Hilbert schemes and simple singularities

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Abstract

The first half of this article is expository; it contains a brief survey of the famous ADE classification, and how it applies to six kinds of objects, some old and some relatively new. The second half is a research article, discussing the two dimensional McKay correspondence from the new point of view of Hilbert schemes.

0 Introduction

There is a whole series of apparently unrelated phenomena that are governed by the so-called ADE Dynkin diagram scheme. It is widely believed that, despite the diverse nature of the objects concerned, there must be some hidden reasons for these coincidences. The ADE Dynkin diagrams provide a classification of the following types of objects (among others):

(a) simple singularities (rational double points) of complex surfaces (Du Val, Artin, Brieskorn),

(b) finite subgroups of $\mathrm{SL}(2,\mathbb{C})$,

(c) simple Lie groups and simple Lie algebras (Elie Cartan, Dynkin),

(d) quivers of finite type ([Gabriel72]),

(e) modular invariant partition functions in two dimensions (Capelli, Itzykson and Zuber [CIZ87]),

(f) pairs of von Neumann algebras of type II_1 ([Ocneanu88]).

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The present article consists of two halves, an expository part and a research part. The expository part occupies the first six sections. In Sections 1–4, we recall briefly the above ADE classifications. Sections 2–3 report in some detail on the relatively new subjects of modular invariant partition functions and type II_1 von Neumann algebras (II_1 factors). In Section 4 we recall the two dimensional McKay correspondence. Section 5 summarizes some of the missing links between the six objects and related problems. We would like to say that while much is known about these, much remains unknown.

Next, in Section 6, we recall some basic facts about Hilbert schemes for use in the research part, and give a quick review on three dimensional quotient singularities in Section 7. Section 7 is not directly related to the rest of the paper, but it provides motivation for further study in the same direction as Sections 8–16. For instance, a natural three dimensional generalization of the McKay correspondence, quite different from that of Theorem 7.2, can be obtained by applying similar ideas. This direction is still under research and we simply mention [Reid97], [INkjm98] and [Nakamura98] as available references for it.

In the second half of the article we discuss the two dimensional McKay correspondence from a somewhat new point of view, namely by applying the technique of Hilbert schemes. Any known explanations for the classical McKay correspondence enables each irreducible component of the exceptional set E to correspond naturally to an irreducible representation of a finite subgroup G. In the present article we do a little more. In fact, to any point of the exceptional set, we associate in a natural way a G-module, irreducible or otherwise, whose equivalence class is constant along the irreducible component of E. We discuss this in outline in Section 8, and in detail in Sections 8–16. Some new progress and related problems are mentioned in Section 17.

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There are a number of excellent reports on the first four topics (a)–(d), for example: Hazewinkel, Hesselink, Siersma and Veldkamp [HHSV77] and [Slodowy95]. See [Slodowy90] and [Gawedzki89] for the topic (e). See also [Ocneanu88], Goodman, de la Harpe and Jones [GHJ89], [Jones91] and Evans and Kawahigashi [EK97], Section 11 for the last topic (f). The authors hope the readers to read or to have a glance at these reports too.

We have in mind both specialists in algebraic geometry and nonspecialists as readers of the expository part. Therefore we have tried to include elementary examples and algebraic calculations, though they are not completely self-contained.

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1 Simple singularities and ADE classification

1.1 Simple singularities (1)

We first recall the definition of simple singularities. A germ of a two dimensional isolated hypersurface singularity is called a *simple singularity* if one of the following equivalent conditions holds:

1. It is isomorphic to one of the following germs at the origin

$$\begin{array}{lll} A_n: & x^{n+1}+y^2+z^2=0 & \text{for } n\geq 1, \\ D_n: & x^{n-1}+xy^2+z^2=0 & \text{for } n\geq 4, \\ E_6: & x^4+y^3+z^2=0, & \\ E_7: & x^3y+y^3+z^2=0, & \\ E_8: & x^5+y^3+z^2=0. & \end{array}$$

2. It is isomorphic to a germ of a weighted homogeneous hypersurface of $(\mathbb{C}^3,0)$ of total weight one such that the sum of weights (w_1,w_2,w_3) of the variables is greater than one. The possible weights are $(\frac{1}{n+1},\frac{1}{2},\frac{1}{2})$, $(\frac{1}{n-1},\frac{n-2}{2n-2},\frac{1}{2})$, $(\frac{1}{4},\frac{1}{3},\frac{1}{2})$, $(\frac{2}{9},\frac{1}{3},\frac{1}{2})$ and $(\frac{1}{5},\frac{1}{3},\frac{1}{2})$.

3. It has a minimal resolution of singularities with exceptional set consisting of smooth rational curves of selfintersection -2 intersecting transversally.

4. It is a quotient of $(\mathbb{C}^2,0)$ by a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ ([Klein]).

5. Its (semi-)universal deformation contains only finitely many distinct isomorphism classes ([Arnold74]).

Many other characterizations of the singularities are given in [Durfee79]. The third characterization of a simple singularity classifies the exceptional set explicitly. In fact, the dual graph of the exceptional set is one of the Dynkin diagrams of simply connected complex Lie groups shown in Figure 1.

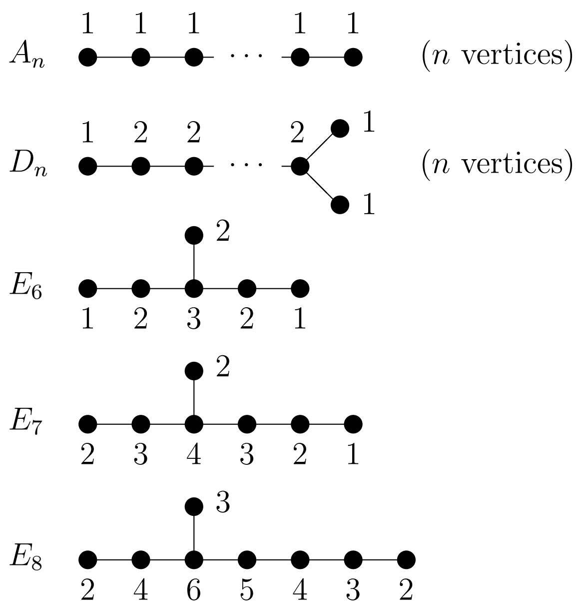

Figure 1: The Dynkin diagrams ADE

1.2 Simple singularities (2)

Let $(S, 0)$ be a germ of simple singularities, $\pi\colon X \to S$ its minimal resolution, $E := \pi^{-1}(0)_{\text{red}}$ and E_i for $1 \le i \le r$ the irreducible component of E. It is known that $E_i \simeq \mathbb{P}^1$ and $(E_i^2)_X = -2$. Let $\operatorname{Irr} E$ be the set $\{E_i; 1 \le i \le r\}$. We see that $H_2 = H_{2,\text{SING}}(S) := H_2(X, \mathbb{Z}) = \bigoplus_{1\le i\le r} \mathbb{Z}[E_i]$. Then H_2 has a negative definite intersection pairing $(\ ,\ )_{\text{SING}}\colon H_2 \times H_2 \to \mathbb{Z}$. Since $(E_iE_j)_{\text{SING}} = 0$ or 1 for $i \ne j$, the pairing $(\ ,\ )_{\text{SING}}$ can be expressed by a finite graph with simple edges. We rephrase this as follows: we associate a vertex $v(E')$ to any irreducible component E' of E, and join two vertices $v(E')$ and $v(E'')$ if and only if $(E'E'')_{\text{SING}} = 1$. Thus we have a finite graph with simple edges, from which in turn the bilinear form $(\ ,\ )_{\text{SING}}$ can be recovered in the obvious manner. We call this graph the *dual graph* of E, and denote it by $\Gamma(E)$ or $\Gamma_{\text{SING}}(S)$. Let $H^2 = H^2_{\text{SING}}(S) := H^2(X, \mathbb{Z})$.

There exists a unique divisor E_{fund}, called the *fundamental cycle* of X, which is the minimal nonzero effective divisor such that $E_{\text{fund}}E_i \le 0$ for all i. Let $E_{\text{fund}} := \sum_{i=1}^r m_i^{\text{SING}} E_i$ and $E_0 := -E_{\text{fund}}$. For the simple singularities we have $E_0E_i = 0$ or 1 for any $E_i \in \operatorname{Irr} E$ except for the case A_1, when $E_0E_1 = 2$. Therefore we can draw a new graph $\widetilde{\Gamma}_{\text{SING}}$ by adding the vertex $v(E_0)$ to $\Gamma_{\text{SING}}(S)$. By a little abuse of notation we denote $\operatorname{Irr} E \cup \{E_0\}$ by $\operatorname{Irr}_* E$.

For instance let us consider the D_5 case. Then $E = \sum_{i=1}^5 E_i$ with $E_i^2 = -2$ and

$$-E_0 = E_{\text{fund}} = E_1 + 2E_2 + 2E_3 + E_4 + E_5.$$

Then $E_0E_2 = E_1E_2 = E_2E_3 = E_3E_4 = E_3E_5 = 1$, and all other $E_iE_j = 0$. Hence $(m_1^{\mathrm{SING}}, \dots, m_5^{\mathrm{SING}}) = (1, 2, 2, 1, 1)$, as indicated in Figure 2.

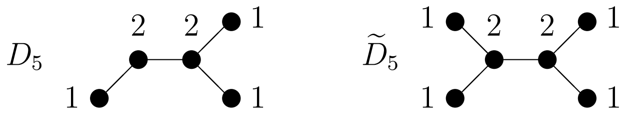

Figure 2: The Dynkin diagrams D_5 and $\widetilde{D}_5$

There are various ways of computing E. We check this starting from the fact that D_5 is the quotient singularity of $\mathbb{A}^2$ by the binary dihedral group $\mathbb{D}_3$ of order 12. The binary dihedral group $G := \mathbb{D}_3$ is generated by σ and τ:

$$\sigma = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\varepsilon := e^{2\pi\sqrt{-1}/6}$. We have $\sigma^6 = \tau^4 = 1$, $\sigma^3 = \tau^2$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$. The ring of G-invariants in $\mathbb{C}[x, y]$ is generated by three elements $F := x^6 + y^6$, $H := xy(x^6 - y^6)$ and $I := x^2y^2$. The quotient $\mathbb{A}^2/G$ is isomorphic to the hypersurface $4I^4 + H^2 - IF^2 = 0$. Since G has a normal subgroup $N := \{\sigma\}$ of order 6, we first take the quotient $\mathbb{A}^2/N$ and its minimal resolution X_N.

Since $P := x^6$, $Q := y^6$ and $R := xy$ are N-invariants, $\mathbb{A}^2/N$ is a hypersurface $PQ = R^6$. Hence X_N has an exceptional set consisting of a chain of 5 smooth rational curves $C_1 + \cdots + C_5$. The action of τ on $\mathbb{A}^2$ induces an action on X_N, which maps C_i into C_{5-i}, so in particular takes C_3 to itself. The action of τ on X_N has exactly two fixed points $\mathfrak{p}_+$ and $\mathfrak{p}_-$ on C_3, which give rise to all the singularities of $X_N/\{\tau\}$.

The images of $\mathfrak{p}_\pm$ give smooth rational curves E_4 and E_5 on the minimal resolution X of $\mathbb{A}^2/G$ by resolving the singularities of $X_N/\{\tau\}$ at $\mathfrak{p}_\pm$. Thus on X we have the images E_i of C_i for $i = 1, 2, 3$ and two new rational curves E_4 and E_5. This gives the exceptional set E of X. We see easily that $(E_i)^2_{\mathrm{SING}} = -2$. The intersection pairing $(\ ,\ )_{\mathrm{SING}}$ is expressed with respect to the basis E_i for $0 \le i \le 5$ as a 6×6 symmetric matrix with diagonal entries equal to -2. We write it down multiplied by -1 for convenience:

$$(-1) \cdot (E_iE_j)_{\mathrm{SING}} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

Let $v_i := v(E_i)$ for $0 \le i \le 5$. Then we obtain the Dynkin diagram D_5 from v_i for $1 \le i \le 5$ and the extended Dynkin diagram $\widetilde{D}_5$ from v_i for $0 \le i \le 5$, as in Figure 2.

1.3 Simple singularities and simple Lie algebras (1)

Let $\mathfrak{G}$ be a simply-laced simple Lie algebra and $\mathfrak{H}$ a Cartan subalgebra of $\mathfrak{G}$. We fix a lexicographical order of the roots of $\mathfrak{H}$ and let Δ (respectively Δ_+, Δ_{simple}) be the set of roots (respectively, positive roots, positive simple roots) of $\mathfrak{G}$ with respect to T. (See [Bourbaki] for more details.) Let r be the rank of $\mathfrak{G}$ ($= \dim \mathfrak{H}$) and $\Delta_{\text{simple}} = \{\alpha_i; 1 \leq i \leq r\}$.

Let Q be the root lattice, namely the lattice spanned by Δ over $\mathbb{Z}$ endowed with the Cartan–Killing form $(\ ,\ )_{\text{LIE}}$ and $P := \text{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$ the dual lattice of Q (the weight lattice):

$$Q := \bigoplus_{\alpha \in \Delta} \mathbb{Z}\alpha = \bigoplus_{\alpha \in \Delta_{\text{simple}}} \mathbb{Z}\alpha.$$

The Cartan–Killing form $(\ ,\ )_{\text{LIE}}$ with respect to the basis Δ_{simple} is a positive definite integral symmetric bilinear form with $(\alpha, \alpha) = 2$ for all $\alpha \in \Delta_{\text{simple}}$. Since $(\alpha, \beta)_{\text{LIE}} = 0$ or -1 for $\alpha \neq \beta \in \Delta_{\text{simple}}$, we can express the bilinear form by a finite graph with simple edges Γ_{LIE} as we did for the dual graph of the set of exceptional curves of simple singularities.

There is a maximal root in Δ with respect to the given order, called the *highest root* of Δ. (This name is justified by the fact that it is the highest root of the adjoint representation of $\mathfrak{G}$. See Table 1.) Let the highest root be $\alpha_0 := \alpha_{\text{highest}} = \sum_{i=1}^r m_i^{\text{LIE}} \alpha_i$. Then $(\alpha_0, \beta) = 0$ or -1 for any $\beta \in \Delta_{\text{simple}}$ (expect for the case A_1, when $(\alpha_0, \beta) = 2$), so that we can draw a new graph $\widetilde{\Gamma}_{\text{LIE}}(\mathfrak{G})$ (called the *extended Dynkin diagram* of $\mathfrak{G}$) by adding the vertex α_0 to $\Gamma_{\text{LIE}}(\mathfrak{G})$.

Type	r	(m_0)	$m_1, m_2, m_3, \ldots, m_{r-1}; m_r$
A_n	n	1	$1, 1, \ldots, 1, 1$
D_n	n	1	$1, 2, 2, \ldots, 2, 1, 1$
E_6	6	1	$1, 2, 3, 2, 1; 2$
E_7	7	1	$1, 2, 4, 3, 2, 1; 2$
E_8	8	1	$2, 4, 6, 5, 4, 3, 2; 3$

Table 1: Multiplicities of the highest root

Let us consider the D_5 case as an example. The Lie algebra $\mathfrak{G} := \mathfrak{G}(D_5)$ is given by $\mathfrak{o}(10) := \{X \in M_{10}(\mathbb{C}); {}^tX + X = 0\}$. Its Cartan subalgebra $\mathfrak{H}$ is spanned by $H_i := E_{i,i+5} - E_{i+5,i}$ for $1 \leq i \leq 5$ where E_{ij} is the matrix with (i, j)th entry equal to 1 and 0 elsewhere. We define $\varepsilon_i \in \text{Hom}_{\mathbb{C}}(\mathfrak{H}, \mathbb{C})$ by $\varepsilon_i(H) := t_i$ for all $H = \sum_{i=1}^5 t_i H_i \in \mathfrak{H}$. Then we can choose simple roots α_i

with order $\alpha_1 > \alpha_2 > \cdots > \alpha_5$ as follows:

$$\alpha_i := \varepsilon_i - \varepsilon_{i+1}, \quad \alpha_5 := \varepsilon_4 + \varepsilon_5 \quad \text{for } 1 \leq i \leq 4.$$

The highest root α_0 is $\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$. For each α_i we define an element $\widetilde{H}_i \in \mathfrak{H}$ by $\alpha_i(H) = -\frac{1}{2}\operatorname{Tr}(\widetilde{H}_i H)$ for all $H \in \mathfrak{H}$. We see that $\widetilde{H}_i = H_i - H_{i+1}$ for $1 \leq i \leq 4$, and $\widetilde{H}_5 = H_4 + H_5$. We define $(\alpha_i, \alpha_j) := \alpha_i(\widetilde{H}_j) = \alpha_j(\widetilde{H}_i)$. Then we have $(\alpha_i, \alpha_j) = -(E_i, E_j)$ for $0 \leq i \leq j \leq 5$ in the notation of 1.1–1.2. This shows that $\Gamma_{\mathrm{SING}}(D_5) = \Gamma_{\mathrm{LIE}}(\mathfrak{G}(D_5))$ and $\widetilde{\Gamma}_{\mathrm{SING}}(D_5) = \widetilde{\Gamma}_{\mathrm{LIE}}(\mathfrak{G}(D_5))$.

We note that $P = \sum_{i=1}^{5} \mathbb{Z}\varepsilon_i$ and $Q = \sum_{i=1}^{5} \mathbb{Z}\alpha_i$.

The first theorem to mention is the following:

Theorem 1.4 *Let S be a simple singularity and* $\mathrm{Lie}(S)$ *a simple Lie algebra of the same type as S. Then there is an isomorphism*

$$i\colon H^2_{\mathrm{SING}}(S) \simeq P(\mathrm{Lie}(S))$$

such that

1. $i(H_{2,\mathrm{SING}}(S)) = Q(\mathrm{Lie}(S))$;
2. $i(\mathrm{Irr}(E(S))) = \Delta_{\mathrm{simple}}(\mathrm{Lie}(S))$;
3. $i(E_{\mathrm{fund}}(S)) = -\alpha_{\mathrm{highest}}(\mathrm{Lie}(S))$;
4. $(\ ,\ )_{\mathrm{SING}} = -i^*(\ ,\ )_{\mathrm{LIE}}$;
5. $\Gamma_{\mathrm{SING}}(S) = \Gamma_{\mathrm{LIE}}(\mathrm{Lie}(S))$ *and* $\widetilde{\Gamma}_{\mathrm{SING}}(S) = \widetilde{\Gamma}_{\mathrm{LIE}}(\mathrm{Lie}(S))$.

1.5 Simple singularities and simple Lie algebras (2)

There are two kinds of similar constructions of simple singularities from simple Lie algebras: first of all, the Grothendieck–Brieskorn–Springer construction and second, the Knop construction. Good references for this topic are for instance [Slodowy80], [Slodowy95] and [Knop87].

1.6 Finite reflection groups and Coxeter exponents

Let V be a vector space over $\mathbb{R}$ endowed with a positive definite bilinear form $(\ ,\ )$. A linear automorphism s of V is called a *reflection* if there is a vector $\alpha \in V$ and a hyperplane H_α orthogonal to α such that $s(\alpha) = -\alpha$, and the restriction of s to H_α is trivial: $s_{|H_\alpha} = \mathrm{id}_{H_\alpha}$. There is a simple formula

$$s(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha. \tag{1}$$

A finite group generated by reflections is called a finite reflection group. For instance, let Q be the root lattice of a simple Lie algebra $\mathfrak{G}$ over $\mathbb{C}$, $(\ ,\ )_{\mathrm{LIE}}$ its Cartan–Killing form, and set $V = Q \otimes \mathbb{C}$. For any simple root $\alpha_i \in \Delta_{\mathrm{simple}}$, we define a reflection $s_i := s_{\alpha_i}$ of V by the formula (1). The group W generated by all reflections s_α for $\alpha \in \Delta_{\mathrm{simple}}$ is finite, and is called the *Weyl group* of $\mathfrak{G}$. The Weyl group W acts on the polynomial ring $\mathbb{C}[V^*]$ generated by $V^* := \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$, the dual of V.

The product $s = \prod_{i=1}^{r} s_i$ of reflections for all the simple roots is called a *Coxeter element* of W. All s defined in this way for different choices of lexicographical order of the roots are conjugate in W. Therefore the order of s in W is uniquely determined, and we denote it by h and we call it the *Coxeter number* of $\mathfrak{G}$.

Theorem 1.7 ([Chevalley55]) *Let W be the Weyl group of a simple Lie algebra $\mathfrak{G}$ over $\mathbb{C}$, and r the rank of $\mathfrak{G}$. Then*

1. *the invariant ring $\mathbb{C}[V^*]^W$ is generated by r algebraically independent homogeneous polynomials $f_1, f_2, \ldots, f_r$. We order the f_i so that $\deg f_i$ is monotonically increasing.*

2. *For any choice of the generators f_i as above, the sequence of degrees $(\deg f_1, \ldots, \deg f_r)$ is uniquely determined.*

Definition 1.8 We define the *Coxeter exponents* e_i by $e_i := \deg f_i - 1$ for $1 \le i \le r$.

