MA4L7 Algebraic curves

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Worksheet 4

Exercise 4.1 Write down half a dozen characterisations of $\mathbb{P}^1$ of $g=0$ and check that you know how to go from one to another. (This is Part 3, Section 8.1 of the notes, but it is more useful for you to do it for yourself.)

Exercise 4.2 Prove that a curve $C\subset\mathbb{P}^3$ that spans $\mathbb{P}^3$ and has degree 3 is projectively equivalent to the twisted cubic, the image of $\mathbb{P}^1$ under the embedding u^3,u^2v,uv^2,u^3.

Exercise 4.3 The same question for $C\subset\mathbb{P}^r$ that spans $\mathbb{P}^r$ and has degree r. Show that C is projectively equivalent to the rational normal curve of degree r parametrised by $S^r(u,v)=\{u^r,u^{r-1}v,\dots,v^r\}$ and defined by the $\binom{r}{2}$ minors of the $2\times r$ matrix $\left(\begin{smallmatrix}x_0&x_1&\dots&x_{r-2}&x_{r-1}\\x_1&x_2&\dots&x_{r-1}&x_r\end{smallmatrix}\right)$.

Exercise 4.4 Use the RR theorem to prove that a curve C of $g=1$ is isomorphic to a plane cubic. More precisely, give $P\in C$, you know $\mathcal{L}(C,P)$, $\mathcal{L}(C,2P)$, $\mathcal{L}(C,3P)$, that you can think of as embedded one in the next. Give basis elements names and calculate multiplication maps $\mathcal{L}(aP)\times\mathcal{L}(bP)$ to $\mathcal{L}((a+b)P)$ for the initial values $(a,b)=2,3,4,\dots$. Derive the Weierstrass normal form $C_3\subset\mathbb{P}^2$ with P as the flex at infinity.

Exercise 4.5 Replacing the $P\in C$ of the previous question with the letter O (for origin or zero). Show how mapping $P,Q\in C$ to the unique effective divisor linearly equivalent to $P+Q-O$ defines a group law on a curve of $g=1$. Find an argument that explains (or proves) that this is “the same” as the popular secant-tangent construction.

Exercise 4.6 Treat the embedding $\varphi_{4P}\colon C\to\mathbb{P}^3$ in a similar manner. Show that the image $C_4=Q_1\cap Q_4$ is the complete intersection of 2 quadrics, and try to write them in a nice normal form analogous to the Weierstrass form of a cubic.

Exercise 4.7 Extended question on $g = 2$ and $\deg D = 5$.

Let C be a curve of $g = 2$. Remind yourself of the properties of K_C and the canonical map φ_{K_C}

(I) Let D be a divisor of degree 3. Prove that one of the following alternatives holds:

- either $|D|$ has a fixed point P and $D \overset{\text{lin}}{\sim} K_C + P$;
- or $|D|$ is a free g^1_3.

Suppose that P_1, P_2, P_3 are three points that φ_{K_C} maps to distinct. Prove that $|D| = P_1 + P_2 + P_3|$ is a free g^1_3.

(II) State without proof the criterion for a divisor D on C to be very ample. Prove that any divisor D of degree 5 on C is very ample.

Prove that the image $\varphi_D(C)$ is contained in a quadric hypersurface $Q \subset \mathbb{P}^3$. (Hint: Map the vector space of quadratic forms on $\mathbb{P}^3$ to $\mathcal{L}(C, 2D)$.)

(III) Challenge question for the committed student. Let $Q \subset \mathbb{P}^3$ be an irreducible quadric surface. Show that Q is either:

(i) A quadric of rank 4, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, having the equation $(xt = yz)$.

(ii) Or an ordinary quadric cone with vertex $P = (0, 0, 0, 1)$, having the equation $xz = y^2$.

In case (i), prove that $D \overset{\text{lin}}{\sim} K_C + A$, with A a free g^1_3 on X.

In case (ii), prove that C passes through the vertex P of Q, and $D \overset{\text{lin}}{\sim} 2K_C + P$.

Conversely, determine a priori which degree 5 divisors belong to which case (i) or (ii).

Exercise 4.8 Let C be a hyperelliptic curve of genus g, A its g^1_2, and $\varphi_A \colon C \to \mathbb{P}^1$ its associated morphism, which is a 2-to-1 cover. Write t_1, t_2 for homogeneous coordinates on the image $\mathbb{P}^1$.

