An elementary approach to effective diophantine approximation on $\mathbb{G}_m$

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To Sir Peter Swinnerton-Dyer, on his 75th birthday

1 Introduction

Effective results in the diophantine approximation of algebraic numbers are difficult to obtain, and for a long time the only general method available was Baker's theory of linear forms in logarithms. An alternative, more algebraic, method was later proposed in Bombieri [2] and Bombieri and Cohen [3]. This new method is quite different from the classical approach through the theory of linear forms in logarithms.

In this paper, we improve on results derived in [3]. These results concern effective approximations to roots of high order of algebraic numbers and their application to diophantine approximation in a number field by a finitely generated multiplicative subgroup. We restrict our attention to the non-archimedean case, although our results and methods should go over *mutatis mutandis* to the archimedean setting.

We do not claim that our theorems are the best that are known in this direction. Linear forms in two logarithms (which are easier to treat than the general case) suffice to prove somewhat better results than our Theorem 5.1, see Bugeaud [6] and Bugeaud and Laurent [7]; we give an explicit comparison in §5, Remark 5.1.

Theorem 5.2, which is useful for general applications, follows from Theorem 5.1 by means of a trick introduced for the first time in [2] and improved in [3]. Thus any improved form of Theorem 5.1 carries automatically an improvement of Theorem 5.2. Note however that Theorem 5.2, in the form given here, is still far from what one can obtain directly from Baker's theory of logarithmic forms in many variables, as in Baker and Wüstholz [1] in the archimedean case and Kunrui Yu [10, 11] in the p-adic case.

Notwithstanding the comparison with Baker's theory, we feel that there is some untapped potential here. For example, one treats with equal ease the archimedean and the p-adic case, while this is not so in Baker's theory because of the bad analytic behaviour of the p-adic exponential.

The auxiliary construction involves a universal family of two-variable polynomials invariant under an action of roots of unity of a certain order. The main new feature in the current paper is the use of an elementary Wronskian argument, involving differentiation only in a single variable, to derive a zero estimate which bypasses former appeals to a more sophisticated two-variable Dyson's Lemma. This was initially inspired by private communication between the first author and David Masser in 1984. We reproduce part of that communication in §6.

Although the method of the current paper is more elementary, the results obtained are sharper than those of [3]. The main results are stated in §5, Theorem 5.1 and Theorem 5.2. Theorem 5.1 represents an improvement over the corresponding result of [3] both in the absolute constants and in the lower bound for r in (H1), where $(\log\frac{1}{\kappa})^7$ is replaced by $(\log\frac{1}{\kappa})^5$, as well as in the lower bound for $h(\alpha')$ in (H2) of [3], which is no longer required. These improvements automatically carry over to Theorem 5.2, which we restate for convenience here in the Main Theorem. We follow the notations of [3], §2. In particular $H(\ )$ denotes the absolute Weil height, $h(\ ) = \log H(\ )$ the absolute logarithmic Weil height and $|\ |_v$ is the absolute value associated to a place $v \in M_K$, normalized so that $h(x) = \sum_{v\in M_K} \max(0, \log |x|_v)$.

We define $\rho(x)$ to be the solution $\rho(x) > e^5$ of $\rho/(\log\rho)^5 = x$ if $x > e^5 5^{-5}$, and $\rho(x) = e^5$ otherwise; for large x we have $\rho(x) \sim x(\log x)^5$.

Main Theorem *Let K be a number field of degree d and v a place of K dividing a rational prime p. We denote by f_v the residue class degree of the extension $K_v/\mathbb{Q}_p$ and set $D_v^* = \max(1, \frac{d}{f_v \log p})$. Define a modified logarithmic height of $x \in K$ by $h'(x) = \max\bigl(h(x), \frac{1}{D_v^*}\bigr)$, and let $H'(x) = \exp h'(x)$.*

Let Γ be a finitely generated subgroup of the multiplicative group K^*, and write $\xi_1, \dots, \xi_t$ for generators of Γ/tors. Let $\xi \in \Gamma$, $A \in K^*$ and $\kappa > 0$ be such that

$$0 < |1 - A\xi|_v < H'(A\xi)^{-\kappa}.$$

Define

$$C = 66 p^{f_v} (D_v^*)^6 \quad \textit{and} \quad Q = \bigl(2t\rho(C/\kappa)\bigr)^t \prod_{i=1}^{t} h'(\xi_i).$$

Then we have

$$h'(A\xi) \ \le\ 16 p^{f_v} \rho(C/\kappa)\, Q \max\Bigl(h'(A), 4p^{f_v} Q\Bigr).$$

It is an interesting problem to try to refine the auxiliary construction of §2 to the point where the nonvanishing of $P(x, y)$ at the point (α, α') is immediate, that is $P(\alpha, \alpha') \neq 0$. In Cohen and van der Poorten [8] it is shown that this would lead to a result comparable with the best known consequences of Baker's method.

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2 Equivariant polynomials

Let $r \geq 2$ be a positive integer and l an integer with $(l, r) = 1$. For $0 \leq j < r$ we define e_j to be the integer with $0 \leq e_j < r$ such that

$$le_j \equiv -j \pmod{r}.$$

Let $0 \leq s < r$ and consider the polynomial

$$P(x, y) = \sum_{j=0}^{s} A_j(x^r) x^{e_j} y^j,$$

where the $A_j(x) \in \mathbb{Q}[x]$ are polynomials in x of degree at most n, not all identically 0. This polynomial is invariant under the action $(x, y) \mapsto (\varepsilon^l x, \varepsilon y)$ of rth roots of unity, in the sense that

$$P(\varepsilon^l x, \varepsilon y) = P(x, y) \quad \text{whenever } \varepsilon^r = 1.$$

We define the *index* $i(P; \xi, \eta)$ of $P(x, y)$ at a point (ξ, η) to be the order of zero of $P(x, \eta)$ at $x = \xi$, namely

$$i(P; \xi, \eta) = \operatorname{ord}_\xi P(x, \eta).$$

In what follows, for a real number t we abbreviate $t^+ = \max(t, 0)$.

