MA4L7 Algebraic curves

Part 3. Applications of RR to geometry of curves

7 Introduction

Part III takes on trust the theorem and some of the characterisations of g, and discusses at some length what RR means and what it can do for us. The main overall application of RR is the following: ensuring that C has enough global functions with given poles allows us to study the possible ways of embedding C into projective space. In good cases, this allows us to go from abstract notions such as a curve of genus g or a curve with a linear system g^r_d (see below) to a subvariety embedded $C \subset \mathbb{P}^n$ in a definite space and defined by explicit equations. For example, a curve of genus 1 is isomorphic to a plane cubic $C_3 \subset \mathbb{P}^3$.

A particularly important general use of RR in complex analysis is to prove that every compact Riemann surface is actually a projective algebraic curve, so an object of algebraic geometry. This idea has many applications, and opens up several branches of research.

7.1 Linear systems and projective embeddings

The RR spaces $\mathcal{L}(C,D)$ provide ways of mapping C to projective space: a basis $f_{1\dots l}$ of $\mathcal{L}(C,D)$ gives the rational map $\varphi_D \colon C \dashrightarrow \mathbb{P}^{l-1}$ that does $P \mapsto \bigl(f_1(P) : \cdots : f_l(P)\bigr)$. Here I study how to establish whether φ_D is an embedding (an isomorphism of C to its image), and if so, what the divisor D has to do with the geometry of $C \subset \mathbb{P}^{l-1}$.

First, some traditional terminology that goes back to antiquity. For C a nonsingular projective curve and $D = \sum d_P P$ a divisor, write

$$|D| = \bigl\{\operatorname{div} f + D \bigm| f \in \mathcal{L}(C,D)\bigr\}$$

for the *linear system* of D. By construction, the divisors $D_f = \operatorname{div} f + D$ for $f \in \mathcal{L}(C,D)$ run through the effective divisors linearly equivalent to

D. The set $|D|$ is parametrised by $\mathbb{P}^{l-1} = \big(\mathcal{L}(C,D) \setminus 0\big)/k^\times$, the projective space of 1-dimensional subspaces of the vector space $\mathcal{L}(C,D)$. We picture this as a bunch of points running around C, parametrised by a projective space, in much the same way as the pencil of plane conics $\lambda Q_1 + \mu Q_2 = 0$ is parametrised by $\mathbb{P}^1_{\langle\lambda,\mu\rangle}$.

The effective divisors of $D_f \in |D|$ may all have a common part $A > 0$. This means that each $f \in \mathcal{L}(C,D)$ satisfies $\operatorname{div} f + D \geq A$, or in other words, $\mathcal{L}(C,D) = \mathcal{L}(C, D-A)$. The biggest such A is the *fixed part* of $|D|$. We write $|D| = A + |D-A|$, where A is the fixed part and $|D-A|$ the *free part.*

We say that $|D|$ is *free* (or *fixed-point free*) if it has no fixed part. Then for every $P \in C$, some $f \in \mathcal{L}(C,D)$ has valuation $v_P(f) = -d_P$. In terms of the sheaf $\mathcal{O}_C(D)$, this means that the global section $f \in \Gamma(C, \mathcal{O}_C(D)) = \mathcal{L}(C,D)$ is $z_P^{-d_P} \times$ unit of $\mathcal{O}_{C,P}$, so that f is a local basis of $\mathcal{O}_C(D)$ as $\mathcal{O}_C$-module near P. Thus $|D|$ free is synonymous with $\mathcal{O}_C(D)$ being *generated by its global sections.*

Remark 7.1 A free linear system $|D|$ of degree d and $\mathcal{L}(C,D) = r+1$ is traditionally called a g^r_d, meaning that $|D|$ consists of effective divisors of degree d moving in an r-dimensional family. For example, the 2-to-1 map $C \to \mathbb{P}^1$ from a hyperelliptic curve to $\mathbb{P}^1$ is given by a g^1_2; the hyperplane linear system $|H|$ on a curve of degree $C_a \subset \mathbb{P}^2$ is a g^2_a.

Two traditional sources of confusion: first, $r+1 = l(C,D)$ is the dimension of $\mathcal{L}(C,D)$ as a vector space, whereas r refers to its projectivisation $\mathbb{P}^r = \big(\mathcal{L}(C,D) \setminus 0\big)/k^\times$, the parameter space of the linear system $|D|$.

