MA4J8 Commutative algebra II

Addenda to tidy up Week 1

2.1 More prerequisites

Zariski topology on $\operatorname{Spec} A$ Refresh your memory of $\operatorname{Spec} A$ with its Zariski topology and principal open sets $X_f = \operatorname{Spec} A \setminus V(f)$, equal to $\operatorname{Spec} A[\frac{1}{f}]$.

The radical $\operatorname{rad} I = \{f \in A \mid f^N \in A \text{ for some } N\}$ is characterised by $\operatorname{rad} I = \bigcap_{P \in V(I)} = \bigcap_{I \subset P} P$. This is a standard exercise using "plenty of primes".

Krull dimension Dimension theory is treated at length in Week 4, but I made simple use of Krull dimension $\dim A$ of a ring A. This is the maximum length n of a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals. The *height* of a prime ideal P of a prime $\operatorname{ht} P$ is the maximum length of a chain up to P. Thus a minimal prime of A has $\operatorname{ht} P = 0$. A standard exercise: show that the localisation A_P of A at a prime P has $\dim A_P = \operatorname{ht} P$.

Localisation $S^{-1}A$ **and** $S^{-1}M$ **of ring and modules** Refresh your memory of partial ring of fractions and localisation $S^{-1}A$ and $S^{-1}M$. In particular, write down the kernel, functoriality and exactness properties of $M \mapsto S^{-1}M$.

Let A be a ring, S a multiplicative set in A, and M an A-module. As you know, the localisation $S^{-1}A$ and $S^{-1}M$ is the set of fractions

$$\begin{aligned} S^{-1}A &= \{\tfrac{a}{s} \mid a \in A \text{ and } s \in S\}/\sim \\ S^{-1}M &= \{\tfrac{m}{s} \mid m \in M \text{ and } \in S\}/\sim \end{aligned} \tag{2.1}$$

Here $\sim$ (pronounced twiddles) is the equivalence relation

$$\begin{aligned} \tfrac{a_1}{s_1} \sim \tfrac{a_2}{s_2} &\iff \exists t \in S \text{ so that } t(s_2a_1 - s_1a_2) = 0 \in A. \\ \text{resp. } \tfrac{m_1}{s_1} \sim \tfrac{m_2}{s_2} &\iff \exists t \in S \text{ so that } t(s_2m_1 - s_1m_2) = 0 \in M. \end{aligned} \tag{2.2}$$

The point of "there exists $t \in S$" is that (in contrast to the integral domains we worked with in Week 1), we allow S to contain elements s that are zerodivisors of A or of M. After we make such an s invertible, any a with $sa = 0$ necessarily maps to zero in $S^{-1}A$.

Once we take that precaution, we get a new ring $S^{-1}A$ with a ring homomorphism $A \to S^{-1}A$ such that every $s \in S$ maps to an invertible element of $S^{-1}A$, resp. an A-module homomorphism $M \to S^{-1}M$ with the property that multiplication by every $s \in S$ is an isomorphism $S^{-1}M \to S^{-1}M$. In fact $A \to S^{-1}A$ and $M \to S^{-1}M$ are characterised by saying that they satisfy the UMP for the condition that s maps as a bijection.

Noetherian conditions, picture of M fibred over $\operatorname{Spec} A$ So far we have had lots of stuff on Noetherian rings and modules. (If in doubt, read the prerequisites.) We frequently assume that we work with Noetherian rings A, and modules M that are finite over them, so also Noetherian.

It is often a good strategy to break something complicated into smaller components. I want to explain the picture at the front of [UCA]. It has $\operatorname{Spec} A$ drawn as a geometric base space, and a module M as an object living over A or preferably over its closed subsets.

I assume the Zariski topology on $\operatorname{Spec} A$ as a prerequisite. The closed sets are $V(I) = \{P \mid I \subset P\}$. The letter V means variety, so $V(I)$ is the locus where every $f \in I$ maps to zero in the residue ring A/P (an integral domain), or in its field of fractions $k(P) = \operatorname{Frac}(A/P)$. One checks this is a topology.

Write $X = \operatorname{Spec} A$. Then for $f \in A$, we have the closed set $V(f)$ (the set of primes P with $f \in P$), and its complement X_f, the principal open set $\{P \in \operatorname{Spec} A \mid f \text{ is a unit of } A_P\}$. One checks that $\operatorname{Spec}(A[1/f]) = X_f$.