Lemma 2.1 *We have*

$$\sum_{\xi \in \mathbb{C}^* / \{\varepsilon^r = 1\}} \max_\eta \bigl(i(P; \xi, \eta) - s\bigr)^+ \leq (s + 1)n.$$

Proof We use the classical Wronskian argument. Let $I \subset \{0, 1, \ldots, s\}$ be the set of indices j such that $A_j(x)$ is not identically 0 and $t+1$ its cardinality. Clearly $t \leq s$. We calculate the $(t+1) \times (t+1)$ Wronskian determinant

$$\begin{aligned} \det\left[\frac{\partial^{h+k}}{\partial x^h \partial y^k} P(x,y)\right]_{0 \leq h,k \leq t} &= \det\left[\sum_{j \in I} \frac{\partial^h}{\partial x^h}\Big(A_j(x^r)x^{e_j}\Big) \cdot \frac{\partial^k}{\partial y^k}\,(y^j)\right]_{0 \leq h,k \leq t} \\ &= \det\left[\frac{\partial^h}{\partial x^h}\Big(A_j(x^r)x^{e_j}\Big)\right]_{0 \leq h \leq t,\, j \in I} \cdot \det\left[\frac{\partial^k}{\partial y^k}\,(y^j)\right]_{j \in I,\, 0 \leq k \leq t}. \end{aligned}$$

Thus this Wronskian is a polynomial $W(x)y^{\Delta}$, with

$$\Delta = \sum_{j \in I} j - \frac{t(t+1)}{2}.$$

Moreover, $W(x)$ is not identically 0, because the polynomials $A_j(x^r)x^{e_j}$ are linearly independent over $\mathbb{Q}$ and the monomials y^j, $j \in I$, are also linearly independent over $\mathbb{Q}$ (the $A_j(x)$ for $j \in I$ are not identically 0 by hypothesis and the exponents in the monomials in $A_j(x^r)x^{e_j}$ belong to different arithmetic progressions as j varies). By looking at the determinant of the matrix $\big[(\mathrm{d}/\mathrm{d}x)^h A_j(x^r)x^{e_j}\big]$ we verify that

$$\operatorname{ord}_0 W(x) \geq \sum_{j \in I} e_j - \frac{t(t+1)}{2} \quad \text{and}$$

$$\operatorname{ord}_\infty W(x) \leq r(t+1)n + \sum_{j \in I} e_j - \frac{t(t+1)}{2}.$$

Now, if we specialize y to any $\eta \neq 0$ (which does not affect the vanishing of $W(x)$) and look at the first column of the Wronskian, we see that

$$\operatorname{ord}_\xi W(x) \geq \big(i(P;\xi,\eta) - t\big)^+;$$

therefore we have

$$\operatorname{ord}_\xi W(x) \geq \max_{\eta \neq 0} \Big(i(P;\xi,\eta) - t\Big)^+.$$

Since $P(x,y)$ is invariant we have $i(P;\varepsilon^l\xi, \varepsilon\eta) = i(P;\xi,\eta)$. Hence

$$\begin{aligned} r \sum_{\xi \in \mathbb{C}^*/\{\varepsilon^r=1\}} \max_\eta \Big(i(P;\xi,\eta) - t\Big)^+ &= \sum_{\xi \in \mathbb{C}^*} \max_\eta \Big(i(P;\xi,\eta) - t\Big)^+ \\ &\leq \sum_{\xi \in \mathbb{C}^*} \operatorname{ord}_\xi W(x) = \operatorname{ord}_\infty W(x) - \operatorname{ord}_0 W(x) \leq r(t+1)n, \end{aligned}$$

concluding the proof. $\square$

Consider now the $\mathbb{Q}$-vector spaces $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k \supseteq \cdots$ defined by

$$\begin{aligned} V_0 &= \{P : P = \sum_{j=0}^{s} A_j(x^r)x^{e_j}y^j\}, \\ V_k &= \{P : P \in V_0 \quad \textit{and} \quad i(P;1,1) \geq k\}. \end{aligned}$$

Lemma 2.2 *The vector space V_k has dimension*

$$\dim V_k = (s+1)(n+1) - k.$$

Proof We abbreviate ∂_k for $(\partial/\partial x)^k$.

It is clear that $\dim V_0 = (s+1)(n+1)$. Also, we have $\dim V_k/V_{k+1} \leq 1$, because

$$V_{k+1} = \big\{P \in V_k : (\partial_k P)(1,1) = 0\big\}.$$

Thus the lemma follows from the statement that

$$\dim V_{(s+1)(n+1)} = 0.$$

Suppose this is not the case. Then there is a polynomial P, not identically 0, with $i(P;1,1) \geq (s+1)(n+1)$. By Lemma 2.1 we get $(s+1)(n+1) - s \leq (s+1)n$, a contradiction. This completes the proof. $\square$

Our next result gives us a small basis of the vector space V_k.

Lemma 2.3 *There is a basis $\{P_l\}$ of V_k such that*

$$\sum_{l=1}^{(s+1)(n+1)-k} h(P_l) \leq \frac{1}{2}k^2 \log\left(\frac{r(n+1)}{4k}\right) + \frac{3}{4}k^2.$$

Proof Consider

$$P(x,1) = \sum_{j=0}^{s} A_j(x^r)x^{e_j} = \sum_{j=0}^{s}\sum_{h=0}^{n} a_{jh}x^{rh+e_j}.$$

Then V_k can be identified with the subspace of $\{a_{jh}\} \in \mathbb{Q}^{(s+1)(n+1)}$ defined by the linear equations

$$\sum_{j=0}^{s}\sum_{h=0}^{n} a_{jh}\binom{rh+e_j}{i} = 0, \quad \text{for} \quad i = 0, 1, \ldots, k-1,$$

which has codimension k by Lemma 2.2. Let $\mathcal{A}$ be the associated matrix

$$\mathcal{A} = \left[\binom{rh+e_j}{i}\right]_{\substack{i=0,1,\dots,k-1\\(j,h)\in\{0,\dots,s\}\times\{0,\dots,n\}}}.$$

By Lemma 2.2, $\mathcal{A}$ has maximal rank k, therefore it is a submatrix of maximal rank of the matrix

$$\mathcal{B} = \left[\binom{l}{i}\right]_{\substack{i=0,1,\dots,k-1\\l=0,\dots,rn+r-1}}.$$

It follows that $H(\mathcal{A}) \le H(\mathcal{B})$ where $H(\ )$ is the height. In our case, where everything is over $\mathbb{Z}$, the height of $\mathcal{B}$ is given by

$$H(\mathcal{B}) = \sqrt{\sum \det\left[\binom{n_j}{i}\right]^2_{\substack{i=0,\dots,k-1\\j=1,\dots,k}}}$$

where the sum ranges over all k-tuples $0 \le n_1 < n_2 < \cdots < n_k < r(n+1)$ (note that the greatest common divisor of the determinants of all maximal minors of $\mathcal{B}$ is 1).