Next, this $\mathbb{P}^r = |D|$ corresponds to 1-dimensional subspaces of $\mathcal{L}(C,D)$, whereas the target space of $\varphi_D \colon C \to \mathbb{P}^{l-1}$ has $\mathcal{L}(C,D)$ as its linear forms, so its points correspond to codimension 1 subspaces of $\mathcal{L}(C,D)$. The divisors of $|D|$ are given by hyperplane sections of $\mathbb{P}^{l-1}$.

7.2 Strategy to prove embedding

How do we establish that $\varphi_D \colon C \dashrightarrow \mathbb{P}^{l-1}$ is an isomorphism to its image $\varphi(C) = \Gamma \subset \mathbb{P}^n$? An algebraic variety is a set of points X with locally defined functions $\mathcal{O}_X$ on it. Thus for $\varphi \colon C \to \Gamma$ to be an isomorphism, we need (1) that it is bijective as a map of point sets, and (2) that pullback of functions on Γ provide all the functions on C.

Definition 7.2 We say that a divisor D or a linear system $|D|$ is *very ample* if $\varphi_D \colon C \to \mathbb{P}^{l-1}$ is an isomorphism to its image $\varphi_D(C) = \Gamma \subset \mathbb{P}^{l-1}$, and the hyperplanes of $\mathbb{P}^{l-1}$ cut out the linear system $|D|$ on C.

The main result is the following theorem.

Theorem 7.3 *Let D be a divisor on a nonsingular projective curve C. Then D is very ample if and only if the RR spaces of D on C satisfy the conditions:*

(1) $l(D-P) = l(D)-1$ for every $P \in C$; equivalently, $\mathcal{L}(D-P) \subsetneq \mathcal{L}(D)$. That is, $|D|$ is free.

(2) $l(D-P-Q) = l(D)-2$ for every $P, Q \in C$; that is, $\mathcal{L}(D-P-Q) \subsetneq \mathcal{L}(D-P) \subsetneq \mathcal{L}(D)$. We say that $|D|$ is free and separates points.

(3) $l(D-2P) = l(D)-2$ for every $P \in C$; equivalently, $\mathcal{L}(D-2P) \subsetneq \mathcal{L}(D-P) \subsetneq \mathcal{L}(D)$. That is, D separates tangent directions.

I start by relating the assumptions of the theorem to the above discussion. (1) is the statement that $|D|$ has no fixed part. In the more general case, passing from D to the free part $D' = D - A$ does not change the morphism $\varphi_{D'} = \varphi_D$. However, in that case, if the free part $|D'|$ defines an embedding $\varphi_{D'}$, the hyperplanes of $\mathbb{P}^{l-1}$ would cut out the free $|D'|$ and take no account of the fixed part A.

(2) is the condition that $\mathcal{L}(D-P-Q) \subset \mathcal{L}(D)$ has codimension 2, so that there is an $f \in \mathcal{L}(D)$ that vanishes at P and not at Q, or in other words, there is a hyperplane of $\mathbb{P}^{l-1}$ through $\varphi_D(P)$ and not through $\varphi_D(Q)$. Thus (2) gives directly that φ_D is bijective on point sets.

To discuss (3), suppose that $P \in C$ appears in D with coefficient d_P, and that z_P is a local parameter of the DVR $\mathcal{O}_{C,P}$. Then by (1) we know that some $f_2 \in \mathcal{L}(D)$ has valuation $v_P(f_2) = -d_P$, so is a basis of $\mathcal{O}_C(D)$ on an affine neighbourhood U of P. Assumption (3) asserts that there is some $f_1 \in \mathcal{L}(D)$ with $v_P(f_1) = -(d_P - 1)$. Then f_1/f_2 is a regular function on U, and is a regular parameter of the local ring $\mathcal{O}_{C,P}$.

In complex analysis, this would complete the proof – we have a injective regular map, and functions on the image include a local analytic parameter at each point P, so the map is an immersion by the implicit function theorem.

