MA4L7 Algebraic curves

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Example Sheet 3

1. Number of forms of degree n Write $S_n = S^n(x,y,z) = k[x,y,z]_n$ for the space of homogeneous forms of degree n in x,y,z. Calculate the dimension of S_n. [Hint: To guess the answer, calculate it for $n = 0,1,2,3$. To prove it, set up an induction on n. (This is an absolutely basic calculation.)]

2. Hyperplane divisor H Let $C_a \subset \mathbb{P}^2_{\langle x,y,z\rangle}$ be a nonsingular curve of degree a, defined by homogeneous form $F_a(x,y,z) = 0$. Define the valuation $v_P(z)$ of the linear form z on C_a to be $d_P = v_P(z/x)$ if P is in the affine piece $x \neq 0$ or $d_P = v_P(z/y)$ if P is in the affine piece $y \neq 0$. Equivalently, it is the multiplicity of the form $F_a(x,y,z)$ restricted to $\mathbb{P}^1_{\langle x,y\rangle}$ as in [UAG, Chap. 1]. (N.B. Valuation starts off as a property of a rational function $f \in k(C)$ at $P \in C$; for a nonsingular projective curve $C \subset \mathbb{P}^n$, this definition extends it to homogeneous forms $f \in k[C]_{\text{homog}}$.)

Write $H = \operatorname{div}(z) = \sum v_P(z)P$ for the divisor of z on C_a (or "divisor at infinity"). It is an effective divisor of degree a by the argument of [UAG, Chap. 1].

If $L \subset \mathbb{P}^2$ is any line, show that $\operatorname{div}(L)$ is a divisor of degree a linearly equivalent to H.

3. Degree of a principal divisor Any rational function $f \in k(C_a)$ can be written as $f = G_m/H_m$ for some $G_m, H_m \in S_m$. Bézout's theorem says that G_m intersects the nonsingular curve C_a in a divisor of degree ma. Deduce the identity $\deg(\operatorname{div} f) = 0$. This is an informal treatment of one proof of Main Proposition (I). (See Shafarevich, 2nd Edition, 2.1, p.168 for details.)

4. The RR space of mH For $G_m \in S_m$ not vanishing on C_a, the rational function $G_m/z^m \in k(C_a)$ defines an element of $\mathcal{L}(C_a, mH)$. Calculate the dimension of the subspace defined by these restricted forms. [Hint: $G_m \in S_m$ to C_a vanishes on C_a if and only if G_m is in the ideal of multiples of F_a. That is, the sequence

$$0 \to S_{m-a} \to S_m \to \mathcal{L}(C_a, mH) \tag{0.1}$$

is exact, where the first map is multiplication by F_a.]

Prove that $l(C_a, mH) = \dim \mathcal{L}(C_a, mH)$ has dimension

$$\geq \binom{m+2}{2} - \binom{m-a+2}{2} \quad \text{if } m \geq a.$$

Show how to rewrite this as $1 - g + \deg(mH)$ for appropriate g.

From now on, assume that (0.1) is also exact at the right end. Deduce the exact formula $l(C_a, mH) = 1 - g + \deg(mH)$ for $m \geq a$.

5. Informal proof of Main Proposition (II) A nonsingular projective curve $C \subset \mathbb{P}^n$ can be mapped birationally to a plane curve $C_a \subset \mathbb{P}^2$, almost always acquiring finitely many singular points along the way. Choose a line H of $\mathbb{P}^2$ that meets C_a transversally in a nonsingular points $\{P_i\}$, and let $D = P_1 + \cdots + P_a$ be the divisor on the original C made up of those points. For a general form $G_d(x, y, z)$ of degree d on $\mathbb{P}^2$ (also meeting C_a transversally), the divisor G_d is linearly equivalent to dH. These forms map to a subspace of $\mathcal{L}(C, dH)$ of dimension $\binom{d+2}{2} - \binom{d-a+2}{2}$. Use this to prove Main Proposition (II).

6. The canonical class of the nonsingular curve C_a of Exc. 4 is $(a-3)H$ For $m = a, a-1, a-2, a-3$, you get into interpreting the binomial coefficient $\binom{n}{2}$ for $n \leq 2$. Show that the exact formula of Q4 works as stated for $m \geq a - 2$.

By considering $m = a - 3$, show that C_a has a divisor K_C so that $\deg K_C = 2g - 2$ and $l(K_C) = g > 1 - g + \deg K_C$.

7. $\mathcal{L}(K_C + P)$ With $C_a \in \mathbb{P}^2$ and $K_C = (a-3)H$ as above, prove that $\mathcal{L}(K_C + P) = \mathcal{L}(K_C)$ for any $P \in C$. [Hint. Let L be any line through P. The divisor $\operatorname{div}(L) - P$ is what you get by taking the intersection $C_a \cap L$ in [UAG, Chap 1] consisting of a points with multiplicity, and decrease the multiplicity of P by 1 (usually from 1 to 0). Now consider $\mathcal{L}(C_a, (a-2)H)$ from Q4 above, and impose the conditions of vanishing on $\operatorname{div}(L) - P$.]

8. Linear systems on elliptic curve The traditional Weierstrass model of an elliptic curve is $y^2 = x^3 + ax + b$ (completed as $y^2 z = x^3 + axz^2 + bz^2$ in $\mathbb{P}^2_{\langle x,y,z\rangle}$, with the single point $O = (0, 1, 0)$ at infinite as the origin. (See 5.1 in the main text for the interpretation using Weierstrass $\wp$-function.)

Show that the functions

$$1, x, \ldots, x^m, y, xy, \ldots, x^{m-2}y \in \mathcal{L}(2mO),$$
$$\text{respectively} \quad 1, x, \ldots, x^m, y, xy, \ldots, x^{m-1}y \in \mathcal{L}((2m+1)O) \quad (0.2)$$

base the RR spaces. Show also that $y^2 \in \mathcal{L}(6O)$ satisfies the Weierstrass equation $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{C}$. [Hint: Argue on their leading term at O.]