MA4J8 Commutative algebra II

Frequently forgotten facts

This is a crib sheet of standard algebraic prerequisites. Each of them has a 2–3 line elementary proof, but does not immediately come to mind when I need them. To remedy this I give the statements names. I assume the Noetherian and Artinian finiteness conditions.

1. "Plenty of primes" If I is an ideal of A and S a multiplicative set disjoint from I then applying Zorn's lemma gives that there is a maximal element P among the ideals containing I and disjoint from S, and P is prime.

2. "$\operatorname{rad} I = \bigcap P$ taken over $P \in V(I)$" If $f \notin \operatorname{rad} I$ then there exists a prime ideal P containing I but disjoint from $\{f^n\}$. Therefore $\operatorname{rad} I$ equals the intersection of all primes $P \supset I$. This set of primes is the closed set $V(I)$ in the definition of the Zariski topology. Two ideals I_1 and I_2 have $V(I_1) = V(I_2)$ if and only if $\operatorname{rad}(I_1) = \operatorname{rad}(I_2)$.

3. "Existence of associated primes" An ideal that is maximal among $\{\operatorname{ann} m \mid \text{nonzero } m \in M\}$ is a prime $P \in \operatorname{Ass} M$, with $Am \cong A/P \subset M$. If A is Noetherian any $\operatorname{ann} m$ is contained in an associated prime of M.

4. "Zerodivisors of M" It follows that every zerodivisor of M is contained in $\bigcup P$ taken over $P \in \operatorname{Ass} M$.

5. "Dévissage of M:" A finite module M over a Noetherian ring has a composition sequence

$$0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M$$

with each $M_i/M_{i-1} \cong A/P_i$ for $P_i \in \operatorname{Ass}(M/M_i)$. It follows from this that $\operatorname{Ass} M$ is a finite set [UCA, 7.6].

6. "Prime avoidance:" For a finite number of prime ideals P_i, if an ideal I is not contained in any of the P_i then it is not contained in $\bigcup P_i$ [A&M, 1.11, (i)]. The other way around, if I is contained in a finite union $\bigcup P_i$ of prime ideals P_i, it is contained in one of them: $I \subset P_i$.

It follows that for a finite module M over a Noetherian ring, if every element of an ideal I is a zerodivisor of M then $I \subset \operatorname{ann} m$ for some nonzero element $m \in M$.

7. "The support of a module" For a finite module,

$$\operatorname{Supp} N = V(I) \iff \operatorname{rad}(I) \subset \operatorname{rad}(\operatorname{ann} N).$$

[UCA, 7.6]

8. "Localisation commutes with Hom:" We know that taking rings of fractions $A \mapsto S^{-1}A$ for a multiplicative set is an exact functor on A-modules. Suppose A is a ring, N, M are A-modules and N is finitely presented. then

$$S^{-1}(\operatorname{Hom}_A(N, M)) = \operatorname{Hom}_{S^{-1}A}(S^{-1}N, S^{-1}M).$$

[Ma, Th 7.1] and [Ei, Prop 2.10].

This was used in showing that a finitely presented module is projective iff it is locally free.

More advance crib-sheet

Several sections of the textbooks can be summarised in this style, although the proofs may be longer than a couple of lines.

From the chapter on dimension theory, $\dim N = \delta(N)$, where the left-hand side is the Krull dimension of $A/\operatorname{ann}(N)$, and $\delta(N)$ is the minimal length of a system of parameters.

Krull's Hauptidealsatz: For a principal ideal $I = (a)$, let P be a minimal prime ideal containing I and minimal with that property. The theorem states that $\operatorname{ht}(P) \leq 1$. In geometric language, if you take a section $a = 0$ of a variety X for $a \in k[X]$, every irreducible component of the section has codimension ≤ 1. It follows that a minimal prime ideal P containing $(a_1, \dots, a_r)$ has $\operatorname{ht}(p) \leq r$.

The modern proof uses the treatment of dimension as the order of growth of the Hilbert–Samuel function. There are proofs in the literature that work directly with ideals and ring elements.

David Rees' 1956 proof is given later. He proves in particular that if a local Noetherian integral domain A, m has a principal ideal (x) that is m-primary, then every nonzero ideal of A is also m-primary, and $\operatorname{Spec} A = m$.

Crib sheet to continue ...