MA4J8 Commutative algebra II. Worksheet 1

Basic refreshers

0. Recall the division with remainder statement for the ring of integers $\mathbb{Z}$, and for the polynomial ring $k[x]$ over a field. Use it to prove that each of the two rings are PIDs and hence UFDs.

1. The definition of local ring is a ring A with a unique maximal ideal m. Prove this holds if and only if the set of all nonunits of A is an ideal. [For one direction you will need Zorn's lemma.]

2. Consider the ring $k[\![x]\!]$ of formal power series over a field k. If

$$f = a_0 + a_1 x + \cdots + a_n x^n + \cdots , \tag{0.1}$$

when is f a unit? a non-unit?. Prove that $k[\![x]\!]$ is a local ring.

In complex analysis, with z a complex variable, prove that the ring of power series

$$\sum_{n\geq 0} c_n z^n \quad \text{with bounded radius of convergence} \tag{0.2}$$

is a local ring.

3. For p a prime, the ring $\mathbb{Z}_p$ of p-adic integers consists of formal power series

$$f = a_0 + a_1 p + \cdots + a_n p^n + \cdots \quad \text{for } a_i \in [0 \ldots p-1], \tag{$*$}$$

where addition $f + g$ and multiplication fg are defined as in the integers mod p^m, for $m \gg 0$. To see the difference between $\mathbb{Z}_p$ and $\mathbb{F}_p[\![x]\!]$, calculate

$$\text{the sum } (p-i) + i, \quad \text{and the product} \quad (1 + ip)(1 + (p-i)p)$$

in the two rings (where $i \in [1 \ldots p-1]$). For practice, calculate

$$\sum_{n=0}^{\infty} p^n \quad \text{and} \quad \sum_{n=0}^{\infty} np^n \quad \text{in } \mathbb{Z}_p. \tag{0.3}$$

Determine when f of (*) is a unit or a non-unit in $\mathbb{Z}_p$. Prove that $\mathbb{Z}_p$ is a local ring.

Remark. You probably won't find any local rings in elementary algebra: as you noticed, the above examples all involve the logic of infinite expressions. Other examples are

$$\begin{aligned}\mathbb{Z}_{(p)} &= \text{localisation of } \mathbb{Z} \text{ at prime ideal } (p)\\ &= \text{rationals } a/b \in \mathbb{Q} \text{ with denominator coprime to } p.\\ k[x]_{(x-a)} &= \text{localisation of } k[x] \text{ at } (x-a) \text{ for some } a \in k\\ &= \text{rational functions } f/g \in k(x) \text{ with } g(a) \neq 0.\end{aligned}$$

These also adds infinitely many possible denominators.

This explains to some extent why local rings rarely come up in early u/g algebra courses, although a large part of commutative algebra, algebraic geometry, and much of algebraic number theory depends on reducing statements and proofs to arguments based on local rings.

4. Work with an integral domain A and its field of fractions $K = \operatorname{Frac} A$. Any $f \in K$ can be written as $f = g/h$ with $h \neq 0$. (Warning: you are not allowed to assume the expression is unique in any sense.)

Let $P \subset A$ be a prime ideal. Set

$$\begin{aligned}A_P &= S^{-1}A \quad \text{where } S = A \setminus P.\\ &= \{f \in K \mid \exists \text{ an expression } f = g/h \text{ with } h \notin P\}.\end{aligned}$$

Prove that this is local ring, with maximal ideal PA_P. This generalises the two cases of (3), and does not involve the added pain of dealing with zero divisors in S.

Prime ideals and Spec A

5. For ideals $I, J \subset A$, it is trivial that the product $I \cdot J$ is contained in $I \cap J$. Give several counterexample to the converse.

If P is a prime ideal and P contains $I \cdot J$, prove that it also contains $I \cap J$. [Once you get the point, the question is much too easy.]

6. If m_1 and m_2 are distinct maximal ideals, prove that $m_1 + m_2 = A$, that $m_1 \cdot m_2 = m_1 \cap m_2$, and that $A/(m_1 \cap m_2) \cong A/m_1 \oplus A/m_2$.

If $m_1, \dots, m_k$ are distinct maximal ideals, prove that

$$\bigcap_{i=1}^{k} m_i \subsetneq \bigcap_{i=1}^{k-1} m_i. \tag{0.4}$$

Deduce that an Artinian ring has only finitely many distinct maximal ideals.