Study the linear system multiples $|bA|$. For $b \le g-1$, show that the $b+1$ monomials $S^b(t_1, t_2) = \{t_1^b, t_1^{b-1}t_2, \dots, t_2^b\}$ base $\mathcal{L}(C, bA)$, and that $(g-1)A$ is a canonical divisor of C.

Show that the linear system $|gA|$ is also composed of the double cover to $\mathbb{P}^1$. (Compare the $g+1$ monomials $S^g(t_1,t_2)$ with the RR equality for the dimension $l(gA)$ in the regular range.)

Write down the RR equality for the dimension $l((g+1)A)$ (in the regular range) and deduce that $|(g+1)A|$ is *not* composed with the morphism φ_A. In more detail

$$\mathcal{L}((g+1)A) = S^{g+1}(t_1,t_2) \oplus k \cdot y \tag{4.1}$$

with $y \in \mathcal{L}((g+1)A)$ a rational function that is not in $k(\mathbb{P}^1) = k(t_1/t_2)$.

Now prove that for every $b \geq g+1$, the monomials $S^b(t_1,t_2)$ and $S^{b-g-1}(t_1,t_2) \cdot y$ form a basis of $\mathcal{L}(bA)$.

Show that the monomial y^2 in the new generator y in $\mathcal{L}((g+1)A)$ is given by a quadratic relation that (in characteristic $\neq 2$) can be written as $y^2 = f(t_1,t_2)$. (Complete the square.)

The double cover $C \to \mathbb{P}^1$ has 2 distinct points of C over most points of $\mathbb{P}^1$. Use y to say where it is ramified. Use this to write the standard model $y^2 = f(t_1,t_2)$ of C, determine the number of branch points $P_i \in \mathbb{P}^1$ where C looks locally like $y^2 = t$. The points Q_i of C over the branch points P_i are called *ramification points* or *Weierstrass* points. How many of them are there? Determine the divisor class of the divisor $\sum Q_i$.

Unsorted extra questions, incl. past exams

Exercise 4.9 For distinct $a_i \in k$, write

$$F = \prod (x - a_i) \quad \text{and} \quad G_i = \prod_{j \neq i} (x - a_j).$$

Find the zeros of $\sum \lambda_i G_i - 1$, and deduce that

$$\sum_i \lambda_i G_i \equiv 1 \quad \text{where} \quad \lambda_i = \prod_{j \neq i} \frac{1}{a_i - a_j}.$$

Exercise 4.10 Let $P_i \in \mathbb{A}^n$ be a finite set of distinct points. Prove that there exists polynomials $f_i \in k[x_{1\dots n}]$ for which

$$f_i(P_j) = \delta_{ij} \quad \text{(Kronecker delta)}.$$

[Hint: Use a projection to $\mathbb{A}^1$ that separates the P_i, and apply Ex. 4.1.]

Deduce that there exists $g_i \in m_{P_j}^N \subset k[x_{1\dots n}]$ (where m_{p_j} is the maximal ideal at P_j) with $g_i(P_i) = 1$.

Exercise 4.11 For k an algebraically closed field and $I \subset k[x_{1\dots n}]$ an ideal defining a finite set $V(I) = \{P_i\} \subset \mathbb{A}^n$, prove that

$$k[x_{1\dots n}]/I \cong \bigoplus_{P_i} \mathcal{O}_{P_i}/I \cdot \mathcal{O}_{P_i}.$$

Use the coprime result of Ex. 4.3 together with the NSS to prove the natural map is surjective and injective. For more details, compare [Fulton, Algebraic curves, Section 9, Prop. 6].

Exercise 4.12 Let $F, G \in k[x, y]$ be polynomials with no common factors. Then $V(F, G) \subset \mathbb{A}^2$ is a finite set, say $\{P_i\}$. Write $\mathcal{O}_P = \mathcal{O}_{\mathbb{A}^2,P}$ for the local ring of $P \in \mathbb{A}^2$, and at each P_i, consider the ideal $\mathcal{I}_{P_i} \subset \mathcal{O}_{P_i}$ for the ideal generated by F, G.

Write $I_{P_i} \subset k[x, y]$ for the inverse image of $\mathcal{I}_{P_i}$ under localisation map. In other words, I_{P_i} is the ideal of polynomials whose image in $\mathcal{O}_{\mathbb{A}^2,P}$ is in the local ideal generated by F, G.

Each $\mathcal{I}_{P_i}$ contains a power of the maximal ideal m_{P_i} (by the NSS).

