MA4J8 Commutative algebra II

6 Chapter 6. Cohen–Macaulay and Gorenstein

6.1 Depth is controlled by Ext^i vanishing in initial range

I assume the above prerequisites. This is based on [Ma, Th 16.6], although I find it clearer to start from the zero-dimensional case, that is the base of the proof by induction:

Lemma 6.1 *Let A be a Noetherian ring with ideal I. Assume that finite A-modules M and N satisfy*

(1) $M \neq 0$ *and* $I\text{-depth}\, M = 0$.

(2) N *has* $\operatorname{Supp} N = V(I)$. *(That is, the prime ideals P with $N_P \neq 0$ are those with $P \supset I$.)*

Then $\operatorname{Hom}(N, M) \neq 0$.

In the converse direction, if $I\text{-depth}\, M \geq 1$ *and* $\operatorname{Supp} N \subset V(I)$ *then* $\operatorname{Hom}(N, M) = 0$.

Lemma 6.1 is the case $n = 0$ of [Ma, Th 16.6].

Proof (1) is the statement that every $f \in I$ is a zerodivisor of M. Then $I \subset \bigcup P$ taken over all $P \in \operatorname{Ass} M$, a finite set, and by prime avoidance $I \subset P$ for some $P \in \operatorname{Ass} M$. Therefore M contains A/P as a submodule. The localisation M_P contains a copy of the residue field $k(P) = A_P/(PA_P)$.

On the other hand, (2) implies that $N_P \neq 0$, so also $N_P/(PN_P) \neq 0$ by Nakayama's lemma. Now $N_P/(PN_P)$ is a nonzero vector spaces over $k(P)$, and M_P contains a copy of $k(P)$, so there exists a nonzero $k(P)$-linear map $N_P/(PN_P) \to M_P$.

Thus $\operatorname{Hom}_{A_P}(N_P, M_P) \neq 0$, and since this equals the localisation at P of $\operatorname{Hom}_A(N, M)$, it follows that $\operatorname{Hom}_A(N, M) \neq 0$.

For the converse, recall $\operatorname{Supp} N \subset V(I)$ means that $I \subset \operatorname{rad}(\operatorname{ann} N)$ [UCA, 7.1]. Thus every $s \in I$ has nilpotent action on N. If some $s \in I$ is a nonzerodivisor for M, it follows that $\operatorname{Hom}_A(N, M) = 0$. □

Theorem 6.2 *Let A be a Noetherian ring with ideal I. Let M be a finite A-module. Write $\operatorname{Ext}^i(N, M)$ for Ext^i_A. Equivalent conditions:*

(0) $I\text{-depth}\, M \geq n$.

(1) $\operatorname{Ext}^i(N, M) = 0$ *for all* $i \le n-1$ *and for all* N *with* $\operatorname{Supp} N \subset V(I)$.

(2) $\operatorname{Ext}^i(A/I, M) = 0$ *for all* $i \le n-1$.

(3) $\operatorname{Ext}^i(N, M) = 0$ *for all* $i \le n-1$ *and for some* N *with* $\operatorname{Supp} N = (I)$.

The case $n = 0$ is Lemma 6.1. If $n \ge 1$ there is an M-regular element $s_1 \in I$, so work with the s.e.s.

$$0 \to M \xrightarrow{s_1} M \to \overline{M} \to 0, \quad \text{where } \overline{M} = M/(s_1 M),$$

and its long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(N,M) \to \operatorname{Hom}(N,M) \to \operatorname{Hom}(N,\overline{M}) \xrightarrow{\delta} \\ \to \operatorname{Ext}^1(N,M) \to \operatorname{Ext}^1(N,M) \to \operatorname{Ext}^1(N,\overline{M}) \cdots . \end{aligned} \tag{6.1}$$

First assume (0). The quotient $\overline{M}$ has depth $\ge n-1$, so $\operatorname{Ext}^i(N, \overline{M}) = 0$ for $i < n-1$ by induction. In the long exact sequence of Exts, this gives

$$0 \to \operatorname{Ext}^{n-1}(N,M) \xrightarrow{s_1} \operatorname{Ext}^{n-1}(N,M) \tag{6.2}$$

so that s_1 is injective.

