MA4J8 Commutative Algebra II. Worksheet 4

I. Frequently Forgotten Facts

Assume the results on Noetherian rings and finite modules. Remind yourself of the prerequisites by completing the following statements (from my FFF crib-sheet), and proving them for yourself:

1. "plenty of primes:" Given ideal I and multiplicative set S ...
2. "existence of associated primes:" Given finite M ...
3. "zerodivisors of M:" Every zerodivisor ...
4. "dévissage of M:" Finite M is a successive extension of A/P_j ...
5. "finiteness of Ass:" Finite M has only finitely many ...
6. "prime avoidance:" If $I \subset \bigcup P_j$ then ...
7. "localisation commutes with Hom" For N s.t. ... $S^{-1} \operatorname{Hom}(N, M) = \dots$
8. "etc." Suggest more for my FFF crib sheet.

I.2 Question on $\operatorname{Supp} N$ Review the material on Supp in [UCA], Chap. 7. In particular, write out proofs of the following:

(1) For $M \cong A/I$ a cyclic module, $\operatorname{Supp}(A/I) = V(I)$.

(2) For a finite module $\operatorname{Supp} N = V(I)$ is equivalent to $I = \operatorname{rad}(\operatorname{ann} N)$.

(3) $\operatorname{Supp} N \subset V(I)$ is equivalent to $I \subset \operatorname{rad}(\operatorname{ann} N)$.

II. Treatment of $\operatorname{Ext}^1(M, N)$ by hand

The name Ext come from the idea that the failure of the contravariant functor $\operatorname{Hom}(-, N)$ to be exact leads to an extension $0 \to N \to E \to M \to 0$. A general exercise or project is to work this out from first principles. This starts from the following idea: given a s.e.s. $N \to B \to C$ and a homomorphism $M \to C$, we can construct a pullback diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & B & \to & C & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & N & \to & E & \to & M & \to & 0 \end{array}$$

where E is the pullback or fibre sum $E = \ker\{(1, -1)\colon M \oplus B \to C\}$. That is, the set of pairs (m, b) such that b and m have the same image in B. The kernel of the first projection $E \to M$ is just a copy of M.

If the top row is split, then $E = N \oplus M$, the trivial extension.

Show how the group and A-module operations on these short exact sequences works, and why $\operatorname{Ext}^1(M,N)$ fits into a 6-term exact sequence. [I haven't had time to work this out as an exercise for an assessed worksheet.]

The set $\operatorname{Ext}^1(M,N)$ consists of extensions of M by N up to isomorphism. An extension is a short exact sequence $0 \to N \to E \to M \to 0$, with isomorphism of extensions defined as commutative diagrams of isomorphism.

How to define the group structure on $\operatorname{Ext}^1(M,N)$?

Given two s.e.s. with extension modules E_1, E_2, cook up a new extension module using the diagonal map $N \to E_1 \oplus E_2$. and the sum map $E_1 \oplus E_2 \to M$.

How to define the A-module structure on $\operatorname{Ext}^1(M,N)$?

Similar, using linear combinations of the maps $N \to E_1$ and $N \to E_2$, together with linear combinations of the maps $E_1 \to M$ and $E_2 \to M$.

Pairs $(m_1, m_2) \in E_1 \oplus E_2$ that map to the same element of m form the fibre product $\ker(\pi_1 - pi_2)$. Set $E = (E_1 \oplus E_2)/\ker(\pi_1 - pi_2)$ and $i\colon N \to E$ to be the composite of the diagonal inclusion $(i_1, i_2)\colon N \to E_1 \oplus E_2$ with the projection modulo the fibre prouct.

II.1. How projective resolutions give $\operatorname{Ext}^1(N,M)$ If M has a projective resolution $M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$, show that $H^1(\operatorname{Hom}(P_\bullet))$ calculates the set of extensions discussed above.

More draft exercises Consider again a s.e.s. of A-modules $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$. We get the exact sequence $0 \to \operatorname{Hom}(C,N) \to \operatorname{Hom}(B,N) \to \operatorname{Hom}(A,N)$

Given $f\colon A \to N$, construct the *pushout* diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & N & \to & B' & \to & C & \to & 0 \end{array}$$

where $B' = (B \oplus N)/\operatorname{im}(\alpha, f)$. If the bottom row is a split s.e.s. of A-modules (this means $B' = N \oplus C$, with arrows the inclusion and projection of the direct sum), we know how to extend f to B by including $B \hookrightarrow B'$ then projecting the direct sum to its first factor.

