THE MCKAY CORRESPONDENCE FOR FINITE SUBGROUPS OF $\mathrm{SL}(3,\mathbb{C})$

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Abstract. Let $G \subset \mathrm{SL}(n,\mathbb{C})$ be a finite subgroup and $\varphi\colon Y \to X = \mathbb{C}^n/G$ any resolution of singularities of the quotient space. We prove that crepant exceptional prime divisors of Y correspond one-to-one with "junior" conjugacy classes of G. When $n=2$ this is a version of the McKay correspondence (with irreducible representations of G replaced by conjugacy classes). In the case $n=3$, a resolution with $K_Y=0$ is known to exist by work of Roan and others; we prove the existence of a basis of $H^*(Y,\mathbb{Q})$ by algebraic cycles in one-to-one correspondence with conjugacy classes of G. Our treatment leaves lots of open problems.

Contents

1. Statement of the results

Let $G \subset \mathrm{SL}(n,\mathbb{C})$ be a finite subgroup and $X = \mathbb{C}^n/G$ the quotient space, an affine variety with $K_X = 0$. A *crepant resolution* $f\colon Y \to X$ is a resolution of singularities such that $K_Y = f^*K_X = 0$. A crepant resolution does not necessarily exist in dimension ≥ 4 (see 4.5); it is known to exist in dimension 2 (classical, Du Val), and in dimension 3 by work of a number of people (see for example [Markushevich], [Ito1–3] and [Roan]), but the proofs are computational rather than conceptual.

This paper contributes some very easy remarks to the following question raised by [Dixon–Harvey–Vafa–Witten], worked out by [Hirzebruch–Höfer], and now famous among algebraic geometers as the "Physicists' Euler number conjecture". See for example [Roan] for the background.

Conjecture 1.1. *$G \subset \mathrm{SL}(n,\mathbb{C})$ is a finite subgroup, $X = \mathbb{C}^n/G$ the quotient space and $f\colon Y \to X$ a crepant resolution. Then there exists a basis of $H^*(Y,\mathbb{Q})$ consisting of algebraic cycles in one-to-one correspondence with conjugacy classes of G.*

It is an elementary fact that Y has no odd-dimensional cohomology, and that $H^{2i}(Y,\mathbb{Q})$ is spanned by algebraic cycles (see 4.1).

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Definition 1.2. Let G be a finite group. Write $\Gamma = \operatorname{Hom}(\boldsymbol{\mu}_R, G)$, where R is any common multiple of the orders of all $g \in G$ and $\boldsymbol{\mu}_R$ is the group of complex Rth roots of 1; thus Γ is a set which becomes a group isomorphic to G after making a choice of roots of 1. In colourful language, you can think of Γ as the "Tate twist" $\Gamma = G(-1) = \operatorname{Hom}(\boldsymbol{\mu}, G)$, where $\boldsymbol{\mu} = \widehat{\mathbb{Z}}(1) = \varprojlim \boldsymbol{\mu}_r$.

Theorem 1.3 (age grading). *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup and Γ as above. Then Γ has a canonical grading*

$$\Gamma = \sqcup_{i=1}^{n-1} \Gamma_i, \qquad \text{with} \quad \operatorname{age} \Gamma_i = i.$$

(Here $\Gamma_0 = \{e\}$, the identity element of Γ.)

Putting a grading on G is a well-known idea in toric geometry; see for example [C3-f], Theorem 3.1, and 2.2 below, and compare [Batyrev–Dais]. The point of the definition of Γ is to make the grading independent of the choice of roots of 1. If you believe Conjecture 1.1, you expect conjugacy classes in Γ_i to give the basis of H^{2i}, so that the age grading of Γ is an analog of the usual weight filtration of $H^*(Y)$, with weight $= 2 \times$ age.

The *junior elements* of Γ (or of G) are the elements $g \in \Gamma_1$ of minimal age 1; the set Γ_1 is called the *junior simplex* of Γ (or of G). The grading of Theorem 1.3 is invariant under conjugacy by G (or by Γ, it's the same thing). The notation Γ_i/G always means conjugacy classes of elements of age i.

Theorem 1.4. *There is a canonical one-to-one correspondence between junior conjugacy classes in Γ and crepant discrete valuations of X.*

The statement is explained in 2.3 below. Roughly, a discrete valuation of X is a prime divisor $E \subset V$ on any normal model V of X up to birational equivalence. Discrete valuations enable us to state Theorem 1.4 in a birationally invariant language, without any assumptions on crepant resolutions or minimal models of X.

A *minimal model* of X (in the sense of Mori theory) is a projective morphism $f\colon Y \to X$, where f is crepant (that is, $K_Y = f^*K_X$) and Y has $\mathbb{Q}$-factorial terminal singularities.

Corollary 1.5. *Suppose that there exists a minimal model of $X = \mathbb{C}^n/G$. Then there are canonical one-to-one correspondences*

$$\Gamma_1/G \longleftrightarrow \text{exceptional prime divisors of } f \longleftrightarrow \text{basis of } H^2(Y, \mathbb{Q}).$$

Now suppose that $n = 3$. Then it is known by work of Mori and others (these are general arguments, applicable to any 3-fold) that a minimal model $f\colon Y \to X = \mathbb{C}^3/G$ exists. Moreover, it has factorial cDV singularities (see [Kollár], 2.1.7.6), and is therefore a $\mathbb{Q}$-cohomology manifold, so that Poincaré duality applies between $H^4(Y, \mathbb{Q})$ and cohomology with compact support $H^2_c(Y, \mathbb{Q})$.

Theorem 1.6. *Let $f\colon Y \to X$ be as just described. Then $H^2_c(Y, \mathbb{Q})$ is based by the compactly supported exceptional prime divisors of Y, that is, the divisors $E \subset f^{-1}(0)$ lying over 0. Write $\Gamma_1^{(0)}$ for the junior elements of Γ corresponding to these (see also 2.10 below).*

Then there are canonical one-to-one correspondences

$$\Gamma_2/G \longleftrightarrow \Gamma_1^{(0)}/G \longleftrightarrow \text{basis of } H^2_c(Y, \mathbb{Q}) \longleftrightarrow \text{dual basis of } H^4(Y, \mathbb{Q}).$$

These results are proved in §2. Some features and implications are discussed in §4.

1.7. Acknowledgements. We are grateful to the organisers of the Trento conference (Jun 1994), where a seminar was held which discussed lots of this stuff (see for example [Batyrev]). A. Corti pointed out that 3-folds with $\mathbb{Q}$-factorial terminal singularities are $\mathbb{Q}$-homology manifolds, so that Poincaré duality holds. Questions by John Moody have helped us to improve the argument in a number of places. The second author wishes to thank S. Mori and Kyoto Univ. Koenkai for generous invitation and financial support. We are also indebted to David Morrison for pointing out that physicists are human beings and have individual names.

2. Proofs

2.1. Proof of Theorem 1.3. Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite group. Any element $g \in G$ has n eigenvalues $\lambda_1, \dots, \lambda_n$, which are rth roots of 1 if $g^r = 1$. In the normal run of things, in order to write $\lambda_i = \varepsilon^{a_i}$, we have to choose ε a primitive rth root of 1. However, the present set-up side-steps this problem by considering elements $g \in \Gamma = \mathrm{Hom}(\boldsymbol{\mu}_R, G)$; then $\varepsilon \to g(\varepsilon) \to \lambda_i(g(\varepsilon))$ is an unordered set of n characters of $\boldsymbol{\mu}_R$, and thus $\lambda_i(g(\varepsilon)) = \varepsilon^{a_i}$. If g has order r then $a_1, \dots, a_n$ correspond to the characters $\varepsilon \mapsto \varepsilon^{a_i}$ of order r, and we write them as the fractional expression $\frac{1}{r}(a_1, \dots, a_n)$ with integers $a_i \in [0, r)$.

Now, since $G \subset \mathrm{SL}(n)$ it follows that $\sum a_i \equiv 0 \mod r$, and hence $\frac{1}{r}\sum a_i = \alpha$ for some integer $\alpha = \operatorname{age} g \in [0, n)$. This defines the age grading of Γ, and so completes the proof. Q.E.D.