Theorem 1.9 *Let $\mathfrak{G}$ be a simple Lie algebra, h its Coxeter number, and e_i its Coxeter exponents. Then we have*

1. *$e_i + e_{r-i} = h$ for all i;*

2. *$|W| = \prod_{i=1}^{r}(e_i + 1)$.*

For the proof, see [Humphreys90], Orlik and Terao [OT92] and [Bourbaki].

Let us look at the D_5 case. From the root system given in 1.2–1.3 we see easily that the Weyl group $W(D_5)$ is a group of order $2^4 \cdot 5! = 1920$ fitting in the exact sequences

$$1 \to W(D_5) \to G \xrightarrow{\psi} \mathbb{Z}/2\mathbb{Z} \to 1$$

and

$$1 \to (\mathbb{Z}/2\mathbb{Z})^{\oplus 5} \to G \xrightarrow{\varphi} S_5 \to 1.$$

The group G, hence the Weyl group $W(D_5)$ as a subgroup of G, acts on $\mathbb{C}[\mathfrak{H}(D_5)^*] \simeq \mathbb{C}[x_1, \ldots, x_5]$ by

$$\sigma^*(x_i) = \varepsilon_i x_{\varphi(\sigma)(i)},$$

where $\sigma \in G$, $\varepsilon_i = \pm 1$ and $\psi(\sigma) = \varepsilon_1 \cdots \varepsilon_5$. Write f_j for the jth elementary symmetric function of 5 variables. Then $\mathbb{C}[\mathfrak{H}(D_5)^*]^{W(D_5)}$ is generated by $g_j := f_j(x_1^2, \ldots, x_5^2)$ for $j = 1, 2, 3, 4$ and $g_5 := f_5 = x_1 \cdots x_5$. It follows that $\{\deg g_j\} = (2, 4, 6, 8, 5)$ so that the Coxeter exponents are $1, 3, 5, 7, 4$. Since the Coxeter number $h(D_5)$ equals 8, we have $8 = 1 + 7 = 3 + 5 = 4 + 4$. Moreover $|W(D_5)| = 1920 = 2 \cdot 4 \cdot 6 \cdot 8 \cdot 5$.

Type	r	$e_1, e_2, e_3, \ldots, e_{r-1}, e_r$	h
A_n	n	$1, 2, \ldots, n-1, n$	$n+1$
D_n	n	$1, 3, 5, \ldots, 2n-3, n-1$	$2n-2$
E_6	6	$1, 4, 5, 7, 8, 11$	12
E_7	7	$1, 5, 7, 9, 11, 13, 17$	18
E_8	8	$1, 7, 11, 13, 17, 19, 23, 29$	30

Table 2: Coxeter exponents and Coxeter numbers

1.10 Quivers (= oriented graphs) of finite type

Let Γ be a connected oriented graph. It consists of a finite set of vertices and (simple) oriented edges joining two vertices. Write $v(\Gamma)$ and $e(\Gamma)$ for the set of vertices and edges of Γ.

For an edge ℓ, we define $\partial(\ell) = \beta(\ell) - \alpha(\ell)$, where $\alpha(\ell)$ and $\beta(\ell)$ are the starting and end points of ℓ.

Definition 1.11 ([Gabriel72]) A *representation* $\mathbf{V} := \{V_\alpha, \varphi_\ell\}$ of Γ is a set of finite dimensional vector spaces V_α, one for each $\alpha \in v(\Gamma)$, coupled with a set of homomorphisms $\varphi_\ell \colon V_{\alpha(\ell)} \to V_{\beta(\ell)}$ for all $\ell \in e(\Gamma)$. We define the *dimension vector* of a representation $\mathbf{V}$ to be $\mathbf{v} = \dim \mathbf{V} := \{\dim V_\alpha; \alpha \in v(\Gamma)\}$.

Two representations $\mathbf{V} = \{V_\alpha, \varphi_\ell\}$ and $\mathbf{W} = \{W_\alpha, \psi_\ell\}$ are *equivalent* if there are isomorphisms $f_\alpha \colon V_\alpha \to W_\alpha$ such that $\psi_\ell \cdot f_{\alpha(\ell)} = f_{\beta(\ell)} \cdot \varphi_\ell$ for any $\ell \in e(\Gamma)$. Two equivalent representations have the same dimension vector.

We say that Γ is a *quiver of finite type* if there are only finitely many equivalence classes of representations of Γ for any fixed dimension vector. This notion is independent of the choice of orientation of Γ.

Theorem 1.12 ([Gabriel72]) *Let Γ be a quiver of finite type. Then Γ with orientation forgotten is one of A_n, D_n and E_n. Conversely, if Γ is one of these types, it is a quiver of finite type.*

Proof (Outline) Suppose that Γ is of finite type. Let $\mathbf{v} = (n_\alpha)_{\alpha\in v(\Gamma)}$ be a vector with positive integer coefficients n_α. We choose and fix a representation $\mathbf{V} := \{V_\alpha, \varphi_\ell\}$ of Γ. Hence $n_\alpha = \dim V_\alpha$. Then the set of representations of Γ is the set $M := \prod_{\ell\in e(\Gamma)} \mathrm{Hom}(V_{\alpha(\ell)}, V_{\beta(\ell)})$. Let $G := \prod_{\alpha\in v(\Gamma)} \mathrm{End}(V_\alpha)$. Then G acts on M by

$$(\varphi_\ell) \mapsto (g_{\beta(\ell)} \cdot \varphi_\ell \cdot g_{\alpha(\ell)}^{-1}) \quad \text{for } g_\alpha \in \mathrm{End}(V_\alpha).$$

The set of equivalence classes of representations of Γ with fixed $\dim \mathbf{V} = \mathbf{v}$ is the quotient of M by the action of G. Since Γ is connected, the centre of G consists of scalar matrices. Therefore $\dim M \le \dim G - 1$ by assumption. It follows that $\sum_{\ell\in e(\Gamma)} n_\alpha n_\beta \le \sum_{\alpha\in v(\Gamma)} n_\alpha^2 - 1$. Since this holds for any $\mathbf{v} \in (\mathbb{Z}_+)^{\mathrm{Card}(v(\Gamma))}$, the bilinear form $\sum_{\alpha\in v(\Gamma)} x_\alpha^2 - \sum_{\ell\in e(\Gamma)} x_{\alpha(\ell)} x_{\beta(\ell)}$ is positive definite. It follows from the same argument as in the classification of simple Lie algebras that the graph Γ is one of ADE. □

Theorem 1.13 ([Gabriel72]) *Let Γ be a quiver of finite type. Then the map $\mathbf{V} \mapsto \dim \mathbf{V}$ is a bijective correspondence between the set of equivalence classes of indecomposable representations and the set of positive roots of the root system corresponding to Γ.*

2 Conformal field theory

2.1 Background from physics

In the study of conformal field theories, under certain physically natural assumptions, if we consider the theory on a real two dimensional torus, or equivalently the theory periodic in one time direction and one space direction, the system turns out to fit into an ADE classification.

We start by telling in very rough terms a story that physicists take for granted. Suppose given an infinite dimensional vector space $\mathcal{H}$ and a finite set of operators A_j on $\mathcal{H}$. The space $\mathcal{H}$ is supposed to be a realization of various physical states. The operators A_j are supposed to be selfadjoint insofar as they correspond to actual physical operators or "observables". In this sense, the vector space $\mathcal{H}$ is required to have a Hermitian inner product, namely, we require $\mathcal{H}$ to be unitary. Rather surprisingly, we will soon see that the unitary assumption picks up mathematically interesting objects.

If we have a kind of Hamiltonian operator in the algebra $\mathcal{A}$, the eigenvalue of the operator would be the energy of the (eigen)-state, and in general any state is an infinite linear combination of eigenstates, like a Fourier series expansion. The operators A_j are supposed to correspond to physical observables such as energy of particles in the system, and they correspond in mathematical terms to irreducible representations of some algebra $\mathcal{A}$ on $\mathcal{H}$, where the system is said to admit $\mathcal{A}$-symmetry.

The system $\{\mathcal{A}, A_j, \mathcal{H}\}$ is called a *conformal field theory* if the algebra $\mathcal{A}$ contains a Virasoro algebra acting nontrivially on $\mathcal{H}$.

The distribution of various energy levels is captured by the so-called *partition function* of the system, which in mathematical terms is the generating function of $\mathcal{H}$ weighted by the values of energy. If the system has space-time symmetry, one proves by a physical argument that the partition function is $\mathrm{SL}(2,\mathbb{Z})$-invariant.

The problem is to determine all possible systems admitting space-time symmetry; hence, as a first step, we consider the problem of classifying all possible modular invariant partition functions, namely $\mathrm{SL}(2,\mathbb{Z})$-invariant partition functions in certain restricted situations. In the situations we are interested in, the algebra $\mathcal{A}$ is either the affine Lie algebra $A_1^{(1)}$ or the minimal unitary series of Virasoro algebras with central charge $c = 1 - 6/m(m+1)$ for $m \geq 3$. Although the minimal unitary series is more interesting, the partition function for $A_1^{(1)}$ is easier to write down and more coherent to the ADE classification. Therefore we limit ourselves to $A_1^{(1)}$. It is not known whether the modular invariant partition functions in the subsequent table (Table 3) are partition functions of some conformal field theory admitting space-time symmetry.

We now rephrase all this in more mathematically rigorous terms.

Definition 2.2 Write

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for the standard generators of $\mathfrak{sl}_2(\mathbb{C})$. The Cartan–Killing form of $\mathfrak{sl}_2(\mathbb{C})$ is given by $(x,y)_{\mathrm{LIE}} = \mathrm{Tr}(xy)$. The affine Lie algebra $A_1^{(1)}$ is an infinite dimensional Lie algebra $\mathcal{A}$ over $\mathbb{C}$ spanned by $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$, together with a central element c, subject to the relations

$$[x(m), y(n)] = [x,y](m+n) + mc\delta_{m+n,0}(x,y)_{\mathrm{LIE}} \quad \text{and} \quad [c, x(m)] = 0,$$

for all $m, n \in \mathbb{Z}$; here t is an indeterminate, and we write $x(m) := x \otimes t^m$ for $x \in \mathfrak{sl}_2(\mathbb{C})$.

Theorem 2.3 *Let k be a positive integer and s an integer with $0 \leq s \leq k$. We define an $A_1^{(1)}$-module $V(s,k) := A_1^{(1)} \cdot v(s,k)$ by*

$$\begin{aligned} &x(n)v(s,k) = 0, \quad e(0)v(s,k) = 0 \quad \textit{for } x \in \mathfrak{sl}_2(\mathbb{C}) \textit{ and } n \geq 1, \\ &h(0)v(s,k) = sv(s,k), \quad cv(s,k) = kv(s,k). \end{aligned}$$

Then $V(s,k)$ is a unitary integrable irreducible $A_1^{(1)}$-module having highest weight vector $v(s,k)$. Conversely, any unitary irreducible integrable highest weight $A_1^{(1)}$-module V is isomorphic to $V(s,k)$ for some pair (s,k) as above.

By convention, we write $v(s,k)$ as the ket $|s,k\rangle$. The integer k is called the *level* of the $A_1^{(1)}$-module $V(s,k)$. By the Kac–Weyl character formula, we have

Theorem 2.4 *The character of $V(s,k)$ is given by*

$$\chi_{s,k}(q,\theta) = \sum_{m\in\mathbb{Z}} q^{(k+2)m^2+(s+1)m}\left(e^{\sqrt{-1}\theta((k+2)m+\frac{s}{2})} - e^{-\sqrt{-1}\theta((k+2)m+\frac{s}{2}+1)}\right)/D,$$

where the denominator is $D = (1-e^{-\sqrt{-1}\theta})\varphi(\tau)\varphi_+(\tau)\varphi_-(\tau)$, and

$$\varphi(q) = \prod_{n\geq 1}(1-q^n), \quad \varphi_\pm(q,\theta) = \prod_{n\geq 1}(1-e^{\pm\sqrt{-1}\theta}q^n).$$

Although this may look different from the usual form of the Kac–Weyl formula, the above form of the character is adjusted to the expression used by physicists to write down partition functions. In Kac's notation ([Kac90], Chapter 6 and p. 173) and the notation in 2.6

$$\begin{aligned}\chi_{s,k} &= \chi_{L((k-s)\Lambda_0+s\Lambda_1))}\\ &= \mathrm{Tr}_{L((k-s)\Lambda_0+s\Lambda_1))}(q^{(k+2)L_0}e^{\sqrt{-1}(k+2)\theta h(0)/2}).\end{aligned}$$

We note that $L_0 = -d$ and $c = K$ in the notation of [Kac90], Chapters 6–7.

Definition 2.5 The *Virasoro algebra* Vir_c with central charge c is the infinite dimensional Lie algebra over $\mathbb{C}$ generated by L_n for $n\in\mathbb{Z}$ and c, subject to the following relations

$$\begin{aligned}[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0},\\ [L_n, c] &= 0 \quad \text{for all } n, m.\end{aligned}$$

There is a way of constructing L_n from the affine Lie algebra $A_1^{(1)}$, called the *Segal–Sugawara construction*:

$$L_n = \frac{1}{2(k+2)}\sum_{m\in\mathbb{Z}}\Big({:}e(n-m)f(m){:} + {:}f(n-m)e(m){:} + \tfrac{1}{2}{:}h(n-m)h(m){:}\Big).$$

Here : : is the *normal ordering* defined by

$${:}x(m)y(n){:} = \begin{cases} x(m)y(n) & \text{if } m<n,\\ \frac{1}{2}(x(m)y(n)+y(n)x(m)) & \text{if } m=n,\\ y(n)x(m) & \text{if } m>n.\end{cases}$$

Then we infer the relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}\cdot\frac{3k}{k+2}(m^3-m)\delta_{m+n,0},$$
$$[L_m, x(n)] = -nx(m+n) \quad \text{and} \quad [L_0, x(-n)] = nx(-n)$$

for all $m, n \in \mathbb{Z}$ and $x \in \mathfrak{sl}_2(\mathbb{C})$.

Thus given a system having $A_1^{(1)}$ symmetry of level k, the system admits a Virasoro algebra Vir_c symmetry with $c = 3k/(k+2)$. Write $v := x(-n_1)x(-n_2)\cdots x(-n_p)|s,k\rangle$; note that $V(s,k)$ is spanned by vectors v of this form for various $n_i > 0$. The element L_0 acts on v by

$$L_0(v) = \Big\{\frac{1}{4(k+2)}(s^2+2s) + (n_1+n_2+\cdots+n_p)\Big\}v$$

This shows that L_0 behaves as if it measures the *energy* of the state v.

2.6 Modular invariant partition functions

Write $\mathcal{A}$ for the affine Lie algebra $A_1^{(1)}$, and $\mathcal{A}^*$ for its complex conjugate. We fix the level k, and consider only unitary irreducible integrable $\mathcal{A}$ or $\mathcal{A}^*$-modules of level k. We consider the following particular $\mathcal{A}\otimes\mathcal{A}^*$-module:

$$\mathcal{H} = \bigoplus_{\ell,\ell'} m_{\ell,\ell'} V(\ell,k)\otimes(V(\ell',k))^*,$$

where $m_{\ell,\ell'}$ is the multiplicity of the copy $V(\ell,k)\otimes(V(\ell',k))^*$.

This is what physicists call Hilbert spaces in such a situation, without further qualifications. We only need to take the completion of $\mathcal{H}$ in order to be mathematically rigorous. Mathematicians might guess why we have to choose $\mathcal{H}$ as above. This is a special case of the factorization principle widely accepted by physicists. Now L_0 is supposed to play the same role as the Hamiltonian operator of the system, and therefore the eigenvalues of L_0 should express the energies. For the (physical) theory it is always important to know the energy level distribution inside the system. Thus it is important to know the eigenvalues of L_0 and to *count* the dimension of the eigenspaces, in other words to determine the *partition function* Z of the system. The partition function Z of the system (= the $A_1^{(1)}$-module) $\mathcal{H}$ is defined by

$$\begin{aligned} Z(q,\theta,\bar{q},\bar{\theta}) &:= \mathrm{Tr}_{\mathcal{H}}\Big(q^{(k+2)L_0}e^{\sqrt{-1}(k+2)\theta h(0)/2}\bar{q}^{(k+2)\bar{L}_0}e^{-\sqrt{-1}(k+2)\bar{\theta}\bar{h}(0)/2}\Big) \\ &= \sum_{\ell,\ell'} m_{\ell,\ell'}\,\chi_{\ell,k}\,\chi^*_{\ell',k}\,, \end{aligned}$$

where $q = e^{2\pi\sqrt{-1}\tau}$ with τ in the upper half plane, and θ is a real parameter. When τ is pure imaginary, $-i\tau$ equals the ratio of sizes of time and one dimensional space. For more details see [Cardy88] and [EY89].

In this situation, the physicists assume

1. $m_{0,0} = 1$;

2. $Z(q, \theta, \bar{q}, \bar{\theta})$ is $\mathrm{SL}(2, \mathbb{Z})$-invariant.

Condition (1) means that the system has a unique state of lowest energy, usually called the *vacuum*. This is one of the principles that physicists take for granted. We therefore follow the physicists' tradition, *doing as the Romans do*. Next, (2) is the condition of discrete space-time symmetry. It means that Z is invariant under the transformations $\tau \mapsto -1/\tau$ and $\theta \mapsto \theta + 1$. See [Cardy86] and [Cardy88] for more details. These assumptions have very surprising consequences.

Theorem 2.7 *Modular invariant partition functions are classified as in Table 3. Write the partition function $Z = \sum a_{ij}\chi_i\chi_j^*$ in terms of $A_1^{(1)}$-characters. Then the indices i with nonzero a_{ii} are Coxeter exponents of the Lie algebra of the same type. Moreover the value $k + 2$ is equal to the Coxeter number.*

Type	$k+2$	partition function $Z(q, \theta, \bar{q}, \bar{\theta})$
A_n	$n+1$	$\sum_{\lambda=1}^{n} \lvert\chi_\lambda\rvert^2$
D_{2r}	$4r-2$	$\sum_{\lambda=1}^{r-1} \lvert\chi_{2\lambda-1} + \chi_{4r+1-2\lambda}\rvert^2 + 2\lvert\chi_{2r-1}\rvert^2$
D_{2r+1}	$4r$	$\sum_{\lambda=1}^{2r} \lvert\chi_{2\lambda-1}\rvert^2 + \sum_{\lambda=1}^{r-1}(\chi_{2\lambda}\bar{\chi}_{4r-2\lambda} + \bar{\chi}_{2\lambda}\chi_{4r-2\lambda}) + \lvert\chi_{2r}\rvert^2$
E_6	12	$\lvert\chi_1 + \chi_7\rvert^2 + \lvert\chi_4 + \chi_8\rvert^2 + \lvert\chi_5 + \chi_{11}\rvert^2$
E_7	18	$\lvert\chi_1 + \chi_{17}\rvert^2 + \lvert\chi_5 + \chi_{13}\rvert^2 + \lvert\chi_7 + \chi_{11}\rvert^2$
		$+\lvert\chi_9\rvert^2 + (\chi_3 + \chi_{15})\bar{\chi}_9 + \chi_9(\bar{\chi}_3 + \bar{\chi}_{15})$
E_8	30	$\lvert\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}\rvert^2 + \lvert\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}\rvert^2$

Table 3: Modular invariant partition functions

For example, for $k = 6$ there are two modular invariant partition functions:

$$Z(A_7) = |\chi_1|^2 + |\chi_2|^2 + \cdots + |\chi_6|^2 + |\chi_7|^2,$$

$$Z(D_5) = \sum_{\lambda} |\chi_{2\lambda-1}|^2 + (\chi_2\chi_6^* + \chi_2^*\chi_6) + |\chi_4|^2,$$

where A_7 (respectively D_5) has Coxeter exponents $\{1, 2, \ldots, 6, 7\}$ (respectively $\{1, 3, 5, 7, 4\}$). Note that the indices $2, 6$ are not among the Coxeter exponents of D_5. For $k = 10$, there are three types of modular invariant partition functions $Z(A_{11})$, $Z(D_7)$ and $Z(E_6)$.

For more details, see Capelli, Itzykson and Zuber [CIZ87], Kato [Kato87], Gepner and Witten [GW86] and Kac and Wakimoto [KW88]. Compare also [Slodowy90]. Pasquier [Pasquier87a] and [Pasquier87b] used Dynkin diagrams to construct some lattice models and rediscovered a series of associative algebras (called the *Temperly–Lieb algebras*) which are expected to appear as some algebra of operators on the Hilbert space in the continuum limit of the models. See also 3.4 and [GHJ89], p. 87, p. 259. Although the relation of the models with modular invariant partition functions remains obscure, the partition function of Pasquier's model is expected to coincide in some sense with those classified in Table 3. See [Zuber90]. The connection of CFT with graphs is studied by Petkova and Zuber [PZ96].

2.8 $N = 2$ super conformal field theories

There are other series of conformal field theories – the $N = 2$ superconformal field theories or (induced) topological conformal field theories, which are more intimately related to the theory of ADE singularities. However, these are a priori close to the theory of singularities. See Blok and Varchenko [BV92].

The following result might be worth mentioning here.

Theorem 2.9 *Suppose that there exists an irreducible unitary* Vir_c*-module, namely an irreducible* Vir_c*-module admitting a* Vir_c*-invariant Hermitian inner product. Then* $c \geq 1$ *or* $c = 1 - 6/m(m+1)$ *for some* $m \in \mathbb{Z}, m \geq 3$.