Now multiplication by s_1 on N is nilpotent because $\operatorname{Supp} N \subset V(I)$. The s_1 in (6.2) can be viewed[1] as the contravariant functor applied to $N \xrightarrow{s_1} N$, so that the multiplication by s_1 on Ext^{n-1} is also nilpotent. As in Lemma 6.1, if a map is both injective and nilpotent, the module it act on is zero.

This proves (0) implies (1), and (2), (3) are trivial.

The proof that (3) implies (0) is a straightforward induction on n. If M satisfies (3) then $\operatorname{Ext}^i(N, \overline{M})$ is sandwiched between $\operatorname{Ext}^{i-1}(N, M)$ and $\operatorname{Ext}^i(N, M)$ in the Ext long exact sequence (6.1), so is zero for all $i \le n-2$. Therefore $\overline{M}$ has depth $\ge n-1$ by induction, so M has depth $\ge n$. $\square$

Corollary 6.3 *Let A be a Noetherian ring, I an ideal and M a finite module with $IM \ne M$. Then I-depth M is determined as the length of any maximal regular sequence in I, or by*

$$I\text{-}\operatorname{depth} M = \inf\{i \mid \operatorname{Ext}^i(A/I, M) \ne 0\}. \tag{6.3}$$

Two ideals I_1, I_2 have $V(I_1) = V(I_2)$ if and only if $\operatorname{rad}(I_1) = \operatorname{rad}(I_2)$, and then Theorem 16.6 gives I_1-depth $M = I_2$-depth M.

[1] This argument uses compatibility between Ext as a covariant functor in M, and as a contravariant functor in N. To spell that out: the s_1 in (6.2) originally came from the covariant functor in M. That is, $\operatorname{Ext}^{n-1}(N, M)$ is $H_{n-1}(\operatorname{Hom}(N, I_\bullet))$ where $M \to I_\bullet$ is an injective resolution; then $s_1 \colon M \to M$ gives rise to $s_{1\bullet} \colon I_\bullet \to I_\bullet$. Now in the context of $\operatorname{Ext}^\bullet(N, M)$, multiplication by s_1 on $M \to I_\bullet$ has the same effect as s_1 acting on N by premultiplication. That is $N \xrightarrow{\alpha} M \xrightarrow{s_1} M$ is the same map as $N \xrightarrow{s_1} N \xrightarrow{\alpha} M$. And the same for homs into $I_\bullet$.

Ischebek's theorem This is not really new. It rewrites the conclusion of Theorem 16.6 for a local Noetherian ring A, m, replacing the restrictions on $\operatorname{Supp} M$ in (1–3) with conditions on the dimension of $A/\operatorname{ann} N$ – this involves an appeal to Krull's Hauptidealsatz or the $\delta = \dim$ implication of the main theorem on dimension.

Lemma 6.4 *Let A, m be a local Noetherian ring. Assume that $\dim N \le d$ and m-$\operatorname{depth} M \ge n$.*

Then $\operatorname{Ext}^i(N, M) = 0$ for $i + d < n$.

Choose a composition sequence $0 \subset N_1 \subset \cdots \subset N_{r-1} \subset N$ with each $N_j/N_{j-1} = A/P_j$ having $\dim A/P_j \le d$. The exact sequences

$$\cdots \to \operatorname{Ext}^i(N_{j-1}, M) \to \operatorname{Ext}^i(N_j, M) \to \operatorname{Ext}^i(A/P_j, M) \to \cdots$$

express $\operatorname{Ext}^i(N, M)$ as a successive extension of the $\operatorname{Ext}^i(A/P_j, M)$, so it is enough to prove that

$$\operatorname{Ext}^i(A/P, M) = 0 \quad \text{for} \quad i < n - \max\dim(A/P_j).$$

Set $P = P_j$ and work by induction on $d = \dim A/P$. For a prime ideal P of a local ring A, m, if $d = 0$ then $P = m$, and Theorem 6.2 gives $\operatorname{Ext}^i(A/I, M) = 0$ for $i < n$ as required.