Remarks. 1. In fixed coordinates $x_1, \dots, x_n$ on $\mathbb{C}^n$, the expression $g = \frac{1}{r}(a_1, \dots, a_n)$ is a well-defined homomorphism $\boldsymbol{\mu}_r \to \mathrm{GL}(n, \mathbb{C})$, namely

$$\varepsilon \mapsto \operatorname{diag}(\varepsilon^{a_1}, \dots, \varepsilon^{a_n}).$$

2. Note that we didn't need to make an explicit choice of eigencoordinates, since the characters ε^{a_i} are obtained as the set of eigenvalues of matrixes, and so are invariant under conjugacy by $\mathrm{GL}(n, \mathbb{C})$. We chose an order $\frac{1}{r}(a_1, \dots, a_n)$ of the characters for convenience, but the order disappears at the end, since the age is the elementary symmetric function $\frac{1}{r}\sum a_i$.

3. The group G itself does not have a well-defined grading: for example, if ε is a primitive 7th root of 1 and $G = \mathbb{Z}/7$ is generated by $\operatorname{diag}(\varepsilon, \varepsilon^2, \varepsilon^4) \in \mathrm{SL}(3, \mathbb{C})$, then choosing the same ε gives

$$\operatorname{age}(\overline{1}, \overline{2}, \overline{4}) = 1 \quad \text{and} \quad \operatorname{age}(\overline{3}, \overline{5}, \overline{6}) = 2,$$

whereas the choice of $\eta = \varepsilon^{-1}$ gives the opposite grading.

2.2. Notation: the unit box $\square$ and junior simplex Δ_1. We now go through the same proof in more conventional terms, and, at the same time, introduce notation that we need later.

Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be as usual and $A \subset G$ an Abelian group; write r for some exponent of A. Choose coordinates $x_1, \dots, x_n$ on $\mathbb{C}^n$ to diagonalise the action, so that A acts on the $x_1, \dots, x_n$ by diagonal matrixes:

$$g \mapsto \operatorname{diag}(\lambda_1(g), \dots, \lambda_n(g)),$$

where $\lambda_i\colon A \to \boldsymbol{\mu}_r$ are characters of A. Then the Pontryagin dual of $A \hookrightarrow \boldsymbol{\mu}_r^n \subset (\mathbb{C}^\times)^n$ is a surjective homomorphism

$$h\colon \mathbb{Z}^n \to A^\vee \qquad \text{defined by } \prod x_i^{m_i} \mapsto \sum m_i\lambda_i.$$

Here $A^\vee = \operatorname{Hom}(A, \mathbb{C}^\times)$ is the character group of A, and we identify $\mathbb{Z}^n$ with the set of Laurent monomials $x^m = \prod x_i^{m_i}$, as usual in toric geometry. The kernel of h is the set of invariant monomials $M \subset \mathbb{Z}^n$, a sublattice of finite index. The dual lattice of M is an overlattice $L \supset \mathbb{Z}^n$. For example, if A is cyclic with generator $\operatorname{diag}(\varepsilon^{a_1}, \dots, \varepsilon^{a_n})$ then $L = \mathbb{Z}^n + \mathbb{Z} \cdot \frac{1}{r}(a_1, \dots, a_n)$. To distinguish the two, M is sometimes called the *lattice of monomials*, and L the *lattice of weights*.

Now in the copy of $\mathbb{R}^n$ corresponding to $\mathbb{Z}^n \subset L$, write

$$\square = \prod [0,1) = \bigl\{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \bigm| 0 \le \alpha_i < 1\bigr\}$$

for the unit cube, and

$$\Delta_1 = \bigl\{(\alpha_1, \dots, \alpha_n) \in \square \bigm| \sum \alpha_i = 1\bigr\}$$

for the unit simplex (see Figure 1). The point is that, provided roots of 1 are chosen, the quotient $L/\mathbb{Z}^n$ is naturally isomorphic to A, and since $\square$ is a fundamental domain for the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$, every element of $L/\mathbb{Z}^n \cong A$ has a unique representative in $\square$, and so $A = L \cap \square$. Thus the age grading of A is just the natural slicing of $\square$ by the planes $\sum \alpha_i = j$ for $j = 1, \dots, n-1$. In particular the set of junior elements of A equals $L \cap \Delta_1$.

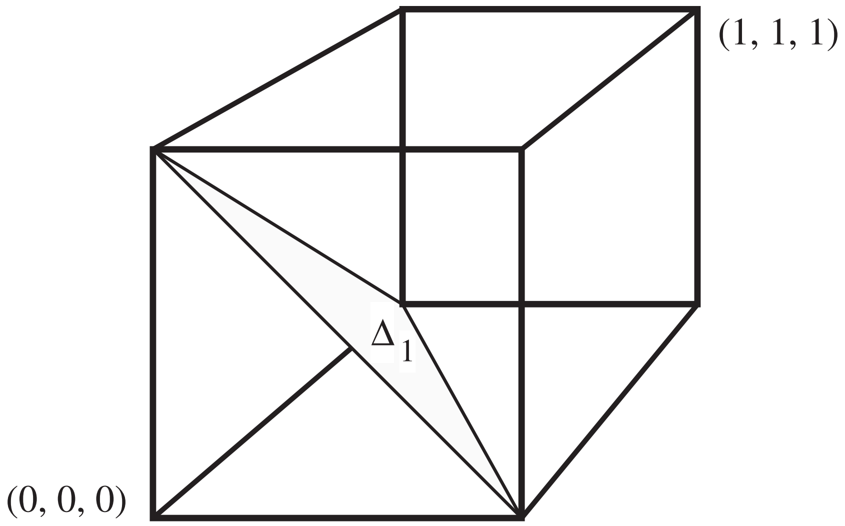

Figure 1. The unit box $\square$ and junior simplex Δ_1

Lemma. *There is a canonical isomorphism* $L/\mathbb{Z}^n = A(1) = \operatorname{Hom}(\boldsymbol{\mu}_r, A)$.

Proof. Given the coordinate system $x_1, \dots, x_n$, the exact sequence

$$0 \to M \to \mathbb{Z}^n \to A^\vee \to 0$$

is intrinsic, and, at first sight, it seems that the isomorphism $L/\mathbb{Z}^n \to A$ should also be: in fact, lattice theory gives a canonical duality pairing $\mathbb{Z}^n/M \times L/\mathbb{Z}^n \to \mathbb{Q}/\mathbb{Z}$. The subtle point is that this is duality with coefficients in $\mathbb{Q}/\mathbb{Z}$, giving

$$L/\mathbb{Z}^n = \mathrm{Hom}(A^\vee, \mathbb{Q}/Z),$$

whereas the character group $A^\vee = \mathrm{Hom}(A, \mathbb{C}^\times)$ is the dual with coefficients in $\mathbb{C}^\times$. Now obviously

$$A(1) = \mathrm{Hom}(\boldsymbol{\mu}_r, A) = \mathrm{Hom}(A^\vee, \mathbb{Q}/\mathbb{Z}).$$

An alternative, slightly bizarre, proof is to apply $\mathrm{Hom}_{\mathbb{Z}}(\text{blank}, \mathbb{Z})$ to the exact sequence $0 \to M \to \mathbb{Z}^n \to A^\vee \to 0$, to get

$$0 = \mathrm{Hom}(A^\vee, \mathbb{Z}) \to \mathbb{Z}^n \to L \to \mathrm{Ext}^1_{\mathbb{Z}}(A^\vee, \mathbb{Z}) \to 0, \quad \text{that is,} \quad L/\mathbb{Z}^n = \mathrm{Ext}^1_{\mathbb{Z}}(A^\vee, \mathbb{Z}),$$

and finally to argue that

$$\mathrm{Ext}^1(A^\vee, \mathbb{Z}) = A \otimes \mathrm{Ext}^1(\widehat{\mathbb{Z}}, \mathbb{Z}) = A \otimes \widehat{\mathbb{Z}}(1). \quad \text{Q.E.D.}$$

2.3. Overview of proof of Theorem 1.4. We start by explaining the statement and the strategy. Let $f\colon V \to X$ be any resolution of singularities, with exceptional divisors $\{E\}$. Write $K_V = f^*K_X + \sum a_E E$. The discrepancy a_E is independent of V. It is known that $a_E \geq 0$ for all E (see 2.4 below, and compare [C3-f], Theorem 3.1). The *crepant* divisors E are those with $a_E = 0$. Every crepant E must appear on any resolution V. The theorem states that the crepant divisors E for any V correspond one-to-one with Γ_1/G.