2.10 Minimal unitary series

Virasoro algebras of the second type are called the minimal $c < 1$ unitary series of Virasoro algebras. They attract attention because of their exceptional characters. There is a series of von Neumann algebras with indices equal to similar values $4\cos^2(\pi/h)$ for $h = 3, 4, \ldots$, where h is the Coxeter number in a suitable interpretation. Conjecturally, the minimal unitary $c < 1$ series of CFTs are deeply related to the class of subfactors which will be introduced in §3. Much is already known about this topic. See [GHJ89], [Jones91], [EK97].

3 Von Neumann algebras

3.1 Factors and subfactors

We give a brief explanation of von Neumann algebras, II_1 factors of finite type, and subfactors. The reader is invited to refer, for instance, to [GHJ89], [Jones91], [EK97]. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and $B(\mathcal{H})$ the space of all bounded $\mathbb{C}$-linear operators on $\mathcal{H}$ endowed with an operator seminorm in some suitable sense. A *von Neumann algebra* M is by definition a closed subalgebra of $B(\mathcal{H})$ containing the identity and stable under conjugation $x \mapsto x^*$. This is equivalent to saying that M is $*$-stable and is equal to its bicommutant. This is von Neumann's bicommutant theorem. See [Jones91], p. 2. The commutant of a subset S of $B(\mathcal{H})$ is by definition the centralizer of S in $B(\mathcal{H})$. The bicommutant of M is the commutant of the commutant of M. If M is a $*$-stable subset of $B(\mathcal{H})$, then the bicommutant of M is the smallest von Neumann algebra containing M.

A *factor* is defined to be a von Neumann algebra M with centre Z_M consisting only of constant multiples of the identity. Let M be a factor. A factor N is called a *subfactor* of M if it is a closed $*$-stable $\mathbb{C}$-subalgebra of M. A *II_1 factor* is by definition an infinite dimensional factor M which admits a $\mathbb{C}$-linear map $\mathrm{tr}\colon M \to \mathbb{C}$ (called the *normalized trace*) such that

1. $\mathrm{tr}(\mathrm{id}) = 1$,
2. $\mathrm{tr}(xy) = \mathrm{tr}(yx) \quad$ for all $x, y \in M$,
3. $\mathrm{tr}(x^*x) > 0 \quad$ for all $0 \neq x \in M$.

We note that the above normalized trace is unique. Let $L^2(M)$ be the Hilbert space obtained by completing M with respect to the inner product $\langle x \mid y\rangle := \mathrm{tr}(x^*y)$ for $x, y \in M$. The normalized trace induces a trace (not necessarily normalized) $\mathrm{Tr}_{M'}$ on the commutant M' of M in $B(\mathcal{H})$, called the *natural trace*. If $\mathcal{H} = L^2(M)$, then $\mathrm{Tr}_{M'}(JxJ) = \mathrm{tr}_M(x)$ for all $x \in M$ where J is the extension to $L^2(M)$ of the conjugation $J(z) = z^*$ of M.

A *finite factor* M is either a II_1 factor or $B(\mathcal{H})$ for a finite dimensional Hilbert space $\mathcal{H}$. Let M be a finite factor, and N a subfactor of M. Then the *Jones index* $[M : N]$ is defined to be $\dim_N L^2(M) := \mathrm{Tr}_{N'}(\mathrm{id}_{L^2(M)})$, where N' is the commutant of N. In general $[M : N] \in [1, \infty]$ is a (possibly irrational) positive number.

For instance, $M = \mathrm{End}_{\mathbb{C}}(W)$ is a factor (a simple algebra) for any finite dimensional $\mathbb{C}$-vector space W. If $N = \mathrm{End}_{\mathbb{C}}(V)$ is a subfactor of M, then we have a representation of $N = \mathrm{End}_{\mathbb{C}}(V)$ on W, in other words, W is an $\mathrm{End}_{\mathbb{C}}(V)$-module. We recall that

1. any $\mathrm{End}_{\mathbb{C}}(V)$-module is completely reducible, and

2. V is a unique nontrivial irreducible $\mathrm{End}_{\mathbb{C}}(V)$-module up to isomorphism.

Therefore $W \simeq V \otimes_{\mathbb{C}} U$ for some $\mathbb{C}$-vector space U. Hence $\dim_{\mathbb{C}} W$ is divisible by $\dim_{\mathbb{C}} V$. Since M is complete with respect to the inner product, we have $[M : N] = \dim_N L^2(M) = \dim_N M = (\dim_{\mathbb{C}} M)(\dim_{\mathbb{C}} N)^{-1} = (\dim_{\mathbb{C}} U)^2$, a square integer. See [GHJ89], p. 38.

The importance of the index $[M : N]$ is explained by the following result:

Theorem 3.2 ([GHJ89], p. 138) *Suppose that M is a finite factor, and let H and H' be M-modules which are separable Hilbert spaces. Then*

1. $\dim_M H = \dim_M H'$ *if and only if H and H' are isomorphic as M-modules.*
2. $\dim_M H = 1$ *if and only if $H = L^2(M)$.*
3. $\dim_M H$ *is finite if and only if* $\mathrm{End}_M(H)$ *is a finite factor.*

Theorem 3.3 ([GHJ89], p. 186) *Suppose that $N \subset M$ is a pair of II_1 factors whose principal graph is finite.*

1. *If $[M : N] < 4$ then $[M : N] = 4\cos^2(\pi/h)$ for some integer $h \geq 3$.*
2. *If $[M : N] = 4\cos^2(\pi/h) < 4$, the principal graph of the pair $N \subset M$ is one of the Dynkin diagrams A_n, D_n and E_n with Coxeter number h. (Only A_n, D_{2n}, E_6 and E_8 can appear, see [Izumi91], p. 972. This was proved independently by Kawahigashi and Izumi.)*
3. *If $[M : N] = 4$ then the principal graph of the pair $N \subset M$ is one of the extended Dynkin diagrams $\widetilde{A}_n$, $\widetilde{D}_n$ and $\widetilde{E}_n$.*
4. *Conversely for any value $\lambda = 4$ or $4\cos^2(\pi/h)$, there exists a pair of II_1 factors $N \subset M$ with $[M : N] = \lambda$.*

See [GHJ89], [Jones91], p. 35. See [GHJ89], p. 186 for principal graphs. See also 3.8-3.10 where to each tower of finite-dimensional semisimple algebras we associate a finite graph Γ analogous to a principal graph for a pair of factors. This will help us to guess principal graphs for factors.

3.4 The fundamental construction and Temperly–Lieb algebras

Why do the constants $4\cos^2(\pi/h)$ appear? Let us explain this briefly.

Given a pair of finite II_1 factors $N \subset M$ with $\beta := [M : N] < \infty$, there exists a tower of finite II_1 factors M_k for $k = 0, 1, 2, \ldots$ such that

1. $M_0 = N$, $M_1 = M$,

2. $M_{k+1} := \mathrm{End}_{M_{k-1}} M_k$ is the von Neumann algebra of operators on $L^2(M_k)$ generated by M_k and an orthogonal projection $e_k\colon L^2(M_k) \to L^2(M_{k-1})$ for any $k \geq 1$, where M_k is viewed as a subalgebra of M_{k+1} under right multiplication.

By Theorem 3.2, (3), M_{k+1} is a finite II_1 factor. The sequence $\{e_k\}_{k=1,2,\dots}$ of projections on $M_\infty := \bigcup_{k\geq 0} M_k$ satisfies the relations

$$\begin{aligned} e_i^2 &= e_i, \quad e_i^* = e_i, \\ e_i &= \beta e_i e_j e_i \quad \text{for} \quad |i-j| = 1, \\ e_i e_j &= e_j e_i \qquad \text{for} \quad |i-j| \geq 2. \end{aligned}$$

We define $A_{\beta,k}$ to be the $\mathbb{C}$-algebra generated by $1, e_1, \dots, e_{k-1}$ subject to the above relations, and $A_\beta := \bigcup_{k=1}^\infty A_{\beta,k}$. The algebra A_β is called the *Temperly–Lieb algebra*. Compare also [GHJ89], p. 259.

Thus given a pair of II_1 factors, the fundamental construction gives rise to a unitary representation of the Temperly–Lieb algebra. However, the condition that the representation is unitary restricts the possible values of β, as Theorem 3.5 shows.

Theorem 3.3, (1) follows from the following result

Theorem 3.5 ([Wenzl87]) *Suppose given an infinite sequence $\{e_k\}_{k=1,2,\dots}$ of projections on a complex Hilbert space satisfying the following relations:*

$$\begin{aligned} e_i^2 &= e_i, \quad e_i^* = e_i, \\ e_i &= \beta e_i e_j e_i \quad \textit{for} \quad |i-j| = 1, \\ e_i e_j &= e_j e_i \qquad \textit{for} \quad |i-j| \geq 2. \end{aligned}$$

If $e_1 \neq 0$, then $\beta \geq 4$ or $\beta = 4\cos^2(\pi/\ell)$ for an integer $\ell \geq 3$.

Proof We give an idea of the proof of Theorem 3.5. Suppose we are given a homomorphism $\varphi\colon A_\beta \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, that is, a unitary representation of A_β. For simplicity we identify $\varphi(x)$ with x for $x \in A_\beta$.

First we see that $0 \leq e_1^* e_1 = e_1^2 = e_1 = \beta e_1 e_2 e_1 = \beta (e_2 e_1)^*(e_2 e_1)$. Hence $\beta \geq 0$. If $\beta = 0$ then $e_1 = 0$, contradicting the assumption. Hence $\beta > 0$.

Next we assume $0 < \beta < 1$ to derive a contradiction by using $A_{\beta,3}$. Let $\delta_2 := 1 - e_1$. Then the assumptions of Theorem 3.5 imply $\delta_2^* = \delta_2$, $\delta_2^2 = \delta_2$. Hence

$$0 \leq (\delta_2 e_2 \delta_2)^*(\delta_2 e_2 \delta_2) = (\delta_2 e_2 \delta_2)^2 = (1-\beta^{-1})(\delta_2 e_2 \delta_2) \leq 0,$$

because $\delta_2 e_2 \delta_2 = (e_2\delta_2)^*(e_2\delta_2) \geq 0$. Thus $e_2\delta_2 = 0$. It follows that $e_2 = e_1e_2$, and $e_2 = e_2^2 = e_2e_1e_2 = \beta^{-1}e_2$, so that $e_2 = 0$. Therefore $e_1 = \beta e_1 e_2 e_1 = 0$, contradicting the assumption. If $4\cos^2(\pi/\ell) < \beta < 4\cos^2(\pi/(\ell+1))$, then we derive a contradiction by using $A_{\beta,\ell+1}$. See [GHJ89], pp. 272–273. $\square$

3.6 Bipartite graphs

A *bipartite graph* Γ with multiple edges is a (finite, connected) graph with black and white vertices and multiple edges such that any edge connects a white and black vertex, starting from a white one (see, for example, Figure 3). If any edge is simple, then Γ is an oriented graph (a quiver) in the sense of Section 1. Let Γ be a connected bipartite finite graph with multiple oriented edges. Let $w(\Gamma)$ (respectively $b(\Gamma)$) be the number of white (respectively black) vertices of Γ. We define the *adjacency matrix* $\Lambda := \Lambda(\Gamma)$ of size $b(\Gamma) \times w(\Gamma)$ by

$$\Lambda_{b,w} = \begin{cases} m(e) & \text{if there exists } e \text{ s.t. } \partial e = b - w); \\ 0 & \text{otherwise.} \end{cases}$$

where $m(e)$ is the multiplicity of the edge e.

We define the norm $\|\Gamma\|$ as follows,

$$\|X\| = \max\{\|Xx\|_{\text{EUCL}}; \|x\|_{\text{EUCL}} \leq 1\};$$

$$\|\Gamma\| = \|\Lambda(\Gamma)\| = \left\| \begin{pmatrix} 0 & \Lambda(\Gamma) \\ \Lambda(\Gamma)^t & 0 \end{pmatrix} \right\|,$$

where X is a matrix, x a vector and $\|\ \|_{\text{EUCL}}$ the Euclidean norm. We note that when X is a square matrix, $\|X\|$ is the maximum of the absolute values of eigenvalues of X.

D_5

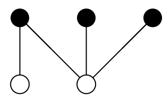

Figure 3: The Dynkin diagram D_5 as a bipartite graph

Lemma 3.7 *Assume Γ is a connected finite graph with multiple edges. Then*

1. *if $\|\Gamma\| \leq 2$ and if Γ has a multiple edge, $\|\Gamma\| = 2$ and $\Gamma = \widetilde{A}_1$.*

2. *$\|\Gamma\| < 2$ if and only if Γ is one of the Dynkin diagrams A, D, E. In this case $\|\Gamma\| = 2\cos(\pi/h)$, where h is the Coxeter number of Γ.*

3. *$\|\Gamma\| = 2$ if and only if Γ is one of the extended Dynkin diagrams $\widetilde{A}, \widetilde{D}, \widetilde{E}$.*

Lemma 3.7 is easy to prove. For instance, if there is a row or column vector of Γ with norm a, then $\|\Gamma\| \geq a$. See also [GHJ89], p. 19.

3.8 The tower of semisimple algebras

Why is Theorem 3.3, (2) true? The interested reader is invited to see [GHJ89]. Here we explain it in a much simpler situation.

Recall that a matrix algebra of finite rank is a finite factor by definition. This is an elementary analogue of a finite II_1 factor with a finite dimensional Hilbert space. So let us see what happens if we consider the fundamental construction for a pair $N \subset M$ of (sums of) matrix algebras. We call N and M (a pair of) semisimple algebras (over $\mathbb{C}$).

Let Γ be a connected bipartite graph with multiple edges, $v(\Gamma)$ and $e(\Gamma)$ its set of vertices and edges. Let $W(w)$ be a $\mathbb{C}$-vector space for a white vertex w. Let $W(b,w)$ be a $\mathbb{C}$-vector space for an edge e with $\partial e = b - w$ and $V(b) = \bigoplus_{\partial e = b-w} W(b,w) \otimes W(w)$ for a black vertex b, where the sum runs over all edges of Γ ending at b. Set

$$
\begin{aligned}
N &:= \bigoplus_{w:\text{white}} \mathrm{End}_{\mathbb{C}}(W(w)),\\
M &:= \bigoplus_{b:\text{black}} \mathrm{End}_{\mathbb{C}}(V(b)),\\
&= \bigoplus_{b:\text{black}} \bigoplus_{\partial e = b-w} \mathrm{End}_{\mathbb{C}}(W(b,w)) \otimes \mathrm{End}_{\mathbb{C}}(W(w)).
\end{aligned}
$$

Now let $\varphi_0 \colon N \to M$ be the homomorphism defined by

$$
\varphi_0 = \bigoplus_b \varphi_{0,b}, \quad \varphi_{0,b} = \bigoplus_{\partial e = b-w} \mathrm{id}_{W(b,w)} \otimes \mathrm{id}_{\mathrm{End}(W(w))},
$$

where $\mathrm{id}_{W(b,w)}$ is the identity homomorphism of $W(b,w)$. This is a representation of the oriented graph Γ in the sense of Definition 1.11 if $m(e) = \dim W(b,w) \leq 1$ for any edge e.

Let $\Lambda(M,N) := \Lambda(\Gamma)$. We call it the inclusion matrix of M in N.

Let us consider a tower of semisimple algebras arising from the fundamental construction for the pair $N \subset M$. We define $M_0 = N$, $M_1 = M$ and $M_{k+1} := \mathrm{End}_{M_{k-1}}(M_k)$ inductively.

Let $M_2 = \mathrm{End}_N M$, φ_1 the monomorphism of M_1 into M_2 by right multiplication. Let $V(b,w) = \mathrm{End}_{\mathbb{C}}(W(b,w))$. Then we see that

$$\begin{aligned}
\mathrm{End}_N M &= \bigoplus_{w:\text{white}} U(w),\\
U(w) &:= \bigoplus_{\partial e=b-w} \mathrm{End}_{W(w)} V(b)\\
&= \bigoplus_{\partial e=b-w} \mathrm{End}_{\mathbb{C}}(V(b,w)) \otimes \mathrm{End}_{\mathbb{C}}(W(w)),\\
\varphi_1 &= \bigoplus_{w} \varphi_{1,w}, \quad \varphi_{1,w} = \bigoplus_{\partial e=b-w} \text{right mult.}_{V(b,w)} \otimes \mathrm{id}_{\mathrm{End}(W(w))}.
\end{aligned}$$

The construction shows that the graph Γ describe the inclusion of M_{k-1} into M_k by interchanging the roles of white and black vertices, and reversing the orientation of edges at each step. We see $\Lambda(M_{2k+1}, M_{2k}) = \Lambda(M,N)^t$, $\Lambda(M_{2k}, M_{2k-1}) = \Lambda(M,N)$.

We set $[M:N] := \lim_{k\to\infty}\big(\dim M_k/\dim M_0\big)^{1/k}$. (This is one of the equivalent definitions of the Jones index $[M:N]$.) We compute this in the simplest case when Γ is a connected graph with two vertices and a single edge e. Let $m(e)$ be the multiplicity of e, and $\partial e = b - w$. Then we see that

$$\begin{aligned}
M_0 &= N = \mathrm{End}_{\mathbb{C}}(W(w)),\\
M_1 &= M = \mathrm{End}_{\mathbb{C}}(V(b)) \simeq \mathrm{End}_{\mathbb{C}}(W(b,w)) \otimes M_0,\\
M_2 &= \mathrm{End}_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}}(W(b,w))) \otimes \mathrm{End}_{\mathbb{C}}(W(w)),\\
&\simeq \mathrm{End}_{\mathbb{C}}(W(b,w)) \otimes \mathrm{End}_{\mathbb{C}}(V(b)) \simeq \mathrm{End}_{\mathbb{C}}(W(b,w)) \otimes M_1.
\end{aligned}$$

Hence we see that $\dim_{\mathbb{C}} M_k/M_{k-1} = \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(W(b,w)) = \dim_{\mathbb{C}}(M/N)$. It follows readily that $[M:N] = \dim_{\mathbb{C}}(M/N)$, as was remarked in 3.1.

In this situation, the following result is proved.

Theorem 3.9 ([GHJ89], pp. 32–33) *1. The following are equivalent:*

(a) there exists a row $b(\Gamma)$-vector s and $\beta \in \mathbb{C}^*$ with $s\Lambda\Lambda^t = \beta s$ such that every coordinate of s and $s\Lambda$ is nonzero,

(b) there exist $\mathbb{C}$-linear maps $e_k\colon M_k \to M_{k-1}$ such that $e_k^2 = e_k$ and

(i) M_k is generated by M_{k-1} and e_k,

(ii) e_k satisfies $e_i = \beta e_i e_j e_i$ if $|i-j| = 1$ and $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.

2. If one of the equivalent conditions in (1) holds, then

$$\beta = \|\Lambda(\Gamma)\Lambda(\Gamma)^t\| = \|\Lambda(\Gamma)\|^2 = [M:N].$$

This is nontrivial, but is just linear algebra. By Theorem 3.9, we have a situation similar to a pair of II_1 factors $N \subset M$ as well as a Temperly–Lieb algebra A_β.

From Lemma 3.7, we infer the following result.

Corollary 3.10 *Let $M_0 = N \subset M_1 = M \subset \cdots \subset M_k \subset \cdots$ be a tower of semisimple algebras. We have a Temperly–Lieb algebra A_β from the tower if and only if $\beta = [M : N]$ and $\beta \geq 4$ or $\beta = 4\cos^2(\pi/h)$ for $h = 3, 4, 5, \ldots$. Moreover*

1. *if $\beta = 4\cos^2(\pi/h)$, then the graph Γ is one of A, D, E;*

2. *if $\beta = 4$, then the graph Γ is one of $\widetilde{A}$, $\widetilde{D}$, $\widetilde{E}$.*

For a pair of II_1 factors $N \subset M$, we can always carry out the same construction as for a pair of semisimple algebras, and we find the same graphs (principal graphs), because the pair in fact satisfies the stronger restrictions of (infinite dimensional) II_1 factors. As a consequence, the cases D_{odd} and E_7 are excluded.

4 Two dimensional McKay correspondence

4.1 Finite subgroups of $\mathrm{SL}(2,\mathbb{C})$

Up to conjugacy, any finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ is one of the subgroups listed in Table 4; see [Klein]. The triple (d_1, d_2, d_3) specifies the degrees of the generators of the G-invariant polynomial ring (compare Section 11).

Type	G	name	order	h	(d_1, d_2, d_3)
A_n	$\mathbb{Z}_{n+1}$	cyclic	$n+1$	$n+1$	$(2, n+1, n+1)$
D_n	$\mathbb{D}_{n-2}$	binary dihedral	$4(n-2)$	$2n-2$	$(4, 2n-4, 2n-2)$
E_6	$\mathbb{T}$	binary tetrahedral	24	12	$(6, 8, 12)$
E_7	$\mathbb{O}$	binary octahedral	48	18	$(8, 12, 18)$
E_8	$\mathbb{I}$	binary icosahedral	120	30	$(12, 20, 30)$

Table 4: Finite subgroups of $\mathrm{SL}(2,\mathbb{C})$

4.2 McKay's observation

As we mentioned in Section 1, any simple singularity is a quotient singularity by a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$, and so has a corresponding Dynkin diagram. McKay [McKay80] showed how one can recover the same graph purely in terms of the representation theory of G, without passing through the geometry of $\mathbb{A}^2/G$.

