A Diophantine system

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Dedicated to Peter Swinnerton-Dyer on the occasion of his seventy-fifth birthday

This note concerns the Diophantine system

$$\begin{aligned} x_1^2+x_2^2+x_3^2 &= y_1^2+y_2^2+y_3^2 \\ x_1^3+x_2^3+x_3^3 &= y_1^3+y_2^3+y_3^3 \\ x_1^4+x_2^4+x_3^4 &= y_1^4+y_2^4+y_3^4 \end{aligned} \tag{1}$$

which represents a surface of degree 24. The system is of interest in that Palamà [7] in 1951 showed that the only *real* points on (1) in the positive quadrant ($x_i > 0$, $y_i > 0$ for $i = 1, 2, 3$) are trivial points, that is, points where (x_1, x_2, x_3) is a permutation of (y_1, y_2, y_3). Geometrically, the only real points on (1) in the positive quadrant lie upon a finite number of planes. Choudhry [2] discovers the nontrivial rational point (which we also refer to as a nontrivial *solution*)

$$(x_1, x_2, x_3; y_1, y_2, y_3) = (358, -815, 1224; -776, 1233, -410),$$

and we observe that this point also satisfies $x_1 + x_2 + y_1 + y_2 = 0$. In this note, we investigate the section of the surface (1) cut by the plane

$$x_1 + x_2 = t(y_1 + y_2) \tag{2}$$

for $t = \pm 1$. There are only finitely many nontrivial points on the section with $t = 1$, but infinitely many nontrivial points on the section with $t = -1$.

First, make the substitution

$$\begin{aligned} x_1 &= a_1 m + b_1 n, & y_1 &= a_1 m - b_1 n \\ x_2 &= a_2 m + b_2 n, & y_2 &= a_2 m - b_2 n \\ x_3 &= a_3 m + b_3 n, & y_3 &= a_3 m - b_3 n \end{aligned} \tag{3}$$

where $mn \neq 0$, so that (2) above becomes

$$m(t-1)(a_1 + a_2) = n(t+1)(b_1 + b_2).$$

Case I: $t = 1$

Then $b_2 = -b_1$, and substituting (3) into (1) gives:

$$\begin{aligned} a_1b_1 - a_2b_1 + a_3b_3 &= 0 \\ 3(a_1^2b_1 - a_2^2b_1 + a_3^2b_3)m^2 + b_3^3n^2 &= 0 \\ (a_1^3b_1 - a_2^3b_1 + a_3^3b_3)m^2 + (a_1b_1^3 - a_2b_1^3 + a_3b_3^3)n^2 &= 0. \end{aligned} \tag{4}$$

Nontrivial solutions demand $a_1 \neq a_2$, $a_3 \neq 0$. Eliminating m, n in (4) and using $b_3 = (-a_1 + a_2)b_1/a_3$ gives

$$\begin{aligned} &a_1^4 - a_1^3a_2 - a_1a_2^3 + a_2^4 - 3a_1^3a_3 + 3a_1^2a_2a_3 + 3a_1a_2^2a_3 \\ &\qquad - 3a_2^3a_3 + 2a_1^2a_3^2 - 4a_1a_2a_3^2 + 2a_2^2a_3^2 + 3a_1a_3^3 + 3a_2a_3^3 - 3a_3^4 = 0. \end{aligned}$$

This quartic curve is singular at the point $(a_1, a_2, a_3) = (1, 1, 0)$, and has genus 2. Put

$$a_1 = u, \quad a_2 = u - v, \quad a_3 = w$$

to give

$$3u^2v^2 - 3(v^3 + 2v^2w - 2w^3)u + (v^2 - w^2)(v^2 + 3vw + 3w^2) = 0. \tag{5}$$

The discriminant (as function of u) being square implies

$$-3(v^6 - 4v^4w^2 + 12v^2w^4 - 12w^6) = \text{square}. \tag{6}$$

This latter curve of genus 2 has of course only finitely many rational points. Its Jacobian is isogenous to the product of the two elliptic curves

$$\begin{aligned} E_1 : \quad -3(V^3 - 4V^2 + 12V - 12) &= S_1^2, \\ E_2 : \quad -3(1 - 4W + 12W^2 - 12W^3) &= S_2^2, \end{aligned}$$

both of which are of rank 1 (with generators $(0, 6)$ and $(1, 3)$ respectively). Accordingly, Chabauty's method (see for example Coleman [3]) for determining the rational points on (6) does not apply. It is possible that methods of Flynn and Wetherell [6] may be effective, but we have not pursued the calculation; see also Bruin and Elkies [1]. In any event, there are only finitely many solutions of the system (1) satisfying $x_1 + x_2 = y_1 + y_2$, and their determination is afforded by finding all rational points on the curve (6). A modest computer search finds only the points $(u, v, w) = (1, 0, 0)$, $(1, 0, 2)$, $(0, 1, 1)$, $(1, 1, 1)$ on (5), corresponding to trivial solutions of (1).