Let V be a normal variety (in applications, a blowup of $\mathbb{C}^n$ or a partial resolution of X). Counting the zeros and poles of rational functions $h \in k(V)$ along a prime divisor $E \subset V$ defines a discrete valuation of the function field $k(V)$, which is a homomorphism $v_E\colon k(V)^\times \to \mathbb{Z}$ satisfying a suitable compatibility with the additive structure of $k(V)$. The theory of discrete valuations is a device for discussing properties of $E \subset V$ in a birational way. In what follows, although not absolutely essential, it is convenient to use the definition of discrete valuation v of $k(V)$, and the fact that every geometric discrete valuation of a function field $k(X)$ is of the form v_E for some prime divisor E on a partial resolution $V \to X$. (*Geometric* means that the residue field k_v has transcendence degree $n-1$; this is a very easy case of Zariski's "local uniformisation", see Zariski and Samuel [Z–S], Vol. II, Chap. VI, §14, Theorem 31.)

It turns out that the only exceptional divisors of partial resolutions $V \to X$ we need arise from weighted blowups of $\mathbb{C}^n$: for each $g \in G$, take eigencoordinates $x_1, \dots, x_n$ for g, and a weighting $\beta = (b_1, \dots, b_n)$ on $x_1, \dots, x_n$ (usually closely related to the eigenvalues of g; see 2.7 for more precise conditions). The exceptional divisor of the weighted blowup $W_\beta \to \mathbb{C}^n$ is a weighted projective space $F_\beta = \mathbb{P}(b_1, \dots, b_n)$, and the valuation $v_\beta = v_{F_\beta}$ of $k(\mathbb{C}^n)$ along $F_\beta \subset W_\beta$ takes x_i to b_i. The corresponding exceptional divisor E of $X = \mathbb{C}^n/G$ is somewhat messy to describe geometrically (see 2.6 below), but the valuation v_E is just the restriction of v_β to the subfield $k(X) = k(\mathbb{C}^n)^G \subset k(\mathbb{C}^n)$, divided by the ramification degree. A valuation of this form is called a *monomial valuation* of $k(X)$.

Theorem 1.4 is a consequence of two easy tricks: (I) if $X = \mathbb{C}^n/G$ is a quotient singularity and $E \subset V \to X$ a prime divisor on a partial resolution, *ramification* *theory* reduces the calculation of the discrepancy of E to the case of a cyclic subgroup $\operatorname{Ram} F = \mathbb{Z}/r \subset G$ (see 2.5–6). (II) A cyclic quotient singularity $\mathbb{C}^n/(\mathbb{Z}/r)$ is a toric variety, so has a toric resolution (see 2.4); every valuation appearing on it is monomial in the sense just discussed. Therefore every nonmonomial valuation of X is discrepant.

The rest of the proof just boils down to calculating monomial valuations and the effect of the group action on them.

2.4. The cyclic case. Consider first the case of a cyclic group $\langle g\rangle$ with $g \in \Gamma$ an element of order r. Choose eigencoordinates to make $g = \frac{1}{r}(a_1, \dots, a_n)$ as described in 2.2. The following toric construction is standard. (For details, compare, for example, the proof of [C3-f], Theorem 3.1 or [YPG], §4.)

The barycentric subdivision of the first octant of $\mathbb{R}^n$ at $\frac{1}{r}(a_1, \dots, a_n)$ (with respect to the two different lattices $\mathbb{Z}^n$ and L) defines two blowups

$$\begin{array}{ccccc} F & \subset & W & \to & \mathbb{C}^n \\ & & \downarrow & & \downarrow \\ E & \subset & V & \to & \mathbb{C}^n/\langle g\rangle. \end{array}$$

Here W is the weighted blowup of $\mathbb{C}^n$ with weights $(a_1, \dots, a_n)$. This has a single exceptional component F, which is a weighted projective space $F = \mathbb{P}(a_1, \dots, a_n)$, and $V = W/\langle g\rangle$. It is easy to see that F has discrepancy $a_F = \sum a_i - 1 = \alpha r - 1$ with $\alpha = \operatorname{age} g$ (see [YPG], Prop. 4.8), and that $\langle g\rangle$ acts biregularly on W, fixing F pointwise, and multiplying a normal coordinate to F by a primitive rth root of 1. Thus the cyclic cover $W \to V$ has ramification of order r along E, and therefore E has discrepancy $\frac{1}{r}(a_F - (r-1)) = \alpha - 1$. Therefore $a_E \ge 0$, and E is crepant if and only if $\alpha = \operatorname{age} g = 1$, that is, g is junior.

In this case, the quotient X is a toric variety, so has a toric resolution $V_\Sigma \to X$, obtained by subdividing the first octant of $\mathbb{R}^n$ into a fan Σ of basic simplexes for the lattice $L = \mathbb{Z}^n + \mathbb{Z} \cdot \frac{1}{r}(a_1, \dots, a_n)$. The exceptional divisors of V_Σ correspond to the 1-skeleton of Σ; if $e = \frac{1}{r'}(b_1, \dots, b_n) \in L$ (for some $r' \mid r$) is in the positive octant then its discrepancy is $\frac{1}{r'}\sum b_i - 1$. Thus (again because $g \in \mathrm{SL}(n, \mathbb{C})$, so that $\sum b_i \equiv 0 \mod r'$), every exceptional divisor has discrepancy ≥ 0, and the crepant divisors correspond to $\Delta_1 \cap L$, that is, to the junior elements Γ_1.

2.5. Ramification groups. The ramification theory of discrete valuations in a Galois field extension $K \subset L$ is very similar to the factorisation of prime ideals in a Galois extension of number fields. (See 2.6 below for the birational geometric picture.) Let $K \subset L$ with $G = \operatorname{Gal}(L/K)$ be a finite Galois extension of function fields. A discrete valuation $v\colon L^\times \to \mathbb{Z}$ of L restricts to a map $v_{|K}\colon K^\times \to Z$ which obviously satisfies all the conditions for a discrete valuation, except that it only maps onto a subgroup $r\mathbb{Z} \subset \mathbb{Z}$ of finite index. We call r the *ramification degree* of v and write $w = \frac{1}{r}(v_{|K})$. Conversely, a discrete valuation w of K extends in finitely many ways to discrete valuations v of L. These extensions are permuted transitively by the Galois group, which acts on the valuations of L by

$$g\colon v \mapsto v \circ g \qquad \text{for } g \in \operatorname{Gal}(L/K).$$

Corresponding to a valuation v of L, there is a tower of subgroups

$$\operatorname{Ram} v \subset \operatorname{Stab} v \subset \operatorname{Gal}(L/K).$$

The *stabiliser group* Stab v and the *ramification group* Ram v are defined by

$$\operatorname{Stab} v = \bigl\{g \in G \bigm| v \circ g = v\bigr\} \quad \text{and} \quad \operatorname{Ram} v = \bigl\{g \in \operatorname{Stab} v \bigm| g^* = \mathrm{id} \text{ on } k(v)\bigr\},$$

where $k(v) = A_v/m_v$ is the residue field of v. (Compare [Z–S], Vol. II, Chap. VI, §12; these are also traditionally called the *decomposition subgroup* G_Z (Zerlegungsgruppe) and the *inertia subgroup* G_T (Trägheitsgruppe) respectively.)

A crucial point for us is that Ram v is a cyclic group. The complex analysis argument is transparent and is given at the end of 2.6. For convenience, we spell out the algebraic proof of Zariski and Samuel: in their notation, by [Z–S], Vol. II, Chap. VI, §12, Theorem 24, Corollary, we have $G_V = 1$; moreover (same book, pp. 75–76), G_T/G_V is dual to the extension of the value groups Γ^*/Γ, which for discrete valuations is a finite cyclic group of order r, where r is the ramification degree. In either case, characteristic zero is used in an essential way.