For a Noetherian ring, the closed set of $X = \operatorname{Spec} A$ have the descending chain property: any chain $V_1 \supsetneqq V_2 \supsetneqq \cdots \supsetneqq V_n$ must eventually stop. It follows that X or $V(I)$ in X contains a finite number of irreducible components, that correspond to the minimal primes of X or of $V(I)$. See [UCA, p.76–77].

Apart from small side-steps in language, this agrees with what we do with affine algebraic sets in $\mathbb{A}^n_k$ over an algebraically closed field, where we only work with k-valued points $P = (a_1, \dots, a_n)$ or the corresponding maximal ideals $m_P = (x_i - a_i)$.

Associated primes, primary decomposition, dévissage Now assume that A is Noetherian. I discussed the idea of *associated prime* in Week 1: $P \in \operatorname{Spec} A$ is an associated prime of M (written $P \in \operatorname{Ass} M$) if there exists $m \in M$ such that $\operatorname{Ann} m = P$, or equivalently, M contains a submodule $\cdot m$ isomorphic to A/P. Think of the integral domain A/P as a solid plank: multiplication by a acting on A/P is either 0 or injective. Therefore any submodule $N \subset A/P$ is either 0 or has P as its only associated prime.

If M is finite over A then M is a Noetherian module. In this case, [UCA, Theorem 7.6] asserts that $\operatorname{Ass} M$ is a finite set of primes of A. The proof is a *dévissage* (disassembly, breaking up into parts). If $M \neq 0$, it has at least one associated prime, so a submodule $M_0 \subset M$ with $M_0 \cong A/P_0$.

Consider the s.e.s. $M_0 \subset M \twoheadrightarrow \overline{M}$. Then one sees that $\operatorname{Ass} M$ is a subset of $\operatorname{Ass} M_0 \cup \operatorname{Ass} \overline{M}$. By the above "solid plank" argument, $\operatorname{Ass} M_0 = \{P_0\}$. Continuing thus gives an increasing filtration

$$M_0 \subset M_1 \subset \cdots M_i \subset M_{i+1} \subset \cdots M_n = M. \tag{2.3}$$

with each $M_{i+1}/M_i \cong A/P_i$ for P_i an associated prime of M_{i+1}. By the Noetherian assumption the filtration must stop, and then $\operatorname{Ass} M$ is contained in the finite set $\{P_0, \ldots, P_n\}$.

Week 2. Regular local rings, Artinian conditions

2.2 Regular local rings

I want to set up regular local rings (slightly breaking rigorous logical order), because they follow on naturally from DVRs, and are central to applications to algebraic number theory and algebraic geometry. I use one little fact about dimension theory that is only proved in Week 3 or 4. (I don't want logic to stand in the way of understanding.)

DVRs I start with a round-up of DVRs to put regular local rings in context. The conditions for a DVR are:

(1–4) A is (1) a Noetherian ring, (2) an integral domain, (3) local with maximal ideal m, and (4) has Krull dimension 1.

(5) The maximal ideal is principal $m = (z)$.

(1–4) mean A is a Noetherian ring with $\operatorname{Spec} A = \{0, m\}$. Condition (5) is more specialised. It gives immediately all the useful properties of DVRs, to do with factorisation with a single prime element z, the valuation $v(f)$ that gives you the powers of primes in numerators and denominators for a number field, and the divisor of zeros and poles of a rational function $f \in K = \operatorname{Frac} A$ in the case of an algebraic curve.

The Main Theorem on DVRs says that (5) holds if (1–4) hold and A is normal (integrally closed in its field of fractions $K = \operatorname{Frac} A$). Taking integral closure is an automatic procedure in practical applications.

Start from a Noetherian integral domain with $\operatorname{Spec} A = \{0, m\}$, and pass to the integral closure B in $K = \operatorname{Frac} A$ (or in a finite extension field $K \subset L$). One proves that B is finite over A using an extra assumption (see [UCA, 8.11]) – but this holds for all the constructions used in number theory and algebraic geometry and we usually ignore it.

The ideal $m \cdot B$ has quotient B/mB a finite dimensional vector space over the residue field $k = A/m$. It follows that B has only finitely many maximal ideals (it is a *semilocal* ring). If A is 1-dimensional then so is B, and all its localisations are DVRs. (There are a couple of little exercises for you implicit in this.)