To be more precise, let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. Clearly, G has a two dimensional representation, which maps G injectively into $\mathrm{SL}(2,\mathbb{C})$; we call this the *natural representation* ρ_{nat}. Let $\mathrm{Irr}_* G$, respectively $\mathrm{Irr}\, G$, be the set of all equivalence classes of irreducible representations, respectively nontrivial ones. (Caution: note that this goes against the familiar notation of group theory.) Thus by definition, $\mathrm{Irr}_* G = \mathrm{Irr}\, G \cup \{\rho_0\}$, where ρ_0 is the one dimensional trivial representation. Any representation of G over $\mathbb{C}$ is completely reducible, that is, is a direct sum of irreducible representations up to equivalence. Therefore for any $\rho \in \mathrm{Irr}_* G$, we have

$$\rho \otimes \rho_{\mathrm{nat}} = \sum_{\rho' \in \mathrm{Irr}_* G} a_{\rho,\rho'} \rho'$$

where $a_{\rho,\rho'}$ are certain nonnegative integers. In our situation, we see that $a_{\rho,\rho'} = 0$ or 1 (except for the case A_1, when $a_{\rho,\rho'} = 0$ or 2).

Let us look at the example D_5, the case of a binary dihedral group $G := \mathbb{D}_3$ of order 12. The group G is generated by σ and τ:

$$\sigma = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{where } \varepsilon = e^{2\pi\sqrt{-1}/6}.$$

We note that $\mathrm{Tr}(\sigma) = 1$, $\mathrm{Tr}(\tau) = 0$, hence in this case, the natural representation is ρ_2 in Table 5.

ρ	$\mathrm{Tr}\,\rho$	1	σ	τ
ρ_0	χ_0	1	1	1
ρ_1	χ_1	1	1	-1
ρ_2	χ_2	2	1	0
ρ_3	χ_3	2	-1	0
ρ_4	χ_4	1	-1	$\sqrt{-1}$
ρ_5	χ_5	1	-1	$-\sqrt{-1}$

Table 5: Character table of D_5

Definition 4.3 The graph $\widetilde{\Gamma}_{\mathrm{GROUP}}(G)$ is defined to be the graph consisting of vertices $v(\rho)$ for $\rho \in \mathrm{Irr}_* G$, and simple edges connecting any pair of vertices $v(\rho)$ and $v(\rho')$ with $a_{\rho,\rho'} = 1$. We denote by $\Gamma_{\mathrm{GROUP}}(G)$ the full subgraph of $\widetilde{\Gamma}_{\mathrm{GROUP}}(G)$ consisting of the vertices $v(\rho)$ for $\rho \in \mathrm{Irr}\, G$ and all the edges between them.

For example, let us look at the D_5 case. Let $\chi_j := \mathrm{Tr}(\rho_j)$ be the character of ρ_j. Then from Table 5 we see that

$$\chi_2(g)\chi_2(g) = \chi_0(g) + \chi_1(g) + \chi_3(g), \quad \text{for } g = 1, \sigma \text{ or } \tau.$$

Hence $\chi_2\chi_2 = \chi_0 + \chi_1 + \chi_3$. General representation theory says that an irreducible representation of G is uniquely determined up to equivalence by its character. Therefore $\rho_2 \otimes \rho_2 = \rho_0 + \rho_1 + \rho_3$. Hence $a_{\rho_2,\rho_j} = 1$ for $j = 0, 1, 3$ and $a_{\rho_2,\rho_j} = 0$ for $j = 2, 4, 5$. Similarly, we see that

$$\begin{aligned} &\chi_0\chi_2 = \chi_2, \quad \chi_1\chi_2 = \chi_2, \\ &\chi_3\chi_2 = \chi_0 + \chi_1 + \chi_4, \\ &\chi_4\chi_2 = \chi_3 \quad \text{and} \quad \chi_5\chi_2 = \chi_3. \end{aligned}$$

In this way we obtain a graph – the extended Dynkin diagram $\widetilde{D}_5$ of Figure 4. It is also interesting to note that the degrees of the characters $\deg \rho_j = \chi_j(1)$ are equal to the multiplicities of the fundamental cycle we computed in Section 1. This is true in the other cases. Namely the graph $\Gamma_{\mathrm{GROUP}}(G)$ turns out to be one of the Dynkin diagrams ADE, while $\widetilde{\Gamma}_{\mathrm{GROUP}}(G)$ is the corresponding extended Dynkin diagram (see Figure 5). This is the observation of [McKay80].

$\widetilde{D}_5$

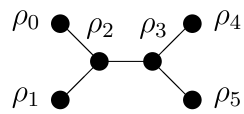

Figure 4: McKay correspondence for $\widetilde{D}_5$

4.4 The Gonzalez-Sprinberg–Verdier construction

Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, X the minimal resolution of $S := \mathbb{A}^2/G$, and E the exceptional set. Gonzalez-Sprinberg and Verdier [GSV83] constructed a locally free sheaf V_ρ on X for any $\rho \in \mathrm{Irr}\, G$ such that there exists a unique $E_\rho \in \mathrm{Irr}\, E$ satisfying

$$\deg(c_1(V_\rho)_{|E_\rho}) = 1 \quad \text{and } \deg(c_1(V_\rho)_{|E'}) = 0 \text{ for } E' \neq E_\rho, E' \in \mathrm{Irr}\, E.$$

Thus the map $\rho \mapsto E_\rho$ turns out to be a bijection from $\operatorname{Irr} G$ onto $\operatorname{Irr} E$.

Their construction of V_ρ is essentially as follows [Knörrer85], p. 178. Let $\rho\colon G \to \mathrm{GL}(V(\rho))$ be a nontrivial irreducible representation of G. Then the associated free $\mathcal{O}_{\mathbb{A}^2}$-module $\mathcal{V}(\rho) := \mathcal{O}_{\mathbb{A}^2} \otimes_{\mathbb{C}} V(\rho)$ admits a canonical G-action defined by $g \cdot (x, v) = (gx, gv)$. Let $\mathcal{V}(\rho)^G$ be the $\mathcal{O}_S$-module consisting of G-invariant sections in $\mathcal{V}(\rho)$. The (locally free) $\mathcal{O}_X$-module V_ρ is defined to be

$$V_\rho := \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{V}(\rho)^G / \mathcal{O}_X\text{-torsion}.$$

Theorem 4.5 *Let G be a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, $S = \mathbb{A}^2/G$, X the minimal resolution of S and E the exceptional set. Then there is a bijection j of $\operatorname{Irr}_* G$ to $\operatorname{Irr}_* E$ such that*

1. $j(\rho_0) = E_0 =: E_{\rho_0}$ *and* $j(\rho) = E_\rho$ *for* $\rho \in \operatorname{Irr} G$;
2. $\deg(\rho) = m^{\mathrm{SING}}_{E_\rho}$ *for all* $\rho \in \operatorname{Irr}_* G$;
3. $a_{\rho,\rho'} = (E_\rho, E_{\rho'})_{\mathrm{SING}}$ *for* $\rho \neq \rho' \in \operatorname{Irr}_* G$.

In particular:

Corollary 4.6 $\Gamma_{\mathrm{GROUP}}(G) = \Gamma_{\mathrm{SING}}(\mathbb{A}^2/G)$ *and* $\widetilde{\Gamma}_{\mathrm{GROUP}}(G) = \widetilde{\Gamma}_{\mathrm{SING}}(\mathbb{A}^2/G)$.

See [McKay80] and [GSV83]. Using invariant theory, [Knörrer85] gave a different proof of Theorem 4.5 based on the construction in [GSV83]. We discuss again the construction of [GSV83] from the viewpoint of Hilbert schemes in Sections 8–16, and give there our own proof of Theorem 4.5.

5 Missing links and problems

5.1 Known links

We review briefly what is known about links between any pair of the objects (a)–(f) – namely,

(a) simple singularities, (b) finite subgroups of $\mathrm{SL}(2, \mathbb{C})$,

(c) simple Lie algebras, (d) quivers, (e) CFT, (f) subfactors.

A very deep understanding of the link from (c) to (a) is provided by work of Grothendieck, Brieskorn, Slodowy and Springer. See [Slodowy80]. However, no intrinsic converse construction of simple Lie algebras starting from (a) is known.

The link from (b) to (a) is on the one hand the obvious quotient singularity construction, and on the other the very nontrivial McKay correspondence.

The construction of [GSV83] gives an explanation for the McKay correspondence. See also [Knörrer85] and Section 4. We will show a new way of understanding the link (the McKay correspondence) in Sections 8-16. Quivers of finite type appear in the course of this, which provides a link from (b) to (d) alongside the link from (b) to (a). This path has already been found in [Kronheimer89] in a slightly different manner.

For a given pair of II_1 factors one can construct a tower of II_1 factors by a certain procedure which specialists call mirror image transformations. In order to have an ADE classification we had better look at the same tower construction for a pair of semisimple algebras (semisimple algebras over $\mathbb{C}$ are sums of matrix algebras). In the tower of semisimple algebras the initial pair $N \subset M$ is described as a representation of an ADE quiver, while the rest of the tower is generated automatically from this. Therefore the link between (d) and (f) is firmly established, though the subfactors are only possible with the exception of D_{odd} and E_7. The link between (e) and (f) does not seem to be perfectly known. See [EK97].

Infinite dimensional Heisenberg/Clifford algebras and their representations on Fock space enter the theory of Hilbert schemes. See [Nakajima96b], [Grojnowski96] and Section 6. This strongly suggests as yet unrevealed relations between the theory of Hilbert schemes with modular invariant partitions and II_1 (sub)factors.

The most desirable outcome would be a theory in which all six kinds of objects (a)–(f) arise naturally in various forms from one and the same object, for instance, from a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$.

5.2 Problems

The following problems are worth further investigation.

1. What are the Coxeter exponents and the Coxeter number for a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, and why? (It is known that the Coxeter number equals the largest degree of the three homogeneous generators of the G-invariant polynomial ring. But why?)

2. What are the multiplicities of the highest weight for (e) and (f)?

3. Why do indices other than Coxeter exponents appear in Table 3 of Theorem 2.7?

4. The link from (b) to (c)? Can we recover the Lie algebras?

5. The link from (a) to (c)? Can we recover the Lie algebras?

6. The links from (b) to (e) and (f)?

7. Theorem 2.9 and Theorem 3.3 hint at an ADE classification of $c < 1$ minimal unitary series. If so, what do they look like? What is the link from (e) to (f) via this route?

6 Hilbert schemes of n points

6.1 Projectivity

Let X be a projective scheme of dimension k over $\mathbb{C}$. Then Hilb^n_X is by definition the universal scheme parametrizing all zero-dimensional subschemes Z of X such that $h^0(Z, \mathcal{O}_Z) = \dim(\mathcal{O}_Z) = n$, which does exist by a theorem of Grothendieck [FGA] Exposé 221. See Theorem 6.2. Set-theoretically

$$\begin{aligned}\mathrm{Hilb}^n_X &= \{Z \subset X; \dim(\mathcal{O}_Z) = n\}\\ &\simeq \{I \subset \mathcal{O}_X; I \text{ an ideal of } \mathcal{O}_X, \dim(\mathcal{O}_X/I) = n\}\end{aligned}$$

Let us show very roughly that Hilb^n_X is a projective scheme.

Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X and $\mathcal{O}_X(m) := \mathcal{O}_X(1)^{\otimes m}$. Let us prove first that for any large m fixed Hilb^n_X is regarded as a subset of the Grassman variety of n-codimensional subspaces of $H^0(X, \mathcal{O}_X(m))$.

Let $Z \in \mathrm{Hilb}^n_X$ be a zero-dimensional subscheme, I the ideal of $\mathcal{O}_X$ defining Z with $h^0(Z, \mathcal{O}_Z) = \dim \mathcal{O}_X/I = n$. Then we have an exact sequence

$$0 \to I\mathcal{O}_X(m) \to \mathcal{O}_X(m) \to \mathcal{O}_Z(m)\ (\simeq \mathcal{O}_Z) \to 0$$

If the support of Z is $P_1, \cdots, P_s$ ($s \le n$), then we see easily $\mathfrak{m}_1^{kn} \cdots \mathfrak{m}_s^{kn} \subset I$ where $\mathfrak{m}_i$ is the maximal ideal of $\mathcal{O}_{X,P_i}$. If $H^1(\mathfrak{m}_1^{kn} \cdots \mathfrak{m}_s^{kn} \mathcal{O}_X(m)) = 0$, then we see that the natural homomorphism

$$H^0(\mathcal{O}_X(m)) \to H^0(\mathcal{O}_X(m)/\mathfrak{m}_1^{kn-1} \cdots \mathfrak{m}_s^{kn} \mathcal{O}_X(m)) \to H^0(\mathcal{O}_X(m)/I\mathcal{O}_X(m))$$

is surjective. Since all the ideals $\mathfrak{m}_1^{kn} \cdots \mathfrak{m}_s^{kn}$ is parametrized by an open subscheme $\{(P_1, \cdots, P_s) \in X^s; P_i \neq P_j, i \neq j\}$ of X^s ($s \le n$), there is a constant m_0 by Serre vanishing and the upper semi-continuity of $\dim H^1$ such that $H^1(\mathfrak{m}_1^{kn} \cdots \mathfrak{m}_s^{kn} \mathcal{O}_X(m)) = 0$ for any $m \ge m_0$. This constant m_0 depends only on n and X. It follows that the natural homomorphism $H^0(\mathcal{O}_X(m)) \to H^0(\mathcal{O}_Z)$ is surjective for any $Z \in \mathrm{Hilb}^n_X$ and any $m \ge m_0$. Thus to each $Z \in \mathrm{Hilb}^n_X$ is associated a point $\mathrm{Grass}^m(Z)$ of the Grassmann variety $\mathrm{Grass}(H^0(\mathcal{O}_X(m)), n)$. Let $n_i = \dim \mathcal{O}_{Z,P_i}$. Then since $\mathfrak{m}_i^{kn} \subset I$ at P_i it is clear that Z is parametrized by P_i and $\mathrm{Grass}(\mathcal{O}_{X,P_i}/\mathfrak{m}_i^{kn}, n_i)$, the Grassman variety of n_i-codimensional subspaces of $\mathcal{O}_{X,P_i}/\mathfrak{m}_i^{kn}$. Therefore again by Serre vanishing plus some standard arguments we may assume that the natural homomorphism $H^0(I\mathcal{O}_X(m)) \otimes O_X \to I\mathcal{O}_X(m)$ is surjective for $m \ge m_1$ ($\ge m_0$)

by choosing a larger m_1 if necessary. We may assume that this m_1 depends only on n and X.

Since $H^0(I\mathcal{O}_X(m))$ is the kernel of the restriction map $H^0(\mathcal{O}_X(m)) \to H^0(\mathcal{O}_X(m) \otimes \mathcal{O}_Z) \simeq H^0(\mathcal{O}_Z)$, $\mathrm{Grass}^m(Z)$ determines $H^0(I\mathcal{O}_X(m))$ uniquely. Since $H^0(I\mathcal{O}_X(m)) \otimes \mathcal{O}_X \to I\mathcal{O}_X(m)$ is surjective, $H^0(I\mathcal{O}_X(m))$ determines $I\mathcal{O}_X(m)$ uniquely, hence $\mathrm{Grass}^m(Z)$ determines Z uniquely. In other words, if $m \geq m_1$, then for any pair of subschemes Z and Z' in Hilb^n_X, $\mathrm{Grass}^m(Z) = \mathrm{Grass}^m(Z')$ if and only if $Z = Z'$. This shows roughly that Hilb^n_X is a subscheme of the Grassman variety.

Next we give a rough proof that Hilb^n_X in is a closed subscheme of the Grassman variety. In fact, if we are given a one parameter flat family of subschemes $Z_t \in \mathrm{Hilb}^n_X$ (say, $0 \neq t \in D := \{|t| < 1\}$), then we have a one parameter family of ideals I_t defining Z_t and a one parameter family of subspaces $H^0(I_t\mathcal{O}_X(m))$ of $H^0(\mathcal{O}_X(m))$. Hence we have a one parameter family of points $\mathrm{Grass}^m(Z_t)$ in $\mathrm{Grass}(H^0(\mathcal{O}_X(m)), n)$ for any $m \geq m_1$. Since $\mathrm{Grass}(H^0(X, \mathcal{O}_X(m)), n)$ is projective, there is a limit $V_m = \lim_{t\to 0} H^0(I_t\mathcal{O}_X(m))$ in $\mathrm{Grass}(H^0(\mathcal{O}_X(m)), n)$ (without taking a nontrivial finite cover of D because $\dim D = 1$). Let $I_m := V_m\mathcal{O}_X(-m)$. Then I_m is an ideal of $\mathcal{O}_X$. We see $I_m \subset I_{m+1}$ because $V_m H^0(\mathcal{O}_X(1)) \subset V_{m+1}$. Since the increasing sequence I_m is stationary by the noetherian property of $\mathcal{O}_X$, there exists m_2 ($\geq m_1$) such that we have $I_m = I_{m+1} = \cdots$ for any $m \geq m_2$. Let $I = I_m = I_{m+1} = \cdots$ and $\mathcal{O}_Z := \mathcal{O}_X/I$. As usual we have an exact sequence

$$0 \to I\mathcal{O}_X(m) \to \mathcal{O}_X(m) \to \mathcal{O}_Z(m) \to 0$$

By Serre vanishing there exists $m_3 \geq m_2$ such that $H^1(I\mathcal{O}_X(m)) = 0$ for any $m \geq m_3$. By their definitions $V_m \subset H^0(I\mathcal{O}_X(m))$. It follows from $H^1(I\mathcal{O}_X(m)) = 0$ that $\dim H^0(\mathcal{O}_Z(m)) \leq n$ for any $m \geq m_3$, hence Z is zero-dimensional. But we can do the same construction of I as above in relative version because $\dim H^0(I_t\mathcal{O}_X(m))$ is constant and equal to $\dim V_m$. As a consequence we see that there exists an ideal sheaf $\mathcal{I}$ of $\mathcal{O}_{X\times D}$ such that $\mathcal{I}_{|X\times t} = I_t$ ($t \neq 0$) and $\mathcal{I}_{|X\times 0} = I$. Therefore by the upper semicontinuity of $\dim H^0$ we see $\dim H^0(\mathcal{O}_Z) = n$. Thus I is the ideal sheaf of $\mathcal{O}_X$ with $\dim \mathcal{O}_X/I = n$, and $Z \in \mathrm{Hilb}^n_X$. This shows that Hilb^n_X is a projective scheme.

Let U be an open subscheme of X. Then Hilb^n_U is an open subscheme of Hilb^n_X consisting of the subschemes of X whose supports are contained in U. We call Hilb^n_U the *Hilbert scheme of n points in U*. Refer [FGA] Exposé 221 for the details on Hilbert schemes.

For later use we quote the theorem of Grothendieck guaranteeing existence and universality of Hilb^n_X. This theorem will be made use of to determine the precise structure of Hilb^G_X defined in § 8 by their universal property.

Theorem 6.2 *Let X be any projective scheme and n any positive integer. Then there exist a projective scheme Hilb^n_X (possibly with finitely many irreducible components) and a universal proper flat family $\pi_{\mathrm{univ}}\colon Z^n \to \mathrm{Hilb}^n_X$ of zero-dimensional subschemes of X such that*

1. *any fibre of π_{univ} belongs to Hilb^n_X,*
2. *$Z^n_t = Z^n_s$ if and only if $t = s$, where $Z^n_t := \pi^{-1}_{\mathrm{univ}}(t)$ for $t \in \mathrm{Hilb}^n_X$,*
3. *given any flat family $\pi\colon Y \to S$ of zero-dimensional subschemes of X with length n, there exists a unique morphism $\varphi\colon S \to \mathrm{Hilb}^n_X$ such that $(Y, \pi) \simeq \varphi^*(Z^n, \pi_{\mathrm{univ}})$.*

6.3 Hilbert-Chow morphism

Write $\mathrm{S}^n(\mathbb{A}^2)$ for the nth symmetric product of the affine plane $\mathbb{A}^2$. This is by definition the quotient of the products of n copies of $\mathbb{A}^2$ by the natural permutation action of the symmetric group S_n on n letters. It is the set of formal sums of n points, in other words, the set of unordered n-tuples of points.

We call $\mathrm{Hilb}^n(\mathbb{A}^2)$ the *Hilbert scheme of n points in* $\mathbb{A}^2$. It is a quasiprojective scheme of dimension $2n$. Any $Z \in \mathrm{Hilb}^n(\mathbb{A}^2)$ is a zero dimensional subscheme with $h^0(Z, \mathcal{O}_Z) = \dim(\mathcal{O}_Z) = n$. Suppose that Z is reduced. Then Z is a union of n distinct points. Since being reduced is an open and generic condition, $\mathrm{Hilb}^n(\mathbb{A}^2)$ contains a Zariski open subset consisting of formal sums of n distinct points. This is why we call $\mathrm{Hilb}^n(\mathbb{A}^2)$ the Hilbert scheme of n points on $\mathbb{A}^2$.