Proof of the theorem In algebraic geometry, write $\Gamma \subset \mathbb{P}^{l-1}$ for the Zariski closure of the image $\Gamma_0 = \varphi_D(C)$. It is an irreducible subvariety, and by (2), the morphism $\varphi_D \colon C \to \Gamma$ is injective on points. I have to prove that φ_D is surjective, and that pullback defines an isomorphism of local rings $\varphi_D^* \colon \mathcal{O}_{\Gamma,Q} \cong \mathcal{O}_{C,P}$ for every $P \in C$.

The proof consists of three parts: (1) reduction to a finite morphism $\varphi_x\colon C_x \to \Gamma_x$ on affine pieces $C_x \subset C$ and $\Gamma_x \subset \Gamma$, with the induced ring homomorphism $\varphi_x^*\colon k[\Gamma_x] \subset k[C_x]$ making $k[C_x]$ finite over $k[\Gamma_x]$; (2) reduction to local commutative algebra with $\varphi_Q^*\colon \mathcal{O}_{\Gamma,Q} \cong \mathcal{O}_{C,P}$ a finite morphism of local rings. (3) Conclusion of the argument using Nakayama's lemma.

Remark 7.4 As with resolution of singularities in Part I, my treatment here works by fitting the morphism $C \to \Gamma$ in diagrams $C \to \Gamma \to \mathbb{P}^1$.

Reduction to affine Write $\Gamma_0 = \varphi_D(C) \subset \mathbb{P}^{l-1}$ and let $\Gamma \subset \mathbb{P}^{l-1}$ be its Zariski closure. Then $\Gamma_0 = \varphi_D(C)$ is an irreducible curve, and Γ adds at most finitely many points $Q \in \Gamma$ (actually none, but that is still to prove). The RR space $\mathcal{L}(C,D)$ gives the linear forms on $\mathbb{P}^{l-1}$, so a choice of coordinates $t_{1\dots l}$ for $\mathbb{P}^{l-1}$ gives a basis $f_{1\dots l}$ of $\mathcal{L}(C,D)$ and vice-versa.

Since Γ is a curve, for general coordinates on $\mathbb{P}^{l-1}$, it is disjoint from the codimension 2 subspace $t_1 = t_2 = 0$. For the corresponding basis of $\mathcal{L}(C,D)$, the first two elements f_1, f_2 give effective divisors $\operatorname{div} f_i + D$ with disjoint support.

Given t_1, t_2 chosen as above, for any $Q \in \Gamma$, I can replace them with appropriate linear combinations so that Q is in the hyperplane $t_1 = 0$ and not in $t_2 = 0$, so that $x = t_1/t_2$ is regular and 0 at Q, that is $x \in \mathcal{O}_{\Gamma,Q}$. Or, for any given point $P \in C$, I can replace the corresponding f_1, f_2 with appropriate linear combinations so that $f_2 \in \mathcal{L}(C,D) \setminus \mathcal{L}(C,D-P)$ and $f_1 \in \mathcal{L}(C,D-P)$ and $x = f_1/f_2 \in \mathcal{O}_{C,P}$.

Now consider the commutative triangle

$$\begin{array}{ccc} C & \xrightarrow{\varphi_D} & \Gamma \\ & \searrow & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

with $C \to \mathbb{P}^1$ the morphism defined by the ratio $(f_1 : f_2)$, and $\Gamma \to \mathbb{P}^1$ the morphism induced by the linear projection $\mathbb{P}^{l-1} \dashrightarrow \mathbb{P}^1_{\langle t_1,t_2\rangle}$.

I now reduce to the construction of Part 1. Set $x = f_1/f_2 \in k(C)$. It is a nonconstant rational function on C, so that $k(x) \subset k(C)$ is a finite field extension. As in Part 1, write A_x for the integral closure of $k[x]$ in $k(C)$ and $C_x = \operatorname{Spec} A_x$ for the corresponding affine curve. I can do the same for $y = x^{-1} = f_2/f_1$, and identify C with the union $C_x \cup C_y$.

Since $\Gamma \subset \mathbb{P}^{l-1}$ is disjoint from $t_1 = t_2 = 0$, it is the union of two standard affine pieces Γ_{t_1} and Γ_{t_2} (with $t_i \neq 0$). The affine curve Γ_{t_2} having a finite morphism to $\mathbb{A}^1_x$ with parameter $x = t_1/t_2$ (respectively Γ_{t_1} to $\mathbb{A}^1_y$ with $y = x^{-1} = t_2/t_1$).