Now for regular local rings of dimension $n \geq 1$. My treatment here short-circuits a small point from dimension theory, which is a main topic later in Week 3 or 4. The analog of the above definition of DVR is this

Definition 2.1 A *regular local ring* (A, m) is a Noetherian local integral domain with $\dim A = n$ for which the maximal ideal m is generated by n elements, $m = (x_1, \ldots, x_n)$.

First,notice that each m^d/m^{d+1} for $d \geq 0$ is a module over A on which m acts by 0, so is a k-vector space where $k = A/m$ is the residue field. Nakayama's lemma gives that $(x_1, \ldots, x_n)$ generate the ideal m if and only if $(x_1, \ldots, x_n)$ span m/m^2 as k-vector space.

It then follows that m^2/m^3 is generated as a k-vector space by quadratic monomials

$$S^2(x_1, \ldots, x_n) = (x_1^2, x_1x_2, \ldots, x_n^2), \tag{2.4}$$

and similarly for m^d/m^{d+1}.

In each degree d there are $\binom{d+n-1}{d}$ monomials, that span the vector space m^d/m^{d+1}.

Exercise 2.2 Recall the properties of binomial coefficients, then recall them again until you have them on board. In $k[x, y, z]$, write down the monomials of degree 0, 1, 2, 3, 4 and count them. Spend 2 minutes writing out Pascal's triangle[1]. More generally, state and prove a formula for the number of monomials of given degree $d \geq 0$.

Now the same question for monomials of degree *up to and including* d. Write $m = (x, y, z)$ for the maximal ideal of $k[x, y, z]$. Calculate the dimension of the quotient $k[x, y, z]/m^{d+1}$.

If you think this is all too trivial to be worth your attention, let me assure you that it is a major item in the cohomology of coherent sheaves on projective varieties, and we use it again for dimension theory in Week 4.

Recall that Definition 2.1 had $n = \dim A$ (Krull dimension), and I write $x_1, \ldots, x_n$ for generators of m. The following result is the main point:

Theorem 2.3 *Let A, m be a Noetherian local integral domain with residue field $A/m = k$.*

If $\dim A = n$ *then* $\dim_k m/m^2 \geq n$. *Thus the maximal ideal m needs* at least $n = \dim A$ *generators.*

Moreover, if m is generated by exactly $n = \dim A$ elements $x_1, \ldots, x_n$ then they are algebraically independent: that is, for each d, the monomials $S^d(x_1, \ldots, x_n)$ (of which there are $\binom{d+n-1}{d}$) are linearly independent over k, and base m^d/m^{d+1}.

[1] or YANG Hui's triangle (11th Century, but it goes back millenia before this.)

Corollary 2.4 *Suppose also that A contains the field k as a subring, such that k maps isomorphically to A/m. Then the monomials of degree $\leq d$ are linearly independent over k and base A/m^{d+1} as k-vector space.*

Exercise 2.5 With your new-found familiarity with binomial coefficients, prove that $\dim_k A/m^{d+1} = \binom{d+n}{d} = \sum_{a=0}^{d} \binom{a+n-1}{a}$.

Remark 2.6 The assumption A contain k with composite $k \to A \to A/m$ an isomorphism always holds when we do algebraic geometry over an algebraically closed field. This assumption is the main reason that algebraic geometry over $\mathbb{C}$ is easier than over $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$.

However, in many cases the assumption of Corollary 2.4 does not hold. For example, $\mathbb{Z}[t]$ or its localisation $\mathbb{Z}[t]_{(p,t)}$ at the maximal ideal (p,t) is not an algebra over *any* field. Or $A = \mathbb{R}[x]$ has maximal ideal $m = (x^2+1)$, and the quotient A_m/m is an extension field of the field of definition $\mathbb{R}$.

The notion of finite length $\ell(m)$ of an A-module remedies the situation. After the definition, we simply replace $\dim_k A/m^{d+1}$ by $\ell(A/m^{d+1})$. The descending chain of ideals

$$A \supset m \supset m^2 \supset \cdots \supset m^d \supset m^{d+1} \tag{2.5}$$

has successive quotients m^d/m^{d+1}, each of which is a f.d. vector space over $k = A/m$, so has the length stated in Theorem 2.3.

2.3 Noetherian and Artinian conditions

We have seen the Noetherian conditions on modules many times: TFAE

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   a.c.c. on submodules
   every submodule of M is f.g.
   any nonempty set of submodules has a maximal element.
Then many results of the type: adjoining something finite
to Noetherian object gives new Noetherian ring or module.
Prerequisites: Hilbert Basis theorem and corollaries:
If A is Noetherian ring, B = f.g. A-algebra again Noetherian.
M a finite A-module is Noetherian. If
   0 -> N -> M -> M/N -> 0
then M is Noetherian <=> N, M/N both are Noetherian.
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We now switch to the Artinian conditions on a module M that are superficially similar: equivalently,

(1) The submodules of M satisfy the d.c.c. condition.