We have a natural morphism π from $\mathrm{Hilb}^n(\mathbb{A}^2)$ onto $\mathrm{S}^n(\mathbb{A}^2)$ defined by

$$\pi\colon Z \mapsto \sum_{p \in \mathrm{Supp}(Z)} \dim(\mathcal{O}_{Z,p})p$$

We call π the Hilbert-Chow morphism (of $\mathbb{A}^2$). Let D be the subset of $\mathrm{S}^n(\mathbb{A}^2)$ consisting of formal sums of n points with at least two coincident points. It is clear that π is the identity over $\mathrm{S}^n(\mathbb{A}^2) \setminus D$, hence is birational. If $n = 2$ and if Z is nonreduced with $\mathrm{Supp}(Z)$ the origin, then Z is a subscheme defined by the ideal

$$I = (ax + by, x^2, xy, y^2), \quad \text{where} \quad (a, b) \neq (0, 0).$$

Thus the set of these subschemes is $\mathbb{P}^1$ parametrizing the ratios $a : b$. It follows that $\mathrm{Hilb}^2(\mathbb{A}^2)$ is the quotient by the symmetric group S_2 of the blowup of the nonsingular fourfold $\mathbb{A}^2 \times \mathbb{A}^2$ along the diagonal $\mathbb{A}^2$. For all n there is a relatively simple description, due to Barth, of $\mathrm{Hilb}^n_{\mathbb{A}^2}$ as a scheme, in terms of monads. See [OSS80] and [Nakajima96b], Chapter 2. We write some of these down explicitly in Sections 12–16.

One of the most remarkable features of $\mathrm{Hilb}^n(\mathbb{A}^2)$ is the following result.

Theorem 6.4 ([Fogarty68]) $\mathrm{Hilb}^n(\mathbb{A}^2)$ *is a smooth quasiprojective scheme, and* $\pi\colon \mathrm{Hilb}^n(\mathbb{A}^2) \to \mathrm{S}^n(\mathbb{A}^2)$ *is a resolution of singularities of the symmetric product.*

A simpler proof of Theorem 6.4 is given in [Nakajima96b]. We note that smoothness of $\mathrm{Hilb}^n(\mathbb{A}^2)$ is peculiar to $\dim \mathbb{A}^2 = 2$. If $n \geq 3$, then a subscheme $Z \subset \mathbb{A}^n$ can be very complicated in general [Göttsche91]. See [Iarrobino77], [Briançon77]. [Göttsche91], p. 60 writes that $\mathrm{Hilb}^n(\mathbb{A}^k)$ is known to be singular for $k \geq 3$ and $n \geq 4$ while it is smooth for any k if $n = 3$. $\mathrm{Hilb}^n(\mathbb{A}^k)$ is connected for any n and k by [Fogarty68], while it is reducible hence singular for any k and any large $n >> k$ by [Iarrobino72].

Besides smoothness, $\mathrm{Hilb}^n(\mathbb{A}^2)$ has various mysterious nice properties. Among others, the following is relevant to our subsequent study of $\mathrm{Hilb}^G(\mathbb{A}^2)$.

Theorem 6.5 ([Beauville83]) $\mathrm{Hilb}^n(\mathbb{A}^2)$ *admits a holomorphic symplectic structure.*

Proof See also [Fujiki83] for $n = 2$, and [Mukai84] for a more general case. The sketch proof below, mostly taken from [Beauville83], shows that the theorem also holds for $\mathrm{Hilb}^n(S)$ if S is a smooth complex surface with a nowhere vanishing holomorphic two form. Let ω be a nowhere vanishing closed holomorphic 2-form on $S := \mathbb{A}^2$, say $\mathrm{d}x \wedge \mathrm{d}y$ in terms of the linear coordinates on S. The product S^n of n copies of S has the holomorphic 2-form $\psi := \sum_{i=1}^n p_i^*(\omega)$, where p_i is the ith projection. We show that ψ induces a symplectic form on $S^{[n]} := \mathrm{Hilb}^n(S)$.

Let $S^{(n)} = \mathrm{S}^n(S)$ for the nth symmetric product of S, that is, by definition the quotient of the products of n copies of S by the natural permutation action of the symmetric group S_n on n letters. Let $\varepsilon\colon S^n \to S^{(n)}$ be the natural morphism. Let D_* be the open subset of D consisting of all 0-cycles of the form $2x_1+x_2+\cdots+x_{n-1}$ with all the x_i distinct. We set $S_*^{(n)} := S^{(n)} \setminus (D \setminus D_*)$, $S_*^{[n]} = \pi^{-1}(S_*^{(n)})$, $S_*^n := \varepsilon^{-1}(S_*^{(n)})$ and $\Delta_* = \varepsilon^{-1}(D_*)$. Then Δ_* is smooth and of codimension 2 in $S_*^{(n)}$. Then by [Beauville83], p. 766, $S_*^{[n]}$ is isomorphic to the quotient of the blowup of $\mathrm{Bl}_{\Delta_*}(S_*^{(n)})$ of $S_*^{(n)}$ along Δ_* by the symmetric group S_n. Hence we have a natural morphism $\rho\colon \mathrm{Bl}_D(S_*^{(n)}) \to S_*^{[n]}$. We see easily that ψ induces a holomorphic 2-form φ on $S_*^{[n]}$, which extends to $S^{[n]}$ because the codimension of the inverse image of $S^{[n]} \setminus S_*^{[n]}$ in $S^{[n]}$ is greater than one.

Let E_* be the inverse image of Δ_* in $\mathrm{Bl}_{\Delta_*}(S_*^{(n)})$. Then the canonical bundle of $\mathrm{Bl}_{\Delta_*}(S_*^{(n)})$ is E_*, because that of S^n is trivial. On the other hand, it is the sum of the divisor $\rho^*(\varphi^n)$ and the ramification divisor R of ρ. Since $R = E_*$ on $\mathrm{Bl}_{\Delta_*}(S_*^{(n)})$, we see that $(\varphi)^n$ is everywhere nonvanishing on $S_*^{[n]}$, hence also on $S^{[n]}$ [Beauville83]. Thus φ is a nowhere degenerate 2-form, that is, a holomorphic symplectic form on $S^{[n]}$. □

Definition 6.6 The *infinite dimensional Heisenberg algebra* $\mathfrak{s}$ is by definition the Lie algebra generated by p_i, q_i for $i \geq 1$ and c, subject to the relations

$$[p_i, q_j] = c\delta_{ij}, \quad [p_i, p_j] = [q_i, q_j] = [p_i, c] = [q_i, c] = 0.$$

It is known that for any $a \in \mathbb{C}^*$, the Lie algebra $\mathfrak{s}$ has the *canonical commutation relations representation* σ_a on *Fock space* $R := \mathbb{C}[x_1, x_2, \ldots]$, that is, the ring of polynomials in infinitely many indeterminates x_i; the representation is defined by

$$\sigma_a(p_i) = a\frac{\partial}{\partial x_i}, \quad \sigma_a(q_i) = x_i, \quad \sigma_a(c) = a \cdot \mathrm{id}_R.$$

We denote this $\mathfrak{s}$-module by R_a. We also define a derivation d_0 of $\mathfrak{s}$ by

$$[d_0, q_i] = iq_i, \quad [d_0, p_i] = -ip_i, \quad [d_0, c] = 0.$$

The following fact is important (see [Kac90], pp. 162–163):

Theorem 6.7 *An irreducible $\mathfrak{s}$-module with generator v_0 is isomorphic to R_a if $p_i(v_0) = 0$ for all i and $c(v_0) = av_0$ for some $a \neq 0$. The character of R_a is given by*

$$\mathrm{Tr}_{R_a}(q^{d_0}) = \prod_{i=1}^{\infty}(1 - q^i)^{-1}.$$

The vector v_0 in the above theorem is called a *vacuum vector* of V. We quote one of the surprising results of [Nakajima96b].

Theorem 6.8 *Let $\mathfrak{s}$ be the infinite dimensional Heisenberg algebra. Then the direct sum of all the cohomology groups $\bigoplus_{n\geq 0} H^*(\mathrm{Hilb}^n(\mathbb{A}^2), \mathbb{C})$ is an irreducible $\mathfrak{s}$-module with $a = 1$ whose vacuum vector v_0 is a generator of $H^0(\mathrm{Hilb}^0(\mathbb{A}^2), \mathbb{C})$.*

By Theorem 6.7, the above theorem gives in a sense the complete structure of the $\mathfrak{s}$-module. However we should mention that its irreducibility follows from comparison with the following Theorem 6.9.

[Nakajima96b] derives a similar conclusion when $\mathbb{A}^2$ is replaced by a smooth quasiprojective complex surface X. Then $\bigoplus_{n\geq 0} H^*(\mathrm{Hilb}^n(X), \mathbb{C})$ is an infinite dimensional Heisenberg/Clifford algebra module. Its irreducibility again follows from Theorem 6.9.

Cell decompositions of $\mathrm{Hilb}^n(\mathbb{P}^2)$ and $\mathrm{Hilb}^n(\mathbb{A}^2)$, and hence complete formulas for the Betti numbers of $\mathrm{Hilb}^n(\mathbb{P}^2)$ and $\mathrm{Hilb}^n(\mathbb{A}^2)$, are known by Ellingsrud and Strømme [ES87]. The formulas for the Betti numbers of $\mathrm{Hilb}^n(\mathbb{P}^2)$ and $\mathrm{Hilb}^n(\mathbb{A}^2)$ are written by [Göttsche91] more generally in the following beautiful manner.

To state the theorem, we define the Poincaré polynomial $p(X,z)$ of a smooth complex variety X by $p(X,z) := \sum_{i=0}^{\infty} \dim H^i(X,\mathbb{Q})z^i$. Moreover we define $p(X,z,t) := \sum_{n=0}^{\infty} p(\mathrm{Hilb}^n(X),z)t^n$ for a smooth complex surface X.

Theorem 6.9 ([Göttsche91]) *Let X be a smooth projective complex surface. Then*

$$p(X,z,t) = \prod_{m=1}^{\infty} \frac{(1+z^{2m-1}t^m)^{b_1(X)}(1+z^{2m+1}t^m)^{b_3(X)}}{(1-z^{2m-2}t^m)^{b_0(X)}(1-z^{2m}t^m)^{b_2(X)}(1-z^{2m+2}t^m)^{b_4(X)}},$$

where $b_i(S)$ is the ith Betti number of S.

7 Three dimensional quotient singularities

7.1 Classification of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$

Threefold Gorenstein quotient singularities have attracted the attention of both mathematicians and physicists in connection with Calabi–Yau threefolds, mirror symmetry and superstring theory. For a finite subgroup G of $\mathrm{GL}(n,\mathbb{C})$, the quotient $\mathbb{A}^n/G$ is Gorenstein if and only if $G \subset \mathrm{SL}(n,\mathbb{C})$; see [Khinich76] and [Watanabe74].

Now we review the classification of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ from the very classical works of [Blichfeldt17], and Miller, Blichfeldt and Dickson [MBD16]. In these works they nearly completed the classification of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ up to conjugacy. Unfortunately, however, there were two missing classes, which were supplemented later by Stephen S.-T. Yau and Y. Yu [YY93], p. 2.

There is an obvious series of finite subgroups coming from subgroups of $\mathrm{GL}(2,\mathbb{C})$. In fact, associating $(\det g)^{-1} \oplus g$ to each $g \in \mathrm{GL}(2,\mathbb{C})$, we have a finite subgroup of $\mathrm{SL}(3,\mathbb{C})$ for any subgroup of $\mathrm{GL}(2,\mathbb{C})$. Including this series, there are exactly four infinite series of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$:

1. diagonal Abelian groups;

2. groups coming from finite subgroups in $\mathrm{GL}(2,\mathbb{C})$;

3. groups generated by (1) and T;

4. groups generated by (3) and Q.

Here

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \text{where } \omega := e^{2\pi\sqrt{-1}/3}.$$

There are exactly eight sporadic classes, each of which contains a unique finite subgroup up to conjugacy, of order 108, 216, 648, 60, 168, 180, 504 and 1080 respectively. Only two finite simple groups appear: A_5 ($\simeq \mathrm{PSL}(2, \mathbb{F}_5)$) of order 60, and $\mathrm{PSL}(2, \mathbb{F}_7)$ of order 168.

The subgroup $\mathrm{PSL}(2, \mathbb{F}_7)$ of $\mathrm{SL}(3, \mathbb{C})$ is the automorphism group of the Klein quartic curve $x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0$. On the other hand, A_5 is realized as a subgroup of $\mathrm{SL}(3, \mathbb{C})$ as follows. Let G be the binary icosahedral subgroup of $\mathrm{SL}(2, \mathbb{C})$ of order 120 (compare Section 16). This acts on the space of polynomials of homogeneous degree two on $\mathbb{A}^2$, with $\pm 1 \in G$ acting trivially. Therefore this is an irreducible representation of $G/\{\pm 1\}$ ($\simeq A_5$) of rank three. This realizes A_5 as a finite subgroup of $\mathrm{SL}(3, \mathbb{C})$. Or, more simply, $A_5 \subset \mathrm{SO}(3)$ is the group of automorphisms of the icosahedron.

In the case of order 108, the quotient $\mathbb{A}^3/G$ is a complete intersection defined by two equations, while it is a hypersurface in the remaining seven cases. The defining equations are completely known; in contrast with the two dimensional case, they are not all weighted homogeneous. The weighted homogeneous ones are the cases of order 108, 648, 60, 180 and 1080 [YY93].

All finite subgroups of $\mathrm{GL}(2, \mathbb{C})$ are known by Behnke and Riemenschneider [BR95]. We note that in the easiest series (1) the quotients are torus embeddings. Therefore their smooth resolutions are constructed through torus embeddings. See [Roan89].

Outstanding in this area is the following theorem, which generalizes the two dimensional McKay correspondence to some extent.

Theorem 7.2 *For any finite subgroup G of $\mathrm{SL}(3, \mathbb{C})$, there exists a smooth resolution X of the quotient $\mathbb{A}^3/G$ such that the canonical bundle of X is trivial (X is then called a* crepant resolution *of $\mathbb{A}^3/G$). For any such resolution X, $H^*(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank equal to the number of the conjugacy classes of G.*

[Ito95a], [Ito95b], [Markushevich92], [Roan94] and [Roan96] contribute to the proof of this theorem. It seems desirable to simplify the proofs for the complicated sporadic classes. Ito and Reid [IR96] generalized the theorem and sharpened it especially in dimension three by finding a bijective correspondence between irreducible exceptional divisors of the resolution and conjugacy classes of G (called *junior*) with certain type of eigenvalues: they defined the notion of *age* of a conjugacy class; the junior conjugacy classes are those of age equal to one. The junior ones play a more important role in the study of crepant resolutions.

8 Hilbert schemes and simple singularities Introduction

The second half of the article starts here. In it, we study the link from (b) to (a).

8.1 Abstract

For any finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ of order n, we consider the *G-orbit Hilbert scheme*, namely, a certain subscheme $\mathrm{Hilb}^G(\mathbb{A}^2)$ of $\mathrm{Hilb}^n(\mathbb{A}^2)$ that parametrizes G-invariant subschemes. We first give a direct proof, independent of the classification of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$, that $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a minimal resolution of a simple singularity $\mathbb{A}^2/G$. Any point of the exceptional set E is a G-invariant 0-dimensional subscheme Z of $\mathbb{A}^2$ with support the origin. Let I be the ideal sheaf defining Z. Then I is an infinite dimensional G-module. Dividing it by a natural G-submodule of I gives a finite G-module $V(I)$, which turns out to be either an irreducible G-module or the sum of two inequivalent irreducible G-modules. This gives the McKay correspondence as described in Section 4.

8.2 Summary of main results

We explain in a little more detail. Let $\mathrm{S}^n(\mathbb{A}^2)$ be the nth symmetric product of $\mathbb{A}^2$ (that is, the Chow variety $\mathrm{Chow}^n(\mathbb{A}^2)$), and $\mathrm{Hilb}^n(\mathbb{A}^2)$ the Hilbert scheme of n points of $\mathbb{A}^2$. By Theorems 6.4 and 6.5, $\mathrm{Hilb}^n(\mathbb{A}^2)$ is a crepant resolution of $\mathrm{S}^n(\mathbb{A}^2)$ with a holomorphic symplectic structure.

Let G be an arbitrary finite subgroup of $\mathrm{SL}(2,\mathbb{C})$; it acts on $\mathbb{A}^2$, and therefore has a canonical action on both $\mathrm{Hilb}^n(\mathbb{A}^2)$ and $\mathrm{S}^n(\mathbb{A}^2)$. Now we consider the particular case where n equals the order of G. Then it is easy to see that the G-fixed point set $\mathrm{S}^n(\mathbb{A}^2)^G$ in $\mathrm{S}^n(\mathbb{A}^2)$ is isomorphic to the quotient $\mathbb{A}^2/G$. The G-fixed point set $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ in $\mathrm{Hilb}^n(\mathbb{A}^2)$ is always nonsingular, but could a priori be disconnected. There is however a unique irreducible component of $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ dominating $\mathrm{S}^n(\mathbb{A}^2)^G$, which we denote by $\mathrm{Hilb}^G(\mathbb{A}^2)$. Since $\mathrm{Hilb}^G(\mathbb{A}^2)$ inherits a holomorphic symplectic structure from $\mathrm{Hilb}^n(\mathbb{A}^2)$, $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a crepant (that is, minimal) resolution of $\mathbb{A}^2/G$ (see Theorem 9.3).

Our aim in this part is to study in detail the structure of $\mathrm{Hilb}^G(\mathbb{A}^2)$ using representations of G defined in terms of spaces of homogeneous polynomials or symmetric tensors.

Let $\mathfrak{m}$ (respectively $\mathfrak{m}_S$) be the maximal ideal of the origin of $\mathbb{A}^2$ (respectively $S := \mathbb{A}^2/G$) and set $\mathfrak{n} = \mathfrak{m}_S\mathcal{O}_{\mathbb{A}^2}$. A point $\mathfrak{p}$ of $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a G-invariant 0-dimensional subscheme Z of $\mathbb{A}^2$, and to it we associate the G-invariant ideal subsheaf I defining Z, and the exact sequence

$$0 \to I \to \mathcal{O}_{\mathbb{A}^2} \to \mathcal{O}_Z \to 0.$$

We assume that $\mathfrak{p}$ is in the exceptional set E of $\mathrm{Hilb}^G(\mathbb{A}^2)$; since G acts freely outside the origin, Z is then supported at the origin, and $I \subset \mathfrak{m}$. As is easily shown, I contains $\mathfrak{n}$ (Corollary 9.6). Let $V(I) := I/(\mathfrak{m}I + \mathfrak{n})$. The finite G-module $V(I)$ is isomorphic to a minimal G-submodule of $I/\mathfrak{n}$ generating the $\mathcal{O}_{\mathbb{A}^2}$-module $I/\mathfrak{n}$.

If $\mathfrak{p}$ is a smooth point of E, we prove that $V(I)$ is a nontrivial irreducible G-module; while if $\mathfrak{p} \in E$ is a singular point, $V(I)$ is the direct sum of two inequivalent nontrivial irreducible G-modules. For any equivalence class of a nontrivial irreducible G-module ρ we define the subset $E(\rho)$ of E consisting of all $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$ such that $V(I)$ contains ρ as a G-submodule. We will see that $E(\rho)$ is naturally identified with the set of all nontrivial proper G-submodules of $\rho^{\oplus 2}$, which is isomorphic to a smooth rational curve by Schur's lemma (Theorem 10.7). The map $\rho \mapsto E(\rho)$ gives a bijective correspondence (Theorem 10.4) between the set $\mathrm{Irr}\, G$ of all the equivalence classes of irreducible G-modules and the set $\mathrm{Irr}\, E$ of all the irreducible components of E, which turns out to be the classical McKay correspondence [McKay80].

We also give an explanation of why it is that tensoring by the natural representation appears as the key ingredient in the McKay correspondence. An outline of the story is given in 13.5. The most remarkable point, in addition to the McKay correspondence itself, is that there are two kinds of dualities (Theorems 10.6 and 12.4) in the G-module decomposition of the algebra $\mathfrak{m}/\mathfrak{n}$. (After completing the present work, we were informed by Shinoda that the dualities also follow from [Steinberg64].) It is the second duality (for instance, Theorem 10.6) that explains why tensoring by the natural representation appears in the McKay correspondence.

Our results hold also in characteristic p provided that the ground field k is algebraically closed and the order of G is coprime to p.

The research part of the article is organized as follows. In Section 9 we prove that $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a crepant (or minimal) resolution of $\mathbb{A}^2/G$. We also give some elementary lemmas on representations of finite groups. In Section 10 we formulate our main theorem and relevant theorems. We give a complete description of the ideals corresponding to the points of the exceptional set E. In Section 11 we prove the dualities independently of the classification of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$. In Sections 12–16 we study $\mathrm{Hilb}^G(\mathbb{A}^2)$ and prove the main theorem separately in the cases A_n, D_n, E_6, E_7 and E_8 respectively.

In Section 17, we raise some unsolved questions.