This gives affine varieties and morphisms $C_x \to \Gamma_x \to \mathbb{A}^1_x$, with coordinate rings $k[x] \subset k[\Gamma_x] \subset k[C_x]$. What I gain is that $k[C_x]$ is finite as a module over $k[x]$, so a fortiori over $k[\Gamma_x]$.

At this point it clarifies the argument to separate the commutative algebra from the geometry.

Proposition 7.5 *Let $A \subset B$ be finitely generated k-algebras that are integral domains and $m \subset A$ a maximal ideal. Assume the following:*

(i) B is finite as A-module.

(ii) The ideal $I = mB$ is contained in a unique maximal ideal $n \subset B$ and $k = A/m = B/n$.

(iii) $m \to n/n^2$ is surjective.

Then on localising, the morphism of local rings $A_m \to B_n$ is surjective.

In the current case, $A = k[\Gamma_x]$ and $B = k[C_x]$. I have arranged that B is finite over A. Next $m = m_Q$ is the maximal ideal of a point $Q \in \Gamma_x$. The variety $V(I)$ of the ideal $I = mB$ consists of the points of C_x that map to Q. This consists of at most one point of C by (2), with $A/m = \mathcal{O}_{C,P}/m_p = k$. It is nonempty by the following lemma.

Lemma 7.6 *$mB \neq B$, so mB is contained in a maximal ideal of B.*

By contradiction, assume $B = mB$ and suppose b_i generate B. Then $b_i = \sum a_{ij}b_j$ with $a_{ij} \in m$, and the usual determinant trick gives $\Delta B = 0$ where $\Delta = \det(\delta_{ij} - a_{ij})$. Then $\Delta = 0$ because $1_A \in B$, but $\Delta \cong 1 \bmod m$, which is a contradiction.

So $C_x \to \Gamma_x$ is surjective, and since φ_D is injective then $Q = \varphi_D(P)$ for a unique P; this implies (b). Finally, (c) holds since (3) implies that some $f \in \mathcal{L}(C, D - P)$ has $v_P(f) = -(d_P - 1)$ which gives $v_P(f/f_2) = 1$.

Reduction to local Replace $A \subset B$ by their localisations $A_m \subset B_n$. One checks that the following still hold.

(i) B_n is still finite as A_m module.

(ii) The ideal $I_n = mB_n$ is contained in nB_n and we still have $k = A/m = A_m/mA_m$, $k = B/n = B_n/nB_n$.

(iii) $nB_n/n^2B_n = n/n^2$, so that $mA_m \to nB_n/n^2B_n$ remains surjective.

Proof of the local statement We have $I_n \subset n$, and by (3), and the image of I_n generates n/n^2. This means that $n = I_n + n^2$, so that Nakayama's lemma (applied to the B-module n) implies that $I_n = n$.

Now B is a finitely generated k-algebra and n a maximal ideal, it follows by the weak NSS that $B/n = k$ (the same k). Therefore 1 generates $B/I = B/mB$, so that Nakayama's lemma (appplied to the A-module B) implies that 1 generatees.

8 Traditional applications of RR

8.1 Characterisation of $g = 0$

Proposition 8.1 *Let C be a curve. Equivalent conditions*

1. $l(D) = 1 + \deg D$ *for some divisor D of degree* ≥ 1*;*
2. $P \sim Q$ *for every* $P, Q \in C$.
3. $g = 0$.
4. $C \cong \mathbb{P}^1$.

This is all easy. If $l(D) = 1 + \deg D$ with $\deg D > 1$, the same continues to hold for $D - P$, and by induction we get a divisor of degree 1 with $l(D) = 2$. Then the linear system $|D|$ contains every $P \in C$ as a divisor, proving 2. The map $\varphi_D \colon C \to \mathbb{P}^1$ is an isomorphism by Theorem 7.3.

8.2 Treatment of $g = 1$

The ideas around RR provides practically the whole of the geometric theory and function theory of elliptic curves. First, to restate RR in the special case $g = 1$, it says that $l(D) = \deg D$ for every divisor D of degree ≥ 1. For D of degree 0, either $D \sim 0 \sim K_C$ or $l(D) = 0$.

