THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: JUNE 2020

ALGEBRAIC CURVES

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. (i) Let $A \subset K$ be a subring of a field. What does it mean to say that an element $y \in K$ is integral over A? If this holds, prove that the subring $A[y] \subset K$ is finitely generated as A-module. Formulate, without proof, a generalisation to a subring $A[y_1, \dots, y_n] \subset K$ generated by finitely many integral elements. **[4]**

(ii) Give the definition of discrete valuation ring (DVR). If $V \subset \mathbb{A}^n$ is an irreducible affine variety, prove that a point $P \in V$ is a nonsingular point of a curve if and only if the local ring $\mathcal{O}_{V,P}$ is a DVR. **[4]**

(iii) Prove that the coordinate ring $k[C]$ of a nonsingular affine curve $C \subset \mathbb{A}^n$ is normal (that is, integrally closed in its function field $k(C)$). [You may use that an integral domain A is normal if and only if the local ring A_P is normal for every prime P.] **[4]**

(iv) If $\Gamma \subset \mathbb{A}^n$ is an irreducible affine curve, explain how to use normalisation (integral closure) of its affine coordinate ring $k[\Gamma]$ to construct a nonsingular model $C \to \Gamma$ of Γ. [Please give brief statements without proofs of the results of commutative algebra that you need.] **[5]**

(v) Describe the normalisation of $\mathbb{C}[\Gamma]$ for the plane curve $\Gamma = V(y^3 - x^2(x-1))$ in $\mathbb{A}^2_{\mathbb{C}}$, and the nonsingular model $C \to \Gamma$. **[8]**

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2. Let C be a nonsingular projective curve. Standard results on nonsingularity and discrete valuation rings may be assumed.

(i) What is meant by a divisor D on C, and what is its degree $\deg D$? Give the definition of principal divisor, linear equivalence of divisors, and state the main results concerning their degrees. [No proofs are required.] **[4]**

(ii) Define the Riemann–Roch space $\mathcal{L}(C, D)$ of a divisor D. Prove that $\deg D < 0$ implies $\mathcal{L}(C, D) = 0$, and deduce that $\dim \mathcal{L}(C, D) \leq 1 + \deg D$ for every D. **[4]**

Let $\mathbb{P}^2$ be the projective plane with coordinates x, y, z, and $C \subset \mathbb{P}^2$ a nonsingular cubic curve.

(iii) By considering rational functions on C of the form F_n/z^n with F_n homogeneous of degree n, or otherwise, show that C has divisors D_n of arbitrarily large degree for which $\dim \mathcal{L}(C, D_n) \geq \deg D_n$. **[4]**

Deduce that $\dim \mathcal{L}(C, D) = \deg D$ for every divisor of degree > 0. (You may assume that C is not rational.) **[5]**

(iv) Now let $O \in C$ be a chosen point. Prove that for any $P, Q \in C$, there is a unique point $R \in C$ for which $O + R$ is linearly equivalent to $P + Q$. **[4]**

Prove that the operation $P, Q \mapsto R$ defines a group law on the set of points of C, with neutral element O. **[4]**

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3. (i) Let D_1, D_2 be effective divisors on a nonsingular projective curve C. Prove that multiplication in $k(C)$ defines a k-bilinear map

$$m\colon \mathcal{L}(C, D_1) \times \mathcal{L}(C, D_2) \to \mathcal{L}(C, D_1 + D_2),$$

and that $m(s_1, s_2) \neq 0$ for nonzero $s_1 \in \mathcal{L}(C, D_1)$ and $s_2 \in \mathcal{L}(C, D_2)$. **[3]**

(ii) Consider the subspace of $\mathcal{L}(C, D_1 + D_2)$ spanned by the image of m. Prove that it has dimension $\geq l(D_1) + l(D_2) - 1$. [Here $l(D_i) = \dim \mathcal{L}(C, D_i)$. Assume as given that a projective subvariety $X \subset \mathbb{P}^n$ of dimension a has nonempty intersection with every projective linear subspace of dimension $n - a$.] **[4]**

(iii) For D a divisor on C and $s_1, s_2 \in \mathcal{L}(C, D)$, assume that the effective divisors $D_1 = D + \operatorname{div} s_1$ and $D_2 = D + \operatorname{div} s_2$ are supported on disjoint sets of points. For any divisor A, determine the intersection

$$s_1 \cdot \mathcal{L}(C, A) \cap s_2 \cdot \mathcal{L}(C, A) \subset \mathcal{L}(C, A + D).$$

Deduce that $s_1 \cdot \mathcal{L}(C, A) + s_2 \cdot \mathcal{L}(C, A)$ is a subspace of $\mathcal{L}(C, A + D)$ of dimension equal to $2l(A) - l(A - D)$. **[12]**