9 The crepant (minimal) resolution

Lemma 9.1 *Let G be a finite subgroup of $\mathrm{GL}(2,\mathbb{C})$, and $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ the subset of $\mathrm{Hilb}^n(\mathbb{A}^2)$ consisting of all points fixed by G. Then $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ is nonsingular.*

Proof By Theorem 6.4, $\mathrm{Hilb}^n(\mathbb{A}^2)$ is nonsingular. Let $\mathfrak{p}$ be a point of $\mathrm{Hilb}^n(\mathbb{A}^2)^G$. Since the action of G on $\mathrm{Hilb}^n(\mathbb{A}^2)$ at $\mathfrak{p}$ is linearized. In other words we see that there exist local parameters x_i of $\mathrm{Hilb}^n(\mathbb{A}^2)$ at $\mathfrak{p}$ and some constants $a_{ij}(g) \in \mathbb{C}$ such that $g^*x_i = \sum a_{ij}(g)x_j$ for any $g \in G$. The fixed locus $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ at $\mathfrak{p}$ is by definition the reduced subscheme of $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ defined by $x_i - \sum a_{ij}(g)x_j = 0$ ($\forall g \in G$). Hence it is nonsingular. $\square$

Lemma 9.2 *Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ of order n, and $\mathrm{S}^n(\mathbb{A}^2)^G$ the subset of $\mathrm{S}^n(\mathbb{A}^2)$ consisting of all points of $\mathrm{S}^n(\mathbb{A}^2)$ fixed by G. Then $\mathrm{S}^n(\mathbb{A}^2)^G \simeq \mathbb{A}^2/G$.*

Proof Let $0 \neq \mathfrak{q} \in \mathbb{A}^2$ be a point. Then since $\mathfrak{q}$ is not fixed by any element of G other than the identity, the set $G \cdot \mathfrak{q} := \{g(\mathfrak{q}); g \in G\}$ determines a point in $\mathrm{S}^n(\mathbb{A}^2)^G$. Conversely, any point of $\mathrm{S}^n(\mathbb{A}^2)^G$ is an unordered G-invariant set Σ in $\mathbb{A}^2$. If Σ contains a point $\mathfrak{q} \neq 0$, it must contain the set $G \cdot \mathfrak{q}$. Since $|\Sigma| = n = |G|$, we have $\Sigma = G \cdot \mathfrak{q}$. Note $G \cdot \mathfrak{q} = G \cdot \mathfrak{q}'$ for a pair of points $\mathfrak{q}, \mathfrak{q}' \neq 0$ if and only if $\mathfrak{q}' \in G \cdot \mathfrak{q}$. Therefore we have the isomorphism $\mathrm{S}^n(\mathbb{A}^2 \setminus \{0\})^G \simeq (\mathbb{A}^2 \setminus \{0\})/G$, which extends naturally to a bijective morphism of $\mathrm{S}^n(\mathbb{A}^2)^G$ onto $\mathbb{A}^2/G$. It follows that $\mathrm{S}^n(\mathbb{A}^2)^G \simeq \mathbb{A}^2/G$ because $\mathbb{A}^2/G$ is normal. $\square$

Theorem 9.3 *Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup of order n. Then there is a unique irreducible component $\mathrm{Hilb}^G(\mathbb{A}^2)$ of $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ dominating $\mathbb{A}^2/G$, which is a crepant (or equivalently a minimal) resolution of $\mathbb{A}^2/G$.*

Proof We have the Hilbert-Chow morphism of $\mathrm{Hilb}^n(\mathbb{A}^2)$ onto $\mathrm{S}^n(\mathbb{A}^2)$ defined by $\pi(Z) = \mathrm{Supp}(Z)$ (counted with the appropriate multiplicities) for a zero-dimensional subscheme Z of $\mathbb{A}^2$. Since $\mathrm{Hilb}^n(\mathbb{P}^2)$ is a projective scheme by Theorem 6.2, the Hilbert-Chow morphism of $\mathrm{Hilb}^n(\mathbb{P}^2)$ is proper. Hence the Hilbert-Chow morphism of $\mathrm{Hilb}^n(\mathbb{A}^2)$ is proper because it is obtained by restricting the image variety $\mathrm{S}^n(\mathbb{P}^2)$ to $\mathrm{S}^n(\mathbb{A}^2)$. This induces a natural morphism of $\mathrm{Hilb}^G(\mathbb{A}^2)$ onto $\mathrm{S}^n(\mathbb{A}^2)^G \simeq \mathbb{A}^2/G$. Any point of $\mathrm{S}^n(\mathbb{A}^2)^G \setminus \{0\}$ is a G-orbit of a point $0 \neq \mathfrak{p} \in \mathbb{A}^2$, which is a reduced zero-dimensional subscheme invariant under G. It follows that $\mathrm{Hilb}^G(\mathbb{A}^2)$ is birationally equivalent to $\mathrm{S}^n(\mathbb{A}^2)^G$, so that it is a resolution of $\mathrm{S}^n(\mathbb{A}^2)^G \simeq \mathbb{A}^2/G$.

By [Fujiki83], Proposition 2.6, $\mathrm{Hilb}^G(\mathbb{A}^2)$ inherits a canonical holomorphic symplectic structure from $\mathrm{Hilb}(\mathbb{A}^2)$. Since $\dim \mathrm{Hilb}^G(\mathbb{A}^2) = \dim \mathbb{A}^2/G = 2$, this implies that the dualizing sheaf of $\mathrm{Hilb}^G(\mathbb{A}^2)$ is trivial. This completes the proof. $\square$

Lemma 9.4 *Let G be a finite subgroup of* $\mathrm{GL}(n,\mathbb{C})$*. Let S be a connected reduced scheme, and $\mathcal{I}$ an ideal of $\mathcal{O}_{\mathbb{A}^n\times S}$ such that $\mathcal{O}_{\mathbb{A}^n\times S}/\mathcal{I}$ is flat over S. Let $\mathcal{I}_s := \mathcal{I}\otimes\mathcal{O}_{\mathbb{A}^n\times\{s\}}$. Suppose that we are given a regular action of G on $\mathbb{A}^n\times S$ possibly depending nontrivially on S. If* $\dim \mathrm{Supp}(\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s) = 0$ *for any $s \in S$, then the equivalence class of the G-module $\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s$ is independent of s.*

Proof By the assumption $h^1(\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s) = 0$. Therefore $h^0(\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s)$ is constant on S because $\chi(\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s)$ is constant by [Hartshorne77], Chapter III. Hence again by [ibid.] $\mathcal{O}_{\mathbb{A}^n\times S}/\mathcal{I}$ is a locally free sheaf of $\mathcal{O}_S$-modules of finite rank. Let $E := \mathcal{O}_{\mathbb{A}^n\times S}/\mathcal{I}$ and $\Delta(g,x) := \det(x\cdot\mathrm{id} - T(g))$ be the characteristic polynomial of the action $T(g)$ of $g\in G$ on E. Clearly $\Delta(g,x)$ is independent of a local trivialization of the sheaf E. It follows that $\Delta(g,x) \in \mathrm{Hom}(\det E, \det E)[x] \simeq \Gamma(\mathcal{O}_S)[x]$, the polynomial ring of x over $\Gamma(\mathcal{O}_S)$. Moreover coefficients of the polynomial $\Delta(g,x)$ in x are elementary symmetric polynomials of eigenvalues of $T(g)$. Since all the eigenvalues of $T(g)$ are n-th roots of unity where $n = |G|$, coefficients of $\Delta(g,x)$ take values in a finite subset of $\mathbb{C}$ over S. Since S is connected and reduced, they are constant. It follows that $\Delta(g,x)\in\mathbb{C}[x]$. In particular the character $\mathrm{Tr}\,T(g)$, the coefficient of x in $\Delta(g,x)$ is independent of $s\in S$. Since any finite G-module is uniquely determined up to equivalence by its character, the equivalence class of the G-module $\mathcal{O}_{\mathbb{A}^n\times\{s\}}/\mathcal{I}_s$ is independent of $s\in S$. $\square$

Corollary 9.5 *Let G be a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$*, and I an ideal of $\mathcal{O}_{\mathbb{A}^2}$ with $I\in\mathrm{Hilb}^G(\mathbb{A}^2)$. Then as G-modules $\mathcal{O}_{\mathbb{A}^2}/I \simeq \mathbb{C}[G]$, the regular representation of G.*

Corollary 9.6 *Let I be an ideal of $\mathcal{O}_{\mathbb{A}^2}$ with $I\in\mathrm{Hilb}^G(\mathbb{A}^2)$. Any G-invariant function vanishing at the origin is contained in I.*

Proof $\mathcal{O}_{\mathbb{A}^2}/I \simeq \mathbb{C}[G]$ by Corollary 9.5. This implies that $\mathcal{O}_{\mathbb{A}^2}/I$ has a unique trivial G-submodule spanned by constant functions of $\mathbb{A}^2$. It follows that any G-invariant function vanishing at the origin is contained in I. □

Remark 9.7 By [Nakajima96b], Theorem 4.4, for $I \in \mathrm{Hilb}^n(\mathbb{A}^2)$, the following conditions are equivalent,

1. $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$;
2. $\mathcal{O}_{\mathbb{A}^2}/I \simeq \mathbb{C}[G]$;
3. $\mathrm{Hom}_{\mathcal{O}_{\mathbb{A}^2}}(I, \mathcal{O}_{\mathbb{A}^2}/I)^G \neq 0$.

10 The Main Theorem

10.1 Stratification of $\mathrm{Hilb}^G(\mathbb{A}^2)$ by $\mathrm{Irr}\, G$

Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. As in Section 4.2, we write $\mathrm{Irr}\, G$ for the set of all the equivalence classes of nontrivial irreducible G-modules, and $\mathrm{Irr}_*\, G$ for the union of $\mathrm{Irr}\, G$ and the trivial one dimensional G-module. Let $V(\rho) \in \mathrm{Irr}\, G$ be a G-module, and $\rho\colon G \to \mathrm{GL}(V(\rho))$ the corresponding homomorphism.

Let $X = X_G := \mathrm{Hilb}^G(\mathbb{A}^2)$ and $S = S_G := \mathbb{A}^2/G$. Write $\mathfrak{m}$ (respectively $\mathfrak{m}_S$) for the maximal ideal of $\mathbb{A}^2$ (respectively S) at the origin 0, and set $\mathfrak{n} := \mathfrak{m}_S\mathcal{O}_{\mathbb{A}^2}$. Let $\pi\colon X \to S$ be the natural morphism and E the exceptional set of π. Let $\mathrm{Irr}\, E$ be the set of irreducible components of E. Any $I \in X$ contained in E (to be exact, the subscheme defined by I belongs to X) is a G-invariant ideal of $\mathcal{O}_{\mathbb{A}^2}$ which contains $\mathfrak{n}$ by Corollary 9.6. For any ρ, ρ', and $\rho'' \in \mathrm{Irr}\, G$, we define

$$\begin{aligned}
V(I) &:= I/(\mathfrak{m}I + \mathfrak{n}),\\
E(\rho) &:= \left\{I \in \mathrm{Hilb}^G(\mathbb{A}^2); V(I) \supset V(\rho)\right\},\\
P(\rho,\rho') &:= \left\{I \in \mathrm{Hilb}^G(\mathbb{A}^2); V(I) \supset V(\rho) \oplus V(\rho')\right\},\\
Q(\rho,\rho',\rho'') &:= \left\{I \in \mathrm{Hilb}^G(\mathbb{A}^2); V(I) \supset V(\rho) \oplus V(\rho') \oplus V(\rho'')\right\}.
\end{aligned}$$

Remark 10.2 Note that we allow $\rho = \rho'$ in the definition of $P(\rho,\rho')$. Of course if $\rho \neq \rho'$, then $P(\rho,\rho') = E(\rho) \cap E(\rho')$.

Definition 10.3 Two irreducible G-modules ρ and ρ' are said to be *adjacent* if $\rho \otimes \rho_{\mathrm{nat}}$ contains ρ' if and only if $\rho' \otimes \rho_{\mathrm{nat}}$ contains ρ.

In fact, since $G \subset \mathrm{SL}(2,\mathbb{C})$, we have $\chi_{\mathrm{nat}}(x^{-1}) = \chi_{\mathrm{nat}}(x)$ $(\forall x \in G)$ where $\chi_{\mathrm{nat}} := \mathrm{Tr}(\rho_{\mathrm{nat}})$. Hence for any characters χ and χ' of G

$$(\chi\chi_{\mathrm{nat}}, \chi') = (1/|G|)\sum_{x\in G}\chi(x)\chi_{\mathrm{nat}}(x)\chi'(x^{-1}) = (\chi, \chi'\chi_{\mathrm{nat}}).$$

Thus the multilicity of ρ' in $\rho \otimes \rho_{\mathrm{nat}}$ equals that of ρ in $\rho' \otimes \rho_{\mathrm{nat}}$.

The *Dynkin diagram* $\Gamma(\mathrm{Irr}\, G)$ or the *extended Dynkin diagram* $\Gamma(\mathrm{Irr}_*\, G)$ of G is the graph whose vertices are $\mathrm{Irr}\, G$ or $\mathrm{Irr}_*\, G$ respectively, with ρ and ρ' joined by a simple edge if and only if ρ and ρ' are adjacent.

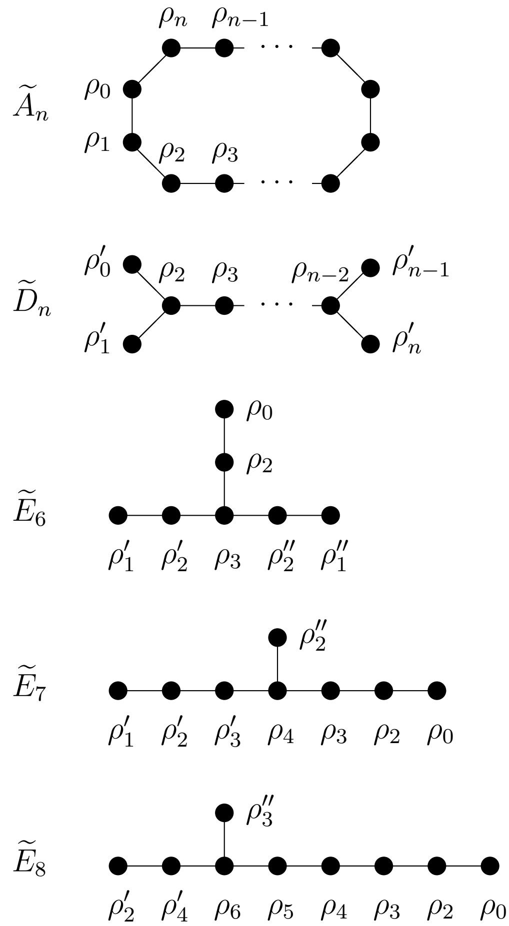

Figure 5: The extended Dynkin diagrams and representations

Then our main theorem is stated as follows.

Theorem 10.4 *Let G be a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$. *Then*

1. *the map $\rho \mapsto E(\rho)$ is a bijective correspondence between* $\mathrm{Irr}\, G$ *and* $\mathrm{Irr}\, E$*;*

2. *$E(\rho)$ is a smooth rational curve with $E(\rho)^2 = -2$ for any $\rho \in \operatorname{Irr} G$;*

3. *$P(\rho, \rho') \neq \emptyset$ if and only if ρ and ρ' are adjacent. In this case $P(\rho, \rho')$ is a single (reduced) point, at which $E(\rho)$ and $E(\rho')$ intersect transversally;*

4. *$P(\rho, \rho) = Q(\rho, \rho', \rho'') = \emptyset$ for any $\rho, \rho', \rho'' \in \operatorname{Irr} G$.*

In the A_n case Theorem 10.4 follows from Theorem 9.3 and Theorems in Section 12. In the other cases Theorem 10.4 follows from Theorem 9.3, Theorem 10.7 and Remark 10.8.

By Theorem 10.4, (3), $\Gamma(\operatorname{Irr} G)$ is the same thing as the dual graph $\Gamma(\operatorname{Irr} E)$ of E, in other words, the Dynkin diagram of the singularity S_G. Let h be the Coxeter number of $\Gamma(\operatorname{Irr} E)$. We also call h the Coxeter number of G. See Table 2 and Subsection 11.1.

We define nonnegative integers $d(\rho)$ for any $\rho \in \operatorname{Irr} G$ as follows. If G is cyclic, choose a character χ of G such that $\rho_{\text{nat}} = \chi \oplus \chi^{-1}$, and define $e(\chi^k) = k$, $d(\chi^k) = |\frac{n+1}{2} - k|$. Although there are two choices of the generator χ, the definition of the pair $\left(\frac{h}{2} - d(\rho), \frac{h}{2} + d(\rho)\right) = \left(e(\rho), n + 1 - e(\rho)\right)$ is independent of the choice. If G is not cyclic, then $\Gamma(\operatorname{Irr} G)$ is star-shaped with a unique centre. For any $\rho \in \operatorname{Irr} G$, we define $d(\rho)$ to be the distance from the vertex ρ to the centre. It is obvious that $d(\rho) = d(\rho') \pm 1$ if ρ and $\rho' \in \operatorname{Irr} G$ are adjacent. Also in the cyclic case if we define the centre to be the midpoint of the graph, then $d(\rho)$ is the distance from the centre.

For any positive integer m let $S_m := S_m(\rho_{\text{nat}})$ be the symmetric m-tensors of ρ_{nat}, that is, the space of homogeneous polynomials of degree m. We say that a G-submodule W of $\mathfrak{m}/\mathfrak{n}$ is *homogeneous* of degree m if it is generated over $\mathbb{C}$ by homogeneous polynomials of degree m.

The G-module $\mathfrak{m}/\mathfrak{n}$ splits as a direct sum of irreducible homogeneous G-modules. If W is a direct sum of homogeneous G-submodules, then we denote the homogeneous part of W of degree m by $S_m(W)$. For any G-module W in some $S_m(\mathfrak{m}/\mathfrak{n})$, we write $S_j \cdot W$ for the G-submodule of $S_{m+j}(\mathfrak{m}/\mathfrak{n})$ generated over $\mathbb{C}$ by the products of $S_j(\mathfrak{m}/\mathfrak{n})$ and W. We denote by $W[\rho]$ the ρ factor of W, that is, the sum of all the copies of ρ in W; and similarly, we denote by $[W : \rho]$ the multiplicity of $\rho \in \operatorname{Irr} G$ in a G-module W.

We define

$$S_{\text{McKay}}(\mathfrak{m}/\mathfrak{n}) = \sum_{\rho \in \operatorname{Irr} G} S_{\frac{h}{2} \pm d(\rho)}(\mathfrak{m}/\mathfrak{n})[\rho].$$

Theorem 10.5 (First duality theorem) *Let G be any finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ and h its Coxeter number. Then as G-modules, we have*

1. *$\mathfrak{m}/\mathfrak{n} = \sum_{\rho \in \operatorname{Irr} G} 2(\deg \rho)\rho$;*

2. *$S_{\text{McKay}}(\mathfrak{m}/\mathfrak{n}) \simeq \sum_{\rho \in \operatorname{Irr} G} 2\rho$;*

3. $S_{\frac{h}{2}-k}(\mathfrak{m}/\mathfrak{n}) \simeq S_{\frac{h}{2}+k}(\mathfrak{m}/\mathfrak{n})$ *for any* k;
4. $S_k(\mathfrak{m}/\mathfrak{n}) = 0$ *for* $k \geq h$.

Theorem 10.6 (Second duality theorem) *Assume that* G *is not cyclic. Let* h *be the Coxeter number of* G *and* $V_{\frac{h}{2}\pm d(\rho)}(\rho) := S_{\frac{h}{2}\pm d(\rho)}(\mathfrak{m}/\mathfrak{n})[\rho]$ *for any* $\rho \in \operatorname{Irr} G$. *Then*

1. $V_{\frac{h}{2}-d(\rho)}(\rho) \simeq V_{\frac{h}{2}+d(\rho)}(\rho) \simeq \rho^{\oplus 2}$ *or* ρ *if* $d(\rho) = 0$, *respectively* $d(\rho) \geq 1$.
2. *If* ρ *and* ρ' *are adjacent with* $d(\rho') = d(\rho) + 1 \geq 2$, *then*
$$\begin{aligned} V_{\frac{h}{2}-d(\rho)}(\rho) &= \{S_1 \cdot V_{\frac{h}{2}-d(\rho')}(\rho')\}[\rho], \\ \text{and} \quad V_{\frac{h}{2}+d(\rho')}(\rho') &= \{S_1 \cdot V_{\frac{h}{2}+d(\rho)}(\rho)\}[\rho']. \end{aligned}$$
3. *If* $d(\rho) = 0$, *we write* $\rho_i \in \operatorname{Irr} G$ *for* $i = 1, 2, 3$ *for the three irreducible representations adjacent to* ρ; *then*
$$\{S_1 \cdot V_{\frac{h}{2}-1}(\rho_i)\}[\rho] \simeq \rho,$$
$$V_{\frac{h}{2}+1}(\rho_i) = \{S_1 \cdot V_{\frac{h}{2}}(\rho)\}[\rho_i] \simeq \rho_i \quad \text{for } i = 1, 2, 3; \text{ and}$$
$$V_{\frac{h}{2}}(\rho) = \{S_1 \cdot V_{\frac{h}{2}-1}(\rho_i)\}[\rho] + \{S_1 \cdot V_{\frac{h}{2}-1}(\rho_j)\}[\rho] \simeq \rho^{\oplus 2} \quad \text{for } i \neq j.$$

See Section 11 for the proof of Theorems 10.5–10.6. It is the detailed form of the duality in Theorems 10.6 and 12.4 that we need for the explanation of the McKay observation in 13.5.