Exercise: Please think about how to prove the converse.

As a covariant functor in N The same question for the covariant functor $\operatorname{Hom}(M, -)$.

How this relates to an injective resolution $N \to I_\bullet$.

Repetition of material on "missing monomials" Calculate from first principles the relations holding between u^4, u^3v, uv^3, v^4 for Macaulay's quartic curve (5.5, Ex 4).

Write down the subring $A \subset k[x,y]$ consisting of all polynomials $f(x,y)$ such that $f(1,0) = f(-1,0)$. [Hint: x is not allowed, but $1 - x^2$ and all its multiples are.] Calculate the relations holding between these polynomials by analogy with (5.5, Ex 3).

III. The Ext groups are well defined

III.1. As a contravariant functor in M Recall the definition of a projective resolution of a module M. Show that two projective resolutions $M \leftarrow P_\bullet$ and $M \leftarrow Q_\bullet$ can be compared by an isomorphism of complexes, that is, a commutative diagram of isomorphisms $\varphi_i \colon Q_i \to P_i$. Moreover any two φ and φ' are homotopy equivalent. [Hint: All the required maps are proved to exist by the projective assumption on the P_i.]

Use this to prove that $\operatorname{Ext}^i(M,N)$ calculated from a projective resolution are well defined up to isomorphism. (Left as exercise in [Ma], p.278.)

III.2. As a covariant functor in N The same question for $\operatorname{Ext}^i(M,N)$ calculated from an injective resolution $N \to I_\bullet$.

III.3. Double complexes and tensor product $K_\bullet \otimes L_\bullet$ Matsumura [Ma] Appendix B, p.275. A double complex is a double indexed array of modules K_{ij} with two sets of differentials d'_i, d''_j where d' lowers i by 1 (that is $d'_i \colon K_{ij} \to K_{i-1,j}$) and d'' lowers j by 1. Assume that each horizontal row and vertical column is a complex (that is, $d'_{i-1} \circ d'_i = 0$ and similarly for d'', and that the squares anticommute (if you start with the squares commuting, put one minus sign in each square for example by editing the d_j to $(-1)^j d_j$).

This is what you get if you take tensor product of two complexes, or a general complex and make a resolution of it. Check that the single complex $K_{\text{sum}} := \sum_{i+j=k} K_{ij}$ is a complex.

[Ma], p. 277 leaves the exercise of proving: if all the rows and columns are exact except at zero, then

- the homology groups of the bottom row K_{i0}
- are isomorphic to those of the associated single complex K_{sum}
- and in turn isomorphic to those of the first column K_{0j}.

  [Hint: This is just an elaborate diagram chase.]

III.4. Ext as a bifunctor Use this to verify that the two calculations of $\operatorname{Ext}^i(M,N)$ by projective resolution of M and by injective resolution of N give isomorphic homology groups.

III.5. A-module action on Ext The two different constructions of $\operatorname{Ext}^i(M,N)$ also give isomorphic cohomology A-modules, with a small high-brow clarification. Everything on the N and $N \to I_\bullet$ side is covariant: N and the I_i are all A-modules, so multiplying $\varphi\colon M \to I_i$ by x in A is just the obvious thing.

However, as a contravariant functor in M, the action of A on the Hom groups is always *premultiplication*: the Hom functor take M to $\operatorname{Hom}(M,N)$, and takes a homomorphism $\psi\colon M_1 \to M_2$ into the homomorphism

$$\operatorname{Hom}(M_2,N) \to \operatorname{Hom}(M_1,N) \quad \text{given by} \quad \varphi \mapsto \varphi \circ \psi.$$

This applies to the module structure multiplying M by $x \in A$. The A-module structure on $\operatorname{Hom}(M,N)$ (as a contravariant functor) is $\varphi(m) \mapsto \varphi(x \cdot m)$ for $m \in M$. In other words, we multiply by x while we are still in M, before applying φ.

It is now an exercise to verify that the A-module structure on $\operatorname{Ext}^i(M,N)$ defined by the two constructions coincide.