Alg curves, Lecture 1

I start with a colloquial description of where we are going.
The contents of the course can be described as very simple,
but depending on sophisticated and in places quite difficult
prerequisites and foundational development.

I treat nonsingular projective curves C in PP^N, assumed
irreducible. Over the complex field CC, this is a Riemann
surface or 1-dimensional complex manifold. At P in C there
is a local analytic coordinate z_P or z so that an analytic
neighbourhood P in U in C is isomorphic to |z| < 1 in CC.

C has a field of rational function k(C). A rational function
h is the quotient h = f/g of two polynomial functions, with
denominator g not identically zero. A polynomial function is
a regular (or holomorphic) function on the Riemann surface
of C, and a rational function is a globally defined
meromorphic function.

For P in C, a rational function h in k(C) can have a pole at
P (so its value is undefined or infinity), or can be regular
and nonzero (so a unit near P), or regular and have a zero
at P. The divisor of h is the formal sum
  div h = zeros of h - poles of h = sum ni*Pi
with Pi in C finitely many points, and ni in ZZ.

In terms of a local parameter z at P,
  h = z^n * unit with n in ZZ,
and h has a zero or order n if n > 0, or a pole of order
m = -n if n < 0. If h has a pole of order m then it has
a Laurent expansion
  h = am z^-m + .. + a1 z^-1 + regular
with m coefficients {a1,..m} making up the principal part.
Allowing h to have a pole of order m thus gives it the
freedom of an m-dimensional principal part to choose from.

One easily takes for granted that h does not have zeros and
poles at the same point P in C, because we are used to
cancelling common factors top and bottom. But that is not
true in dimension >= 2 (consider the rational function y/x
on AA^2), or if C is singular (consider the rational
function y/x on the nodal curve y^2 = x^2*(x+1)).

After the foundational work of establishing nonsingular
projective curves as a sensible object of study, the first
main aim of the course is the Riemann-Roch theorem. RR
addresses the question: how many rational functions are
there on C?

0. If you don't allow any poles, you don't get any
functions.

1. If you allow any number of poles, you get the whole

of k(C), which is of course infinite dimensional.

2. If you allow only a finite set of poles of given
degree, you get a finite dimensional space of rational
functions.

3. The dimension of the space of rational functions
with poles at most D = sum ni*Pi grows linearly with
deg D = sum ni.

More precisely, introduce the notion of divisor and RR
space. Divisor
  D = sum ni*Pi a finite sum with Pi in C and ni in ZZ.
A divisor is effective, or D >= 0 means that all its
coefficients ni >= 0.

Given D, its RR space is
  L(C, D) = { h in k(C) | div h + D >= 0 }.
The definition intends that L(C, D) is a k-vector subspace
of k(C), so by convention, I add the function 0 to L(C, D).
The condition div h + D >= 0 is a clever way of combining
two statements "poles at most D^+" if D is effective, and
"zeros at least D^-" if D has some negative part.

Write l(C, D) = dim L(C, D). The first part of RR says that
  l(C, D) <= 1 + deg D   and
  l(C, D) >= 1 - g + deg D.   (*)
Here g = g(C) is some constant depending on C, its genus. It
has several different definitions, and is one of the main
topics of the course.

Finally, the complete RR concerns the difference in (*). The
result is that there exists a divisor K = KC such that
  l(C, D) - l(C, K-D) = 1 - g + deg D. (*)
There are several different treatments of KC, and this is
also a main component of the course.

In applications, RR gives all kinds of implications for the
geometry of curves C and their embeddings C into PP^n.