (1) must be zero. Therefore, there exists a principal homogeneous space C_1 of E_1 with points everywhere locally, that lifts C. This means that the map $\alpha^t_*\colon H^1(\mathbb{Q}, E_1) \to H^1(\mathbb{Q}, E)$ sends $[C_1]$ to $[C]$. There is a finite étale morphism $C_1 \to C$ that represents C as the quotient of C_1 by the action of $\ker(\alpha^t)$. Let $Y = C \times D$, $Y_1 = C_1 \times D$. This gives rise to a finite étale morphism $Y_1 \to Y$ which is the identity on D. Let f_1 be the composition of the finite étale maps $Y_1 \to Y \to X$, and let $\pi\colon Y_1 \to D$ be the projection to the second factor. In this notation we have the following key property analogous to ([S1], Theorem 1):

$$f_1^*(\mathrm{Br}\, X) \subset \pi^*(\mathrm{Br}\, D). \tag{2}$$

To prove this we note that for any smooth and projective surface X with $p_g = 0$, in particular, for a bielliptic surface, we have an isomorphism of Galois modules $\mathrm{Br}\, \overline{X} = \mathrm{Hom}(\mathrm{NS}(\overline{X})_{\mathrm{tors}}, \mathbb{Q}/\mathbb{Z})$ (see [G], II, Corollary 3.4, III, (8.12)). As in the proof of Corollary 2 one shows that the Albanese variety of X is D/μ_3. The same argument as in ([S1], pp. 403–404) works in our situation, and we obtain $\mathrm{NS}(\overline{X})_{\mathrm{tors}} = E^{\mu_3}$. Then (i) implies that $(\mathrm{Br}\, \overline{X})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = 0$. Therefore, $\mathrm{Br}\, X = \ker[\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}]$. A well known Leray spectral sequence shows that the quotient of this group by the image of $\mathrm{Br}\, \mathbb{Q}$ is naturally isomorphic to $H^1(\mathbb{Q}, \mathrm{Pic}\, \overline{X})$ ([S2], (2.23); here we use the fact that $H^3(\mathbb{Q}, \overline{\mathbb{Q}}^*) = 0$). The analysis of the morphism of Galois modules $f_1^*\colon \mathrm{Pic}\, \overline{X} \to \mathrm{Pic}\, \overline{Y}_1$ is carried out in the same way as in the proof of Lemma 2 of [S1], where the multiplication by 2 on E has now to be replaced by the isogeny $\alpha\colon E \to E_1$. The result is that the image $f_1^*(H^1(\mathbb{Q}, \mathrm{Pic}\, \overline{X}))$ in $H^1(\mathbb{Q}, \mathrm{Pic}\, \overline{Y}_1)$ is contained in $\pi^*(H^1(\mathbb{Q}, \mathrm{Pic}\, \overline{D}))$. Formula (2) now follows from the functoriality of the Leray spectral sequence.

Let us construct an adelic point on X satisfying all the Brauer–Manin conditions. Take a rational point $R \in D(\mathbb{Q})$, and a collection $\{P_v\} \in C_1(\mathbf{A}_\mathbb{Q})$. Then $f_1(\{(P_v, R)\}) \in X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$, as follows from (2) and the Hasse reciprocity law.

It remains to show that there are no $\mathbb{Q}$-points on X. Indeed, rational points on X come from twists of Y given by $a \in H^1(\mathbb{Q}, \mu_3) = \mathbb{Q}^*/\mathbb{Q}^{*3}$. Any such twist of Y is the product $C_a \times D_a$, where C_a and D_a are curves of genus 1. Moreover, D_a is a principal homogeneous space of D of the kind described in Condition (iv) of the theorem. By that condition, if D_a has points everywhere locally, then a is trivial, so that $D_a = D$. Thus there are no $\mathbb{Q}$-points on the nontrivial twists of Y. On the other hand, Y has no $\mathbb{Q}$-points since by Condition (iii) there are no $\mathbb{Q}$-points on C. Therefore, $X(\mathbb{Q}) = \emptyset$. This completes the proof.

More details can be found in the thesis of the first author [Ba]. The preparation of this paper was speeded up by John Voight's notes of the conference "Rational and Integral Points on Higher Dimensional Varieties" at the American Institute of Mathematics in December, 2002. We thank him for the notes, and the organizers for stimulating atmosphere. We are very grateful to Tom Fisher for telling us about the curve $x^3 + 11y^3 + 43z^3 = 0$. We thank Ekaterina Amerik for useful discussions.

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