David Rees' 1956 paper

0.1 Definition of the Rees ring $R(A, \mathfrak{a})$

Write $A[t, t^{-1}]$ for the Laurent polynomial ring over a ring A with t an indeterminate of degree 1. For $\mathfrak{a} = (a_1, \dots, a_m)$ an ideal in A, the Rees ring $R(A, \mathfrak{a})$ is the $\mathbb{Z}$-graded subring $R(A, \mathfrak{a}) \subset A[t, t^{-1}]$ generated by $a_1 t, \dots, a_m t$ and t^{-1}. It has degree $-k$ graded piece $t^{-k} A$ for $-k < 0$.

A Laurent polynomial $c = \sum c_r t^r \in A[t, t^{-1}]$ is in $R(A, \mathfrak{a})$ if and only if $c_r \in \mathfrak{a}^r$ for $r \geq 0$. Identify A with the degree 0 piece $A = R_0 \subset R$. If we set $u = t^{-1}$ for the negative generator[1] of $A[t, t^{-1}]$ then multiplication by $u^n = t^{-n}$ takes the degree n piece R_n of $R(A, \mathfrak{a})$ into $t^{-n} R_n \cap A = (\mathfrak{a}^n) \subset A$.

The Rees ring $R(A, \mathfrak{a})$ is Noetherian if A is. The quotient ring $R(A, \mathfrak{a})_{\geq 0}/(t)$ is the graded ring $\operatorname{gr}_{\mathfrak{a}} A = \bigoplus \mathfrak{a}^n/\mathfrak{a}^{n+1}$ as discussed in [Ma, p. 120].

0.2 Krull's intersection theorem

Theorem 0.1 *For an ideal $\mathfrak{a}$ of a Noetherian ring*

$$x \in \bigcap\nolimits_0^\infty \mathfrak{a}^n \iff x = ax \quad \textit{for some } a \in \mathfrak{a}$$

Proof The implication $\Leftarrow$ is trivial. To prove the converse $\Rightarrow$, Step 1 is the special case with $\mathfrak{a} = (u)$ principal, generated by a nonzerodivisor u. Since $x \in \mathfrak{a}^i$ for every i, we can write $x = u^i y_i$. The Noetherian assumption applied to the ascending chain $\cdots \subset (y_i) \subset (y_{i+1}) \subset \cdots$ gives $(y_n) = (y_{n+1})$ for some n. Thus $y_{n+1} = b y_n$, and hence $y_n = a y_n$ where $a = bu \in \mathfrak{a}$. Then $ax = u^n a y_n = u^n y_n = x$.

The Rees ring $R(A, \mathfrak{a})$ reduces the general case $\mathfrak{a} = (a_1, \dots, a_m)$ to the case of a principal ideal (u). The element $u = t^{-1} \in R(A, \mathfrak{a})$ is a nonzerodivisor. If $x \in A$ is contained in $\mathfrak{a}^i$ then $x \in u^i R$. So by Step 1 there exists $c = \sum c_r t^r \in R(A, \mathfrak{a})$ for which $x = xcu$. Now $x \in A$, so that $x = ax$, where $a = c_1 \in \mathfrak{a}$. This proves the theorem. $\square$

[1] Including the negatively graded part of R allows u as a ring element; its main role is simply to relabel an element of R_n as an element of $\mathfrak{a} \cdot R_{n-1}$. We sometimes tacitly work only with $\bigoplus_{n \geq 0} R_n$.

Preparation for the Principal Ideal Theorem

Lemma 0.2 (Prototype for the Artin–Rees lemma) *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of a Noetherian ring A. Then there exists an integer k such that*

$$\mathfrak{a}^n \cap \mathfrak{b} = (\mathfrak{a}^k \cap \mathfrak{b})\mathfrak{a}^{n-k} \quad \textit{for all } n \geq k.$$