We have

$$\det\left[\binom{n_j}{i}\right]_{\substack{i=0,\dots,k-1\\j=1,\dots,k}} = \frac{\prod_{h<j}(n_j-n_h)}{1!2!\cdots(k-1)!},$$

as one sees by transforming the determinant into $\det(n_j^i/i!)$ and computing the Vandermonde determinant $\det(n_j^i)$. For the logarithmic height, this gives

$$h(\mathcal{B}) = \frac{1}{2}\log\Bigl(\frac{1}{1!2!\cdots(k-1)!}\sum_{0\le n_1<\cdots<n_k<r(n+1)}\prod_{h<j}(n_j-n_h)^2\Bigr).$$

An exact calculation based on the theory of orthogonal polynomials can be found in Bombieri and Vaaler [5]. Writing for simplicity $N = r(n+1)$, we have

$$h(\mathcal{B}) = \frac{1}{2}\sum_{m=-k+1}^{k-1}(k-|m|)\log\left(\frac{N+m}{k+m}\right) \le N^2 u\left(\frac{k}{N}\right),$$

where[1]

$$\begin{aligned}
u(\theta) &= \frac{1}{4}\theta^2\log\frac{1-\theta^2}{16\theta^2} + \frac{1}{2}\theta\log\frac{1+\theta}{1-\theta} + \frac{1}{4}\log(1-\theta^2)\\
&= \frac{1}{2}\theta^2\log\frac{1}{4\theta} + \frac{3}{4}\theta^2 - \sum_{h=2}^{\infty}\frac{\theta^{2h}}{4(h-1)h(2h-1)}\\
&\le \frac{1}{2}\theta^2\log\frac{1}{4\theta} + \frac{3}{4}\theta^2.
\end{aligned}$$

[1] The series expansion is given incorrectly in [5], p.57 with h^2-2h+2 in place of $(h-1)h$.

To conclude the proof of Lemma 2.3 we simply apply the main theorem in Bombieri and Vaaler [4] which, in our case over the rational field $\mathbb{Q}$, gives the existence of a basis $\{P_l\}$ for V_k such that

$$\sum_{l=0}^{(s+1)(n+1)-k} h(P_l) \le h(\mathcal{A}) \le h(\mathcal{B}). \quad \square$$

Let $a \in K$ and suppose that a is neither 0 nor a root of unity. We fix an rth root $\alpha = a^{1/r}$ and set $\alpha' = \gamma^{-1}\alpha$ with $\gamma \in K$ and $\gamma \neq 0$.

Lemma 2.4 *Let $M \ge 1$. There is an invariant polynomial $P \in V_k$ with rational integral coefficients such that*

$$\sum_{\substack{m=-M \\ m\neq 0}}^{M} \big(i(P;\alpha^{lm},(\alpha')^m) - s\big)^+ \le (s+1)(n+1) - k - 1$$

and

$$h(P) \le \frac{1}{2}\frac{k^2}{(s+1)(n+1)-k}\left[\log\left(\frac{r(n+1)}{4k}\right) + \frac{3}{2}\right].$$

Proof By Lemma 2.3, there is an invariant polynomial $P(x,y) \in V_k$ with rational integral coefficients such that

$$h(P) \le \frac{1}{2}\frac{k^2}{(s+1)(n+1)-k}\left[\log\left(\frac{r(n+1)}{4k}\right) + \frac{3}{2}\right].$$

Lemma 2.1 gives

$$\sum_{m=-M}^{M} \big(i(P;\alpha^{lm},(\alpha')^m) - s\big)^+ \le (s+1)n,$$

while on the other hand $i(P;1,1) \ge k$ because $P \in V_k$. It follows that

$$\sum_{\substack{m=-M \\ m\neq 0}}^{M} \big(i(P;\alpha^{lm},(\alpha')^m) - s\big)^+ \le (s+1)n - k + s,$$

completing the proof. $\square$

3 The Thue–Siegel method

In this section we prove

Lemma 3.1 *Let $v \in M_K$ be a finite place for which*

$$\alpha \in K_v, \quad |\alpha^l - 1|_v < 1 \quad \text{and} \quad |\alpha' - 1|_v < 1.$$

Let also k, s, n be positive integers such that $s < k < (s+1)(n+1)$, $s < r$, and Λ a positive real number such that

$$\log \frac{1}{|1-\alpha^l|_v} \geq \Lambda \quad \text{and} \quad \log \frac{1}{|1-\alpha'|_v} \geq (k-s)\Lambda. \tag{A1}$$

Define $D = \frac{1}{2M}\big((s+1)(n+1) - k - 1\big) + s + 1$. Then we have

$$\begin{aligned}(k - D + 1)\Lambda \;\leq\; & M + \frac{1}{2}\big[(n+1)|l|\, h(a) + s\, h(\gamma)\big] \\ & + \frac{1}{2}\frac{k^2}{(s+1)(n+1)-k}\left[\log\left(\frac{r(n+1)}{4k}\right) + \frac{3}{2}\right] \\ & + D\left[\log\left(\frac{r(n+1)}{D}\right) + 1\right].\end{aligned}$$

Proof Fix m with $-M \leq m \leq M$, $m \neq 0$. We write for simplicity $(\xi, \eta) = (\alpha^{lm}, (\alpha')^m)$, $i_m = i(P; \xi, \eta)$.

Let $P(x,y) = \sum a_{hj} x^{rh+e_j} y^j$ and $Q(x,y) = x^{i_m} (\partial_{i_m} P)(x,y)$. We have

$$Q(x,y) = \sum_{hj} a_{hj} \binom{rh+e_j}{i_m} x^{rh+e_j} y^j$$

whence, setting $\beta = Q(\xi, \eta)$, we have

$$\beta = \sum_{hj} a_{hj} \binom{rh+e_j}{i_m} a^{lhm + [(le_j + j)/r]m} \gamma^{-mj} \in K.$$

The fact that $\beta \in K$ rather than an extension of K is essential for our next argument.

By definition of i_m we have

$$(\partial_{i_m} P)(\xi, \eta) \neq 0,$$

therefore $\beta \neq 0$ and the product formula in K yields

$$\sum_{w \in M_K} \log |\beta|_w = 0.$$

Now we estimate each term $\log|\beta|_w$ as follows.