For P with $\dim A/P > 0$ there is an $x \in m \setminus P$ (a nonzerodivisor of the integral domain A/P). Consider

$$0 \to A/P \xrightarrow{x} A/P \to A/(P, x) \to 0$$

Now $\dim(A/(P, x) = \dim(A/P) - 1 \le d - 1$ by dimension theory (the Hauptidealsatz). By induction, this gives $\operatorname{Ext}^i(A/(P, x), M) = 0$ for $i + d - 1 \le n$. By the long exact sequence of Exts

$$\begin{aligned} &\cdots \to \operatorname{Ext}^i(A/(P, x), M) \to \operatorname{Ext}^i(A/P, M) \xrightarrow{x} \operatorname{Ext}^i(A/P, M) \to \\ &\to \operatorname{Ext}^{i+1}(A/(P, x), M) \to \cdots, \end{aligned}$$

multiplication by x is surjective on $\operatorname{Ext}^i(A/P, M)$ for $i + d \le n$. Nakayama's lemma then implies that $\operatorname{Ext}^i(A/P, M) = 0$, which proves the result.

Corollary 6.5 *A finite module M over a Noetherian local ring A, m has m-$\operatorname{depth} M \le \dim M$.*

System of parameters and regular sequences Let A, m be local. Recall one of the characterisations of dimension: a *system of parameters* (s.o.p.) is a sequence $x_1, \dots, x_n \in m$ that generates an m-primary submodule. This means that $A/(x_1, \dots, x_n)$ is an Artinian quotient ring, so of finite length or zero dimensional. We set $\delta(A) =$ minimum length of a s.o.p., and eventually proved that $\delta(A) = \dim A$.

We define A to be Cohen–Macaulay if it has $m\text{-depth}\, A = n = \dim A$. Thus A has a regular sequence in m that is a s.o.p. In geometric terms, we can cut A down by a regular sequence to an Artinian quotient ring, with each step the quotient by a principal ideal.

Definition 6.6 (Cohen–Macaulay) A nonzero finite A-module M over a Noetherian local ring A, m is *Cohen–Macaulay* if $m\text{-depth}\, M = \dim M$. The local ring A is a CM ring if it is CM as an A-module.

The module $A/(x_1, \dots, x_n)$ depends (of course) on the regular s.o.p. we choose – for example, we should be able to do the exercise of proving that $A/(x_1^s, x_2, \dots, x_n)$ has length s times the length of $A/(x_1, x_2, \dots, x_n)$. However, the condition that the s.o.p. be a regular sequence is independent of the choice. If one s.o.p. is a regular sequence, so is every other.

The length, or dimension over $k = A/m$ of the final Hom module

$$\mathrm{Hom}_A(k, A/(x_1, x_2, \dots, x_n)) = \mathrm{Ext}^n_A(k, A).$$

is also independent of the choice of s.o.p.

Macaulay unmixedness (1912) Cohen–Macaulay rings and modules have miraculous properties:

Corollary 6.7 *Let A be a Cohen–Macaulay ring of dimension n and $I = (x_1, \dots, x_r)$ an ideal generated by r elements. Then A/I has dimension $n - r$ if and only if $(x_1, \dots, x_r)$ is a regular sequence, and the quotient ring A/I is again Cohen–Macaulay.*

If M is a Cohen–Macaulay A-module then every $P \in \mathrm{Ass}\, M$ has the same height, dimension and depth:

$$\dim(A/P) = \dim M = \operatorname{depth} M.$$

If M is a Cohen–Macaulay A-module and $x_1, \dots, x_r$ a regular sequence for M then the quotient $M/(x_1, \dots, x_r)M$ is again CM.

If M is Cohen–Macaulay then M_P is a CM module over A_P for every P in $\mathrm{Supp}\, M$, and they all have the same depth

$$P\text{-}\operatorname{depth} M = \operatorname{depth}_{M_P} M_P.$$

Lots of other easy corollaries [Ma, Sect 17] on the theme of unmixedness. [Macaulay 1912] worked with graded polynomial rings, [Cohen 1946] proved the same result for regular local rings.

