MA4L7 Algebraic curves

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Worksheet 5

Exercise 5.1 Curve C of genus 2 with a g^1_3. Assume k has characteristic $\neq 2$. Describe the morphism φ_{K_C}. Let P_1, P_2, P_3 be points mapping to different points under φ_{K_C}. Prove that $|P_1 + P_2 + P_3$ is a free g^1_3.

Now let D be a divisor of degree 5. Prove that $|D|$ is a free linear system and embeds $C \hookrightarrow \mathbb{P}^3$. Prove that image of C in $\mathbb{P}^3$ is contained in a quadric $Q \subset \mathbb{P}^3$. [Hint: Restrict quadrics of $\mathbb{P}^3$ to C and count dimensions.]

If Q is of rank 4, it is projectively equivalent to $Q : xt = yz$ and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Deduce that $D \overset{\text{lin}}{\sim} K_C + A$ where A is a free g^1_3.

If Q is of rank 3, it is projectively equivalent to $Q : xz = y^2$, and is an ordinary cone with vertex $P = (0, 0, 0, 1)$. In this case, prove that C passes through P and that $|D| = |P + 2K_C|$. [Hint: The generators of the cone cut out a linear system on C.]

Exercise 5.2 Curve of genus 5 with a g^1_3. Let C be a curve of genus 5, and suppose D is a divisor on C with $\deg D = 3$ and $l(D) = 2$.

Write s_1, s_2 for a basis of $\mathcal{L}(C, D)$. Prove that $l(2D) \geq 3$, and deduce that $K_C - 2D$ is linearly equivalent to an effective divisor B of degree 2.

Assume that $|K_C - D|$ is a free linear system, and the divisor $B = P_1 + P_2$ consists of distinct points $P_1 \neq P_2$. Prove that $|K_C - D|$ defines a birational morphism $\varphi_{K_C - D} \colon C \to \overline{C} \subset \mathbb{P}^2$ where $\overline{C}$ is a plane quintic.

Prove that $\varphi_{K_C - D}$ maps the two points P_1, P_2 of B to the same point $Q \in \overline{C}$, and that this is a singular point of $\overline{C}$. [Hint: It is enough to prove that $l(K_C - D - P_1 - P_2) = 2$, given that $l(K_C - D) = 3$.] Interpret the given D and its g^1_3 as the pencil of lines of $\mathbb{P}^2$ through Q. What difference does ir make if $P_1 = P_2$?

Conversely, if $\overline{C}_5$ is a plane quintic with a node or a cusp P, show that the nonsingular model of C has $g = 5$ a has a g^1_3 cut out by the birational transform of the pencil of lines through P.

Exercise 5.3 The 2-torsion of a hyperelliptic curve of genus $g \geq 2$.

Let be the hyperelliptic double cover $C \to \mathbb{P}^1$, or the hypersurface $y^2 = f_{2g+2}(x_1, x_2)$ in the weighted projective plane $\mathbb{P}(1,1,g+1)_{\langle x_1, x_2 \rangle}$. Write A for the g^1_2. There are $2g+2$ ramification points $x = a_i$ in $\mathbb{P}^1$ (with $a_i \neq a_j$), and over each a *Weierstrass point* P_i where the curve is locally $y^2 = (x - a_i)$.

Show that $2P_i \overset{\text{lin}}{\sim} A$. Show also that $\sum P_i \overset{\text{lin}}{\sim} (g+1)A$.

Show that $B_{ij} = P_i - P_j$ is a 2-torsion divisor class up to linear equivalence, that is, a divisor with $2B_{ij} \overset{\text{lin}}{\sim} 0$. Also any linear combination of the B_{ij} is 2-torsion. These correspond to taking difference of disjoint subsets of the $2g+2$ Weierstrass points $\{P_i\}$ with appropriate parity conditions, modulo the linear equivalence relations mentionned above. Identify 2^{2g} of these.

These are the 2-torsion points of the Jacobian variety $\operatorname{Jac} C$, which is a g-dimension Abelian variety that I will describe briefly in extra time at the end of the course.

Exercise 5.4 Let $C \subset \mathbb{P}^2$ be the plane quintic curve defined by

$$F = x^2y^3 - yz^4 - x^3z^2$$

Write down the 3 standard affine pieces. Prove that the piece $z = 1$ is nonsingular. [Hint: calculate $yF_y - 3F$ and set $z = 1$.] Also C meets the line $z = 0$ in two singular points.

The point $(1,0,0)$ is a cusp $z^2 = y^3 - yz^4$ (locally $z^2 = y^3 + \text{h.o.t.}$).

The point $(0,1,0)$ is a tacnode $x^2 = z^4 - x^3x^2$ (locally $x^2 = z^4 + \text{h.o.t.}$, that is $(x+z^2)(x-z^2)) = \text{h.o.t.}$.