We have

$$|le_j + j| \le |l|(r-1) + (r-1) = (|l|+1)(r-1) < (|l|+1)r,$$

and since the left-hand side of this inequality is divisible by r we find $|le_j+j| \le |l|\,r$. This gives $|lm|h + [(le_j+j)/r]\,|m| \le |lm|(n+1)$. Hence for every $w \ne v$ we have

$$\begin{aligned} \log|\beta|_w \;\le\; & |lm|(n+1)\log^+|a|_w + |m|s\log^+|1/\gamma|_w \\ & + \varepsilon_w \max_{hj}\log|a_{hj}|_w + \varepsilon_w \log\left|\binom{r(n+1)}{i_m+1}\right|_w \end{aligned}$$

where as usual $\varepsilon_w = [K_w : \mathbb{Q}_w]/[K:\mathbb{Q}]$ if $w \mid \infty$ and $\varepsilon_w = 0$ otherwise. In the proof of the above estimate, we have used the obvious majorization

$$\sum_{hj}\binom{rh+e_j}{i_m} \;\le\; \sum_{k=0}^{rn+r-1}\binom{k}{i_m} \;=\; \binom{r(n+1)}{i_m+1}.$$

If instead $w = v$, we note that since

$$\alpha \in K_v, \quad |\alpha^l - 1|_v < 1 \quad \text{and} \quad |\alpha' - 1|_v < 1,$$

we also have $|\xi|_v = 1$, $|\eta|_v = 1$ and

$$|\xi - 1|_v \le |\alpha^l - 1|_v < 1, \quad |\eta - 1|_v \le |\alpha' - 1|_v < 1.$$

The Taylor series of $Q(x,y)$ with center at $(1,1)$ has rational integral coefficients because $Q(x,y) \in \mathbb{Z}[x,y]$. Moreover, by construction, the polynomial $Q(x,1) = x^{i_m}(\partial_{i_m}P)(x,1)$ has a zero of order $\ge (k-i_m)^+$ at $x=1$, therefore

$$\begin{aligned} |\beta|_v &= |\xi^{i_m}(\partial_{i_m}P)(\xi,\eta)|_v \\ &\le \max\left(|\xi-1|_v^{(k-i_m)^+}, |\eta-1|_v\right) \\ &\le \max\left(|\alpha^l-1|_v^{(k-i_m)^+}, |\alpha'-1|_v\right). \end{aligned}$$

If we combine these estimates with the product formula we find

$$\begin{aligned} \min&\left((k-i_m)^+\log\frac{1}{|\alpha^l-1|_v}, \log\frac{1}{|\alpha'-1|_v}\right) \\ &\le |lm|(n+1)h(a) + |m|s\,h(\gamma) + h(P) + \log\binom{r(n+1)}{i_m+1}. \end{aligned}$$

Using $(x-y)^+ \geq (x-z)^+ - (y-z)^+$ and (A1), this implies the new inequality

$$(k-s)^+\Lambda \;\leq\; |lm|(n+1)\,h(a) + |m|s\,h(\gamma) + h(P) + \log\binom{r(n+1)}{(i_m-s)^+ + s + 1} + (i_m - s)^+\Lambda.$$

We take the average of this inequality for $-M \leq m \leq M$, $m \neq 0$. In view of the easy estimate

$$\log\binom{p}{q} \leq q\log\frac{p}{q} + q$$

we obtain

$$\begin{aligned}(k-s)^+\Lambda \;\leq\; & \frac{M+1}{2}\left[(n+1)|l|\,h(a) + s\,h(\gamma)\right] + h(P) \\ & + \frac{1}{2M}\sum_{\substack{m=-M\\ m\neq 0}}^{M} \big((i_m-s)^+ + s + 1\big)\left[\log\left(\frac{r(n+1)}{(i_m-s)^+ + s + 1}\right) + 1\right] \\ & + \frac{1}{2M}\sum_{\substack{m=-M\\ m\neq 0}}^{M} (i_m - s)^+\Lambda.\end{aligned}$$

In order to bound the right-hand side of this inequality we replace $(i_m - s)^+$ by a positive continuous variable z_m subject to $\sum z_m \leq (s+1)(n+1) - k - 1$ and estimate the maximum using Lagrange multipliers. The maximum is achieved if z_m is constant, hence $z_m + s + 1 = D$ with D as in the statement of the lemma. Since $(k-s)^+ - (D-s-1) \geq k - D + 1$, this completes the proof of Lemma 3.1. $\square$

4 Simplification of the main inequality

In order to apply Lemma 3.1 we make some further assumptions and introduce new variables, with the aim of tidying up the inequality stated in the conclusion of the lemma.

First, we remove the condition that k be a positive integer. To this end, it suffices to note that the right-hand side of our inequality increases in k and D for $4k \leq r(n+1)$; thus we may drop the integrality condition on k, replacing D by $D' = \big((s+1)(n+1) - k\big)/(2M) + s + 1$ throughout and $k+1$ by k in the left-hand side of the inequality. Note also that dropping the integrality condition on k makes condition (A1) even more stringent. This gives

$$\begin{aligned}(k-D')\Lambda \;\le\; & M+\frac{1}{2}\Big[|l|(n+1)h(a)+s\,h(\gamma)\Big]\\ &+\frac{1}{2}\frac{k^2}{(s+1)(n+1)-k}\left[\log\left(\frac{r(n+1)}{4k}\right)+\frac{3}{2}\right]\\ &+D'\left[\log\left(\frac{r(n+1)}{D'}\right)+1\right].\end{aligned} \tag{4.1}$$

Next, we choose $k=\lambda(s+1)(n+1)$. Then

$$D'=(1-\lambda)(s+1)(n+1)/(2M)+s+1.$$

After dividing by $(s+1)(n+1)$ the resulting inequality becomes

$$\begin{aligned}\left(\lambda-\frac{1-\lambda}{2M}-\frac{1}{n+1}\right)\Lambda \;\le\; & \frac{M+1}{2}\left[|l|\frac{h(a)}{s+1}+\frac{s}{s+1}\frac{h(\gamma)}{n+1}\right]\\ &+\frac{1}{2}\frac{\lambda^2}{1-\lambda}\left[\log\left(\frac{r}{4\lambda(s+1)}\right)+\frac{3}{2}\right]\\ &+\frac{1}{2M}\left[\log\left(\frac{2Mr}{s+1}\right)+1\right].\end{aligned} \tag{4.2}$$