(2) Every nonempty set of modules of M has a minimal element.

Exercise 2.7 In a short exact sequence of A-modules

$$0 \to N \to M \to M/N \to 0, \tag{2.6}$$

M is Artinian if and only if N and M/N are both Artinian.

As with Corollary 2.4, a key case is when A is a k-algebra. Then we may view M as a k-vector space. It is clear that M finite dimensional over k implies both Noetherian and Artinian.

Remark 2.8 If A, m is a local ring, it follows from Akizuki and Hopkins' Theorem 2.5 that

$$A \text{ is Artinian } \iff m \text{ is finitely generated and } m^n = 0 \text{ for some } n.$$

2.4 Jordan-Hoelder filtration and modules of finite length

Theorem Let M be an A-module. Then the following are equivalent:
(1) M is both Artinian and Noetherian
(2) There exists a finite filtration
    0 = M0 inneq M1 inneq .. M_{n-1} inneq Mn = M.
such that for each i, there is no A-module strictly intermediate between Mi and M{i+1}.

The last condition is: if Mi inneq N inneq M_{i+1} then either Mi = N or N = M_{i+1}. You can also say that the module M_{i+1}/M_i is simple (has no nontrivial submodule). Each is a 1-dimensional vector space of a field k(P) for some P in Spec A.

(1) => (2) is straightforward. M is Noetherian so (if <> 0) there is a maximal submodule M' in M. The inclusion M' in M has not no strictly intermediate property by construction. Apply the same to M' (assuming <>) and get a descending chain. Since M is Artinian, this must terminate, giving (2).

Jordan-Hoelder filtration is not unique. Any two filtrations have successive simple quotients M_{i+1}/M_i that up to permutation are isomorphic. (The proof that length is well defined and additive in s.e.s. is a useful exercise, that may be familiar to you from the case of finite groups.) In particular, the length of the chain is well defined, and is called the _length of M_, \ell(M).

This section has 2 aims:
(I) to treat modules that satisfy both chain conditions, and characterise them as modules of finite length.
(II) To discuss Artinian rings, that are analogous to k-algebras that are finite dimensional as vector spaces over a field k. As opposed to modules, Artinian rings are necessarily also Noetherian. Definition: Simple module. M is a simple A-module if its only submodules are 0,M. One sees that then M iso A/m = k(m) for some maximal ideal k.
Obviously, if M is a simple module and N any module,
  any homomorphism M -> N is 0 or injective
  any homomorphism N -> M is 0 or surjective
If M, N are both simple, any M -> N is 0 or an isomorphism.

Definition: A a ring, and M an A-module. A Jordan-Hoelder filtration is a chain of submodules
  0 = M0 in M1 in .. M_{n-1} in M_n = M (*)
with no strictly intermediate modules between M_{i-1} and M_i. The latter condition holds if and only if M_i/M_{i-1} is simple.

Theorem Equivalent conditions on M:
  (1) M is Artinian and Noetherian.
  (2) M has a Jordan-Hoelder filtration.
Moreover, if they hold, the set of simple quotient modules Mi/M_{i-1} in any JH filtration is unique up to isomorphism and permutation. This means that M is just a bunch of residue fields k(m) tied together in a successive extension module.

If the conditions hold we say M has _finite length_, or has length n as an A-module where n is in (*). (This of course depends on A -- if we view M as a module over a smaller ring, it might lose the finite length property, and its length will usually increase.)

(1) => (2) is the obvious follow-your-nose argument:

First, define Si = set of all nonzero submodules. If M <> 0 this is nonempty, so by the Artinian condition it has a minimal element M1. Next, set Si = set of all submodules strictly bigger that M1. If this is empty, M1 = M and we are finished. If nonempty, it has a minimal element M2. Continue by induction: we construct

    0 = M0 < M1 < .. Mi <= M

where each step cannot be refined. At any stage, either we have reached M or we can take another step. So far, this has only used Artinian. Now since M is also Noetherian the increasing chain must terminate, so at some point Mn = M. You can also do this from the other end, working down from M and taking maximal nontrivial M' < M.