The canonical system of the resolved curve is the quadrics of $\mathbb{P}^2$ passing through the node, and through the tacnode and its "infinitely near double point", or in more explicit language, passing through $(0,1,0)$ and tangent there to $x = 0$. That is, quadrics contained in ideal $(y,z) \cap (x, z^2)$, which gives (xy, xz, z^2). Setting $u = xy, v = xz, w = z^2$ and eliminating x, y, z gives the canonical image as the nonsingular quartic defined by $G = u^3w - uw^3 - v^4$.

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R<x,y,z> := PolynomialRing(Rationals(),3);
PP2 := Proj(R);

Bs := Scheme(PP2, [y,z]) join Scheme(PP2, [x,z^2]);
Basis(Ideal(Bs));
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F := x^2*y^3 - y*z^4 - x^3*z^2;
C0 := Curve(PP2,F);
KPP2<u,v,w> := ProjectiveSpace(Rationals(),2);
phi := map< PP2 -> KPP2 | [ x*y, x*z, z^2 ] >;
Image(phi,C0,4);
C := Curve(KPP2, -v^4 + u^3*w - u*w^3);
IsNonsingular(C);
Genus(C);
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Exercise 5.5 [past exam question — be brief on the bookwork]

a) Let C be a nonsingular projective curve and D a divisor. Define the associated rational map $\varphi_D\colon C \to \mathbb{P}^{l(D)-1}$, and explain why it is a morphism.

   Say what it means for $|D|$ to be a free linear system, and when this holds, explain how $|D|$ relates to the projective space $\mathbb{P}^{l(D)-1}$ (the proof is not required).

   Give the definition of a very ample divisor D. State the criterion for D to be very ample in terms of Riemann–Roch spaces $\mathcal{L}(D')$ with D' divisors related to D and points of C. Explain why the conditions are necessary. (The proof of sufficiency is not required).

   In (c, d) below you may assume that a curve C of genus g has a divisor K_C with $\deg K_C = 2g-2$ and $l(K_C) = g$, and K_C is unique up to linear equivalence.

b) Prove that a divisor D of degree $d \geq 2g+1$ on C of genus g is very ample.

c) Prove that a divisor D of degree $2g$ linearly equivalent to $K_C + P + Q$ is not very ample.

d) Prove that a divisor D of degree $2g$ that is not linearly equivalent to $K_C + P + Q$ for any $P, Q \in C$ is very ample.

   In (e) you may assume that a nonsingular plane curve $C_a \subset \mathbb{P}^2$ of degree a has genus $g = \binom{a-1}{2}$. A divisor D on C is regular if equality holds in the RR formula $l(D) = 1 - g + \deg D$.

e) Prove that on a curve of genus g, a divisor D that is regular and very ample must have $\deg D \geq g+3$, except for the three cases $g = 0$ and $d = 1$ or 2, and $g = 1$ and $d = 3$.

Exercise 5.6 [past exam question — be brief on the bookwork]

a) Let $A \subset K$ be a subring of a field. If $y \in K$ is integral over A, prove that the subring $A[y] \subset K$ is a finite A-module (finitely generated as A-module).

   Generalise the statement to the subring $A[y_1, \ldots, y_n] \subset K$ generated by finitely many integral elements. [The proof is not required.]

b) Suppose now that $A \subset B \subset K$ with the subring B a finite A-module. Prove that any $b \in B$ is integral over A. [Hint: Choose generators of B over A, and write out the A-linear map of multiplication by b as a matrix with entries in A. Argue on the determinant of b times the identity minus this matrix.]

c) Deduce from (a–b) that the sum and products of elements of K that are integral over A are again integral over A, hence that the integral closure of A in K is a subring.

d) Calculate the integral closure of the ring $A = k[x, y]/(y^3 - x^8)$ in its field of fractions.

   Explain briefly how taking normalisation (integral closure) provides the nonsingular model of the curve $C \subset \mathbb{A}^2$ given by $y^3 = x^8$.

Exercise 5.7 [past exam question — be brief on the bookwork]

a) Let C be a nonsingular curve of genus 1 over $\mathbb{C}$ and $P \in C$ any point. Give the dimension of the spaces $\mathcal{L}(C, nP)$ for $n \geq 1$. (No proofs are required.)

b) Prove that for $n \geq 2$ an element $x \in \mathcal{L}(nP) \setminus \mathcal{L}((n-1)P)$ has a pole of order exactly n at P.

c) Write down monomials in x and y that provide bases of the spaces $\mathcal{L}(4P)$ and $\mathcal{L}(5P)$.

d) Write down 7 monomials in x and y contained in $\mathcal{L}(6P)$ and deduce that there must be a linear dependence relation between them.

e) Use change of coordinate in the x and y to reduce the linear dependence relation to the form

$$y^2 = x^3 + ax + b.$$

f) Deduce that there exists an isomorphism φ of C with the plane cubic $E \subset \mathbb{P}^2$ given by

$$y^2 z = x^3 + axz^2 + bz^3, \quad \text{for some } a, b \in \mathbb{C}$$

where $\varphi\colon C \to E$ takes $\varphi(P) = (0,1,0)$.

g) Prove that $\varphi_{4P}\colon C \to \mathbb{P}^3$ defines an isomorphism of C onto the curve given by the two quadratic equations

$$uv = x^2 \quad \text{and} \quad y^2 = xv + aux + bu^2.$$

[Hint: Write u, x, y, v for the 4 monomials you found in $\mathcal{L}(4P)$.]