Now let

$$r>A\ge h(a),\;\; G\ge h(\gamma),\;\; r=\rho A,\;\; s+1=\sigma A\le r,\;\; n+1=\nu G, \tag{A2}$$

and set

$$\lambda\log\rho=\Lambda;$$

decreasing λ if needed, we also assume that

$$\lambda<1,\qquad 0<\Lambda\le 1. \tag{A3}$$

If we suppose

$$G>\frac{2M}{\nu\lambda}, \tag{A4}$$

which implies $\lambda/(2M)\ge 1/(n+1)$, the above inequality simplifies to

$$\begin{aligned}\left(\lambda-\frac{1}{2M}\right)\lambda\log\rho \;\le\; & \frac{M+1}{2}\left(\frac{|l|}{\sigma}+\frac{1}{\nu}\right)\\ &+\frac{1}{2}\frac{\lambda^2}{1-\lambda}\left[\log\left(\frac{\rho}{4\sigma\lambda}\right)+\frac{3}{2}\right]+\frac{1}{2M}\left[\log\left(\frac{2M\rho}{\sigma}\right)+1\right].\end{aligned} \tag{4.3}$$

This inequality is obtained under assumptions (A1), (A2), (A3), (A4) and the further assumption, implicitly made along the way, that M, σA and νG are integers. We choose[2]

$$M = \lceil 2\lambda^{-2} \rceil. \tag{4.4}$$

With this choice, (4.3) can be replaced by

$$\begin{aligned}\left(\lambda - \frac{1}{4}\lambda^2\right)\lambda \log\rho \;\le\; & \left(\frac{1}{\lambda^2}+1\right)\left(\frac{|l|}{\sigma}+\frac{1}{\nu}\right) \\ & + \frac{1}{2}\frac{\lambda^2}{1-\lambda}\left[\log\left(\frac{\rho}{4\sigma\lambda}\right)+\frac{3}{2}\right] + \frac{\lambda^2}{4}\left[\log\left(\frac{\rho}{\sigma}\frac{2+2\lambda^2}{\lambda^2}\right)+1\right]. \end{aligned}\tag{4.5}$$

We now choose

$$\nu = \frac{1}{G}\left\lceil \frac{G\sigma}{|l|} \right\rceil, \qquad \sigma = A^{-1}\left\lceil \frac{8|l|A}{\lambda^4 \log\rho} \right\rceil; \tag{4.6}$$

note that M, σA and νG are integers, hence our implicit assumption is verified. An easy majorization of the right-hand side of (4.5) shows that

$$\begin{aligned}\left(\lambda - \frac{1}{4}\lambda^2\right)\lambda \log\rho \;\le\; & \left(\frac{1}{\lambda^2}+1\right)\frac{1}{4}\lambda^4\log\rho + \frac{\lambda^2(3-\lambda)}{4(1-\lambda)}\log\rho \\ & + \frac{\lambda^2}{4(1-\lambda)}\left[\log\left(\frac{1+\lambda^2}{8\sigma^3\lambda^4}\right)+4\right]. \end{aligned}\tag{4.7}$$

Since $\sigma \ge 8\lambda^{-4}(\log\rho)^{-1}$, we see that (4.7) implies

$$\begin{aligned}\left(\lambda - \frac{1}{4}\lambda^2\right)\lambda \log\rho \;\le\; & \left(\frac{1}{\lambda^2}+1\right)\frac{1}{4}\lambda^4\log\rho + \frac{\lambda^2(3-\lambda)}{4(1-\lambda)}\log\rho \\ & + \frac{\lambda^2}{4(1-\lambda)}\left[\log\left(8^{-4}(1+\lambda^2)\lambda^8(\log\rho)^3\right)+4\right]. \end{aligned}\tag{4.8}$$

Since $\lambda\log\rho \le 1$, inequality (4.8) yields

$$\begin{aligned}\left(\lambda - \frac{1}{4}\lambda^2\right)\lambda \log\rho \;\le\; & \left(\frac{1}{\lambda^2}+1\right)\frac{1}{4}\lambda^4\log\rho + \frac{\lambda^2(3-\lambda)}{4(1-\lambda)}\log\rho \\ & + \frac{\lambda^2}{4(1-\lambda)}\left[\log(\lambda^5+\lambda^7)-4.317\right]. \end{aligned}\tag{4.9}$$

Note that $\log(\lambda^5+\lambda^7)-4.317 < \log 2 - 4.317 < -3.623 < 0$. Dividing both sides of (4.9) by $\lambda^2\log\rho$ and using the lower bound $1/\log\rho \ge \lambda$ gives

$$1 - \frac{1}{4}\lambda \le (1+\lambda^2)\frac{1}{4} + \frac{3-\lambda}{4(1-\lambda)} - 3.623\,\frac{\lambda}{4(1-\lambda)}$$

[2] We use here the ceiling function $\lceil x \rceil = \min_{n\in\mathbb{Z}}\{n : n \ge x\}$.

and after multiplication by $4(1-\lambda)$ and an easy simplification we find

$$0 \leq -0.623\lambda - \lambda^3 < 0.$$

This is a contradiction, and shows that one of the hypotheses (A1) to (A4), together with the choices (4.4) and (4.6), is untenable. Therefore, (A1) does not hold if we assume (A2), (A3), (A4) and (4.4) and (4.6). Our choice of parameters in (4.4) and (4.6) guarantees that (A2), (A3), (A4) are verified, except possibly for the condition $s+1 \leq r$ in (A2) that must be compatible with our choice of σ in (4.6). Let us assume for the time being that this is the case. Then if we assume the first half of (A1), namely $\log|1-\alpha^l|_v \leq -\Lambda$, we conclude that the second half of (A1) does not hold. Note also that by (4.6) we have

$$\sigma \geq 8|l|\lambda^{-4}(\log\rho)^{-1} \quad \text{and} \quad \nu \geq \sigma/|l| \geq 8\lambda^{-4}(\log\rho)^{-1}; \tag{4.10}$$

therefore, $2\lceil 2\lambda^{-2}\rceil/(\nu\lambda) \geq \frac{1}{2}\lambda\log\rho$ and *a fortiori* (A4) can be replaced by $G \geq \lambda\log\rho$.