Therefore the quotient $\mathcal{O}_{P_i}/\mathcal{I}_{P_i}$ is a finite dimensional algebra over k (Artinian algebra)/ Also, the $\mathcal{I}_{P_i}$ are strongly coprime: for each i one can find a polynomial $f \in k[x, y]$ with $f(P_i) = 1$ and $f \in \mathcal{I}_{P_j}$ for $j \neq 1$.

Exercise 4.13 Let $C = C_a \subset \mathbb{P}^2$ be a nonsingular curve of degree a defined by the homogeneous polynomial F_a. Assume that C meets the line $z = 0$ transversally in a points and set H for the divisor of z.

A rational function $h \in k(C)$ can be written $h = A/B$ with A, B homogeneous forms of the same degree d. If $h = \mathcal{L}(C, nH)$, prove that B is in the ideal (z^n, F_a), so that f can also be written as A'/z^n. This implies that the map $k[x, y, z]_n \to \mathcal{L}(C, nH)$ discussed in Ex. 2.7 is surjective. [Hint: The assumption $h \in k(C)$ is that its poles on C are cancelled locally by multiplication by z^n.]

Exercise 4.14 (Genus 1 curve as $Q_1 \cap Q_2 \subset \mathbb{P}^3$) Let E be a genus 1 curve and $P \in E$. Assume known the treatment of $R(E, P) = \bigoplus \mathcal{L}(E, nP)$ as $k[x, y, z]/(f_6)$ with x, y, z of degree $1, 2, 3$ and f_6 the relation

$$z^2 = y^3 + ax^4y + bx^6.$$

Calculate the subring $R(E, 4P) \subset R(E, P)$. [Hint: Give names to the monomials of degree 4. Find two quadratic relations between these in $\mathcal{L}(8P)$, either as trivial coincidences between monomials, or involving multiples of f_6.]

Deduce that $4P$ is very ample and that the image of $\varphi_{4P}\colon E \to \mathbb{P}^3$ is an intersection of two quadrics.

Exercise 4.15 (Curve of genus 3) Let C be a curve of $g = 3$. Write out the dimensions $l(nK_C)$ for $n = 0$, $n = 1$ and $n \geq 2$. The canonical ring of C is

$$R(C, K_C) = \bigoplus_{n \geq 0} \mathcal{L}(C, nK_C).$$

Complete the calculations of $R(C, K_C)$ given in lectures:

If $\varphi_{K_C}\colon C \to \mathbb{P}^2$ is an embedding, prove that the hypersurface ring $k[x_{0\dots2}]/f_4$ has the right dimension in each degree. Then the graded homomorphism $k[x_{0\dots2}] \to R(C, K_C)$ is surjective by Max Noether's theorem, and the kernel consists exactly of the multiples of f_4.

If there is a quadratic relation $q(x_{0\dots2}) = 0$, prove that there is only one. The space $\mathcal{L}(2K_C)$ needs a further relation y. You have to figure out the number of monomials in $x_{0\dots2}, y$ in each degree, and the number of monomials multiples of the relation q in each degree. You need to prove that there is a relation F_4 involving y^2, and finally check that the dimension of $k[x_{0\dots2}, y]/(q_2, F_4)$ matches the dimension of $\mathcal{L}(nK_C)$ in each degree $n \geq 4$.

Exercise 4.16 (Half-canonical divisor) Let C be a curve of $g = 3$ and $P, Q \in C$ two points such that $2(P + Q) \overset{\text{lin}}{\sim} K_C$. Set $A = P + Q$, so that $K_C = 2A$. Assume that $l(A) = 1$ (the cases $l(A) = 0$ and $l(A) = 2$ are also possible, and interesting, but left for another day).

Consider the sections ring $R(C, A) = \bigoplus_{n \geq 0} \mathcal{L}(C, nA)$. This clearly contains the canonical ring $R(C, K_C) = R(C, 2A)$. The question is to determine the possibilities for generators and relations for $R(C, A)$, by analogy with Ex. 4.5. Geometrically, there are two cases: if $C = C_4 \subset \mathbb{P}^2$, the line joining P, Q is a bitangent of C_4. If C is hyperelliptic then P, Q are ramification points of the double cover $C \to \mathbb{P}^1$.

Write out the dimension of $\mathcal{L}(C, nA)$ for $n = 0, 1, 2$ and $n \geq 3$.

Check that these dimensions coincide with those of the graded ring $k[x, y_1, y_2, z]/(f_4, g_6)$, where the generators have degrees $1, 2, 2, 3$ and the relations have the indicated degrees.

In the hyperelliptic case, there is an extra quadratic relation $q(x^2, y_1, y_2) = 0$ (of weighted degree 4), and an extra generator t of degree 4, giving $k[x, y_1, y_2, z, t]/(q_4, f_4, g_6)$.

