MA4L7 Algebraic curves. Example sheet 2

Week 2 of lectures were on integral extensions, finite A-modules, normalisation, characterisation of DVR. The material is standard, covered in many commutative algebra textbooks. I mostly follow [UCA, esp. Chap. 8].

Recall that *finite A-module* means finitely generated as A-module: every element can be written as a *linear combination* of finitely many generators $e_1, \dots, e_n$. (As opposed to a finitely generated A-algebra $A \subset B$, when every $b \in B$ is a *polynomial* combination of generators $x_1, \dots, x_n$.)

1. Tower law. Let $A \subset B_1 \subset B_2$ are integral domains. If B_1 is finite as A-module and B_2 is finite as B_1-module prove that B_2 is finite as A-module.

Given the determinant trick [UCA, 2.7], modify the argument to prove the same statement for integral extensions.

2. Standard open sets X_g. If X is an affine algebraic variety with coordinate ring $k[X]$ and $g \in k[X]$, it is known that the open subvariety $X_g = \{P \in X \mid g(P) \neq 0\}$ is also affine, and has coordinate ring $k[X_g] = k[X][\frac{1}{g}]$. The X_g, called *standard open sets*, form a basis of the Zariski topology of X.

Prove that $k[X_g]$ is a finite $k[X]$-module if and only if $1/g$ is integral over $k[X]$. If $k[X]$ is already normal (integrally closed in $k(X)$), this happens only if g is a unit of $k[X]$, so that $X_g = X$. Thus the inclusion $X_g \subset X$ is usually not a finite morphism.

3. Finite and nonfinite extension. The nodal cubic $C \subset \mathbb{A}^2$ given by $y^2 = x^2(x+1)$ has the usual parametrisation $f \colon \mathbb{A}^1 \to C \subset \mathbb{A}^2$ given by $x = t^2 - 1$, $y = t(t^2 - 1)$. Show that f is finite, that is, $k[\mathbb{A}^1]$ is a finite $k[C]$-module. [Hint: $k[C] \cdot 1_{k[t]}$ contains x, y; what more do you need to get $k[\mathbb{A}^1]$? You might start by finding a basis of the vector space $k[t]/k[x,y]$.]

Now replace $\mathbb{A}^1$ by the hyperbola $H : s(t-1) = 1 \subset \mathbb{A}^2_{\langle t,s \rangle}$ and consider the polynomial map $f : H \to C$ given by $x = t^2 - 1$, $y = t(t^2 - 1)$. Show that f is a bijective map. Show that it is not finite (that is, $k[H]$ is not a finite $k[C]$-module).

4. Similar exercise. The cuspidal cubic $\Gamma : y^2 = x^3$ has parametrisation $x = t^2$, $y = t^3$. Show that it is finite. On the other hand $H = \mathbb{A}^1 \setminus 0$ defined by $st = 1$ is a nonsingular curve, and $x = t^2$, $y = t^3$ maps H isomorphically to $\Gamma \setminus (0,0)$. Show that $H \to \Gamma$ is not finite. (It misses the singular point, so we don't allow it as a resolution of singularities.)

5. Explicit normalisation. Let A be a UFD with field of fractions $K = \operatorname{Frac} A$, and assume $1/2 \in A$. For square-free $a \in A$, consider the quadratic field $K(\alpha)/K$ where $\alpha = \sqrt{a}$. Show that $A[\alpha] \subset K(\alpha)$ is integrally closed. [Hint: find the minimal polynomial of $c + d\alpha$ and show $d \in A$.]

Let A be a UFD with $K = \operatorname{Frac} A$, and assume $1/3 \in A$. Let $a, b \in A$ be square-free coprime elements. Consider the cubic extension field $L = K(\sqrt[3]{a^2b})$ generated by y with minimal polynomial $y^3 = a^2b$. Prove that y and $z = y^2/a$ are integral over A, and show that the ideal of all relations holding between y, z is generated by 3 quadratic relations in y, z. [Hint: $y^3 = a^2b$ is a linear combination of these 3.] Now given that $X = e + cy + dz \in L$ has minimal polynomial $(X - e)^3 - 3abcd(X - e) - ab(ac^3 + bd^3)$, deduce that $A[y, z]$ is the integral closure of A in L.

If $a = (x - 1)(x - 2)$ and $b = x(x + 1)$, determine the normalisation of the affine plane curve $y^3 = ab^2$.

6. Normalisation of monomial curve. Following on from the cuspidal cubic $y^2 = x^3$, determine the normalisation of $k[x, y]/(y^2 - x^5)$. Same question for $k[x, y]/(y^3 - x^7)$. More generally, if a, b are coprime, find the normalisation of $x^a = y^b$. [Hint: If you want to write $x = t^a$ and $y = t^b$ you are on the right track. However, for this to be a normalisation, you still have to establish that $t \in \operatorname{Frac}(A)$ where $A = k[x, y]/(x^a - y^b)$. In other words, express t in terms of x and y.]

7. Trace in a finite field extension. Let $K \subset L$ be a finite field extension. Recall from Galois theory that any $y \in L$ has a *minimal polynomial*, an irreducible polynomial

$$p(T) = T^d + c_{d-1}T^{d-1} + \cdots + c_1T + c_0 \in K[T]$$

such that $p(y) = 0$, so that $K[y] = K[T]/(p(T))$; it follows that $K[y] = K(y)$ is a field, since $(p(T))$ is a maximal ideal. We say that L/K is a *primitive extension* with generator y if $L = K(y)$.

Consider the multiplication map $\mu_y \colon L \to L$ consisting of multiplication by y. If L/K is a primitive extension, write out the matrix of μ_y in the basis $1, y, \ldots, y^{d-1}$, and deduce that its trace is $\operatorname{Tr}_{L/K} \mu_y = -c_{d-1}$.

In general, prove that the trace of μ_y equals $-c_{d-1}[L : K(y)]$. [Hint: let b_j for $j = 1, \ldots, [L : K(y)]$ be any basis of $L/K(y)$, and calculate the trace of μ_y in the basis $y^i b_j$ of L/K.