6.2 Start of Gorenstein: the 0-dimensional case

Throughout this section, (A, m, k) is local Artinian. Recall that this implies $\operatorname{Spec} A = \{m\}$. Define the *socle* of an A-module M to be the submodule

$$\operatorname{Socle} M = \{x \in M \mid mx = 0\}.$$

It is the biggest k-vector space of M. When we view $k = A/m$ as an A-module, the socle is identified with $\operatorname{Hom}_A(k, M)$ – in fact, an A-homomorphism $\varphi\colon k \to M$ must take $1 \in k$ to an element $\varphi(1) \in \operatorname{Socle} M$, and $\varphi(1)$ determines φ.

Example 6.8 Start with $A = k[x, y]/(x^{n+1}, y^{m+1})$. It is a finite dimensional k-vector spaces with basis the monomials $x^i y^j$ for $i \le n, j \le m$. It is also a local ring with maximal ideal (x, y). For $i < n$ multiplication by x is nonzero on $x^i y^j$, and for $j < m$ multiplication by y is nonzero. Therefore $\operatorname{Socle} A$ is the submodule $k \cdot x^n y^m$, and *is* 1*-dimensional as* k*-vector space.*

It follows that multiplication of monomials

$$\begin{aligned}(f, g) &\mapsto \text{coefficient of } x^n y^m \text{ in } fg \\ &= fg \bmod m\end{aligned} \tag{6.4}$$

defines a k-bilinear perfect pairing $A \times A \to k$, with $\{x^{n-i} y^{m-j}\}_{i,j}$ the dual basis of $A^\vee$ to the basis $\{x^i y^j\}_{i,j}$ of A. Here $A^\vee = \operatorname{Hom}_A(A, k)$ (made into an A-module by premultiplication). Under this pairing, premultiplication in $A^\vee$ by x, y is the dual or transpose map of multiplication by x, y in A. Thus A has the vector space basis, with x and y mapping to the right and up, whereas the dual basis of $A^\vee$ has dual multiplication maps pointing left and down:

$$\begin{array}{ccccccccc}
y^m & \to & xy^m & \to & & & \cdots & & x^n y^m \\
\uparrow & & \uparrow & & & & & & \uparrow \\
\vdots & & \vdots & & & & & & \vdots \\
y & \to & xy & \to & & & \cdots & \to & x^n y \\
\uparrow & & \uparrow & & & & & & \uparrow \\
1 & \to & x & \to & x^2 & \to & \cdots & \to & x^n
\end{array} \tag{6.5}$$

For a 0-dimensional local ring, Gorenstein is a condition that makes sense of this kind of self-duality in the slightly more general context, when monomial basis in the sense of linear algebra over a field is not meaningful.

Also, dual to[2] the statement that A is projective, $A^\vee$ *is an injective A-module.*

6.3 Self duality and self-injectivity of an Artinian ring

The standard textbook result is that an Artinian local ring A, m, k is injective as a module over itself if and only if its socle is 1-dimensional as a k-vector space.

Standard proof An Artinian ring is of finite length, so has a Jordan–Hölder sequence

$$0 \subset N_1 \subset \cdots N_{i-1} \subset N_i \subset \cdots N_{r-1} \subset A \tag{6.6}$$

with $N_i/N_{i-1} = k$ for each i. The last module N_{r-1} is necessarily the maximal ideal $N_{r-1} = m$ and the first N_1 is a 1-dimension k-vector subspace of the socle.

There are r steps, and at each the s.e.s. $0 \to N_{i-1} \to N_i \to k \to 0$ gives a long exact sequence

$$\begin{aligned} &0 \to \operatorname{Hom}_A(k, A) \to \operatorname{Hom}_A(N_i, A) \to \operatorname{Hom}_A(N_{i-1}, A) \\ &\xrightarrow{\delta_i} \operatorname{Ext}^1_A(k, A) \to \operatorname{Ext}^1_A(N_i, A) \to \cdots \end{aligned} \tag{6.7}$$

All the modules here have finite length, and (6.7) gives

$$\ell(\operatorname{Hom}_A(N_i, A)) - \ell(\operatorname{Hom}_A(N_{i-1}, A)) = \ell(\operatorname{Hom}_A(k, A)) - \ell(\operatorname{im}(\delta_i)). \tag{6.8}$$

Summing over i gives that

$$\operatorname{Hom}_A(A, A) = r \times \dim_k \operatorname{Hom}_A(k, A) - \sum \ell(\operatorname{im}(\delta_i)). \tag{6.9}$$

Now we are assuming that $\operatorname{Hom}_A(A, A) = A$ has length r. We also know that at the top end, $\operatorname{Ext}^1(N_r, A) = 0$, because $N_r = A$ is a free A-module.