2.6. The geometric picture. For a finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$, the set-up is as follows. Write $L = k(\mathbb{C}^n)$ and $K = k(X) = L^G$. A discrete valuation v of L is of the form $v = v_F$, where $F \subset W$ is a prime divisor on a model $W \to \mathbb{C}^n$. There is a partial resolution $V \to X$ of X such that F maps generically onto a prime divisor $E \subset V$. You can prove this by applying the local uniformisation result quoted in 2.3. The alternative is to construct V as a quotient of some blowup of W on which G acts biregularly. For this, let $\varphi_g \colon W^g \to \mathbb{C}^n$ be the morphism with $W^g = W$ and φ_g the composite $W \to \mathbb{C}^n \xrightarrow{g} \mathbb{C}^n$, and define W' to be the birational fibre product of all the W^g. In other words, take the closure in the fibre product of the graph of the locus where all the g are regular, normalised if necessary.

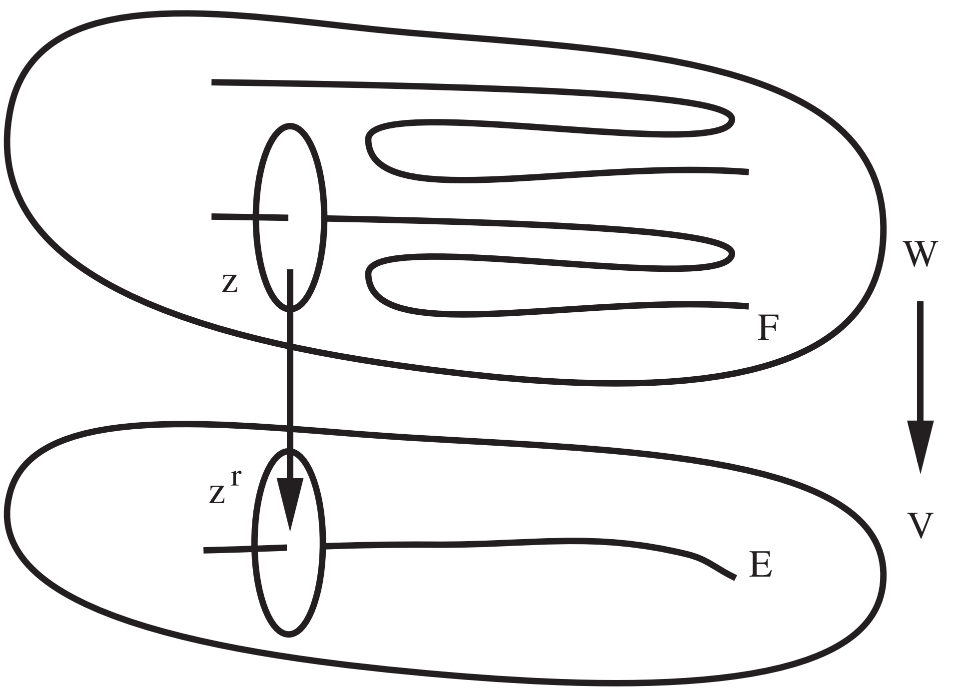

Figure 2. The cover $W \to V$ over the generic point of E.

Thus we can assume that $E \subset V \to X$ is a prime divisor on a partial resolution of X, and that W is a blowup of $\mathbb{C}^n$ that dominates V:

$$\begin{array}{ccc} W & \longrightarrow & \mathbb{C}^n \\ \downarrow & & \downarrow \\ V & \to & X = \mathbb{C}^n/G \end{array}$$

(see Figure 2). Write $\mathcal{F} = \{F\}$ for the set of all prime divisors $F \subset W$ which dominate $E \subset V$. Under these assumptions, the valuations extending $w = v_E$ are of the form v_F where $F \in \mathcal{F}$. Then G acts transitively on $\mathcal{F}$, and if $v = v_F$ then

$$\operatorname{Stab} v = \operatorname{Stab} F = \bigl\{g \in G \bigm| g(F) = F\bigr\}$$

and

$$\operatorname{Ram} v = \operatorname{Ram} F = \bigl\{g \in G \bigm| g_{|F} = \mathrm{id}_F\bigr\}.$$

Note: the action of G on W is only birational; however, it is regular at the generic point of every component F, and $g(F) = F$ and $g_{|F} = \mathrm{id}_F$ are to be interpreted in this sense.

Now since G acts transitively, the orbit space $G/\operatorname{Stab} F$ equals $\mathcal{F}$. Moreover, $\operatorname{Stab} F$ acts birationally on F with birational quotient E, so that $\operatorname{Ram} F \lhd \operatorname{Stab} F$ is a normal subgroup, and

$$\operatorname{Stab} F/\operatorname{Ram} F = \operatorname{Gal}(F/E) = \operatorname{Gal}(k(F)/k(E)).$$

Finally, $\operatorname{Ram} F$ is the cyclic group describing the behaviour of the cover $W \to V$ along the general point of F. Since we are dealing there with a codimension 1 submanifold of a complex manifold, the covering is locally of the form $z \mapsto w = z^r$, so that $\operatorname{Ram} F = \mathbb{Z}/r$ is cyclic.

Ramification and discrepancy. If s is a canonical form (that is, n-form) on V with $v_E(s) = m$ then

$$s = (\text{unit}) \cdot w^m \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}w$$

with $x_1, \dots, x_{n-1}$ local coordinates along E, and its pullback to W is of the form

$$\psi^* s = (\text{unit}) \cdot z^{mr+r-1} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}z,$$

so that $v_F(\psi^* s) = mr + r - 1 = r v_E(s) + r - 1$. Therefore $a_E = \frac{1}{r}(a_F - (r-1))$.

As discussed in 2.3, a crucial point for us is that the discrepancy of E is already present in the cyclic quotient $\mathbb{C}^n/\operatorname{Ram} F$. In our picture, the quotient $W/\operatorname{Ram} F$ contains a divisor $\overline{F}$, and the subsequent map $W/\operatorname{Ram} F \to V$ is etale at a general point of $\overline{F}$.

2.7. Ramification for monomial valuations. The definitions of $\operatorname{Stab} v$ and $\operatorname{Ram} v$ for a discrete valuation v of $k(\mathbb{C}^n)$ extend to the whole of $\mathrm{GL}(n, \mathbb{C})$ (without reference to a finite subgroup) as follows:

$$\operatorname{Stab}' v = \operatorname{Stab}' F = \bigl\{g \in \mathrm{GL}(n,\mathbb{C}) \bigm| v \circ g = v\bigr\} = \bigl\{g \in \mathrm{GL}(n,\mathbb{C}) \bigm| g(F) = F\bigr\}$$

and

$$\operatorname{Ram}' v = \left\{ g \in \operatorname{Stab}' v \;\middle|\; g^* = \mathrm{id} \text{ on } k(v) \right\} = \left\{ g \in \operatorname{Stab}' F \;\middle|\; g_{|F} = \mathrm{id}_F \right\},$$

where $F \subset W$ is a prime divisor of a blowup $W \to \mathbb{C}^n$ such that $v = v_F$. Here $\operatorname{Stab}' v = \operatorname{Stab}_{\mathrm{GL}} v$ denotes (temporarily) the stabiliser of v in GL, and similarly for Ram'.

We now determine these subgroups for monomial valuations. We first fix some ideas and notation. Consider $\mathbb{C}^n$ with a fixed coordinate system $x_1, \dots, x_n$. A primitive nonzero vector $\beta = (b_1, \dots, b_n) \in \mathbb{Z}^n$ with all $b_i \geq 0$ corresponds to the $\mathbb{C}^\times$ action on $\mathbb{C}^n$ given by

$$\lambda\colon (x_1, \dots, x_n) \mapsto (\lambda^{b_1} x_1, \dots, \lambda^{b_n} x_n).$$

The action defines a decomposition of $\mathbb{C}^n$ as a direct sum of eigenspaces; we simply group together the x_i with the same b_i:

$$\mathbb{C}^n = \bigoplus_{b \in \mathbb{Z}} S_b, \quad \text{where} \quad S_b = \operatorname{Spec} k\big[x_i \,|\, b_i = b\big]. \tag{2.7.1}$$

There is a corresponding filtration by increasing b_i:

$$T_0 \subset \cdots \subset T_c \subset \cdots \subset \mathbb{C}^n, \quad \text{where} \quad T_c = \operatorname{Spec} k\big[x_i \,|\, b_i \leq c\big] = \bigoplus_{b \leq c} S_b, \tag{2.7.2}$$

and, of course, $\operatorname{Gr}_b T = T_b / T_{b-1} = S_b$.