Proof Setting $\mathfrak{b}^* = \mathfrak{b}A[t, t^{-1}] \cap R$ defines a homogeneous ideal $\mathfrak{b}^*$ of $R = R(A, \mathfrak{a})$. It consists of all sums $\sum b_r t^r$ with $b \in \mathfrak{a}^r \cap \mathfrak{b}$. Since R is Noetherian, $\mathfrak{b}^*$ is generated by finitely many elements of the form $b_r t^r$. Taking k as the largest exponent of t involved among these generators gives at once

$$\mathfrak{a}^n \cap \mathfrak{b} = (\mathfrak{a}^k \cap \mathfrak{b}^*) = (\mathfrak{a}^k \cap \mathfrak{b})\mathfrak{a}^{n-k} \quad \text{for all } n \geq k.$$

Corollary 0.3 *Suppose $x \in A$ is a nonzerodivisor. Write $\mathfrak{a}^n : x$ for the colon ideal $\{c \in A \mid xc \in \mathfrak{a}^n\}$. There exists an integer k for which*

$$\mathfrak{a}^n : x \subset \mathfrak{a}^{n-k} \quad \textit{for all } n \geq k.$$

Proof By Lemma 0.2, there exists k such that

$$\mathfrak{a}^n \cap xA = (\mathfrak{a}^k \cap xA)\mathfrak{a}^{n-k} \subset x\mathfrak{a}^{n-k} \quad \text{for all } n \geq k.$$

But $\mathfrak{a}^n \cap xA = x(\mathfrak{a}^n : x)$. Now since x is a nonzerodivisor, $(\mathfrak{a}^n : x) \subset \mathfrak{a}^{n-k}$.

0.3 Krull's Hauptidealsatz (Principal Ideal Theorem)

Theorem 0.4 *Let A be a Noetherian local domain with maximal ideal m. Assume some principal ideal Ax is m-primary. Then every nonzero ideal of A is m-primary. In other words, m is the unique nonzero prime ideal of A or* $\operatorname{Spec} A = \{0, m\}$.

Proof Let $y \in A$ be a nonzero element. Apply Lemma 0.2 to $\mathfrak{a} = (x)$ and $\mathfrak{b} = (y)$ to get an integer k such that

$$x^{k+1}A \cap yA = x(x^k A \cap yA). \tag{1}$$

Claim (1) implies that $(x^{k+1}, y) = (x^k, y)$. The claim implies the theorem: it gives $x^k = ax^{k+1} + by$ for some $a, b \in A$, that we rewrite as

$$(1 - ax)x^k = by \in yA.$$

Now $(1-ax)$ is a unit of A, so that $x^k = by \in yA$, and yA is m-primary. Thus every nonzero ideal of A is m-primary.

To prove the claim, use the fact that since (x^n) is m-primary, $A/(x^n)$ and any of its subquotients are modules of finite length. There is an obvious inclusion

$$(x^{k+1}, y) \subset (x^k, y).$$

Calculating lengths of $A/(x^{k+1}, y)$ and $A/(x^k, y)$, we find that they are equal, and hence $(x^{k+1}, y) = (x^k, y)$. Start from

$$(x^{k+1}A + yA)/x^{k+1}A \cong yA/(x^{k+1}A \cap yA)$$

by the Third Isomorphism theorem $(M+N)/N \cong M/(M \cap N)$. Now

$$\begin{aligned}
\ell\bigl(yA/(x^{k+1}A \cap yA)\bigr) &= \ell\bigl(yA/x(x^kA \cap yA)\bigr) \quad \text{by (1)} && (2)\\
&= \ell\bigl(yA/xyA\bigr) + \ell\bigl(xyA/x(x^kA \cap yA)\bigr) && (3)\\
&= \ell\bigl(A/xA\bigr) + \ell\bigl(yA/(x^kA \cap yA)\bigr) && (4)\\
&= \ell\bigl(x^kA/x^{k+1}A\bigr) + \ell\bigl((x^kA + yA)/x^kA\bigr) && (5)\\
&= \ell\bigl((x^kA + yA)/x^{k+1}A\bigr). && (6)
\end{aligned}$$

Step-by-step: (2) to (3) inserts the intermediate ideal xyA between yA and $x(x^kA \cap yA)$. (3) to (4) uses multiplication by the nonzero element y in the domain A to give an isomorphism $A/xA \cong yA/xyA$ and similarly with x for the second summand. (4) to (5) multiplies by x^k on the first summand, and applies the Third Isomorphism theorem for the second. Then (5) to (6) omits the intermediate ideal x^kA. (Kaplansky's more structured interpretation is discussed below.)

