MA4J8 Commutative Algebra II

Worksheet 0

These issues are intended as easy reminders of the course
prerequisites. Please think about them and let me know at once
if you have insuperable difficulties with them.

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(I) Exercises on B\'ezout's identity that illustrate the
Dedekind-Weber parallelism.

(1) Let a,b be coprime integers. Prove that a*x + b*y modulo a*b
takes all possible values for x in [0..b-1] and y in [0..a-1].
Deduce in particular that
   a*x + b*y = 1 + a*b
holds for some positive integers x,y.

(2) Let a,b in k[x] be coprime polynomials of degree d = deg a and
e = deg b. Prove that the polynomials
   a*f + b*g with degree f <= e-1 and deg b <= d-1
provide d-e linearly independent polynomials in k[x]_{<= d+e-1},
so base that vector space. In particular, there exist
a*f + b*g = 1 with (deg f, deg g) in that range

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(II) Reminders on localisation: Let A be a ring, M an A-module
and S in A a multiplicative set. Does f/s twiddles g/t define
an equivalence relation on pairs f/s?

Define the ring of fractions S^-1A and the module of fractions
S^-1M. If j: M -> S^-1M is the standard map taking m to m/s,
determine the kernel of j.

An ideal I in A is prime if and only if the complement S = A - I
is multiplicative. Explain why we include 1 in S as part of the
definition of multiplicative set.

Prove that M -> S^-1M has the property that S^-1M is an A-module
on which S acts bijectively, and is universal for that property.
(You need to figure out what that means.)

Recall the result of MA3G6 that localisation commutes with
taking quotient modules. Deduce that if M1 -a-> M2 -b-> M3 is
an exact sequence of A-modules then S^-1M1 -> S^-1M2 -> S^-1M3
is also an exact sequence. (Hint: write out ker a and im a
and the same for b, then join their localisations together.)
This can be restated as the slogan that S^-1 preserves exact
sequences, or S^-1 is an exact functor.

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(III) Basic exercises on local rings. Prove that the different definitions of _local ring_ are equivalent:
(i) A ring A in which the set of non-units form an ideal.
(ii) A ring A with a unique maximal ideal m in A.
(iii) A ring with an ideal m such that 1+x is a unit for every x notin m.

Prove that the ring k[[x]] of formal power series over a field is a local ring. This is a precursor to the definition of completion (or formal completion) later in the course.

If A is a ring, P a prime ideal with complement S = A - P, prove that the localisation A_P = S^-1A is a local ring with maximal ideal P*A_P.

Prove Nakayama's lemma: For A,m a local ring and M a finite A-module, m*M = M implies that M = 0. [Hint: let u1,.. un be a set of generators of M. Use the given relation M = m*M to show that M is generated by only n-1 of the ui.]

This has the effect of reducing many issues in linear algebra over a local ring A into linear algebra over the residue field k = A/m.

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(IV) For ideals I,J in A, it is trivial that the product I*J is contained in I intersect J. Give several counterexample to the converse.

Now let P be a prime ideal. Prove that P contains I*J implies that that it also contains I intersect J. This seems like a highbrow mystery until you get the point, when it is trivial.

Let m1 and m2 be distinct maximal ideals of a ring A. Prove that
 (i) m1 + m2 = A,
 (ii) m1*m2 = m1 intersect m2, and
 (iii) A/(m1 intersect m2) \iso A/m1 \directsum A/m2.

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(V) As you know, a maximal ideal is prime. Many arguments use the idea that an ideal that is maximal w.r.t. some property is prime.

(a) A basic result, "sufficiently many primes". If A is a ring, I a nontrivial ideal of A and S a multiplicative set, consider an ideal J that is maximal among ideals containing I and disjoint from S. Prove that J is prime.

(b) Let M be an A-module. Define the annihilator of m in M by
  ann(m) = {f in A | fm = 0}.

Prove that an ideal that is maximal among the ann(m) for nonzero m in M is prime. [Hint: if fg in ann(m), argue on gm and ann(gm).]

Deduce that if A is Noetherian and M <> 0 then M contains a nonzero element m with prime annihilator, that is, a submodule A*m isomorphic to A/P for some prime P in Spec A. This is called an _associated prime_ of M. It is an key ingredient in many parts of the course because A/P is an integral domain, so a solid plank for arguments.

(c) As a pretty illustration of the same principle, see the online MA4J8_Exa1.pdf, Exc 12 for Cohen's theorem: an ideal of A that is not finitely generated and maximal among all such is prime. It follows that if every prime ideal is finitely generated, the same goes for every ideal A, so that A is Noetherian.

(d) The MA3G6 worksheet contains the challenge question: Show that an ideal that is not principal but is maximal among nonprincipal ideals is prime. I currrently can't see how to prove this (despite worrying about it for 2 or 3 days and nights), and would be interested if anyone can help me out.

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Let I be an ideal such that every strictly bigger J is principal. Then for f,g notin I both I+(f) and I+(g) are principal, as is I+(f,g). Say

  I+(f) = (x), I+(g) = (y), I+(f,g) = (z)

for some x,y,z.

When we deal in principal ideals, inclusions are the same as divisibility, and I hope that intersection of coprime ideals is the same thing as multiplication.

The prime ideals containing (x) intersect (y) equal the primes containing (x*y). Thus if f*g in I we deduce that the set of primes containing I -- so what? We might be able to work with radicals of ideals. Maybe for ideal I, it might be equivalent that I is principal, or rad I is principal?
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