The weighting $\beta = (b_1, \dots, b_n)$ defines a grading on $k[x_1, \dots, x_n]$, and a weighted blowup $\sigma_\beta\colon W_\beta \to \mathbb{C}^n$, whose exceptional locus $F_\beta \subset W_\beta$ is $\operatorname{Proj} k[x_1, \dots, x_n] = \mathbb{P}(b_1, \dots, b_n)$. Note that some of the b_i may be zero, and then Proj must be taken as a product of a conventional weighted projective space with the affine space S_0, the centre of the blowup in $\mathbb{C}^n$. The weighting $(b_1, \dots, b_n)$ also defines a monomial discrete valuation $v_{F_\beta} = v_\beta$, taking $x_i \mapsto b_i$.

Lemma. *As above, $\beta = (b_1, \dots, b_n)$ is a primitive nonzero vector in the positive octant; suppose that $g \in \mathrm{GL}(n, \mathbb{C})$.*

(1) *$g \in \operatorname{Stab}' v_\beta$ if and only if it preserves the filtration (2.7.2), that is, g^* takes each x_i to a linear combination of x_j with $b_j \leq b_i$:*

$$g^*(x_i) = \sum m_{ij}(g) x_j \quad \text{with } m_{ij} = 0 \text{ whenever } b_i > b_j. \tag{2.7.3}$$

(2) *Suppose that $g \in \operatorname{Stab}' v_\beta$; then the action of g on $F_\beta = \mathbb{P}(b_1, \dots, b_n)$ is given by the induced map $\overline{g}$ on the graded vector space $\bigoplus T_b / T_{b-1}$, that is,*

$$\overline{g}^*(x_i) = \sum \overline{m}_{ij}(g) x_j \quad \text{with } \overline{m}_{ij} = \begin{cases} m_{ij} & \text{if } b_j = b_i, \\ 0 & \text{if } b_j < b_i. \end{cases} \tag{2.7.4}$$

In particular, $g \in \operatorname{Ram}' v_\beta$ if and only if $\overline{g} = \operatorname{diag}(\lambda^{b_1}, \dots, \lambda^{b_n})$ for some $\lambda \in \mathbb{C}^\times$.

Proof. This is really obvious, because the valuation v_β determines the filtration (2.7.2). Indeed, the valuation of a linear form $\sum \alpha_i x_i$ for v_β is just the smallest b_i for which $\alpha_i \neq 0$. For (1), $g \in \mathrm{GL}(n, \mathbb{C})$ preserves the space of linear forms, and if it also preserves v_β then it preserves the filtration (2.7.2).

If g preserves the filtration (2.7.2), it also preserves the filtration of $k[x_1, \dots, x_n]$ by the ideals of polynomials vanishing k times along F_β:

$$I_k = \bigl\{f \bigm| v_\beta(f) \geq k\bigr\}.$$

Therefore its action on $\mathbb{C}^n$ extends to a biregular action on the weighted blowup $W_\beta = \mathrm{Proj} \bigoplus_{k \geq 0} I_k \to \mathbb{C}^n$. The restriction to $F_\beta = \mathbb{P}(b_1, \dots, b_n)$ is in terms of the associated $\mathrm{Gr} = \bigoplus_{k \geq 0} I_k / I_{k+1}$, so the first part of (2) is clear. For the final part, the condition $g \in \mathrm{Ram}' v_\beta$ is that g fixes F_β pointwise (on a dense open set). Thus this is the usual homogeneous coordinate business: $(x_1 : \cdots : x_n) = (y_1 : \cdots : y_n)$ in $\mathbb{P}(b_1, \dots, b_n)$ if and only if $x_i = \lambda^{b_i} y_i$ for some $\lambda \in \mathbb{C}^\times$. Q.E.D.

Corollary. *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite group, and $g \in G$ an element of order r, written as usual $g = \frac{1}{r}(b_1, \dots, b_n)$ in eigencoordinates $x_1, \dots, x_n$. Assume that $\beta = (b_1, \dots, b_n)$ is a primitive vector of $\mathbb{Z}^n$, and let $v = v_\beta$ be the corresponding monomial valuation.*

Then an element $h \in G$ is in $\mathrm{Stab}\, v_\beta$ if and only if it preserves the direct sum decomposition (2.7.1). *Moreover, $h \in \mathrm{Ram}\, v_\beta$ if and only if $h = \mathrm{diag}(\varepsilon^{b_1}, \dots, \varepsilon^{b_n})$, where ε is a root of* 1.

Proof. Choose a G-invariant Hermitian metric on $\mathbb{C}^n$, so that all the elements of G are unitary. Then the direct sum (2.7.1) is the eigenspace decomposition of the unitary operator g, and is thus orthogonal. Moreover, h is also a unitary operator, so it preserves the filtration (2.7.2) if and only if it preserves the direct sum decomposition (2.7.1). The final part follows directly from (2) of the lemma. Q.E.D.

2.8. Conclusion of proof of Theorem 1.4. We divide up the proof into 5 easy steps.

Step 1. Construction of $g \mapsto v_g$. As we hinted above, the correspondence

$$\Gamma_1 \to \bigl\{\text{valuations of } k(X)\bigr\},$$

sends g to the monomial valuation v_g of $k(X)$ corresponding to g. That is, as described in 2.2, write $g = \frac{1}{r}(a_1, \dots, a_n) \in \square$, to mean that g has order exactly r, and is $\varepsilon \mapsto \mathrm{diag}(\varepsilon^{a_1}, \dots, \varepsilon^{a_n})$ in eigencoordinates $x_1, \dots, x_n$. Now set $\widetilde{v}_g = v_\beta$, where $\beta = (a_1, \dots, a_n)$, and

$$v_g = \frac{1}{r}(\widetilde{v}_{g|k(X)})\colon k(X) \to \mathbb{Z} \cup \{\infty\}.$$

It's clear from the definition of discrete valuation that v_g is (possibly a multiple of) a discrete valuation of $k(X)$.

Step 2. Ramification and discrepancy. We claim that for $g \in \Gamma_1$, the ramification group of $\widetilde{v}_g = v_\beta$ in G is exactly $\langle g \rangle$, and the ramification degree is r. It follows from this that v_g is in fact a discrete valuation of $k(X)$, and, by the calculation in 2.4, that its discrepancy is $\mathrm{age}\, g - 1 = 0$.

Since $g = \frac{1}{r}(a_1, \dots, a_n)$, Corollary 2.7 says that the ramification group of $\widetilde{v}_g$ is the subgroup of G of elements of the form $h = \operatorname{diag}(\varepsilon^{a_1}, \dots, \varepsilon^{a_n})$, where ε is a root of 1. If the order of ε divides r then $h \in \langle g \rangle$, which is what we want. If not, replacing h by a suitable combination of g, h, we can assume that ε is a root of unity of order larger than r; but then, because $\sum a_i = r$, such an h is not in $\mathrm{SL}(n, \mathbb{C})$.

Step 3. *v_g is well defined on conjugacy classes.* If $g_1 = hgh^{-1}$, and $x_1, \dots, x_n$ are eigencoordinates for g then $h^* x_i$ are eigencoordinates for g_1 with the same eigenvalues. Since $\widetilde{v}_g$ and $\widetilde{v}_{g_1}$ are constructed as monomial valuations with respect to these eigencoordinates, it follows at once that $\widetilde{v}_g \circ h = \widetilde{v}_{g_1}$, and therefore they restrict to the same valuations of $k(X) = k(\mathbb{C}^n)^G$.

In geometric terms, the weighted blowup of $\mathbb{C}^n$ corresponding to g and g_1 fit together into a diagram of morphisms:

$$\begin{array}{ccc} W_g & \xrightarrow{h} & W_{g_1} \\ \downarrow & & \downarrow \\ \mathbb{C}^n & \xrightarrow[h]{} & \mathbb{C}^n \end{array}$$

That is, h takes the exceptional prime divisor F_g into F_{g_1}, so that they both map birationally to the same exceptional divisor of X.