7. Spec A is the set of prime ideals of A. Its Zariski topology has closed sets $V(I)$ for I an ideal. Run through the proof that it is a topology several times until you can remember it – there is nothing hard about this, but it is abstract and not memorable, so it takes time to get used to it.

8. Write $X = \operatorname{Spec} A$ with its Zariski topology. For $f \in A$, write X_f for the set of primes P with $f \notin P$. This is just the complement of $V(f) \subset X$. Prove that the $\{X_f \mid f \in A\}$ form a basis for the open sets of X. They are called the *principal* open sets.

9. Suppose that $\{f_\lambda\}_{\lambda\in\Lambda}$ is a subset of A such that the principal open sets X_{f_λ} cover X, that is

$$X = \bigcup_{\lambda\in\Lambda} X_{f_\lambda}. \tag{0.5}$$

Prove that there exists a finite subset $\{\lambda_1, \dots, \lambda_n\}$ such that $X = \bigcup_{i=1}^n X_{f_{\lambda_i}}$. That is *with no assumptions on* A, the Zariski topology on Spec A is compact (but not Hausdorff except in trivial cases).

Integral ring extensions and integral closure

Do the examples of [UCA 4.1-4.6], or find some more in other textbooks.

10. Let a, b be coprime and square-free elements of either ring $A = \mathbb{Z}$ or $A = k[x]$. Consider the field extension of $\mathbb{Q}$ or $k(x)$ given by adjoining $\xi = \sqrt[3]{a^2b}$. Calculate the integral closure of A. [Hint: (1) Notice that you can take a factor out of ξ^2 and still get an integral element η. (2) Find all the polynomial relations between ξ and η. (3) Consider a rational linear combination of $1, \xi, \eta$ and determine the conditions on it coefficients for it to be integral.]

The point here is that although it is easy in abstract algebra to say "integral closure", calculating it by hand is hard, and there are comparatively few cases where you can do it convincingly. That is why quadratic number fields are popular with number theorists.

Conditionally maximal implies prime

11. I illustrate the principle

an ideal that is maximal in some class is prime, with an important argument from primary decomposition.

Let A be a Noetherian ring and M a nonzero finite A-module. For any nonzero $m \in M$, write $\operatorname{Ann} m$ (annihilator) for the ideal of A given by

$$\operatorname{Ann} m = (0 : m) = \{f \in A \bigm| fm = 0\}. \tag{0.6}$$

Use the Noetherian assumption to prove that the class $\Sigma = \{\operatorname{Ann} m\}_{0 \neq m \in M}$ of annihilator ideals has a maximal element. Prove that any maximal element of this set is a prime ideal.

[Hint: Unsurprisingly, take $z_1, z_2 \notin \operatorname{Ann} m$. Start by making some useful deduction about the ideal $\operatorname{Ann}(z_1 m)$.]

Conclude that for any nonzero module M there exists a prime ideal P of A and a submodule of M isomorphic as A-module to the integral domain A/P. In primary decomposition, P is called an *associated prime* of M. Does the argument actually uses Noetherian conditions on A and/or M?

12. This is another pretty use of the same principle.

Cohen's theorem: *Let A be a ring. Assume that every prime ideal is finitely generated. Then A is Noetherian.*

Use Zorn's lemma: Let Σ be the set of ideals of A that are not finitely generated. Prove that if Σ is nonempty then it has a maximal element I. (This is a little mind-twister, but it is obvious once you've got the point!) Thus we can assume we have an ideal I that is not finitely generated, but such that $I + Ax$ is finitely generated for every $x \notin I$.

By contradiction, suppose $x, y \notin I$ but $xy \in I$.

Then by maximality $I + Ay$ is f.g. Show that we can write

$$I + Ay = (s_1, \dots, s_n, y) \quad \text{with } s_i \in I. \tag{0.7}$$

Also use $xy \in I$ to deduce that

$$x \in (I : y) = \{a \in A \bigm| ay \in I\}. \tag{0.8}$$

Deduce that $(I : y)$ is strictly bigger than I, so is also f.g., say $(I : y) = (r_1, \dots, r_m)$.

Finally, prove that $I = (s_1, \dots, s_n, yr_1, \dots, yr_m)$.