If we recall that we had chosen $k = \lambda(s+1)(n+1)$, we conclude that

Proposition 4.1 *Let K, v, r, a, $\alpha = a^{1/r}$, γ be as before. Assume that A, ρ, G, λ satisfy $r > A \geq h(a)$, $\rho = r/A$, $G \geq \max(h(\gamma), \lambda\log\rho)$ and $0 < \lambda < \min(1, 1/\log\rho)$. Suppose further that*

$$\log\left|1-\alpha^l\right|_v \leq -\lambda\log\rho.$$

Let

$$\sigma = A^{-1}\left\lceil \frac{8|l|A}{\lambda^4\log\rho}\right\rceil.$$

Then if $\sigma \leq \rho$ we have

$$\log|1-\gamma^{-1}\alpha|_v > -\lambda^2 \left\lceil\frac{G\sigma}{|l|}\right\rceil \left\lceil\frac{8|l|A}{\lambda^4\log\rho}\right\rceil \log\rho.$$

5 Applications to diophantine approximation in a number field by a finitely generated multiplicative group

As a corollary of Proposition 4.1, we derive in this section improvements of Theorem 1 and Theorem 2 of [3]. As in that paper, we let $K(v)$ be the residue field of K_v and f_v, e_v the residue class degree and ramification index of the extension $K_v/\mathbb{Q}_v$. We abbreviate

$$d_v^* = \frac{d}{f_v \log p}, \qquad D_v^* = \max(1, d_v^*).$$

We assume that $|a|_v = 1$, so that if we choose $l = p^{f_v} - 1$, then $|a^l - 1|_v < 1$. From Lemma 1 of [3] we may suppose that

$$\log \frac{1}{|1 - \alpha^l|_v} = \log \frac{1}{|1 - a^l|_v} \geq \frac{f_v \log p}{d} = \frac{1}{d_v^*} \geq \frac{1}{D_v^*}. \tag{5.1}$$

Continuing with the notations of §4, we suppose that $r > 2A$ and choose

$$\lambda = (D_v^* \log \rho)^{-1}. \tag{5.2}$$

Then we can apply Proposition 4.1 and deduce that

$$\log |1 - \gamma^{-1}\alpha|_v > -\lambda^2 \left\lceil \frac{G\sigma}{|l|} \right\rceil \left\lceil \frac{8|l|A}{\lambda^4 \log \rho} \right\rceil \log \rho \tag{5.3}$$

provided that $G \geq \max(h(\gamma), 1/D_v^*)$ and also $\sigma \leq \rho$.

With the modified height $h'(x)$ defined in the statement of the Main Theorem, the condition on G becomes $G \geq h'(\gamma)$. Our choice for A will be $A = h'(a)$.

For the application we have in mind, r must be relatively large compared to $h'(a)$ if we want a nontrivial conclusion for our final result. Thus to begin with we assume that

$$r > e^4 D_v^* h'(a). \tag{5.4}$$

In particular, $\log \rho \geq 4$.

The next step in simplifying (5.3) consists in removing the brackets in the ceiling function. By (4.10), (5.4), $A \geq 1/D_v^*$ and our choice of λ we have

$$\left\lceil \frac{8|l|A}{\lambda^4 \log \rho} \right\rceil = A\sigma \geq 8|l|(D_v^*)^3 (\log \rho)^3 \geq 512,$$

hence we may remove the brackets at the cost of multiplying by $1 + 1/512$, at most. In a similar way, we have

$$\left\lceil \frac{G\sigma}{|l|} \right\rceil \geq 8G(D_v^*)^3 (\log \rho)^3 \geq 512,$$

because $G \geq 1/D_v^*$. Therefore, the cost of removing the brackets is at most a factor of $1 + 1/512$. Again, removing the brackets from σ will not cost us more than a further factor $1+1/512$. Thus the total cost in this simplification is at most a factor $(1+1/512)^3 < 1.006$.

We can replace $h'(\gamma)$ by $h'(\gamma^{-1}\alpha)$, at a small cost. Indeed, $h(\gamma) \leq h(\gamma^{-1}\alpha) + h(\alpha)$, hence using $h' \geq 1/D_v^*$ we find

$$h'(\gamma) \leq h'(\gamma^{-1}\alpha) + \frac{h(a)}{r} < h'(\gamma^{-1}\alpha) + \frac{1}{e^4 D_v^*} < (1+e^{-4})h'(\gamma^{-1}\alpha). \tag{5.5}$$

Thus the total cost of these simplifications is a factor of at most $1.006 \times (1+e^{-4}) < 1.03$. Therefore, after removing the brackets, taking into account this small correction and making a further rounding off of constants, (5.3) becomes the simpler

$$\log|1-\gamma^{-1}\alpha|_v > -66p^{f_v}(D_v^*)^6 h'(a)\left(\log\frac{r}{h'(a)}\right)^5 h'(\gamma^{-1}\alpha). \tag{5.6}$$

This inequality has been obtained under the assumption that $s+1 \leq r$. If however $s+1 \geq r+1$, we must have

$$\left\lceil \frac{8|l|A}{\lambda^4 \log\rho} \right\rceil = A\sigma = s+1 \geq r+1 = \rho A + 1,$$

hence $8|l| \geq \lambda^4 \rho \log\rho$. With our choice of λ and l, this means that if

$$\rho(\log\rho)^{-3} \geq 8p^{f_v}(D_v^*)^4 \tag{A5}$$

then the condition $\sigma \leq \rho$ in Proposition 1 is verified.

We now summarize our results as follows.

Theorem 5.1 *Let K be a number field of degree d and v an absolute value of K dividing a rational prime p. Let $a \in K$ with a not 0 or a root of unity, and suppose that a satisfies $|a|_v = 1$.*

Let r be a positive integer coprime to p. Then a has an rth root $\alpha \in K_v$ satisfying $0 < |1-\alpha^{p^{f_v}-1}|_v < 1$. Let $\alpha' = \alpha\gamma^{-1}$ with $\gamma \in K$, $\gamma \neq 0$. Let $C = 66\, p^{f_v}\, (D_v^*)^6$ and $0 < \kappa$, and suppose that

$$r \geq \rho\left(\frac{C}{\kappa}\right) h'(a). \tag{H1}$$

Then

$$|\alpha' - 1|_v \geq H'(\alpha')^{-r\kappa}.$$

Moreover, if $|a-1|_v < 1$ then a has an rth root $\alpha \in K_v$ satisfying

$$0 < |\alpha - 1|_v < 1,$$

and (H1) can be further improved by replacing C with the smaller constant $C' = 66(D_v^*)^6$.