Step 4. Injectivity. This is the converse of Step 3: if $g, g' \in \Gamma_1$ and $v_g = v_{g'}$ then g and g' are conjugate in G. The assumption is that $\widetilde{v}_g$ and $\widetilde{v}_{g'}$ restrict to the same valuation of $k(X)$. Recall that the Galois group $G = \operatorname{Gal}(k(\mathbb{C}^n)/X)$ acts transitively on the set of extensions of a valuation of $k(X)$. Therefore $\widetilde{v}_{g'} = \widetilde{v}_g \circ h$ for some $h \in G$. We've seen in Step 3 that $\widetilde{v}_g \circ h = \widetilde{v}_{g_1}$ where $g_1 = hgh^{-1}$. Now $\widetilde{v}_{g_1} = \widetilde{v}_{g'}$ clearly implies that $g' = g_1$. (Because they both induce the same filtration of the vector space of linear forms, and give x_i the same weights b_i.)

Step 5. Surjectivity: every crepant divisor of X comes from this construction. If v_E is a crepant valuation of $k(X)$ and $\widetilde{v} = v_F$ a valuation of $k(\mathbb{C}^n)$ extending v then $\operatorname{Ram} \widetilde{v}$ is a cyclic group of order r, the ramification degree. The discrepancy of E depends only on r and that of F, and is given by

$$a_E = \frac{1}{r}(a_F - (r-1)).$$

As explained in 2.3, only monomial valuations of $\mathbb{C}^n$ with nontrivial ramification groups can give crepant valuations of X. These are all of the form v_g with $g \in \Gamma$, and only those with $g \in \Gamma_1$ are crepant.

This completes the proof of Theorem 1.4. Q.E.D.

Remark. One can also consider monomial valuations v_g for conjugacy classes $g \in \Gamma_i/G$ with $i \geq 2$. To get the right discrepancy $a_{E_g} = \text{age} - 1$, and to keep the injectivity proved in Step 3, we restrict to *primitive* elements g (corresponding to a primitive vector $g = \frac{1}{r}(a_1, \dots, a_n) \in \Box$). It's an exercise to generalise Steps 2–4 to these valuations. For example, the cyclic group $\frac{1}{4}(1,1,1,1)$ (the Gorenstein cone on the 4-fold Veronese embedding of $\mathbb{P}^3$) has only one crepant divisor.

2.9. Proof of Corollary 1.5. This is clear. If a minimal model Y exists then the exceptional divisors occurring on it are the crepant divisors and no others. Next,

$H^2(Y,\mathbb{Q})$ is spanned by algebraic cycles, the first Chern classes of divisors (just use the exponential sequence and $R^1 f_* \mathcal{O}_Y = 0$). But every divisor D on Y pushes down to divisor on X some multiple of which is linearly equivalent to zero, because a quotient singularity is analytically $\mathbb{Q}$-factorial. Therefore some multiple of D is linearly equivalent to an exceptional divisor. Q.E.D.

2.10. Proof of Theorem 1.6. For $g = \frac{1}{r}(a_1, \dots, a_n) \in \Box$, the fixed locus $\operatorname{Fix} g \subset \mathbb{C}^n$ is the linear subspace S_0 corresponding to the coordinates x_i with $a_i = 0$. In particular, 0 is an isolated point of $\operatorname{Fix} g$ if and only if all $a_i > 0$, or equivalently, g is in the interior of $\Box$. Thus we set

$$\Gamma_1^{(0)} = \bigl\{g \bigm| \operatorname{Fix} g = \{0\}\bigr\}.$$

Note in passing that if $n = 3$ and $g = \frac{1}{r}(a_1, a_2, a_3)$ then $a_1 = 0$ implies $a_2 + a_3 = r$, so that

$$\operatorname{Fix} g \neq \{0\} \implies g \text{ is junior.}$$

We now restrict attention to the case $n = 3$. Then $g \in \Gamma_1^{(0)}$ is equivalent to $g^{-1} \in \Gamma_2$: in eigencoordinates, if $g = \frac{1}{r}(a_1, a_2, a_3)$ with all $a_i > 0$ then

$$g^{-1} = (1,1,1) - \frac{1}{r}(a_1, a_2, a_3).$$

Thus $g \mapsto g^{-1}$ gives the bijection $\Gamma_1^{(0)} \to \Gamma_2$, which is clearly compatible with conjugacy. This proves the first bijection in Theorem 1.6.

The second bijection is clear, since a divisor E has compactly supported class in H^2 if and only if E contracts to $0 \in X$. The third bijection is just Poincaré duality. Q.E.D.

3. Examples

We start with the examples of the Klein quotient singularities or Du Val surface singularities, and explain why our result Theorem 1.4 gives the exceptional components *together with their configuration.* It is perhaps remarkable that our methods lead naturally to a “McKay quiver” structure on the conjugacy classes of G (rather than irreducible representations).

The Du Val singularity D_{n+2} is the quotient $\mathbb{C}^2 / \mathrm{BD}_{4n}$, where BD_{4n} is the binary dihedral group of order $4n$ generated by

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and ε is a primitive $2n$th root of 1. Thus $A^n = B^2 = -1$, and the elements of BD_{4n} are the $2n$ “rotations” A^i, and the $2n$ “reflections” $A^i B = B A^{2n-i}$.

Conjugacy in G gives $A^i \sim A^{2n-i}$, so there are n conjugacy classes of nonidentity rotations; and $ABA^{-1} = A^2 B$, so that $B \sim A^{2i} B$ and $AB \sim A^{2i+1} B$. The effect of conjugacy on the cyclic subgroups $\bigl\langle A^i B \bigr\rangle \cong \mathbb{Z}/4$ depends on n even or odd: if n is even then $\langle B \rangle$ and $\langle AB \rangle$ are not conjugate, but $B \sim A^n B = -B = B^3$ and $AB \sim (AB)^3 = BA$. If n is odd then all the subgroups $\bigl\langle A^i B \bigr\rangle$ are conjugate, but the elements of order 4 within any subgroup are not conjugate.

Example 1 (Binary dihedral group BD_8, singularity D_4). In this case the two matrixes are

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}),$$

where $i = \sqrt{-1}$. Thus G has three cyclic subgroups of order 4, $\langle A\rangle$, $\langle B\rangle$ and $\langle AB\rangle$, with $A^2 = B^2 = (AB)^2 = -1$, which we draw as follows:

$$\begin{array}{rccccc} & A & & A^2 & & A^3 \\ \langle A\rangle : & \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc \\ \\ & B & & B^2 & & B^3 \\ \langle B\rangle : & \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc \\ \\ & AB & & (AB)^2 & & (AB)^3 \\ \langle AB\rangle : & \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc \end{array} \tag{1}$$

The pictures are the exceptional divisors in the minimal resolution of $\mathbb{C}^2 / \langle g\rangle$ as given by toric geometry. The nodes represent elements of $\langle g\rangle$, and they are joined by an edge if they are neighbouring points in the junior simplex (interval) Δ_1, that is, if the corresponding curves intersect on the minimal resolution.

Conjugacy in G gives

$$A \sim A^3, \quad B \sim B^3, \quad AB \sim (AB)^3.$$

Thus identifying equal and conjugate group elements gives the familiar D_4 diagram:

$$\begin{array}{ccccc} A & & -1 & & B \\ \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc \\ & & | & & \\ & AB & \bigcirc & & \end{array} \tag{2}$$

Example 2 (Binary dihedral group BD_{8l+4}, singularity D_{2l+3}). In this case A has order $4l + 2$. We should draw all the maximal cyclic subgroups, then consider the effect of conjugacy. As noted above, the cyclic subgroups $\langle A^i B\rangle$ are all conjugate, so we only draw $\langle A\rangle$ and $\langle B\rangle$:

$$\begin{array}{rcccccc} & A & & A^2 & & A^{4l} & & A^{4l+1} \\ \langle A\rangle : & \bigcirc & \text{---} & \bigcirc & \cdots & \bigcirc & \text{---} & \bigcirc \\ \\ & B & & B^2 & & B^3 & & \\ \langle B\rangle : & \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc & & \end{array} \tag{3}$$

Now $A^{2l+1} = B^2 = -1$, and conjugacy gives $A^i \sim A^{4l+2-i}$, so that identifying equal and conjugate elements gives

$$\begin{array}{ccccccccc} B & & -1 & & A^{2l} & & A^2 & & A \\ \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc & \cdots & \bigcirc & \text{---} & \bigcirc \\ & & | & & & & & & \\ & B^3 & \bigcirc & & & & & & \end{array} \tag{4}$$

The case of the binary dihedral group BD_{8l}, giving the singularity D_{2l+2} is similar. The only difference is that to see the action of conjugacy, we have to draw both $\langle B\rangle$ and $\langle AB\rangle$, and then fold each of them in two.