A curve of genus 1 is isomorphic to a plane cubic $C \cong C_3 \subset \mathbb{P}^2$. Just choose any divisor D of degree 3. The $l(D) = 3$, whereas $l(D - P) = 2$ and $l(D-P-Q) = 1$ for every $P, Q \in C$, so that $\varphi_D \colon C \to \mathbb{P}^2$ is an isomorphism to its image.

Next, for the group law, the basic point is that a divisor D of degree 1 on C has $l(D) = 1$, so is linearly equivalent to a uniquely specified effective divisor of degree 1, necessarily a point $P \in C$. This makes the set of points of C into a coset of the group $\operatorname{Pic}^0 C$ of divisor classes of degree 0. We need to specify a point $O \in C$ as the neutral element to get out of the coset and into the group.

This construction is important, so I spell it out: write $\operatorname{Div} C$ for the group of all divisors of C (that is, the free Abelian group generated by the points $\{P \in C\}$), and $\deg\colon \operatorname{Div} C \to \mathbb{Z}$ for the degree map. Its kernel is the group $\operatorname{Div}^0 C$ of divisors of degree 0. The principal divisors $\operatorname{PDiv} C = \{\operatorname{div} f \mid f \in k(C)^\times\}$ also form a group (isomorphic to $k(C)^\times/k^\times$), which is a subgroup of $\operatorname{Div}^0 C$, because by Main Proposition (I) a principal divisor has degree 0.

Now define $\operatorname{Pic}^0 C$ to be the quotient group $\operatorname{Pic}^0 C = \operatorname{Div}^0 C/\operatorname{PDiv} C = \operatorname{Div}^0 C/\sim$. The group law on this is just addition of divisors mod linear equivalence, and the zero element is the class of the zero divisor.

Along with $\operatorname{Pic}^0 C$, consider its coset $\operatorname{Pic}^1 C$ formed by divisors of degree 1 up to linear equivalence. As we have seen, this is in bijection with C itself. Now choosing any point $O \in C$ provides a bijective map $\operatorname{Pic}^0 C \to \operatorname{Pic}^1 C \to C$ by $[D] \mapsto [D+O]$. That is, a divisor class D of degree 0 maps to the divisor class $D+O$, which is linearly equivalent to a unique $P \in C$; the inverse bijection $C \to \operatorname{Pic}^0 C$ takes P to the class of $P-O$. In conclusion, the group law on C is

$$(P,Q) \mapsto (P-O, Q-0) \mapsto (P+Q-2O) \mapsto (P +_C Q),$$

where the middle step is addition in Pic^0, and $P +_C Q$ is the unique effective divisor linearly equivalent to $P+Q-O$.

There are a couple of exercises concerned with interpreting the geometric $P+Q+R$ form of the group law [UAG, Chap. 2] within the current treatment.

8.3 $g \geq 2$: canonical embedding versus hyperelliptic

A curve C of genus g has a canonical divisor K with $\deg K = 2g-2$ and $l(K) = g$. In the main case $g \geq 2$, we have the following dichotomy.

Theorem 8.2 *Consider the map $\varphi_K\colon C \to \mathbb{P}^{g-1}$ defined by the canonical divisor. Then* either *φ_K is an isomorphic to its image $C \subset \mathbb{P}^{g-1}$ and the hyperplanes of $\mathbb{P}^{g-1}$ cut out the canonical system $|K|$ on C.* Or *C has a linear system g^1_2, and φ_K is obtained as the composite $C \to \mathbb{P}^1 \cong \Gamma_{g-1} \subset \mathbb{P}^{g-1}$ where the first map is the double cover $C \to \mathbb{P}^1$ defined by the g^1_2, and Γ_{g-1} is the rational normal curve of degree $g-1$.*

Proof Equality $\mathcal{L}(K-P) = \mathcal{L}(K)$ holds only for $g = 0$ (when both spaces are zero). For RR would give $l(P) - g = 1 - g + \deg P$, that is, $l(P) = 2$.