Proof of $\Rightarrow$ If $\dim \operatorname{Socle} A = 1$ then (6.9) implies that all the $\delta_i = 0$. Together with $\operatorname{Ext}^1(N_r, A) = 0$ this gives that $\operatorname{Ext}^1_A(k, A) = 0$, and hence A is an injective module by Baer's criterion.[3]

[2] I believe there is more to say on this topic. Saying that injective is "categorically" dual conceals the possibility of a more substantive module-theoretic duality.

[3] Recall that Baer's criterion states that an A-module M is injective if and only if $\operatorname{Ext}^1_A(A/I, M) = 0$ for every M and every ideal I. If A is Noetherian, this can be reduced to requiring that $\operatorname{Ext}^1_A(A/P, M) = 0$ for every $P \in \operatorname{Spec} A$. In our case, there is only one prime $\operatorname{Spec} A = \{m\}$.

Proof of $\Leftarrow$ Injective means that $\operatorname{Ext}^1_A(M, A) = 0$ so that all the $\delta_i = 0$, and then (6.9) obviously implies $r = 1$.

Self-duality

A third equivalent condition is a more precise form of self-duality: Any increasing JH sequence

$$0 \subset \cdots \subset N_{i-1} \subset N_i \subset \cdots \subset A$$

gives a decreasing sequence

$$A \supset \cdots \supset N_{i-1}^\perp \supset N_i^\perp \supset \cdots \supset 0$$

that is also a JH sequence.

Here $N_i^\perp = \operatorname{ann}(N_i) = [0 : N_i]$ is the annihilator ideal of N_i in A. The inclusion $N_{i-1} \subset N_i$ makes $N_i^\perp \subset N_{i-1}^\perp$ a tautology, but there is no a priori reason why it should always be nontrivial and of relative length 1. The argument following (6.9) means that the single condition that the $\ell(\operatorname{Socle} A) = 1$ already guarantees this.

Dual basis

If we assume also that A is a k-algebra (and $k \subset A$ and $A/m = k$, with the same k), then the symmetric bilinear map $A \times A \to k$ given by multiplication $(a, b) \mapsto ab \bmod m$ is a perfect pairing, so that A and $A^\vee = \operatorname{Hom}_k(A, k)$ are isomorphic. There is a dual k-basis as in the above example $k[x, y]/(x^{n+1}, y^{m+1})$.

In the more general case, each step $N_{i-1} \subset N_i$ is given by adding one new generator n_i. The conclusion is that there is an element $q_i \in A$ that multiplies the new generator n_i to give $q_i n_i$ a unit of A (that is, $q_i n_i \notin m = N_{r-1}$), but multiplies the submodule N_{i-1} into m.

In the general case A may not contain a field. Then it does not make sense to refer to the $\{n_i\}$ as a "monomial basis".

6.4 Definition of Gorenstein

Now let (A, m, k) be a local Noetherian ring, with $n = \dim A \geq 0$. The simplest way of defining Gorenstein is to say that A is Cohen–Macaulay, with a regular sequence $x_1, \ldots, x_n$ and that the Artinian quotient $\overline{A} = A/(x_1, \ldots, x_n)$ is Gorenstein as discussed above in 6.2.

I don't have time to take this discussion much further. The conclusion is that the above definition does not depend on the choice of regular sequence $x_1, \dots, x_n$, and that questions involving Ext^i and injective resolutions of A-modules can be reduced to similar questions for $\overline{A}$. One sees that A Gorenstein is equivalent to A having finite injective dimension, with the injective dimension equal to $n = \dim A$. Matsumura [Ma, Theorem 18.1] gives half-a-dozen equivalent conditions, any of which could be taken as the definition.