Case II: $t = -1$

Now $a_1 + a_2 = 0$, and

$$\begin{aligned} a_1b_1 - a_1b_2 + a_3b_3 &= 0, \\ 3(a_1^2b_1 + a_1^2b_2 + a_3^2b_3)m^2 + (b_1^3 + b_2^3 + b_3^3)n^2 &= 0, \\ (a_1^3b_1 - a_1^3b_2 + a_3^3b_3)m^2 + (a_1b_1^3 - a_1b_2^3 + a_3b_3^3)n^2 &= 0. \end{aligned} \tag{7}$$

Nontrivial solutions demand $b_1 \neq b_2$, $a_3 \neq 0$. Eliminating m, n at (7) and using $b_3 = (-b_1 + b_2)a_1/a_3$ gives

$$a_1(b_1 - b_2)(a_1b_1 - a_3b_1 - a_1b_2 - a_3b_2)P(a_1, a_3, b_1, b_2) = 0,$$

where

$$\begin{aligned} P(a_1, a_3, b_1, b_2) = {} & a_1^4b_1^2 - 2a_1^3a_3b_1^2 + 2a_1a_3^3b_1^2 - a_3^4b_1^2 - 2a_1^4b_1b_2 \\ & - 8a_1^2a_3^2b_1b_2 + a_3^4b_1b_2 + a_1^4b_2^2 + 2a_1^3a_3b_2^2 - 2a_1a_3^3b_2^2 - a_3^4b_2^2. \end{aligned}$$

Consequently, either $a_1 = 0$, or $b_1 = b_2$, or $(a_1 - a_3)b_1 = (a_1 + a_3)b_2$, all of which lead to trivial solutions of the original system, or

$$(a_1 - a_3)^3(a_1 + a_3)b_1^2 - (2a_1^4 + 8a_1^2a_3^2 - a_3^4)b_1b_2 + (a_1 - a_3)(a_1 + a_3)^3b_2^2 = 0.$$

The discriminant of the latter is

$$3a_3^2(2a_1^2 + a_3^2)^2(4a_1^2 - a_3^2)$$

which accordingly is square precisely when

$$4a_1^2 - a_3^2 = 3 \times (\text{square}).$$

Put

$$a_1 = 3u^2 + v^2 \quad \text{and} \quad a_3 = 6u^2 - 2v^2,$$

so that (without loss of generality, on changing the sign of v if necessary)

$$\frac{b_1}{b_2} = -\frac{(u - v)(3u + v)^3}{3(3u - v)(u + v)^3}$$

and

$$\frac{b_3}{b_2} = \frac{2(3u^2 + v^2)(3u^2 + 2uv + v^2)}{3(3u - v)(u + v)^3}.$$

Then from (7),

$$-9(3u - v)^2(u + v)^6m^2 + b_2^2(9u^4 - 24u^3v - 26u^2v^2 - 8uv^3 + v^4)n^2 = 0,$$

that is,

$$U^2 = V^4 - 8V^3 - 26V^2 - 24V + 9, \tag{8}$$

where

$$V = v/u, \qquad U = \frac{3(3u-v)(u+v)^3}{b_2 u^2} m/n.$$

A Weierstrass model for (8) is

$$y^2 = x^3 + x^2 - 4x + 32, \tag{9}$$

and using the APECS program [4] of Ian Connell, or the tables of Cremona ([5]), where (9) is numbered as the curve 552E1, we discover that (8) is of rank 1, with generator $P(V,U) = (-9/4, -111/16)$. Accordingly, we can construct infinitely many rational points (equivalently, integer points) on (1). Indeed, from

$$\begin{gathered} a_1 = 3u^2 + v^2, \qquad a_2 = -3u^2 - v^2, \qquad a_3 = 6u^2 - 2v^2, \\ b_1 = -(u-v)(3u+v)^3, \qquad b_2 = 3(3u-v)(u+v)^3 \\ \text{and} \quad b_3 = 2(3u^2+v^2)(3u^2+2uv+v^2), \end{gathered}$$

with

$$U^2 = V^4 - 8V^3 - 26V^2 - 24V + 9, \qquad V = \frac{v}{u}, \qquad U = \frac{1}{u^2}\, m/n,$$

we obtain

$$\begin{aligned} x_1 &= (3+V^2)U - 27 + 18V^2 + 8V^3 + V^4, \\ x_2 &= -(3+V^2)U + 9 + 24V + 18V^2 - 3V^4, \\ x_3 &= 2(3-V^2)U + 18 + 12V + 12V^2 + 4V^3 + 2V^4, \\ y_1 &= (3+V^2)U + 27 - 18V^2 - 8V^3 - V^4, \\ y_2 &= -(3+V^2)U - 9 - 24V - 18V^2 + 3V^4, \\ y_3 &= 2(3-V^2)U - 18 - 12V - 12V^2 - 4V^3 - 2V^4. \end{aligned} \tag{10}$$

The point $P(V,U) = (-9/4, -111/16)$ pulls back to the solution

$$(-815, 358, 1224; -776, 1233, -410),$$

and the point $2P(V,U) = (-148/33, 29219/1089)$ to the solution

$$(378382959, -931219912, -156845590; 357088490, 195748463, -932263416).$$

References

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