(iv) Now assume in addition that $\deg A - \deg D \geq 2g - 1$. Prove that the image of the multiplication map $\mathcal{L}(C, A) \times \mathcal{L}(C, D) \to \mathcal{L}(C, A + D)$ spans $\mathcal{L}(C, A + D)$. **[6]**

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4. (i) For a divisor D on a nonsingular projective curve C, give the definition of the linear system $|D|$, and say what it means for it to be free (or base-point free). Define the notion of a very ample divisor D. **[3]**

(ii) If D is very ample, prove that the Riemann–Roch spaces satisfy **[3]**

$$\dim \mathcal{L}(C, D - P - Q) = \dim \mathcal{L}(C, D) - 2 \quad \text{for every } P, Q \in C. \tag{4.1}$$

(iii) Conversely, prove that condition (4.1) implies D is very ample. (Standard results on finiteness of normalisation and finite modules over a local ring may be assumed without proof.) **[6]**

In what follows C is a nonsingular projective curve of genus $g \geq 3$ and C is not hyperelliptic. The full statement of the Riemann–Roch theorem may be assumed.

(iv) Prove that K_C is very ample. **[3]**

(v) Let $P_1, P_2, P_3 \in C$ be distinct points. Explain what it means for $|P_1 + P_2 + P_3|$ to be a g^1_3, and state and prove the condition for this in terms of the geometry of the canonical image $\varphi_{K_C}(C)$. **[4]**

(vi) It is known that the canonical image C_{2g-2} is contained in $\binom{g-2}{2}$ linearly independent quadric hypersurfaces of $\mathbb{P}^{g-1}$. If $|P_1 + P_2 + P_3|$ is a g^1_3, deduce from (v) that the intersection of all the quadrics through the canonical image of C contains a surface ruled by straight lines. **[3]**

(vii) Prove that a nonhyperelliptic curve C of genus 4 has either one or two g^1_3s, and explain how the two cases are distinguished. **[3]**

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5. (i) Give the definition of graded ring used in the course. **[2]**

For a curve C and an effective divisor D of degree $d > 0$ that gives a free linear system as in Question 4, (i), give the definition of the graded ring $R(C, D)$. Discuss briefly its main properties (detailed proofs are not required). **[3]**

In what follows, you may use the full form of the Riemann–Roch theorem and its corollaries such as Clifford's theorem $\deg D \geq 2(l(D) - 1)$ for irregular D.

(ii) Let C be a hyperelliptic curve of genus g and D its g^1_2. Choose a basis $s_1, s_2 \in \mathcal{L}(C, D)$ and write $S^n(s_1, s_2) = \{s_1^n, s_1^{n-1}s_2, \dots, s_2^n\} \subset \mathcal{L}(C, nD)$ for the $n + 1$ monomials of degree n.

Prove that the $S^n(s_1, s_2)$ are linearly independent in $\mathcal{L}(C, nD)$, and that they form a basis of $\mathcal{L}(C, nD)$ for all $n \leq g$. **[2]**

Prove that $\mathcal{L}(C, (g + 1)D)$ contains an element z that is complementary to the space spanned by $S^{g+1}(s_1, s_2)$, and that the monomials

$$S^n(s_1, s_2) \quad \text{and} \quad S^{n-g-1}(s_1, s_2)z$$

then form a basis of $\mathcal{L}(C, nD)$ for every $n \geq g + 1$. **[4]**

(iii) Now suppose that C is not hyperelliptic, and let D be a divisor on C with $\deg D = 3$, $l(D) = 2$ and $K \overset{\text{lin}}{\sim} 2D$. Prove that $g = 4$, and that the Riemann–Roch space $\mathcal{L}(C, nD)$ have dimension given by **[4]**

$$l(nD) = \begin{cases} 1 & \text{if } n = 0; \\ 2 & \text{if } n = 1; \\ 4 & \text{if } n = 2; \\ 3n - 3 & \text{if } n \geq 3. \end{cases}$$

Write s_1, s_2 for a basis of $\mathcal{L}(C, D)$ as before, and $y \in \mathcal{L}(C, 2D)$ for a complementary basis element. Prove that the monomials

$$S^n(s_1, s_2), \quad S^{n-2}(s_1, s_2) \cdot y, \quad S^{n-4}(s_1, s_2) \cdot y^2$$

form a basis of $\mathcal{L}(C, nD)$ for every $n \geq 0$. **[6]**

(iv) Deduce that the graded ring $R(C, L)$ has the form $k[x_1.x_2, y]/(f_6)$ where f_6 is a weighted polynomial of degree 6 in the variables x_1, x_x, y of degree $(1, 1, 2)$. **[4]**