The exceptional sets of $\operatorname{Hilb}^G(\mathbb{A}^2)$ are described in Theorems 10.7 and 12.3.

Theorem 10.7 *Assume that* G *is not cyclic.*

1. *Assume that* ρ *is one of the endpoints of the Dynkin diagram. Then* $I \in E(\rho) \setminus \left(\bigcup_{\rho'} P(\rho, \rho')\right)$ *if and only if* $V(I)$ *is a nonzero irreducible* G*-submodule* ($\simeq \rho$) *of* $V_{\frac{h}{2}-d(\rho)}(\rho) \oplus V_{\frac{h}{2}+d(\rho)}(\rho)$ *different from* $V_{\frac{h}{2}+d(\rho)}(\rho)$.
2. *Assume* $d(\rho) \geq 1$ *and that* ρ *is none of the endpoints of the Dynkin diagram. Then* $I \in E(\rho) \setminus \left(\bigcup_{\rho'} P(\rho, \rho')\right)$ *if and only if* $V(I)$ *is a nonzero irreducible* G*-submodule* ($\simeq \rho$) *of* $V_{\frac{h}{2}-d(\rho)}(\rho) \oplus V_{\frac{h}{2}+d(\rho)}(\rho)$ *different from* $V_{\frac{h}{2}-d(\rho)}(\rho)$ *and* $V_{\frac{h}{2}+d(\rho)}(\rho)$.
3. *Let* ρ *and* ρ' *be an adjacent pair with* $d(\rho') = d(\rho) + 1 \geq 2$. *Then* $I \in P(\rho, \rho')$ *if and only if*
$$V(I) = V_{\frac{h}{2}-d(\rho)}(\rho) \oplus V_{\frac{h}{2}+d(\rho')}(\rho').$$
We define the latter to be $W(\rho, \rho')$.

4. *Assume $d(\rho) = 0$.*
   (a) *$I \in E(\rho) \setminus \left(\bigcup_{\rho'} P(\rho, \rho')\right)$ if and only if $V(I)$ is a nonzero irreducible G-module of $V_{\frac{h}{2}}(\rho)$ different from $\{S_1 \cdot V_{\frac{h}{2}-1}(\rho')\}[\rho]$ for any ρ' adjacent to ρ where we note $V_{\frac{h}{2}}(\rho) \simeq \rho^{\oplus 2}$.*
   (b) *$I \in P(\rho, \rho') \neq \emptyset$ if and only if*
$$V(I) = \{S_1 \cdot V_{\frac{h}{2}-1}(\rho')\}[\rho] \oplus V_{\frac{h}{2}+1}(\rho').$$
   We define the latter to be $W(\rho, \rho')$.

The proofs of Theorems 10.4–10.7 are given in Sections 12–16 in the respective cases.

Remark 10.8 One can recover I from $V(I)$ by defining $I = V(I)\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$. By Theorem 10.7, the curve $E(\rho)$ is identified with $\mathbb{P}(\rho \oplus \rho) \simeq \mathbb{P}^1$, the projective space of nontrivial proper G-submodules ρ in $\rho \oplus \rho$.

Remark 10.9 The relations in Theorem 10.6, (2)–(3) as well as the following observation explain why tensoring by ρ_{nat} enters the McKay correspondence. We observe

$$\begin{aligned}
W(\rho, \rho') &= V_{\frac{h}{2}-d(\rho)}(\rho) \oplus V_{\frac{h}{2}+d(\rho')}(\rho') \qquad \text{for } d(\rho) \geq 1, d(\rho') = d(\rho) + 1\\
&= \{S_1 \cdot V_{\frac{h}{2}-d(\rho')}(\rho')\}[\rho] \oplus V_{\frac{h}{2}+d(\rho')}(\rho')\\
&= V_{\frac{h}{2}-d(\rho)}(\rho) \oplus \{S_1 \cdot V_{\frac{h}{2}+d(\rho)}(\rho)\}[\rho'],\\
W(\rho, \rho') &= \{S_1 \cdot V_{\frac{h}{2}-1}(\rho')\}[\rho] \oplus V_{\frac{h}{2}+1}(\rho') \qquad \text{for } d(\rho) = 0, d(\rho') = 1\\
&= \{S_1 \cdot V_{\frac{h}{2}-1}(\rho')\}[\rho] \oplus \{S_1 \cdot V_{\frac{h}{2}}(\rho)\}[\rho'].
\end{aligned}$$

11 Duality

11.1 Degrees of homogeneous generators

Let G be a noncyclic finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. In this section we prove Theorem 10.5, (3) and (4). Also assuming Theorem 10.6, (1) we prove Theorem 10.6, (2) and the first half of (3). Theorem 10.5, (2) follows readily from Theorem 10.6, (1). It remains to prove Theorem 10.5, (1), Theorem 10.6, (1) and the second half of (3), which we prove by case by case examinations in Sections 13–16. The cyclic case is treated in Section 12.

There are three G-invariant homogeneous polynomials φ_i for $i = 1, 2, 3$ which generate the ring of all G-invariant polynomials. Let $d_i := \deg \varphi_i$. We may assume that $d_1 \leq d_2 \leq \deg d_3 = h$, where h is the Coxeter number of G. We know that $d_1 + d_2 = d_3 + 2$. We note that the triple d_i can computed without using the classification of G, using instead the method of [Pinkham80]. See Section 4, Table 4 for the values of the d_i. We set $\overline{S}_m := S_m(\mathfrak{m}/\mathfrak{n})$.

Lemma 11.2 *$\overline{S}_m \neq 0$ for $1 \leq m \leq h-1$ and $\overline{S}_m = 0$ for $m \geq h$.*

Proof Choosing suitable φ_i we may assume that the quotient space $\mathbb{A}^2/G$ is defined by the equation $\varphi_3^2 = F(\varphi_1, \varphi_2)$ as given in Subsection 1.1. See [Klein] and [Pinkham80]. We also see $h = \deg\varphi_3 = \deg\varphi_1 + \deg\varphi_2 - 2$ by [Pinkham80]. Now we prove that there are no trivial common factors of φ_1 and φ_2 as polynomials in x and y. For otherwise, there is $\varphi \in \mathbb{C}[x, y]$ such that $\deg\varphi < d_1$, and φ divides φ_i. Therefore φ divides φ_3 by the relation $\varphi_3^2 = F(\varphi_1, \varphi_2)$. This implies that a one-dimensional subscheme of $\mathbb{A}^2$ defined by $\varphi = 0$ is mapped to the origin of $\mathbb{A}^2/G$. This contradicts that $\mathbb{A}^2$ is finite over $\mathbb{A}^2/G$.

Thus there are no common factors of φ_1 and φ_2. Hence $\varphi_1 S_{m-d_1} \cap \varphi_2 S_{m-d_2} = \varphi_1\varphi_2 S_{m-d_1-d_2} = 0$ for $m \leq h$. Hence $\dim \overline{S}_m = \dim S_m - \dim S_{m-d_1} - \dim S_{m-d_2}$ for $m < h$. It follows that

$$\dim \overline{S}_m = \begin{cases} m+1 & \text{for } 1 \leq m \leq d_1 - 1, \\ d_1 & \text{for } d_1 \leq m \leq d_2 - 1, \\ d_1 + d_2 - m - 1 & \text{for } d_2 \leq m \leq d_3 - 1. \end{cases}$$

Similarly we have

$$\begin{aligned} \dim \overline{S}_h &= \dim S_h/\mathbb{C}\varphi_3 - \dim S_{h-d_1} - \dim S_{h-d_2} \\ &= h - (h+1-d_1) - (h+1-d_2) = d_1 + d_2 - h - 2 = 0. \end{aligned}$$

□

Corollary 11.3 $\dim \mathfrak{m}/\mathfrak{n} = d_1 d_2 - 2 = 2|G| - 2$.

This corollary is not used elsewhere.

Proof The first equality is clear from the proof of Lemma 11.2. The second $d_1 d_2 = 2|G|$ follows from the classfication of G. □

11.4 The bilinear form (f, g) on $\mathfrak{m}/\mathfrak{n}$

Let $f, g \in \mathfrak{m}$ be homogeneous. Then we define a bilinear form (f, g) as follows. First we define $(f, g) = 0$ if $\deg(f) + \deg(g) \neq h$. If $\deg(f) + \deg(g) = h$, then in view of Lemma 11.2 we can express fg as a linear combination of φ_i with coefficients in $\mathcal{O}_{\mathbb{A}^2}$, say $fg = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3$ where a_i is homogeneous and a_3 is a constant. We define

$$(f, g) := a_3.$$

This is well defined. In fact, assume that $fg = b_1\varphi_1 + b_2\varphi_2 + b_3\varphi_3$. Then we have $(a_3 - b_3)\varphi_3 = (b_1 - a_1)\varphi_1 + (b_2 - a_2)\varphi_2$. By the proof of Lemma 11.2, φ_3 is not a linear combination of φ_1 and φ_2 with coefficients in $\mathcal{O}_{\mathbb{A}^2}$. It follows that $a_3 = b_3$. Moreover if either $f \in \mathfrak{n}$ or $g \in \mathfrak{n}$, then $(f, g) = 0$. Therefore the bilinear form is well defined on $\mathfrak{m}/\mathfrak{n}$.

Lemma 11.5 *1. $(fg, h) = (f, gh)$ for all $f, g, h \in \mathfrak{m}$;*

2. $(f, g) = (\sigma^*(f), \sigma^*(g))$ and $(\sigma^*(f), g) = (f, (\sigma^{-1})^*(g))$ for all $f, g \in \mathfrak{m}$, and all $\sigma \in G$;

3. $(\ ,\ ) \colon f \times g \mapsto (f, g)$ is a nondegenerate bilinear form on $\mathfrak{m}/\mathfrak{n}$.

Proof (1) and (2) are clear. We prove (3). For it, we prove the following claim.

Claim 11.6 *Let $f(x, y)$ be a homogeneous polynomial of degree $p < h$. If $xf(x, y) = yf(x, y) = 0$ in $\mathfrak{m}/\mathfrak{n}$, then $f(x, y) = 0$ in $\mathfrak{m}/\mathfrak{n}$.*

In fact, by the assumption, there exist homogeneous a_i and $b_i \in \mathfrak{m}$ such that $xf = a_1\varphi_1 + a_2\varphi_2$ and $yf = b_1\varphi_1 + b_2\varphi_2$. Hence we have

$$(ya_1 - xb_1)\varphi_1 + (ya_2 - xb_2)\varphi_2 = 0.$$

We see that $\deg(ya_i - xb_i) = p + 2 - d_i < h + 2 - d_i \le d_1 + d_2 - d_i$ for $i = 1, 2$, because $h + 2 = d_1 + d_2$. Meanwhile φ_1 and φ_2 have no nontrivial common factors. It follows that $ya_i - xb_i = 0$. This implies that $x \mid a_i$ and $y \mid b_i$. Hence $f = 0$ in $\mathfrak{m}/\mathfrak{n}$. □

We now proceed with the proof of Lemma 11.5, (3). Let $f \in \mathfrak{m}$ be homogeneous. Assume that $(f, g) = 0$ for any $g \in \mathfrak{m}/\mathfrak{n}$. We prove that $f = 0$ in $\mathfrak{m}/\mathfrak{n}$ by descending induction on $p := \deg f$. If $p = h-1$, then $f = 0$ by Claim 11.6. Assume $p < h - 1$. By the assumption, we get $(xf, g) = (f, xg) = 0$ and $(yf, g) = (f, yg) = 0$ for any $g \in \mathfrak{m}/\mathfrak{n}$. By the induction hypothesis, $xf = 0$ and $yf = 0$ in $\mathfrak{m}/\mathfrak{n}$. Then by Claim 11.6 we have $f = 0$ in $\mathfrak{m}/\mathfrak{n}$. □

Lemma 11.7 *Let V be a G-submodule of $\overline{S}_{(h/2)-k}$, and V^* a G-submodule of $\overline{S}_{(h/2)+k}$ dual to V with respect to the bilinear form $(\ ,\ )$, in the sense that $(\ ,\ )$ defines a perfect pairing between V and V^*. Then V is isomorphic to the complex conjugate of V^* as G-modules.*

Proof Let V^c be an arbitrary G-module of $\overline{S}_{(h/2)-k}$ complementary to V. Then we define V^* to be the orthogonal complement in $\overline{S}_{(h/2)+k}$ to V^c. By Lemma 11.5, (2), $\sigma^*(V^*) \subset V^*$ for any $\sigma \in G$. Moreover by Lemma 11.5, (2) $\mathrm{Tr}(\sigma^*_{|V}) = \mathrm{Tr}((\sigma^{-1})^*_{|V^*})$, which is equal to the complex conjugate of $\mathrm{Tr}(\sigma^*_{|V^*})$ because any eigenvalue of $\mathrm{Tr}(\sigma^*_{|V^*})$ is a root of unity. Although the definition of V^* depends on the choice of V^c, we always have $V \simeq$ the complex conjugate of V^*. □

Corollary 11.8 *Let V, V' be G-submodules of $\mathfrak{m}/\mathfrak{n}$. If V and the complex conjugate of V' are not isomorphic as G-modules, then V and V' are orthogonal.*

Lemma 11.9 *Let ρ and ρ' be equivalence classes of irreducible G-modules with $\rho \neq \rho'$. Let $V \simeq \rho$ and $W \simeq \rho'$ be G-submodules in $\overline{S}_{(h/2)-k}$ and $\overline{S}_{(h/2)-k+1}$ respectively, and $W^* \simeq (\rho')^*$ a dual to W in $\overline{S}_{(h/2)+k-1}$. If $W \subset S_1 \cdot V$, there is a G-submodule V^* of $S_1 \cdot W^*$ dual to V. If $[\rho_{\mathrm{nat}} \otimes (\rho')^* : \rho] = 1$, then V^* is uniquely determined.*

Proof Let V^c and W^c be (homogeneous) complementary G-submodules to V and W respectively. Thus by definition,

$$V \oplus V^c = \overline{S}_{(h/2)-k} \quad \text{and} \quad W \oplus W^c = \overline{S}_{(h/2)-k+1}.$$

Let W^* be the orthogonal complement to W^c in $\overline{S}_{(h/2)+k-1}$ with respect to $(\ ,\ )$. If $W \subset S_1 V$, then there exists $g, h \in V$ such that $xg + yh \in W$. By Lemma 11.5, (3), there exists $f^* \in W^*$ such that $(f^*, xg + yh) \neq 0$ so that we first assume that $(xf^*, g) = (f^*, xg) \neq 0$. Let U be a minimal G-submodule of $\mathfrak{m}/\mathfrak{n}$ containing xf^*. Then U contains V^* dual to V by Lemma 11.5, (3) and $(xf^*, g) \neq 0$. Obviously $V^* \subset S_1 W^*$ and $V^* \simeq$ the complex conjugate of V by Lemma 11.7. If $[S_1 \cdot W^* : \rho'] \leq [\rho_{\mathrm{nat}} \otimes (\rho')^* : \rho] = 1$, then uniqueness of V^* is clear. If $(yf^*, g) = (f^*, yg) \neq 0$, then we see the same by the same argument. □

Remark 11.10 For any $\rho'' \in \mathrm{Irr}\, G$, $\rho_{\mathrm{nat}} \otimes \rho''$ is a sum of G-submodules *with multiplicity one* [McKay80] (recall that $G \subset \mathrm{SL}(2, \mathbb{C})$), so that ρ has multiplicity at most one in $S_1 \cdot W^*$. Therefore the dual V^* is uniquely determined and it is the orthogonal complement of V^c in $(S_1 \cdot W^*) \cap \overline{S}_{(h/2)+k-1}$.

Lemma 11.9 implies the following. In the case of E_6, since

$$S_1 \cdot \overline{S}_3[\rho'_2] = \overline{S}_4[\rho'_1] + \overline{S}_4[\rho_3] \quad \text{and} \quad S_1 \cdot \overline{S}_3[\rho''_2] = \overline{S}_4[\rho''_1] + \overline{S}_4[\rho_3],$$

we have $S_1 \cdot \overline{S}_8[\rho'_1] = \overline{S}_9[\rho'_2]$, $S_1 \cdot \overline{S}_8[\rho''_1] = \overline{S}_9[\rho''_2]$ and $S_1 \cdot \overline{S}_8[\rho_3] = \overline{S}_9[\rho'_2] + \overline{S}_9[\rho'_2]$, and vice versa. See Section 14.

11.11 Partial proofs of Theorems 10.5 and 10.6.

Since $\mathrm{Tr}_{\overline{S}_k}$ is real for any k, $\overline{S}_k$ contains any G-module and its complex conjugate with equal multiplicities. Theorem 10.5, (3) is clear from Lemma 11.5, (3) and Lemma 11.7. Theorem 10.5, (4) follows from Lemma 11.2. Theorem 10.6, (2) as well as the first half of (3) are clear from Lemma 11.9.

12 The cyclic groups A_n

12.1 Characters

Let x, y be coordinates on $\mathbb{A}^2$ and $\mathfrak{m} = (x, y)$ be the maximal ideal of $\mathbb{A}^2$ at the origin. Let G be the cyclic group of order $n+1$ with generator σ. Let ε be a primitive $(n+1)$st root of unity. We define the action of the generator σ on $\mathbb{C}^2$ by $(x, y) \mapsto (x, y)\sigma = (\varepsilon x, \varepsilon^{-1} y)$. The simple singularity of type A_n is the quotient $S_G = \mathbb{A}^2/G$. Let $\mathfrak{m}_S$ be the maximal ideal of S_G at the origin and $\mathfrak{n} := \mathfrak{m}_S \mathcal{O}_{\mathbb{A}^2}$.

The Coxeter number h of A_n is equal to $n+1$. Let ρ_0 be the trivial character, and ρ_i for $1 \le i \le n$ the character with $\rho_i(\sigma) = \varepsilon^i$. Then $e(\rho_i) = i$ and $h - e(\rho_i) = n + 1 - i$.

Lemma 12.2 *Any $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$ is one of the following ideals of colength $n+1$:*

$$I(\Sigma) := \prod_{\mathfrak{p} \in \Sigma} \mathfrak{m}_{\mathfrak{p}} = (x^{n+1} - a^{n+1}, xy - ab, y^{n+1} - b^{n+1}),$$

where $\Sigma = G \cdot (a, b)$ is a G-orbit of $\mathbb{A}^2$ disjoint from the origin; or

$$I_i(p_i : q_i) := (p_i x^i - q_i y^{n+1-i}, xy, x^{i+1}, y^{n+2-i}),$$

for some $1 \le i \le n$ and some $[p_i, q_i] \in \mathbb{P}^1$.

Proof Let $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$ with $I \subset \mathfrak{m}$. Then by Corollary 9.5, $\mathcal{O}_{\mathbb{A}^2}/I \simeq \mathbb{C}[G] \simeq \bigoplus_{i=0}^n \rho_i$ as G-modules. Thanks to Corollary 9.6, we define $N := \mathfrak{m}/\mathfrak{n}$ and $M := I/\mathfrak{n}$, and for each $i \neq 0$, let $M[\rho_i]$ and $N[\rho_i]$ be the ρ_i-part of M, respectively N. Then $N[\rho_i] \simeq \rho_i^{\oplus 2}$, spanned by x^i and y^{n+1-i}, while $M[\rho_i] \simeq \rho_i$ for all $i \neq 0$. It follows that for each i, there exists $[p_i, q_i] \in \mathbb{P}^1$ such that $p_i x^i - q_i y^{n+1-i} \in M$. If $p_i q_i \neq 0$ for some i, then setting $u := p_i x^i - q_i y^{n+1-i}$, we have $M = (u) + \mathfrak{n}/\mathfrak{n}$ and $I = (u, xy)$ where i is obviously uniquely determined by I. If M contains no $p_i x^i - q_i y^{n+1-i}$ with $p_i q_i \neq 0$ for any i, then $I = (x^j, y^{n+2-j}, xy)$ for some j. $\square$

Theorem 12.3 *Let a and b be the parameters of $\mathbb{A}^2$ on which the group G acts by $g(a,b) = (\varepsilon a, \varepsilon^{-1} b)$.*

Let $S = \mathbb{A}^2/G := \operatorname{Spec} \mathbb{C}[a^{n+1}, ab, b^{n+1}]$ and $\widetilde{S} \to S$ its toric minimal resolution, with affine charts U_i defined by

$$U_i := \operatorname{Spec} \mathbb{C}[s_i, t_i] \quad \text{for } 1 \leq i \leq n+1,$$

where $s_i := a^i/b^{n+1-i}$ and $t_i := b^{n+2-i}/a^{i-1}$. Then the isomorphism of $\widetilde{S}$ with $\operatorname{Hilb}^G(\mathbb{A}^2)$ is given by (the morphism defined by the universal property of $\operatorname{Hilb}^n(\mathbb{A}^2)$ from) two dimensional flat families of subschemes defined by the G-invariant ideals of $\mathcal{O}_{\mathbb{A}^2}$

$$\mathcal{I}_i(s_i, t_i) := (x^i - s_i y^{n+1-i}, xy - s_i t_i, y^{n+2-i} - t_i x^{i-1})$$

for $1 \leq i \leq n+1$.