Next, if $\mathcal{L}(K-P-Q)=g-2$ for every $P,Q\in C$ then φ_K is an embedding by Theorem 7.3. The alternative possibility is that $\mathcal{L}(K-P-Q)=g-1$ for some $P+Q$. Then RR gives

$$l(P+Q)-(g-1)=1-g+2,\quad \text{that is,}\quad l(P+Q)=2.$$

Thus $|P+Q$ is a g^1_2. It follows again by Theorem 7.3 that it defines a 2-to-1 morphism $\varphi_{P+Q}\colon C\to\mathbb{P}^1$, so that C is hyperelliptic. Every divisor $D\in|P+Q|$ is mapped to a single point by φ_K, so that φ_K factors via φ_{P+Q}. On the other hand, its image must span $\mathbb{P}^{g-1}$, so is Γ_{g-1}. Q.E.D.

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Example sheet 4, Deadline Tue 26th Feb

1. Function theory on a hyperelliptic curve Assume that $\frac{1}{2} \in k$, and let C be a hyperelliptic curve of genus $g \geq 2$. It comes with a divisor $|D|$ that gives a g_2^1 and a double cover $\varphi_D \colon C \to \mathbb{P}^1$. Write $f_1, f_2 \in \mathcal{L}(C, D)$ for a basis, where $x = f_1/f_2$ is a parameter on $\mathbb{P}^1$.

The field extension $k(\mathbb{P}^1) \subset k(C)$ is a quadratic extension defined by $z^2 = F_{2g+2}(x)$, and has a hyperelliptic involution that does $i \colon z \mapsto -z$.

The monomials $S^n(f_1, f_2) = \{f_1^n, f_1^{n-1}f_2, \dots, f_2^n\}$ are linearly independent in $\mathcal{L}(nD)$ for each n, because x is transcendental over k. Calculate the dimension of $\mathcal{L}(nD)$ for $n = 1, \dots, g$. [Hint: Start by using the above to show that $(g-1)D$ must be irregular, and deduce that $K_C \sim (g-1)D$. On the other hand, gD must be regular.]

Next, use RR to show $\mathcal{L}((g+1)D)$ is strictly bigger than $S^{g+1}(f_1, f_2)$. We can choose the complementary basis element g so that $z = g/f_2^{g+1}$ is anti-invariant under the hyperelliptic involution, giving the new generator with $z^2 = F_{2g+2}(x)$.

Show the monomials $S^n(f_1, f_2)$ and $S^{n-g-1}(f_1, f_2) \cdot g$ form a basis of $\mathcal{L}(nD)$ for every n.

2. Curves of genus $g = 4$ Let C be a curve of genus 4, assumed to be nonhyperelliptic. Write $\varphi_K \colon C \hookrightarrow \mathbb{P}^3$ for its canonical embedding and identify C with its image $C \subset \mathbb{P}^3$.

By construction of the canonical embedding, the hyperplanes of $\mathbb{P}^3$ cut out $|K|$ on C. In the same way, quadric surfaces in $\mathbb{P}^3$ cut out divisors of $|2K|$. Calculate the dimension of the space of quadrics in $\mathbb{P}^3$ and $l(2K) = \dim \mathcal{L}(C, 2K)$, and conclude that C is contained in a unique quadric hypersurface $Q \subset \mathbb{P}^3$.

As an irreducible quadric, Q necessarily has rank 3 or 4. If Q has rank 4 (so is $x_1x_2 = x_3x_4$ in appropriate coordinates), prove that C has two different linear systems g_3^1, D_1 and D_2, with $K_C = D_1 + D_2$. Prove that $C \subset Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree $(3, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and so $C \subset Q$ is cut out by a cubic hypersurface, $C = Q \cap F_3$.

If D_1 is a g_3^1 on C, use RR to deduce that $D_2 = K - D_1$ is also a g_3^1. Therefore $K = D_1 + D_2$ is the sum of two linear systems g_3^1. We distinguish two cases: $D_1 \not\sim D_2$, or $D_1 \sim D_2$. Show that the first case corresponds to the canonical image C contained in a quadric of rank 4.

In the second case, write $K = 2D$ with $D = D_1 = D_2$. Write t_1, t_2

for homogeneous coordinates on the target $\mathbb{P}^1$ of $\varphi_D\colon C \to \mathbb{P}^1$. Show that $\mathcal{L}(C, K)$ is based by $x_1, x_2, x_3 = t_1^2, t_1t_2, t_2^2$ and a new variable y. In $\mathcal{L}(2K)$ there is a quadratic relation between the x_1, x_2, x_3, providing the quadric of rank 3 $x_1x_3 = x_2^2$. Calculate the dimension of $\mathcal{L}(3K)$ and show that there must be a cubic relations $y^3 + A_2(x_1, x_2, x_3)y + B_3(x_1, x_2, x_3)$ (here we need $1/3 \in k$ to do the Tschirnhausen transformation).