Remark 5.1 Before completing the proof of Theorem 5.1, a comparison with the explicit result in [6] is in order. To avoid undue complications, we only consider asymptotic bounds as $h(\gamma) \to \infty$ and $r/h'(a) \to \infty$. Then, with the optimal choice of κ, the bound given by our Theorem 5.1 is

$$\log \frac{1}{|\alpha' - 1|_v} \leq \big(66 + o(1)\big)\, p^{f_v}\, (D_v^*)^6 h'(a) \left(\log \frac{r}{h'(a)}\right)^5 h(\gamma).$$

On the other hand, from [6] we may show that

$$\log \frac{1}{|\alpha' - 1|_v} \leq \big(24 + o(1)\big)\, p^{f_v}\, (d_v^*)^4 h'(a) \left(\log \frac{r}{h'(a)}\right)^2 h(\gamma),$$

which is better than Theorem 5.1. Thus the interest of Theorem 5.1 is more in the method of proof than in the result itself.

Proof By (5.6) it suffices that r be so big that

$$\kappa r \geq C h'(a) \left(\log \frac{r}{h'(a)}\right)^5,$$

that is $\rho(\log \rho)^{-5} \geq C/\kappa$. Note that, with our value for C, this condition takes care of (A5) as soon as $\kappa \leq 8(D_v^*)^2$.

On the other hand, we have the Liouville lower bound

$$|\alpha' - 1|_v \geq (2H'(\alpha'))^{-r},$$

while $H'(\alpha')^{D_v^*} \geq e > 2$, hence in any case we have $|\alpha' - 1|_v \geq H'(\alpha')^{-2D_v^* r}$. This shows that the conclusion of Theorem 5.1 is trivial as soon as $\kappa > 2D_v^*$. Thus condition (A5) is of no consequence for the verification of Theorem 5.1, completing the proof. $\square$

In applications, condition (H1) is the most important. A direct comparison with Theorem 1 of [3] shows a big improvement in the absolute constant of (H1) and a reduction in the power of the logarithmic term from 7 to 5. The condition (H2) of [3] is now eliminated.

Theorem 5.2 *Let K be a number field of degree d and v a place of K dividing a rational prime p.*

Let Γ be a finitely generated subgroup of K^* and let $\xi_1, \ldots, \xi_t$ be generators of Γ/tors. *Let $\xi \in \Gamma$, $A \in K^*$ and $\kappa > 0$ be such that*

$$0 < |1 - A\xi|_v < H'(A\xi)^{-\kappa}.$$

Define

$$C = 66p^{f_v}(D_v^*)^6 \quad \textit{and} \quad Q = \big(2t\rho(C/\kappa)\big)^t \prod_{i=1}^{t} h'(\xi_i).$$

Then we have

$$h'(A\xi) \ \leq \ 16p^{f_v}\rho(C/\kappa)\, Q \ \max\big(h'(A), 4p^{f_v}Q\big)\,.$$

Proof The main idea is to find r coprime to p, and $a \in K$ not a root of unity and $\gamma \in \Gamma$ such that $A\xi = a\gamma^{-r} = (\alpha')^r$, without $h(a)$ being too large and with some control on the range of r. In [3], Lemma 6.2 uses a geometry of numbers argument to show that if $|1 - A\xi|_v < H(A\xi)^{-\kappa}$ we can do this with

$$h'(a) \leq h'(A) + rt\left(Q^{-1}\prod_{i=1}^{t} h'(\xi_i)\right)^{1/t} + \frac{4}{r}\,h(\xi) \tag{5.7}$$

and r in a range[3]

$$\frac{N}{2(p^{f_v}-1)Q} - 1 \leq r \leq N + 3, \tag{5.8}$$

for any $Q \geq (tD_v^*)^t \prod h'(\xi_i)$ and $N \geq 8p^{f_v}D_v^* h'(A)Q$. By (5.8), this lower bound for N implies $r \geq 4$.

Since r and p are coprime, we have $|1 - A\xi|_v = |1 - \alpha'|_v$ for some choice of the rth root α; note also that Lemma 6.2 of [3] also guarantees that α is not a root of unity.

In what follows, we abbreviate ρ for $\rho(C/\kappa)$; note that we must have $\kappa < 2D_v^*$ (see the end of the proof of Theorem 5.1), hence $\rho \geq 33D_v^*$.

Suppose now that

$$r \geq \rho h'(a). \tag{5.9}$$

Then Theorem 5.1 yields

$$|1 - \alpha'|_v \geq H'(\alpha')^{-r\kappa}.$$

This contradicts

$$|1 - \alpha'|_v = |1 - A\xi|_v < H(A\xi)^{-\kappa} = H(\alpha')^{-r\kappa}$$

[3] Lemma 6.2 in [3] has $2\sqrt{2}$ in place of 2, but a more accurate evaluation of constants appearing in Lemma 6.1 of [3] yields the cleaner bound stated here.

unless $H(\alpha') < H'(\alpha')$, *i.e.* $h(\alpha') < 1/D_v^*$ or equivalently

$$h(A\xi) < r/D_v^*. \tag{5.10}$$

We have shown that (5.9) implies the bound (5.10) for $h(A\xi)$. It remains to localize r by choosing Q and N appropriately so as to satisfy the hypothesis $r \geq \rho h'(a)$ of Theorem 5.1.

We begin by choosing Q as

$$Q = (2\rho t)^t \prod_{i=1}^{t} h'(\xi_i), \tag{5.11}$$

which we may because $2\rho t > tD_v^*$.

We need to bound $h'(a)$ and for this we use (5.7). In view of (5.11), $r \geq 4$ and $h(\xi) \leq h(A\xi) + h'(A)$, we have

$$h'(a) \leq h'(A) + \frac{1}{2\rho} r + \frac{4}{r} h(\xi) \leq 2\, h'(A) + \frac{1}{2\rho} r + \frac{4}{r} h(A\xi). \tag{5.12}$$

Now we choose N to be

$$N = \left\lceil 2(p^{f_v} - 1)Q\left(1 + \max\left(8\rho h'(A), \sqrt{16\rho h(A\xi)}\right)\right)\right\rceil .$$

Then (5.8) implies that

$$r \geq \max\left(8\rho h'(A), \sqrt{16\rho h(A\xi)}\right)$$

hence (5.12) yields

$$h'(a) \leq \frac{1}{4\rho} r + \frac{1}{2\rho} r + \frac{1}{4\rho} r = \frac{1}{\rho} r,$$

hence (5.9), and *a fortiori* (5.10), holds with this choice of N.