Example 3 (Binary tetrahedral group BT_{48}, singularity E_6). The singularity E_6 is the quotient of $\mathbb{C}^2$ by the group BT_{48} generated by the three matrixes

$$A=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad B=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\quad\text{and}\quad C=\frac{1}{\sqrt{2}}\begin{pmatrix} e^7 & e^7\\ e^5 & e\end{pmatrix}\in \mathrm{SL}(2,\mathbb{C}),$$

where e is a primitive 8th root of 1 (for example, $e=\frac{1+i}{\sqrt{2}}$). Thus

$$\underset{48}{\mathrm{BT}}=\left\langle A,B,C \;\middle|\; A^2=B^2=C^3=-1,\ BA^iB^{-1}=A^{-i},\ CAC^{-1}=B^3\right\rangle.$$

We draw the cyclic subgroups $\langle A\rangle$ and $\langle C\rangle$ of BT_{48} following the previous recipe:

$$\begin{array}{lccccccccc}
 & A & & A^2 & & A^3 & & & & \\
\langle A\rangle: & \bigcirc & — & \bigcirc & — & \bigcirc & & & & \\
 & C & & C^2 & & C^3 & & C^4 & & C^5\\
\langle C\rangle: & \bigcirc & — & \bigcirc & — & \bigcirc & — & \bigcirc & — & \bigcirc
\end{array}\qquad (5)$$

It is easy to see the effect of conjugacy in BT_{48}:

$$A\sim A^3\sim B\sim B^3\sim AB\sim BA,\quad\text{and}\quad A^kC^j\sim C^j \text{ for all } j,k,$$

and the elements of $\langle C\rangle$ are not conjugate to one another. Thus identifying equal and conjugate elements of these two groups gives the resolution graph of E_6:

$$\begin{array}{ccccccccc}
C & & C^2 & & -1 & & C^4 & & C^5\\
\bigcirc & — & \bigcirc & — & \bigcirc & — & \bigcirc & — & \bigcirc\\
 & & & & | & & & & \\
 & & & A & \bigcirc & & & &
\end{array}\qquad (6)$$

Now we see some 3-dimensional examples. (More examples are furnished by the crepant resolutions of $\mathbb{C}^3/G$ of [Roan], [Ito1–3] and [Markushevich].)

Example 4 (Trihedral group). A *trihedral* group is a subgroup G generated by a diagonal subgroup $H\subset \mathrm{SL}(3,\mathbb{C})$, together with the cyclic permutation matrix

$$T=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}.$$

The Abelian subgroup H is generated by one or more diagonal matrixes

$$A_i=\frac{1}{r_i}(a_i,b_i,c_i),\quad\text{with}\quad a_i+b_i+c_i=r_i,$$

meaning as usual that $A_i = \operatorname{diag}(\varepsilon^{a_i}, \varepsilon^{b_i}, \varepsilon^{c_i})$ where ε is a primitive r_ith root of 1. We can of course assume that H has cyclic symmetry, and then it is a normal subgroup $H \triangleleft G$. The quotient of $\mathbb{C}^3$ by G is a *trihedral quotient singularity.* A crepant resolution of singularities $Y \to X$ is described in [Ito1–2].

We work out the special case $H = \left\langle \frac{1}{3}(0,1,2), \frac{1}{3}(1,2,0), \frac{1}{3}(2,0,1) \right\rangle$. Obviously, $G = \left\langle \frac{1}{3}(0,1,2), T \right\rangle$, and is a group of order 27.

It turns out that G has only one senior element $\frac{1}{3}(2,2,2)$, and 25 juniors. Note that, as we saw in 2.10, for subgroups of SL(3, $\mathbb{C}$), all elements with nonisolated fixed points are junior; this applies to every element of G except $\frac{1}{3}(1,1,1)$ and $\frac{1}{3}(2,2,2)$ because, for example, $\operatorname{Fix} \frac{1}{3}(0,1,2) = x$-axis, and every element hT is obviously conjugate in GL(3, $\mathbb{C}$) to T, which fixes the line $\{(x,x,x)\} \subset \mathbb{C}^3$. Thus

$$\Gamma_1 = \left\{ \begin{matrix} \frac{1}{3}(1,1,1),\ \frac{1}{3}(0,1,2),\ \frac{1}{3}(1,2,0),\ \frac{1}{3}(2,0,1), \\ \frac{1}{3}(0,2,1),\ \frac{1}{3}(2,1,0),\ \frac{1}{3}(1,0,2) \end{matrix} \right\} \cup HT \cup HT^2$$

Under conjugacy in G,

$$\frac{1}{3}(a,b,c) \sim \frac{1}{3}(b,c,a), \quad \text{for all } a,b,c \in \{0,1,2\},$$

and

$$T \sim \frac{1}{3}(a,b,c)T, \quad \text{and} \quad T^2 \sim \frac{1}{3}(a,b,c)T^2 \quad \text{for all } a \neq b.$$

Thus the exceptional divisors in the minimal resolution $Y \to X = \mathbb{C}^3/G$ correspond one-to-one with the following junior conjugacy classes:

$$\begin{gathered} \tfrac{1}{3}(0,1,2),\ \tfrac{1}{3}(0,2,1),\ \tfrac{1}{3}(1,1,1), \\ T,\ \tfrac{1}{3}(1,1,1)T,\ \tfrac{1}{3}(2,2,2)T, \\ T^2,\ \tfrac{1}{3}(1,1,1)T^2,\ \tfrac{1}{3}(2,2,2)T^2. \end{gathered}$$

These elements correspond to a basis of $H^2(Y,\mathbb{Q})$, so that $h^2(Y,\mathbb{Q}) = 9$. On the other hand, by Corollary 1.6, $H^4(Y,\mathbb{Q})$ has a basis corresponding to the single element $\Gamma_2 = \{\frac{1}{3}(2,2,2)\}$, and its Poincaré dual $H^2_c(Y,\mathbb{Q})$ has a basis corresponding to $\Gamma_1^{(0)} = \{\frac{1}{3}(1,1,1)\}$. In particular the Euler number $e(Y)$ is given by

$$\begin{aligned} e(Y) &= h^0(Y,\mathbb{Q}) + h^2(Y,\mathbb{Q}) + h^4(Y,\mathbb{Q}) = 1 + 9 + 1 = 11 \\ &= \sharp\{\text{conjugacy class in } G\}. \end{aligned}$$

Example 5 (Icosahedral group). $G_{60} \subset \mathrm{SO}(3)$ is the group of rotations of the icosahedron, acting on $\mathbb{C}^3$ by the inclusion $\mathrm{SO}(3) \subset \mathrm{SL}(3)$. Generating matrixes and defining relations are given for example in [Miller–Blichfeldt–Dickson], p. 250. The reader may eke out our poor words of explanation by imagining the picture of the icosahedron.

It is easy to see that G_{60} has just 5 conjugacy classes:

(1) the identity;
(2) a rotation through π about an axis through the midpoint of an edge;
(3) a rotation through $2\pi/3$ about an axis through the midpoint of a face;
(4) a rotation through $2\pi/5$ about an axis through a vertex;
(5) a rotation through $4\pi/5$ about the same axis.

Each rotation fixes an axis, and is therefore junior by the remark in 2.10. The quotient $\mathbb{C}^3/G_{60}$ has 3 curves of transversal Du Val singularities: a curve of A_1 under the axis of (2); a curve of A_2 under the axis of (3), and a curve C of transversal A_4 singularities under the common axis of (4) and (5). Resolving these curves gives 4 crepant divisors. For example, blowing up the curve C of A_4 singularities gives two surfaces which (outside the origin) are $\mathbb{P}^1$ bundles over a curve that is a double cover of C.

