Alg curves, Lecture 1 (Jan 2020)

I start with a colloquial description of where we are going. The contents of the course can be described as very simple, but depending on sophisticated and in places quite difficult prerequisites and foundational development.

I treat nonsingular projective curves C in PP^N, assumed irreducible. Over the complex field CC, this is a Riemann surface or 1-dimensional complex manifold. At P in C there is a local analytic coordinate z_P or z so that an analytic neighbourhood P in U in C is isomorphic to |z| < 1 in CC.

C has a field of rational functions k(C). A rational function h is the quotient h = f/g of two polynomial functions, with denominator g not identically zero. A polynomial function is a regular (or holomorphic, analytic, etc.) function on the Riemann surface of C, and a rational function is a globally defined meromorphic function.

For P in C, a rational function h in k(C) can have a pole at P (so its value is undefined or infinity), or can be regular and nonzero (that is, a unit near P), or regular and have a zero at P. The divisor of h is the formal sum

    div h = zeros of h - poles of h = sum ni*Pi

with Pi in C finitely many points, and ni in ZZ.

In terms of a local parameter z = z_P at P,

    h = z^n * unit with n in ZZ,

and h has a zero or order n if n > 0, or a pole of order m = -n if n < 0. If h has a pole of order m then it has a Laurent expansion

    h = am z^-m + .. + a1 z^-1 + regular

with m coefficients {a1,..am} making up the principal part. Allowing h to have a pole of order m thus gives it the freedom of an m-dimensional vector space of principal parts to choose from (modulo the regular functions at P).

You easily take for granted that h does not have zeros and poles at the same point P in C, because we are used to cancelling common factors top and bottom. But that fails in dimension >= 2 (consider the rational function y/x at (0,0) in AA^2), or if C is singular (consider the rational function y/x on the nodal curve y^2 = x^2*(x+1)).

After the foundational work of establishing nonsingular projective curves as a sensible object of study, the course revolves around the Riemann-Roch theorem (its statement, proof and many applications). RR addresses the question: how many rational functions are there on C?

0. If you don't allow any poles, you don't get any functions beyond the constants (Liouville's theorem).

1. If you allow any number of poles, you get the whole of k(C), which is of course infinite dimensional, so not what you want.

2. If you allow only a finite set of poles of given degree, you get a finite dimensional space of rational functions.

3. The dimension of the space of rational functions with poles at most D = sum ni*Pi grows linearly with deg D = sum ni.

More precisely, introduce the notion of divisor and RR space.
Divisor
D = sum ni*Pi a finite sum with Pi in C and ni in ZZ.
A divisor is effective, or D >= 0 if all its coefficients ni >= 0.

Given D, its RR space is
L(C, D) = { h in k(C) | div h + D >= 0 }.
The definition intends that L(C, D) is a k-vector subspace of k(C), so by convention, I add the function 0 to L(C, D). The condition div h + D >= 0 is a clever way of combining two statements "poles at most D^+" where D is effective, and "zeros at least D^-" where D has some negative part.

Write l(C, D) = dim L(C, D). The first part of RR says that
l(C, D) <= 1 + deg D   and
l(C, D) >= 1 - g + deg D.   (*)
Here g = g(C) is some constant depending on C, its genus. It has several different definitions, and untangling them is one of the main purpose of the course. The RR formula (*) is equality for deg D >> 0.

Finally, the complete RR concerns the difference in (*). The result is that there exists a divisor K = KC such that
l(C, D) - l(C, K-D) = 1 - g + deg D. (*)
There are several different treatments of KC in topology and complex analysis, and this is also a main part of the course.

In applications, RR gives all kinds of implications for the geometry of curves C and their maps to PP^n. Riemann's initial motivation was to show that a compact Riemann surface embeds into PP^n, by constructing a suitable n+1 dimensional space of global meromorphic functions with pole at most D.