On the other hand, $r \leq N + 3$ and finally from (5.10) we have

$$h(A\xi) \leq (D_v^*)^{-1} \left\lceil 2(p^{f_v} - 1)Q\left(1 + \max\left(8\rho h'(A), \sqrt{16\rho h(A\xi)}\right)\right)\right\rceil + 3.$$

The first alternative for the maximum easily yields

$$h(A\xi) \leq 16 p^{f_v} \rho h'(A) Q,$$

because $\rho h'(A)Q$ is fairly large (use $\rho \geq 33D_v^*$ to get $\rho h'(A)Q \geq (66t)^t$), hence the small corrections in going from $1 + \max$ to $\max$ and in removing the ceiling brackets and the constant 3 are easily absorbed in replacing $p^{f_v} - 1$ by p^{f_v}.

The second alternative for the maximum yields

$$h(A\xi) \leq 2p^{f_v} Q\sqrt{16\rho h(A\xi)}$$

and finally

$$h(A\xi) \leq 64 p^{2f_v} \rho Q^2,$$

completing the proof of Theorem 5.2. $\square$

6 Appendix: from a private communication by David Masser

In this appendix, we reproduce material from a letter of David Masser to the first author dated 8th January 1984. These ideas of Masser inspired our §2 and are reproduced here with his permission.

" ... My own method was based on zero estimates rather than heights, using a 'dividing out' trick from transcendence. It gives the following general result.

Theorem *Suppose θ is algebraic of degree $d \geq 2$ and of absolute height $H \geq 1$. Fix an integer e with*

$$1 \leq e < d$$

and real ε with

$$0 < \varepsilon < \frac{1}{e+1}.$$

Put

$$\delta = \frac{d}{e+1} + \varepsilon, \qquad \alpha = \frac{d\delta}{(e+1)\varepsilon},$$
$$\beta = d\delta + \alpha, \qquad \gamma = 1 - (e+1)\varepsilon.$$

Suppose the integers $p_0, q_0 \geq 1$ satisfy

$$\Lambda = (4H)^{-\beta} q_0^{-\delta} \left| \theta - \frac{p_0}{q_0} \right|^{-\gamma} > 1.$$

Then the effective strict type of θ is at most

$$\frac{-e\gamma \log |\theta - \frac{p_0}{q_0}|}{\log \Lambda}.$$

I didn't try to improve the constant 4, although this could certainly be done by using asymptotics for binomial coefficients.

The proof can be expressed in three lemmas, where $c_1, c_2, \ldots$ denote constants depending only on d, H, ε. For $P(x, y)$ in $\mathbb{C}[x, y]$ write as in Siegel's set-up

$$P_l(x, y) = \frac{1}{l!} \left(\frac{\partial}{\partial x} \right)^l P(x, y).$$

Lemma 1 *For each $k \geq 1$ there exists a nonzero polynomial $P(x,y)$ in $\mathbb{Z}[x,y]$, of degree at most δk in x and at most e in y, with coefficients of absolute value at most $c_1(4H)^{\alpha k}$, such that*

$$P_l(\theta,\theta) = 0, \qquad (0 \leq l < k).$$

Furthermore $P(x,y)$ is not divisible by any nonconstant element of $\mathbb{C}[y]$.

Without the last sentence this is routine (I myself like to use the version of Siegel's Lemma proved as the Proposition (p. 32) of the enclosed offprint[4]). Then one simply divides $P(x,y)$ by its greatest monic factor in $\mathbb{C}[y]$. It is not hard to see that the resulting quotient also satisfies the conditions of the lemma.

Lemma 2 *Suppose $k \geq e$, and let ξ, η be arbitrary numbers with ξ not a conjugate of θ. Then there exists l with*

$$0 \leq l \leq (e+1)\varepsilon k + ed$$

such that

$$P_l(\xi,\eta) \neq 0.$$

Again the proof is essentially routine, on taking a minimal representation

$$P(x,y) = A_0(x)B_0(y) + \cdots + A_f(x)B_f(y).$$

The point is that

$$B_0(\eta) = \cdots = B_f(\eta) = 0$$

is impossible by the last sentence of Lemma 1. This is the step usually done by Gauss's Lemma. It is interesting that the Dyson Lemma appears to give only $\sqrt{2e\delta - d}$ in place of $(e+1)\varepsilon$ multiplying k.

Lemma 3 *Suppose $k \geq ed/\gamma$ and let p_0, q_0, p, q be integers with $q_0 \geq 1$, $q \geq 1$. Then we have*

$$q_0^{-\delta k} q^{-e} \leq c_2(4H)^{\beta k}\left(\left|\theta - \frac{p_0}{q_0}\right|^{\gamma k - ed} + \left|\theta - \frac{p}{q}\right|\right).$$

[4] M. Anderson, D. W. Masser, *Lower bounds for heights on elliptic curves*, Math. Z. **174** (1980) 23–34

This follows from a straightforward comparison of estimates for $P_l(\frac{p_0}{q_0}, \frac{p}{q})$ with l chosen as in Lemma 2.

The Theorem now follows by taking k asymptotic to $e \log q / \log \Lambda$. The usual ineffective arguments give any exponent

$$\lambda > \frac{d}{e+1} + e$$

as in Siegel. The optimal choice $e = 10$, $\varepsilon = \frac{\sqrt{2}-1}{11}$ gives any exponent

$$\lambda > \frac{55}{14}\left(4 + \sqrt{2}\right) = 21.270\ldots$$

for the real root $\theta(m, d)$ of $x^d - mx^{d-1} + 1 = 0$ provided $d \geq d_0(\lambda)$ and $m \geq m_0(d)$.

I briefly looked at a similar approach in the Gelfond–Dyson set-up, with a fixed integer t and derivatives $(\partial/\partial x)^l(\partial/\partial y)^s P(x, y)$ for

$$\frac{l}{k} + \frac{s}{t} < 1$$

But even if the analogous zero estimate could be made to work, it seems as if $t = 1$ (i.e. Siegel) gives the best results for $\theta(m, d)$. So I didn't try too hard with this."

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