The proof of (2) => (1) and the "moreover" final clause involve standard arguments using the isomorphism theorems. Given one JH filtration (*), and any chain of submodules {Ni}, if none of the Ni contain M1, they are part of a chain for M/M1, which has a JH chain of length n-1. In the contrary case, there is some i s.t. N_{i-1} does not contain M1 but Ni does, so necessarily Ni/N{i-1} iso M1. etc. (Clean up the proof for yourself.)

2.5 Artin implies Noetherian for rings

Theorem [Akizuki and Hopkins] An Artinian ring A is Noetherian

Step 1 A has only finitely many maximal ideals m_i.

Proof: An exercise that is "easy", but I always have to do again from first principles. (Please do it for yourself before reading the hints in 2.6 below.)

Step 2 Write J = product mi = intersection mi. (The Jacobson radical.) Then every x in J is nilpotent.

Since every prime ideal is maximal, J is also the
intersection of all prime ideals of A, which is its
nilradical.

Step 3
Claim. J^n = 0.

J = intersection mi = intersect of all primes,

so every x in J is nilpotent. Consider the descending chain
  J > J^2 > .. > J^n.
This must terminate (again by the d.c.c.), say in N = J^n
with
  J*N = N^2 = N.
We prove that N = 0 by contradiction.

If N <> 0 the set Si of ideals b such that b*N <> 0 is
nonempty, so has a minimal element c (again by the d.c.c.).

Now some x in c has x*N <> 0, and c = (x), else (x) would
be smaller than minimal.

Now (x*N)*N = x*N^2 = x*N <> 0. So x*N is an ideal contained
in (x), and again by minimality, x*N = (x). Therefore x = x*y
for some y in N, and hence x = x*y = x*y^2 = .. x*y^n.

But y is an element of N, and every element of J is
nilpotent, so x = x*y^n = 0. This is a contradiction, hence
N = 0.

Step 4. An Artinian ring A has a JH filtration, and so is
Noetherian as A-module.

In fact we proved that the finitely many prime ideals m_i
have prod m_i^n = 0.

The Artinian condition on A implies that m_i^n/m_i^{n+1} is
an Artinian module; but it is a vector space over the

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residue field k(m_i) = A/m_i, so finite dimensional.
Therefore taking prod m_i^{ni} with one of the exponents
increasing by 1 at a time, we get a decreasing sequence with
each a finite dimensional vspace over one of the k(m_i).
Q.E.D.

The above argument divides into a number of fairly tricky
steps, making repeated use of minimality of ideals in a
sequence. Is there an improved argument with fewer appeals
to d.c.c.? [A&M] and [Matsumura] give essentially the same
proof (possibly cribbed from a common source? [Ma, p.16]
refers to Akizuki 1935 and Hopkins 1939), and as far as I
know, no-one seems to have found a shorter argument.
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2.6 The converse as an exercise

This result in the converse direction is straightforward:

Theorem 2.9 *Let A be a Noetherian ring of Krull dimension* $\dim A = 0$. *Then A is Artinian.*

The assumption on the dimension means that all primes are maximal. As in the preceding argument, the intersection J of all maximal ideals is the nilradical, that is, the set of all nilpotent elements. However, A Noetherian means that its ideal J is finitely generated, so it is clear that J^n for some n. Now A/J is a product of fields, and Noetherian implies there are only finitely many of them.

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Exc.1: m1, m2 distinct maximal ideals => m1+m2 = A (Easy).
Also m1 intersect m2 = m1*m2.

Suppose e1 in m1 and e2 in m2 satisfy e1+e2 = 1_A. Then for
x in m1 intersect m2 we get x = e1*x + e2*x. The first term
e1*x is in m1*m2 (because x in m2) and e2*x similarly.

The e1,e2 map to complementary idempotents of
A/(m1 intersect m2).

Exc.2: Now m1,m2,m3 distinct maximal ideals. Claim: there is
an x in m1 interect m2 such that x notin m3.
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Proof: Take y in m1 notin m3 and z in m2 notin m3. Then y*z
does what I claim.

Exc.3: Similar for m1,..mn, m_{n+1}. Compare for example
[A&M] Lemma 1.11.
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2.7 Macaulay inverse systems

This is an Artinian modules, *inverse polynomials*, that you may never have seen: let $A = k[x]$ and consider the ring of Laurent polynomials $k[x, x^{-1}]$, and its quotient module

$$M = k[x, x^{-1}]/k[x]. \tag{2.7}$$

As a vector space, this is countable dimensional, with basis $\{x^{-i}\}$.

