MA4L7 Algebraic curves. First example sheet

The first week's lectures talked around the prerequisites. (Many students who did the course MA4A5 will find this too easy.)

Exercise in Nakayama's lemma Let A be a local ring and M a finite A-module (the same assumptions as in Lemma 2.4), suppose that $m_1, \dots, m_n \in M$ generate M mod m (in other words, $M = mM + \sum Am_i$). Then $m_1, \dots, m_n$ generate M.

Integrally closed is a local condition If $A \subset L$ is an integral domain contained in a bigger field L (that is L is an extension field of $K = \operatorname{Frac}(A)$), show that A is integrally closed in L implies that $A[\frac{1}{g}]$ is also integrally closed in L. If each of its localisation A_p at prime ideals is integrally closed in L then so is A.

1. Affine varieties $X \subset \mathbb{A}^n$

Reread UAG, Chap. 2 up to the proof of NSS. I mainly work with varieties X that are 1-dimensional and irreducible. For these, the Zariski topology is the cofinite topology if X, which is one less thing to worry about.

2. Affine coordinate ring and function field

The coordinate ring is defined as $k[X] = k[x_{1\dots n}]/I_X$ [UAG, Chap. 4]. For irreducible X, the ideal I_X is prime, so that $k[X]$ is an integral domain, and $k(X) = \operatorname{Frac} k[X]$ is ints field of fractions.

Exercise 1.1 Use the NSS to establish the bijections

$$\Big\{\text{maximal ideals of } k[X]\Big\} \longleftrightarrow \Big\{\text{maximal ideal of } k[x_{1\dots n}] \text{ containing } I_X\Big\}$$
$$\longleftrightarrow \Big\{m_P = (x_i - a_i \bigm| i \in [1..n]), \text{ where } P = (a_{1\dots n}) \in X\Big\}.$$

and

$$\Big\{\text{prime ideals of } k[X]\Big\} \longleftrightarrow \Big\{\{\text{prime ideal of } k[x_{1\dots n}] \text{ containing } I_X\}\Big\}$$
$$\longleftrightarrow \Big\{\ I_Y \text{ with } Y \subset X \text{ irreducible subvariety.}\Big\}$$

These have the flavour "the ring $k[X]$ knowns everying about X", and will justify writing $X = \operatorname{Spec} X$ (with a small abuse of terminology concerning the single prime ideal $0 \subset k[X]$).

Exercise 1.2 X affine irreducible with affine coordinate ring $k[X]$ and function field $k(X)$. Prove that if $f \in k(X)]$ is regular at every $P \in X$, then $f \in k[X]$. That is, rational plus everywhere regular implies polynomial.

Moreover for $0 \neq g \in k[X]$, if $f \in k(X)]$ is regular at every $P \in X$ with $g(P) \neq 0$ then $f \in k[X][\frac{1}{g}]$.

In either case, you need to use NSS. This will be used later as a step in going from birational (geometry up to birational equivalence) to biregular (geometry up to isomorphism).

3. DVR

Recall the definition of DVR from lectures or one of the textbooks.

Exercise 1.3 Prove that $P \in X$ is a nonsingular point of a curve if and only if the local ring $\mathcal{O}_{X,P}$ is a DVR.

This is more or less the definition, but you have to get all the words right.

4. Integral closure.

Exercise 1.4 Show that $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$. More generally, if A is a UFD, prove that A is integrally closed. (That is, any element of $K = \operatorname{Frac} A$ that satisfies a monic polynomial equation over A is actually in A.)

Deduce that a DVR is integrally closed.

I proved above that a 1-dimensional Noetherian local ring A that is integrally closed in its field of fractions $K = \operatorname{Frac} A$ is a DVR.

Exercise 1.5 Prove the following lemma: consider $A \subset A[x]/f$ where A is a ring and $f \in A[x]$ a monic polynomial. Then

$$A \text{ is a field} \iff A[x]/f \text{ is a field.}$$

5. Rational functions on $\mathbb{P}^1$ and the "baby case"' of RR.

Let u, v be homogeneous coordinates on $\mathbb{P}^1$, and write $x = v/u$ for the affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$, and write $P = (1:0)$ and $Q = (0:1) \in \mathbb{P}^1$. The vector space $k[x]_{\leq d}$ has dimension $1+d$ (with a basis you can easily guess). If we view it as the space of rational fuctions with pole $\leq dQ$, it is the ideal first case of RR space $\mathcal{L}(\mathbb{P}^1, dQ)$. The equality $l(\mathbb{P}^1, dQ) = 1 - g + d$ (with $g = 0$) holds for all $d \geq -1$, and fails by 1 for $d = -2$.

By considering $(x-a)/(x-b)$, show that $k(\mathbb{P}^1)$ contains a function with $\operatorname{div} f = P_1 - P_2$ for any $P_1, P_2 \in \mathbb{P}^1$. More generally, if $\sum m_i P_i$ and $\sum n_j Q_j$ have $\sum m_i = \sum n_j$, then there exists $f \in k(\mathbb{P}^1)$ with $\operatorname{div} f = \sum m_i P_i - \sum n_j Q_j$.

Prove that $l(\mathbb{P}^1, D) = 1 - g + \deg D$ for any $D = \sum m_i P_i$ of degree $d = \sum m_i$.