3. Clifford's theorem Prove that $d \geq 2r$ for any irregular divisor D defining a g_d^r (here irregular means that the irregularity $l(K - D) \neq 0$). In other words, the fastest growth of $l(D)$ among all curves C and divisors D is given by the hyperelliptic curves discussed in Q1.

[Hints: (1) use the following *linear-bilinear lemma*: let $\varphi\colon V_1 \times V_2 \to W$ be a bilinear map from vector spaces V_1, V_2 of dimension l_1, l_2. Suppose $\varphi(v_1, v_2) \in W$ is nonzero for every nonzero $v_1 \in V_1$ and $v_2 \in V_2$. Then the image of φ spans a subspace of dimension $\geq l_1 + l_2 - 1$ in W. Proof: Tensors of rank 1 $\{v_1 \otimes v_2\}$ form a subvariety of dimension $l_1 + l_2 - 1$ in $V_1 \otimes V_2$. The kernel of $\varphi\colon V_1 \otimes V_2 \to W$ intersects it in 0 only.

(2) Consider the multiplication map $\mathcal{L}(D) \times \mathcal{L}(K - D) \to \mathcal{L}(K)$, and put together the inequality of the lemma with the RR formula.]

4. Degree 4 divisor on curve of genus 2 Let $\Gamma_4 \subset \mathbb{P}^2_{\langle x,y,z\rangle}$ be a plane quartic curve with a node or cusp at $(1, 0, 0)$ and no other singularities. We can assume that its equation is $x^2a_2 + xb_3 + c_4$, with a, b, c forms in y, z of the stated degree. Show that projection from P defines a 2-to-1 cover from the resolution $C \to \mathbb{P}^1_{\langle y,z\rangle}$ ramified in the discriminant sextic $b^2 - 4ac$, so that C is a hyperelliptic curves of genus 2.

Recall that K_C is the final irregular divisor. Prove that for any curve C of genus ≥ 2 and any $P, Q \in C$, we have $l(K + P + Q) - l(K) = 1$, so the morphism φ_D corresponding to $D = K + P + Q$ cannot distinguish the two points P, Q, that is, $\varphi_D(P) = \varphi_D(Q)$.

Now suppose that $g = 2$, and let D be any divisor of degree 4. Show that $l(D - K_C) > 0$, so that D is linearly equivalent to $K + P + Q$. Prove that $\varphi_D\colon C \to \mathbb{P}^2$ either maps C to a quartic curve $\Gamma_4 \subset \mathbb{P}^2$ with a node at $\varphi(P) = \varphi(Q)$ (resp., cusp if $P = Q$), or is a double cover of a plane conic (in the case $D - K_C = g_2^1$, that is, $D = 2g_2^1$).

5. Genus 6 Let C be a curve of $g = 6$, and assume it has no g_2^1, g_3^1 or g_5^2. If D is a g_4^1, show that $K - D$ has degree 6 and $l(K - D) = 3$. Show that $|K - D$ is a g_6^2, so defines a morphism $\varphi_{K-D}\colon C \to \mathbb{P}^2$.

Let $\Gamma_6 \subset \mathbb{P}^2$ be a sextic having double points (nodes or cusps) at the 4 points $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$ of the standard projective frame of reference. By considering the linear system of cubics of $\mathbb{P}^2$ passing through the 4 points, show that the resolution C has a linear system of dimension ≥ 6 and degree ≤ 10.

Given that its resolution $C \to \Gamma_6$ is a curve of genus 6. Show that C has 5 g^1_4s and complementary g^2_6s. [Hint: Four of them are fairly obvious. The fifth comes from the pencil of conics through the 4 points.]

It is a fact that any curve of genus 6 is given either by this construction, or a different construction adapted to the case that C has a g^1_2, g^1_3 or g^2_5, or is a double cover of curve of $g = 1$. (The g^2_5 case correspond to a plane quintic $C_5 \subset \mathbb{P}^2$.) Unfortunately, it would be something of a detour from the main course to discuss this rigorously or comprehensibly.