Exercise 5.8 (Deforming hyperelliptic curves to nonhyperelliptic) It is easy to put the two models for $g = 3$ curves together in a parametrised family: the nonhyperelliptic curves are plane quartic $C_4 \subset \mathbb{P}^2$.

The hyperelliptic ones with marked divisor $A = g^1_2$ are $C_8 \subset \mathbb{P}(1,1,2)$. In this case, if we mark the curve with $K_C = 2A$ for compatibility, we get the complete intersection $(q_2, F_4) \subset \mathbb{P}(1,1,1,2)$ where the variables of degree 1 are $\{x_1, x_2, x_3\} = S^2(s_1, s_2)$ with the relation $x_1 x_3 = x_2^2$. The new generator y of degree 2 is needed to make up for the relation.

To join them up, take the family of complete intersections

$$(q_2, F_4) \subset \mathbb{P}(1,1,1,2) \quad \text{where} \quad \begin{cases} q_2 = \lambda y - (xz - y^2), \\ F_4 = y^2 - f_4(x_1, x_2, x_3), \end{cases}$$

with λ a deformation parameter of weight 0. If $\lambda = 0$ the curve is a double cover of the plane conic $x_1 x_3 - x_2^2$, with the 8 points of intersection $q_2 = f_4 = 0$ as branch points. whereas if λ is invertible, the ring element is no longer needed as a generator, and we can view the relation in degree 4 as $\lambda f_4 = (xz - y^2)^2$. As $\lambda \to 0$ we can imagine this as a plane quartic f_4 wrapping itself ever more tightly two-to-one over a plane conic.

This leads to many interesting exercises. If you try it for $g = 4, 5, \ldots$ – you get at once into graded rings of codimension $5, 6, \ldots$ with very complicated algebra.

$g = 4$ The canonical model of a nonhyperelliptic $g = 4$ is the codimension 2 complete intersection $Q_2 \cap C_3 \subset \mathbb{P}^3$.

In the hyperelliptic case C has A as its g^1_2 and $K_C = 3A$. If (s_1, s_2) base $\mathcal{L}(A)$ then $S^3(s_1, s_2) = \{x_0, x_1, x_2, x_3\}$ base $\mathcal{L}(K_C)$, with the 3 quadratic equations between them that generate the homogenous ideal of the twisted cubic $\Gamma_3 \subset \mathbb{P}^3$. We can write them as the minors of the 2×3 matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \tag{5.1}$$

Write v for the new generator of $\mathcal{L}(6A)$ with divisor the sum of the 10 Weierstrass points. Show that $\mathcal{L}(2K_C)$ needs two more generators $y_1 = vs_1, y_2 = vs_2$ in degree 2 and one more $z = v^2$ in degree 3. Show the relations involving y_1, y_2 include the 3 new 2×2 minors of the extended matrix $\left(\begin{smallmatrix} x_0 & x_1 & x_2 & y_2 \\ x_1 & x_2 & x_3 & y_2 \end{smallmatrix}\right)$ and 3 relations for y_1^2, y_1y_2, y_2^2 derived from $v^2 = f_{10}(s_1, s_2)$. These relations are

$$\begin{aligned} y_1^2 &= [s_1^2 f_{10}(s_1, s_2)] \\ y_1y_2 &= [s_1s_2f_{10}(s_1, s_2)] \\ y_2^2 &= [s_2^2 f_{10}(s_1, s_2)] \end{aligned}$$

where on the right-hand side, the square brackets [ ] mean *rendered* in $S^4(x_0, \ldots, x_3)$. In other words, write a polynomial of weight 12 in s_1, s_2 as a homogeneous polynomial of degree 4 in $x_0, \ldots, x_4$. In the simple case $f_{10} = s_1^{10} + s_2^{10}$, the final 3 relations boil down to

$$\begin{aligned} y_1^2 &= x_0^4 + x_2^2x_3^2, \\ y_1y_2 &= x_0^3x_1 + x_2x_3^3, \\ y_2^2 &= x_0^2x_1^2 + x_3^4. \end{aligned}$$

Putting this set of 9 relations in a deformation family with the $(2, 3)$ complete intersection of the nonhyperelliptic canonical curve is a beautiful exercise (a challenge).