Exercise 4.17 (Past exam question) Part 1. The proof of RR used in the course was based on three main propositions. The first two of these are:

(I) A principal divisor has degree zero: $\deg(\operatorname{div} f) = 0$ for all $f \in k(C)^\times$.

(II) There exists a sequence of divisors D_n of degree tending to $+\infty$ such that the difference $\deg D_n + 1 - l(D_n)$ is bounded.

Use (I) and (II) together with the standard methods of argument to prove the following results:

(i) The maximum $g = \max_D \bigl\{\deg D + 1 - l(D)\bigr\}$ taken over all divisors D is well defined, so that the Riemann–Roch inequality $l(D) \geq 1 - g + \deg D$ is satisfied for every divisor D.

(ii) With g as in (i), every divisor D of degree $\geq g$ has $l(D) > 0$, so is linearly equivalent to an effective divisor.

(iii) There exists a divisor D of degree $g - 1$ for which $l(D) = 0$, so that the RR inequality is equality.

(iv) $l(D) = 1 - g + \deg D$ holds for every divisor D of degree $\geq 2g - 1$.

Part 2. Suppose $g(C) = 2$ and $\deg D = 4$. Prove that $l(D - K_C) \neq 0$, and deduce that φ_D is not an embedding. Show that φ_D is either a generically 2-to-1 map of C to a plane conic, or maps C birational to a quartic curve $\overline{C}$ with a node or cusp as its only singularity. Explain which divisors D correspond to each case. [You may use the criteria on embeddings, and standard properties of the canonical map of a genus 2 curve.]

Status: Bookwork. The whole proof of RR is too long for an exam question, but it is fair to state parts of it as given, and ask for the proof of the next part. The rider Part 2 is too hard for an exam question, but was set as an earlier question.

Exercise 4.18 (Harder question: $g(C) = 3$ and $\deg D = 5$ divisor) Let C be a nonsingular curve of $g = 3$ and D a divisor of degree 5, with D not linearly equivalent to $K_C + P$. Prove that the linear system $|D|$ defines a birational map φ_D of C to a plane quintic $\overline{C}_5 \subset \mathbb{P}^2$. [Hint: Prove that $|D|$ is a free g^2_5. Now, because 5 is a prime, the map φ_D cannot be a multiple cover of another curve, so must be generically one-to-one.]

A nonsingular plane quintic would have $g = 6$, so that $\overline{C}_5$ must be singular. Consider the two main cases: (i) $\overline{C}$ has distinct 3 nodes; and (ii) $\overline{C}$ has an ordinary triple point.

If $\overline{C}$ is a plane quintic with nodes at the three coordinate points, show that the standard quadratic transformation $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $(x, y, z) \mapsto$

$(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}) = (yz : xz : xy)$ takes $\overline{C}$ to a nonsingular quartic, and the composite $\psi \circ \varphi_D$ is the canonical embedding φ_{K_C} of a nonhyperelliptic C of $g = 3$.

If $\overline{C}$ is a plane quintic with an ordinary triple point P, show that the linear projection from P is a double cover, so that $\overline{C}$ is birational to a hyperelliptic curve of $g = 3$.

Exercise 4.19 (Genus 6) Let C be a curve of $g = 6$, and assume it has no g^1_2, g^1_3 or g^2_5. If D is a g^1_4, show that $K - D$ has degree 6 and $l(K - D) = 3$. Show that $|K - D|$ is a g^2_6, so defines a morphism $\varphi_{K-D} \colon C \to \mathbb{P}^2$.

Let $\Gamma_6 \subset \mathbb{P}^2$ be a sextic having double points (nodes or cusps) at the 4 points $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$ of the standard projective frame of reference. By considering the linear system of cubics of $\mathbb{P}^2$ passing through the 4 points, show that the resolution C has a linear system of dimension ≥ 6 and degree ≤ 10.

Given that its resolution $C \to \Gamma_6$ is a curve of genus 6. Show that C has 5 g^1_4s and complementary g^2_6s. [Hint: Four of them are fairly obvious. The fifth comes from the pencil of conics through the 4 points.]

It is a fact that any curve of genus 6 is given either by this construction, or a different construction adapted to the case that C has a g^1_2, g^1_3 or g^2_5, or is a double cover of curve of $g = 1$. (The g^2_5 case correspond to a plane quintic $C_5 \subset \mathbb{P}^2$.) Unfortunately, it would be something of a detour from the main course to discuss this rigorously or comprehensibly.