Putting everything together gives

$$\ell\bigl((x^{k+1}A + yA)/x^{k+1}A\bigr) = \ell\bigl((x^kA + yA)/x^{k+1}A\bigr).$$

Since $x^kA + yA \supset x^{k+1}A + yA$, the claim follows. $\square$

Corollary 0.5 (Krull's Hauptidealsatz) *In a Noetherian ring, if P is a minimal prime containing a principal ideal Ax then P has height* $\operatorname{ht} P \le 1$.

Theorem 0.4 is the statement in the case of a local domain. Two straightforward steps reduce the general Noetherian case to this: we get the local case since a chain of primes $P_0 \subsetneq P_1 \subsetneq P$ in A would give a chain of prime ideals $P_0A_P \subsetneq P_1A_P \subsetneq PA_P$ in the local ring (A_P, PA_P). And by passing to the quotient rings $0 \subsetneq P_1/P_0 \subsetneq A/P$ reduces to a domain. This proves the theorem.

With hindsight, Kaplansky explains what is going on in Rees' display (2–6) more simply and convincingly. The same appeal to Lemma 0.2 gives (1). He now sets $u = x^k$ and interprets (1) as saying

$$tu^2 \in (y) \implies tu \in (y). \tag{!}$$

That is, the basic form of Artin–Rees allows us to cancel a power of u.

Now consider the submodule

$$(u^2, y)/u^2 \subset (u, y)/u^2. \tag{7}$$

Claim Assumption (!) implies equality in (7). The part equals the whole.

In fact on the rhs, inserting the submodule (u) gives the composition series $(u, y) \supset (u) \supset (u^2)$ with factors $A = (u, y)/u$ followed by $B = u/u^2$.

The lhs has composition series $(u^2, y) \supset (u^2, uy) \supset (u^2)$ with factors $C = (u^2, y)/(u^2, uy)$ and $D = (u^2, uy)/(u^2)$.

Now $A = (u, y)/(u) \cong D = (u^2, uy)/u^2$ (multiplying by u in a domain as in (3) to (4) of Rees' display). And

$$B = (u)/(u^2) \cong C = (u^2, y)/(u^2, uy),$$

follows using the magic implication (!).

We are in the Artinian set-up. The two modules in the claim both have the same finite length, and this proves the claim. $\square$

Commentary

Let A be a Noetherian ring. The Zariski topology on $\operatorname{Spec} A$ is Noetherian. Minimal prime ideals $P \in \operatorname{Spec} A$ correspond to its finitely many irreducible components.

Krull's 1928 Hauptidealsatz: *Suppose A is Noetherian and $x \in A$. Then a prime ideal $P \in \operatorname{Spec} A$ minimal among prime ideals containing x has $\operatorname{ht} P \leq 1$.*

If $\operatorname{ht} P = 0$ then P itself is a minimal prime of A. The alternative $\operatorname{ht} P = 1$ means that any $q \in \operatorname{Spec} A$ with $q \subsetneq P$ is a minimal prime ideal of A (and there exists at least one such). The result is nontrivial (Melvyn Hochster says it caused amazement in 1928). It is a corollary of the main theorem of dimension theory for Noetherian local rings.

There are a number of fairly unreadable proofs online, including that in Wikipedia (someone should beat that up).

References

[1] David Rees, Two classical theorems of ideal theory, Proc. Cambridge Philos. Soc. bf52 (1956) 155–157

[2] Irving Kaplansky, Commutative rings, Allyn and Bacon, Inc., Boston, MA (1970) x+180 pp.

[3] Hideyuki Matsumura, Commutative ring theory, Cambridge Stud. Adv. Math. **8** CUP (1986) xiv+320 pp.

[4] Melvin Hochster, Lecture notes

`https://dept.math.lsa.umich.edu/~hochster/615W17/supDim.pdf`

[5] Google search for Krull Hauptidealsatz gives

`https://en.wikipedia.org/wiki/Krull%27s_principal_ideal_theorem`