This M is Artinian: every A-submodule of $k[x, x^{-1}]$ that is proper (not the whole of $k[x, x^{-1}]$) only involves finitely many x^{-n}. The same applies to every f.g. A-submodule $N \subsetneq M$. Hence if x^{-n} is the last of these (with the biggest negative exponent),

$$N = N_n = k\text{-vector space of } M \text{ based by } x^i \text{ for } i \in [-n, -1]. \tag{2.8}$$

The $k[x]$-module multiplication by x does

$$x^{-i} \mapsto x^{-(i-1)}, \quad \text{and in particular, } x^{-1} \mapsto 0. \tag{2.9}$$

A decreasing chain is a chain of f.d. vector spaces, so terminates. All the N_n are in the chain

$$\cdots N_n \supsetneq N_{n-1} \supsetneq \cdots N_{-1} \supsetneq N_0 = 0. \tag{2.10}$$

Whereas multiplication by x in $A = k[x]$ is injective, in M it is surjective, so that M is infinitely divisible by x: for any $x \in M$, we can find a predecessor $x' \in M$ with $x \cdot x' = x$.

There is no longest chain (you can always take a bigger N_n), but any proper submodule stops at some n.

Multiplication by x is nilpotent when restricted to any A-submodule of N_n. The submodule N_n has the single associated prime m, that is $\operatorname{Ass} N_n = m$, the annihilator of x^{-1}. Whereas $A = k[x]$ has 1 as the single generator, the only unit monomial, that maps to a basis of the residue field $k = A/m$, in N_n, the element x^{-1} is the *socle*, the last element to go under nilpotent multiplication. That is, it is the submodule of N_n annihilated by m, which is the same thing as $\operatorname{Hom}_A(A/m, N_n)$.

The $k[x]$-module M is Artinian: although it has infinite basis $\{x^{-n}\}$, every proper submodule $N \subset M$ only involves finitely many of the x^{-n}, and is the k-vector space based by $\{x^i \mid -n < i \leq -1\}$. As with other things in life, you have any number of choices, but you have to take one, and whichever you go for involves excluding almost all the others.

On the other hand, M it is not finitely generated as module, and not Noetherian. As n gets bigger, the submodules (x^{-n}) get bigger, so infinite ascending sequences are the order of the day.

We view M as a tempered dual of the polynomial ring $k[x]$. Since $k[x]$ is infinite dimensional, we really don't want to say the dual vector space $k[x] = \mathrm{Hom}(k[x], k)$, which is uncountable dimensional. Instead, think of $k[x]$ as the union of the finite dimensional spaces $k[x]_{\leq n}$ of polynomials of degree $\leq n$. Then M is the union of their duals

$$\mathrm{Hom}(k[x]_{\leq n}, k) \quad \text{based by } \{x^{-n-1}, \dots, x^{-1}\}. \tag{2.11}$$

The duality between M and $k[x]$ is the analog of Cauchy residue of a meromorphic function – take inverse polynomial q and polynomial f into the residue of product $f \cdot q$, that is, the coefficient of x^{-1}. The free module $k[x]$ has basis $\{x^i\}$, whereas $M = k[x, x^{-1}]/k[x]$ has basis $\{x^{-(i+1)}\}$, which is the dual basis under the residue pairing.

M is called the *module of Macaulay inverse systems.* It contains a unique submodule $N = k \cdot x^{-1}$ isomorphic to the residue field $k[x]/(x)$, and M is *injective* as $k[x]$ module in a sense to be discussed later. The map $k[x]/(x) \hookrightarrow M$ given by the basis x^{-1} of the socle is the injective hull of the residue field $k = k[x]/(x)$.

We can do the same thing with other rings. For example $k[x_1, \dots, x_n]$ and the module with basis $\left\{\prod x_i^{-a_i-1}\right\}$ (Laurent monomials with strictly negative exponents) so the socle is based by $\prod x_i^{-1}$.

The same trick applies to the localisation $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at a prime p. The module $M = \mathbb{Q}/\mathbb{Z}_{(p)}$ is generated by the negative powers p^{-n} of p. Any proper $\mathbb{Z}_((p))$-submodule $N \subsetneq M$ has only finitely many p^{-n}, and the chains of modules have the same shape as above.

The module $\mathbb{Q}/\mathbb{Z}$ is the direct sum of $\mathbb{Q}/\mathbb{Z}_(p)$ taken over all p. It is Artinian but not Noetherian.