4. Discussion

4.1. Theorem 1.6 implies directly Conjecture 1.1 for $n = 3$, with the proviso that you have to fix roots of 1 to get a canonical correspondence. To see that Y cannot have any odd-dimensional homology, it is enough to note that there exists a resolution of X dominated by a variety W obtained by a sequence of blow-ups of $\mathbb{C}^n$ in linear subspaces. Then $H^*(W)$ has no odd-dimensional cohomology, and $H^*(Y) \subset H^*(W)$.

4.2. The minimal model is in fact nonsingular: indeed, by standard results of Mori theory, although the minimal model is usually not unique, all minimal models have the same local singularities. Moreover, Roan and others have proved that a smooth crepant resolution exists in all cases. Unfortunately, the present abstract point of view does not (as yet) offer any clues as to why this should be so.

4.3. There should be a theoretical proof that the minimal model $Y \to X$ is nonsingular, even though we are not clever enough to find it at present. For example, if Y has factorial cDV points, then these have local deformations. It should follow from Friedman's local-to-global spectral sequence that Y has deformations inducing all possible deformations of the local singularities, and in particular, should have a smoothing Y_t; then Y_t also contracts to a variety X_t. Alternatively the Milnor fibre shows that Y_t must have a middle cohomology of Hodge type $H^{1,2} \oplus H^{2,1}$—but so what? Or maybe the singularities would have homotopy properties that contradict that the link of X is 2-connected.

4.4. The shape of Theorems 1.4 and 1.6 is very suggestive: the canonical basis for $H^2(Y, \mathbb{Z})$ involves junior conjugacy classes of $\Gamma = G(-1)$, which is probably the only thing that makes sense in terms of motivic weights. The basis of $H^4(Y, \mathbb{Q})$ is dual to a basis of $H^2_c(Y, \mathbb{Q})$, so to take account of the roots of 1 correctly, you presumably have to take Hom into $\mathbb{Z}(-3)$, and end up with another Tate twist of the set of conjugacy classes Γ_2/G before getting a canonical identification with the basis of $H^4(Y, \mathbb{Q})$ (?).

4.5. Crepant resolutions in higher dimension. In dimension ≥ 4, when does $X = \mathbb{C}^n/G$ have a crepant resolution $Y \to X$? The known counterexamples are the terminal quotient singularities, for example, $\frac{1}{r}(i, r-i, j, r-j)$ (see also [Mori–Morrison–Morrison]). For these, there are not enough junior elements: the unit cube $\square$ in $\mathbb{R}^4$ contains $r-1$ elements on the Γ_2 hyperplane $\sum \alpha_i = 2$, and no junior elements at all. Thus there are no crepant divisors to pull out.

It seems plausible that the following 2 conditions to the effect that G has lots of junior elements are necessary conditions: for each *maximal* cyclic subgroup $\mathbb{Z}/r \subset G$, choose eigencoordinates and write $L = \mathbb{Z}^n + \mathbb{Z} \cdot \frac{1}{r}(a_1, \ldots, a_n)$ as usual. Then

(i) every point of $L \cap \square$ is a positive integral combination of points of $L \cap \Delta_1$.

(ii) the positive octant of L has a subdivision into basic cones with 1-skeleton contained in Δ_1.

Obviously (ii) implies (i), and, for Abelian groups, by toric geometry, (ii) is a necessary and sufficient condition.

4.6. Multiplicative structure. Assume that a crepant resolution $Y \to X$ exists, and that $n = 4$ or 5, to fix ideas. There is one respect in which Conjecture 1.1 looks bad: to get a canonical correspondence, as we have seen, you have to introduce an age grading, and this will only be canonical if you twist G to $G(-1)$. No other twist has a canonical age grading. But then the theory of weights in the philosophy of motives make it ungrammatical to put elements of $G(-1)$ in correspondence with elements of H^4.

One way of trying to rescue the conjecture might be to put elements of H^4 in correspondence with quadratic monomials in Γ_1, maybe with relations between them deduced from a multiplicative structure; this seems to be good for Abelian quotients (compare [Batyrev–Dais]). Some natural questions arise in this connection. Presumably it is true that the cohomology ring $H^*(Y, \mathbb{Q})$ is generated by H^2? Is there a reasonable multiplicative structure involving the conjugacy classes of G?

4.7. Remark. The easy arguments in this paper have left to one side the most important aspect of the McKay correspondence for SL(2, $\mathbb{C}$), namely, the relation with irreducible representations. Understanding this point might begin to explain a lot of things. Some indications:

(1) In 2.5 we needed to find an extraction $V \to X$ pulling out exactly the divisor E corresponding to a conjugacy class, by analogy with the weighted blowup of 2.3 in the cyclic case. This could be provided by quiver-theoretic methods: view $X = \mathbb{C}^n/G$ as a moduli space of representations of a quiver (e.g. the McKay quiver of G). Then X is a categorical quotient of a big vector space by a big algebraic group, and birational modifications of X are naturally obtained by "varying the linearisation".

(2) The age of $g \in \Gamma$ is defined as the elementary symmetric function $\operatorname{age} g = \frac{1}{r}\sum a_i$. Do the other elementary symmetric functions (e.g. $\frac{1}{r^2}\sum a_i a_j$) also have some significance for the McKay correspondence?

References

V. Batyrev, Strong McKay correspondence, lecture at Trento conference, Jun 1994

Victor V. Batyrev and Dimitrios I. Dais, Strong McKay correspondence, String-theoretic Hodge Numbers and Mirror Symmetry, Duke server alg-geom/9410001, to appear in Topology **35** (1996)

V. Danilov, The geometry of toric varieties, Uspekhi Mat. Nauk **33**:2 (1978) 85–134 = Russian Math. Surveys **33**:2 (1978) 97–154

V. Danilov, The birational geometry of toric 3-folds, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 972–981 = Math. USSR – Izv. **21** (1983)

L. Dixon, J. Harvey, C. Vafa and E. Witten, Strings on orbifolds, Nucl. Phys. B **261** (1985) 678–686. II, Nucl. Phys. B **274** (1986) 285–314

G. González-Sprinberg and J.-L. Verdier, Points doubles rationnels et représentations de groupes, C. R. Acad. Sci. Paris, Sér. I, Math. **293** (1981), 111–113

G. González-Sprinberg and J.-L. Verdier, Construction géometrique de la correspondance de McKay, Ann. Sci. École Norm. Sup. (4) **16** (1983), 409–449

F. Hirzebruch and T. Höfer, On the Euler number of an orbifold, Math. Ann. **286** (1990), 255–260

Y. Ito, Crepant resolutions of trihedral singularities, Proc. Japan Acad. **70** (1994), 131–136

Y. Ito, Crepant resolution of trihedral singularities and the orbifold Euler characteristic, Internat. J. Math. **6**:1, 1995, 33–43

Y. Ito, Gorenstein quotient singularities of monomial type in dimension three, J. Math. Sci. Univ. Tokyo **2** (1995), 419–440. Duke server alg-geom/9406001

J. Kollár, Flips, flops etc., in Surveys in differential geometry, **1** (1991) 113–199

D. Markushevich, Resolution of $\mathbb{C}^3/H_{168}$, Technion preprint, 1993

G. A. Miller, H. F. Blichfeldt and L. E. Dickson, Theory and applications of finite groups, Wiley, London, 1917, reprinted Dover 1961

J. McKay, Graphs, singularities, and finite group, in Santa Cruz conference on finite groups (Santa Cruz, 1979), Proc. Symp. Pure Math. **37**, AMS (1980), 183–186

S. Mori, D. R. Morrison and I. Morrison, On four-dimensional terminal singularities, Math. Comp. **51** (1988), 769–786

S-S. Roan, Minimal resolutions of Gorenstein orbifolds in dimension three, Preprint R940606–1, Acad. Sinica, Taibei, Jun 1994, to appear in Topology **35** (1996)

[C3-f] M. Reid, Canonical 3-folds, in Géométrie algébrique, (Angers 1979) (A. Beauville, Ed.), Sijthoff and Noordhoof, Alphen aan den Rijn, 1980, 273–310

[YPG] M. Reid, Young person's guide to canonical singularities, in Algebraic geometry (Bowdoin 1985), Proc. Symp. Pure Math. **46**, AMS, 1987, 345–416

[Z–S] O. Zariski and P. Samuel, Commutative Algebra, Vols. I and II, Van Nostrand, 1960

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