

## MA4L7 Algebraic curves. First example sheet (not assessed)

The first week's lectures talked around the prerequisites. (Many students who did Christian Boehning's lecture course MA4A5 will find this too easy.)

### 1. Affine varieties $X \subset \mathbb{A}^n$

Reread UAG, Chap. 2 up to the proof of NSS. Don't worry too much about the Zariski topology, because it is just the cofinite topology if  $X$  irreducible and 1-dimensional.

### 2. Affine coordinate ring

and function field  $k(X)$  from UAG, Chap. 3.

The coordinate ring is defined as  $k[X] = k[x_{1..n}]/I_X$ .

**Exercise 0.1** Use the NSS to establish the bijections

$$\begin{aligned} \left\{ \text{maximal ideals of } k[X] \right\} &\longleftrightarrow \left\{ \text{maximal ideal of } k[x_{1..n}] \text{ containing } I_X \right\} \\ &\longleftrightarrow \left\{ m_P = (x_i - a_i \mid i \in [1..n]), \text{ where } P = (a_{1..n}) \in X \right\}. \end{aligned}$$

and

$$\begin{aligned} \left\{ \text{prime ideals of } k[X] \right\} &\longleftrightarrow \left\{ \text{prime ideal of } k[x_{1..n}] \text{ containing } I_X \right\} \\ &\longleftrightarrow \left\{ I_Y \text{ with } Y \subset X \text{ irreducible subvariety.} \right\} \end{aligned}$$

These have the flavour “the ring  $k[X]$  knows everything about  $X$ ”, and will justify writing  $X = \text{Spec } k[X]$  (with a small abuse of terminology concerning the single prime ideal  $0 \subset k[X]$ ).

**Exercise 0.2**  $X$  affine irreducible with affine coordinate ring  $k[X]$  and function field  $k(X)$ . Prove that if  $f \in k(X)$  is regular at every  $P \in X$ , then  $f \in k[X]$ . That is rational plus everywhere regular implies polynomial.

Moreover for  $0 \neq g \in k[X]$ , if  $f \in k(X)$  is regular at every  $P \in X$  with  $g(P) \neq 0$  then  $f \in k[X][\frac{1}{g}]$ .

In either case, you need to use NSS. This will be used later as a step in going from birational (geometry up to birational equivalence) to biregular (geometry up to isomorphism).

### 3. DVR

Recall the definition of DVR from lectures or one of the textbooks.

**Exercise 0.3** *Prove that  $P \in X$  is a nonsingular point of a curve if and only if the local ring  $\mathcal{O}_{X,P}$  is a DVR.*

This is more or less the definition, but you have to get all the words right.

### 4. Integral closure.

**Exercise 0.4** *Show that  $\mathbb{Z}$  is integrally closed in  $\mathbb{Q}$ . More generally, if  $A$  is a UFD, prove that  $A$  is integrally closed. (That is, any element of  $K = \text{Frac } A$  that satisfies a monic polynomial equation over  $A$  is actually in  $A$ .)*

*Deduce that a DVR is integrally closed.*

I will prove later that a 1-dimensional Noetherian local ring  $A$  that is integrally closed in its field of fractions  $K = \text{Frac } A$  is a DVR.

**Exercise 0.5** *Prove the following lemma: consider  $A \subset A[x]/f$  where  $A$  is a ring and  $f \in A[x]$  a monic polynomial. Then*

$$A \text{ is a field} \iff A[x]/f \text{ is a field.}$$

### 5. Rational functions on $\mathbb{P}^1$ and the “baby case” of RR.

Let  $u, v$  be homogeneous coordinates on  $\mathbb{P}^1$ , and write  $x = v/u$  for the affine coordinate on  $\mathbb{A}^1 \subset \mathbb{P}^1$ , and write  $P = (1 : 0)$  and  $Q = (0 : 1) \in \mathbb{P}^1$ . The vector space  $k[x]_{\leq d}$  has dimension  $1 + d$  (with a basis you can easily guess). If we view it as the space of rational functions with pole  $\leq dQ$ , it is the ideal first case of RR space  $\mathcal{L}(\mathbb{P}^1, dQ)$ . The equality  $l(\mathbb{P}^1, dQ) = 1 - g + d$  (with  $g = 0$ ) holds for all  $d \geq -1$ , and fails by 1 for  $d = -2$ .

By considering  $(x - a)/(x - b)$ , show that  $k(\mathbb{P}^1)$  contains a function with  $\text{div } f = P_1 - P_2$  for any  $P_1, P_2 \in \mathbb{P}^1$ . More generally, if  $\sum m_i P_i$  and  $\sum n_j Q_j$  have  $\sum m_i = \sum n_j$ , then there exists  $f \in k(\mathbb{P}^1)$  with  $\text{div } f = \sum m_i P_i - \sum n_j Q_j$ .

Prove that  $l(\mathbb{P}^1, D) = 1 - g + \deg D$  for any  $D = \sum m_i P_i$  of degree  $d = \sum m_i$ .