Proof Note first that $\mathcal{I}_i(s_i, 0) = I_i(1 : s_i)$ and $\mathcal{I}_i(0, t_i) = I_{i-1}(t_i : 1)$ for $i \geq 2$.

If $ab = s_i t_i \neq 0$, we see $\mathcal{I}_i(s_i, t_i) = (x^{n+1} - a^{n+1}, xy - ab, y^{n+1} - b^{n+1})$. In fact, let $p = (a,b) \neq (0,0) \in \mathbb{A}^2$ and $\Sigma := \{p \cdot g; g \in G\}$. It is clear that $\mathcal{I}_i(s_i, t_i) \subset \mathfrak{m}_p$ so that $\mathcal{I}_i(s_i, t_i) \subset I_\Sigma$ by the G-invariance of $\mathcal{I}_i(s_i, t_i)$. Since the colengths of $\mathcal{I}_i(s_i, t_i)$ and I_Σ in $\mathcal{O}_{\mathbb{A}^2}$ are equal to $n+1$, $\mathcal{I}_i(s_i, t_i) = I_\Sigma = (x^{n+1} - a^{n+1}, xy - ab, y^{n+1} - b^{n+1})$.

By the universality of $\operatorname{Hilb}^n(\mathbb{A}^2)$ and by Lemma 12.2, we have a finite birational morphism of $\widetilde{S}$ onto a smooth surface $\operatorname{Hilb}^G(\mathbb{A}^2)$. It follows that $\widetilde{S} \simeq \operatorname{Hilb}^G(\mathbb{A}^2)$. $\square$

Theorem 12.4 (Duality for A_n) *Assume that G is cyclic. Then for any $\rho \in \operatorname{Irr} G$ there exists a unique pair $V^+_{e(\rho)}(\rho)$ and $V^-_{n+1-e(\rho)}(\rho)$ of homogeneous G-submodules of $S_{e(\rho)}(\mathfrak{m}/\mathfrak{n})[\rho]$ and $S_{n+1-e(\rho)}(\mathfrak{m}/\mathfrak{n})[\rho]$ such that*

1. *$V^+_{e(\rho)}(\rho) \simeq V^-_{n+1-e(\rho)}(\rho) \simeq \rho$, and*
2. *if ρ and ρ' are adjacent with $e(\rho) = e(\rho') + 1$, then*

$$V^+_{e(\rho)}(\rho) = \{S_1 \cdot V^+_{e(\rho')}(\rho')\}[\rho], \quad V^-_{n+1-e(\rho')}(\rho') = \{S_1 \cdot V^-_{n+1-e(\rho)}(\rho)\}[\rho'].$$

Proof First we prove uniqueness of $V^\pm_j(\rho)$. Since $S_1 = \rho_1 \oplus \rho_n$, we have unique choices $V^+_1(\rho_1) = S_1[\rho_1] = \{x\}$ and $V^-_1(\rho_n) = S_1[\rho_n] = \{y\}$. Then we have

$$\begin{aligned} V^+_{i+1}(\rho_{i+1}) &= \{S_1 \cdot V^+_i(\rho_i)\}[\rho_{i+1}] = \{x^{i+1}\} \\ V^-_{n+1-i}(\rho_i) &= \{S_1 \cdot V^-_{n-i}(\rho_{i+1})[\rho_i] = \{y^{n+1-i}\}. \end{aligned}$$

In fact, this follows from (2) by induction. This proves Theorem 12.4. $\square$

Theorem 10.4 for G cyclic follows from setting $E(\rho_i) = E_i$. There is a way of understanding $I_i(p_i, q_i)$ similar to that of Theorem 10.7.

13 The binary dihedral groups D_n

13.1 Binary dihedral group

Let G be the subgroup of $\mathrm{SL}(2,\mathbb{C})$ of order $4n-8$ generated by two elements σ and τ:

$$\sigma = \begin{pmatrix} \varepsilon, & 0 \\ 0, & \varepsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix},$$

where ε is a primitive $\ell := (2n-4)$th root of unity. Then we have

$$\sigma^{2n-4} = 1, \quad \tau^4 = 1, \quad \sigma^{n-2} = \tau^2, \quad \tau\sigma\tau^{-1} = \sigma^{-1}.$$

The group G is called the *binary dihedral group* $\mathbb{D}_{n-2}$. The Coxeter number h of D_n is equal to $2n-2$. See Table 6 for the characters of D_n.

G acts on $\mathbb{A}^2$ from the right by $(x, y) \mapsto (x, y)g$ for $g \in G$. The ring of all G-invariant polynomials is generated by $x^\ell + y^\ell$, $xy(x^\ell - y^\ell)$ and x^2y^2. By Theorem 9.3, $X_G := \mathrm{Hilb}^G(\mathbb{A}^2)$ is a minimal resolution of $S_G := \mathbb{A}^2/G$ with a simple singularity of type D_n.

Remark 13.2 We note that if we let H be the (normal) subgroup of G generated by σ and $N := G/H$, N acts on $\mathrm{Hilb}^H(\mathbb{A}^2)$ so that we have a minimal resolution $\mathrm{Hilb}^N(\mathrm{Hilb}^H(\mathbb{A}^2))(\simeq X_G)$ of S_G.

13.3 Symmetric tensors modulo $\mathfrak{n}$

Recall $\ell := 2n-4$. Let S_m be the space of symmetric m-tensors of $\rho_{\mathrm{nat}} := \rho_2$, that is, the space of homogeneous polynomials of degree m and $\overline{S}_m$ the images of S_m in $\mathfrak{m}/\mathfrak{n}$. They decompose into irreducible G-modules as follows. Let $\rho_1 := \rho'_0 + \rho'_1$, $\rho_{n-1} := \rho'_{n-1} + \rho'_n$ and $\rho_k := \rho_j$ if $k \equiv j \mod 2n-4$. Then we have

$$S_m = \begin{cases} \rho'_0 + \rho_3 + \rho_5 + \cdots + \rho_{m-1} + \rho_{m+1} & \text{for } m \equiv 0 \mod 4, \\ \rho'_1 + \rho_3 + \rho_5 + \cdots + \rho_{m-1} + \rho_{m+1} & \text{for } m \equiv 2 \mod 4, \\ \rho_2 + \rho_4 + \rho_6 + \cdots + \rho_{m-1} + \rho_{m+1} & \text{for } m \equiv 1, 3 \mod 4. \end{cases}$$

13.4

By Table 7 we see that $\mathfrak{m}/\mathfrak{n} \simeq (\mathbb{C}[G] \ominus \rho_0)^{\oplus 2}$. This isomorphism is realized by giving G-submodules $2\rho'_i$ for $i = 1, n-1, n$ and $4\rho_i$ for $2 \le i \le n-2$ explicitly as follows. We define a G-submodule of $\mathfrak{m}/\mathfrak{n}$ by $\bar{V}_i(\rho_j) := S_i(\mathfrak{m}/\mathfrak{n})[\rho_j]$, and define $V_i(\rho_j)$ to be a G-submodule of S_i such that $V_i(\rho_j) \simeq \bar{V}_i(\rho_j)$ and $V_i(\rho_j) \equiv \bar{V}_i(\rho_j) \mod \mathfrak{n}$. We use $V_i(\rho_j)$ and $\bar{V}_i(\rho_j)$ interchangeably whenever this is

ρ	1	σ	τ	d	$(\frac{h}{2} \pm d)$
ρ'_0	1	1	1	$(n-3)$	–
ρ'_1	1	1	-1	$n-3$	$(2, \ell)$
ρ_2	2	$\varepsilon + \varepsilon^{-1}$	0	$n-4$	$(3, \ell - 1)$
ρ_k	2	$\varepsilon^{k-1} + \varepsilon^{-(k-1)}$	0	$n-2-k$	$(k+1, \ell+1-k)$
ρ_{n-2}	2	$\varepsilon^{n-3} + \varepsilon^{-(n-3)}$	0	0	$(n-1, n-1)$
ρ'_{n-1}	1	-1	i^n	1	$(n-2, n)$
ρ'_n	1	-1	$-i^n$	1	$(n-2, n)$

Table 6: Character table of D_n

m	$\overline{S}_m$	m	$\overline{S}_m$
0	0	$\ell+2$	0
1	ρ_2	$\ell+1$	ρ_2
2	$\rho'_1 + \rho_3$	ℓ	$\rho'_1 + \rho_3$
3	$\rho_2 + \rho_4$	$\ell-1$	$\rho_2 + \rho_4$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
k	$\rho_{k-1} + \rho_{k+1}$	$\ell-k+2$	$\rho_{k-1} + \rho_{k+1}$
$n-2$	$\rho_{n-3} + \rho'_{n-1} + \rho'_n$	n	$\rho_{n-3} + \rho'_{n-1} + \rho'_n$
$n-1$	$2\rho_{n-2}$		

Table 7: Irreducible decompositions of $\overline{S}_m(D_n)$

$V_2(\rho'_1)$	xy	$V_\ell(\rho'_1)$	$x^\ell - y^\ell$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$V_{k-1}(\rho_k)$	x^{k-1}, y^{k-1}	$V_{k+1}(\rho_k)$	$x^k y, xy^k$
$V_{\ell-k+1}(\rho_k)$	$x^{\ell-k+1}, y^{\ell-k+1}$	$V_{\ell-k+3}(\rho_k)$	$x^{\ell-k+2}y, xy^{\ell-k+2}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$V_{n-3}(\rho_{n-2})$	x^{n-3}, y^{n-3}	$V_{n+1}(\rho_{n-2})$	$x^n y, xy^n$
$V_{n-1}(\rho_{n-2})$	$x^{n-1}, y^{n-1}, x^{n-2}y, xy^{n-2}$		
$V'_{n-1}(\rho_{n-2})$	x^{n-1}, y^{n-1}	$V''_{n-1}(\rho_{n-2})$	$x^{n-2}y, xy^{n-2}$
$V_{n-2}(\rho'_{n-1})$	$x^{n-2} - i^n y^{n-2}$	$V_n(\rho'_{n-1})$	$xy(x^{n-2} + i^n y^{n-2})$
$V_{n-2}(\rho'_n)$	$x^{n-2} + i^n y^{n-2}$	$V_n(\rho'_n)$	$xy(x^{n-2} - i^n y^{n-2})$

Table 8: $V_m(\rho)(D_n)$

harmless. We see easily that $V_i(\rho_j) \simeq \rho_j$ or 0 except for $(i,j) = (n-1, n-2)$, while $V_{n-1}(\rho_{n-2}) \simeq \rho_{n-2}^{\oplus 2}$. We list the nonzero G-submodules of $\mathfrak{m}/\mathfrak{n}$.

It is easy to see that $\mathfrak{n}$ is generated by $x^\ell + y^\ell$, $(x^\ell - y^\ell)xy$ and x^2y^2. We also note that $x^{\ell+2}, y^{\ell+2} \in \mathfrak{n}$ and that $\mathfrak{m}/\mathfrak{n}$ is spanned by x^i, y^i, x^iy and xy^i for $1 \le i \le \ell$ with the single relation $x^\ell + y^\ell \equiv 0 \mod \mathfrak{n}$. Hence we see easily that $\mathfrak{m}/\mathfrak{n}$ is the sum of the above $V_i(\rho_j)$. It follows that $\mathfrak{m}/\mathfrak{n} \simeq \sum_{\rho \in \operatorname{Irr} G} 2\deg(\rho)\rho \simeq (\mathbb{C}[G] \ominus \rho_0)^{\oplus 2}$.

13.5 A sketch for D_5

Before starting on the general case, we sketch the case of D_5 without rigorous proofs. First we recall

$$\begin{aligned} &V_2(\rho_1') = \{xy\}, \quad V_6(\rho_1') = \{x^6 - y^6\}, \\ &V_3(\rho_2) = \{x^2y, xy^2\}, \quad V_5(\rho_2) = \{x^5, y^5\}. \end{aligned}$$

We consider the case $\mathcal{I}(W) \in E(\rho_1') \setminus P(\rho_1', \rho_2)$. Let $\mathcal{I}(W) := W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$ for any nonzero G-module $W \in \mathbb{P}(V_2(\rho_1') + V_6(\rho_1')) = \mathbb{P}(\{xy, x^6 - y^6\})$ such that $W \neq V_6(\rho_1')$, that is, $W \neq \{x^6 - y^6\}$. Then we see that

$$\begin{aligned} \mathcal{I}(W)/\mathfrak{n} &= W + \sum_{k=1}^{5} S_1 W + \mathfrak{n}/\mathfrak{n} = W + \sum_{k=1}^{5} S_k V_2(\rho_1') + \mathfrak{n}/\mathfrak{n} \\ &\simeq W + \rho_2 + \rho_3 + (\rho_4' + \rho_5') + \rho_3 + \rho_2 \simeq \sum_{\rho \in \operatorname{Irr} G} \deg(\rho)\rho. \end{aligned}$$

Therefore $\mathcal{I}(W) \in \operatorname{Hilb}^G(\mathbb{A}^2)$. It is clear $V(\mathcal{I}(W)) := \mathcal{I}(W)/\mathfrak{m}\mathcal{I}(W) + \mathfrak{n} \simeq W \simeq \rho_1'$. It follows that $\mathcal{I}(W) \in E(\rho_1') \setminus P(\rho_1', \rho_2)$. Hence we have

$$\begin{aligned} \lim_{W \to V_6(\rho_1')} \mathcal{I}(W) &= V_6(\rho_1') + \sum_{k \ge 1} S_k V_2(\rho_1') \\ &= \mathcal{I}(V_6(\rho_1') \oplus S_1 V_2(\rho_1')) = \mathcal{I}(V_6(\rho_1') \oplus V_3(\rho_2)) \in P(\rho_1', \rho_2), \end{aligned}$$

where $S_1 \otimes V_2(\rho_1') \simeq S_1 V_2(\rho_1') \simeq V_3(\rho_2) \simeq \rho_2$. The factor $S_1 \otimes V_2(\rho_1') \simeq \rho_2$ among generators of $P(\rho_1', \rho_2)$ explains the relation between tensoring by $S_1 \simeq \rho_2$ and the intersection of $E(\rho_1')$ with $E(\rho_2)$ in McKay's observation.

Next we consider $W \in \mathbb{P}(V_3(\rho_2) \oplus V_5(\rho_2))$ with $W \neq V_3(\rho_2), V_5(\rho_2)$. We have

$$\begin{aligned} \mathcal{I}(W)/\mathfrak{n} &:= W + \textstyle\sum_{k \ge 1} S_k W + \mathfrak{n}/\mathfrak{n} \\ &= W + \textstyle\sum_{k \ge 1}^{2} S_k V_3(\rho_2) + \overline{S}_6 + \overline{S}_7 + \mathfrak{n}/\mathfrak{n} \\ &\simeq W + \rho_3 + (\rho_4' + \rho_5') + (\rho_1' + \rho_3) + \rho_2 \simeq \textstyle\sum_{\rho \in \operatorname{Irr} G} \deg(\rho)\rho. \end{aligned}$$

Since $\overline{S}_6 = V_6(\rho_1') + S_3V_3(\rho_2) \neq S_3V_3(\rho_2)$, we have

$$\begin{aligned}\lim_{W\to V_3(\rho_2)} \mathcal{I}(W) &= V_6(\rho_1') + V_3(\rho_2) + \sum_{k\geq 1} S_kV_3(\rho_2)\\ &= \mathcal{I}(V_6(\rho_1') \oplus V_3(\rho_2)) \in P(\rho_1', \rho_2)\\ &= \mathcal{I}(\{S_1V_5(\rho_2)\}[\rho_1'] \oplus V_3(\rho_2)),\end{aligned}$$

where $V_6(\rho_1') = \{S_1V_5(\rho_2)\}[\rho_1'] \simeq \rho_1'$, and $\{S_1V_5(\rho_2)\}[\rho_1'] = V_6(\rho_1') \simeq \rho_1'$ is by definition the sum of all the ρ_1' factors of $S_1V_5(\rho_2) \simeq S_1 \otimes V_5(\rho_2)$. Hence

$$\lim_{\substack{W\to V_6(\rho_1')\\ W\simeq\rho_1'}} \mathcal{I}(W) = \lim_{\substack{W\to V_3(\rho_2)\\ W\simeq\rho_2}} \mathcal{I}(W) \in P(\rho_1', \rho_2).$$

The above argument explains the relation between tensoring by $\rho_2 = \rho_{\mathrm{nat}}$ and the intersection of two rational curves. The argument also shows that $E(\rho)$ is naturally identified with $\mathbb{P}(V_{4-d(\rho)}(\rho) + V_{4+d(\rho)}(\rho))$, the set of all nontrivial proper G-submodules of $V_{4-d(\rho)}(\rho) + V_{4+d(\rho)}(\rho) \simeq \rho^{\oplus 2}$, which is isomorphic to $\mathbb{P}^1$ by Schur's lemma.

Now we consider the general case. We restate Theorem 10.7 as follows.

Theorem 13.6 *Let E be the exceptional set of the morphism $\pi\colon X_G \to S_G$, and $\mathrm{Sing}(E)$ the singular points of E. Let $E(\rho)$ be an irreducible component of E for $\rho \in \mathrm{Irr}\, G$ and $E^0(\rho) := E(\rho) \setminus \mathrm{Sing}(E)$. Then $E^0(\rho)$ and $\mathrm{Sing}(E)$ are as follows:*

$$\begin{aligned}
E^0(\rho_1') &= \left\{\mathcal{I}(W);\ \begin{matrix} W \subset V_2(\rho_1') \oplus V_\ell(\rho_1') \\ W \neq 0, V_\ell(\rho_1') \end{matrix}\right\},\\
E^0(\rho_k) &= \left\{\mathcal{I}(W);\ \begin{matrix} W \subset V_{k+1}(\rho_k) \oplus V_{\ell-k+1}(\rho_k) \\ W \neq 0, V_{k+1}(\rho_k), V_{\ell-k+1}(\rho_k) \end{matrix}\right\} \quad \textit{for } 2 \leq k \leq n-3,\\
E^0(\rho_{n-2}) &= \left\{\mathcal{I}(W);\ \begin{matrix} W \subset V_{n-1}(\rho_{n-2}), W \neq 0, V''_{n-1}(\rho_{n-2}) \\ W \neq S_1 \cdot V_{n-2}(\rho_j') \quad \textit{for } j = n-1, n \end{matrix}\right\},\\
E^0(\rho_j) &= \left\{\mathcal{I}(W);\ \begin{matrix} W \subset V_{n-2}(\rho_j') \oplus V_n(\rho_j') \\ W \neq 0, V_n(\rho_j') \end{matrix}\right\} \quad \textit{for } j = n-1, n;
\end{aligned}$$

and

$$\mathrm{Sing}(E) = \left\{\begin{matrix} P(\rho_1', \rho_2), & P(\rho_k, \rho_{k+1}) \quad \textit{for } 2 \leq k \leq n-3 \\ P(\rho_{n-2}, \rho_{n-1}'), & P(\rho_{n-2}, \rho_n') \end{matrix}\right\},$$

where

$$\begin{aligned} P(\rho_1', \rho_2) &= \mathcal{I}(V_\ell(\rho_1') \oplus V_3(\rho_2)), \\ P(\rho_k, \rho_{k+1}) &= \mathcal{I}(V_{\ell-k+1}(\rho_k) \oplus V_{k+2}(\rho_{k+1})) \quad \textit{for } 2 \leq k \leq n-4, \\ P(\rho_{n-3}, \rho_{n-2}) &= \mathcal{I}(V_n(\rho_{n-3}) \oplus V''_{n-1}(\rho_{n-2})), \\ P(\rho_{n-2}, \rho_j') &= \mathcal{I}(S_1 V_{n-2}(\rho_j') \oplus V_n(\rho_j')). \end{aligned}$$

13.7 Proof of Theorem 13.6 – Start

For $2 \leq k \leq n-2$, write $C(\rho_k)$ for the set of all proper G-submodules of $V_{k+1}(\rho_k) \oplus V_{\ell-k+1}(\rho_k)$; similarly, let $C(\rho_1')$ be the set of all proper G-submodules of $V_2(\rho_1') \oplus V_\ell(\rho_1')$ and for $i = n-1, n$, let $C(\rho_i')$ be the set of all proper G-submodules of $V_{n-2}(\rho_i') \oplus V_n(\rho_i')$. It is clear that the $C(\rho_k)$ and $C(\rho_i')$ are rational curves. As we will see in the sequel, they are embedded naturally into $\mathrm{Grass}(\mathfrak{m}/\mathfrak{n}, 2\ell - 2)$.

Case $\mathcal{I}(W) \in E(\rho_1') \setminus P(\rho_1', \rho_2)$ Let $\mathcal{I}(W) := W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$ for any nonzero G-module $W \in C(\rho_1')$ with $W \neq V_\ell(\rho_1')$. First assume $W = V_2(\rho_1')$. Then it is easy to see that $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{k+1}(\rho_k)$, $V_{\ell-k+3}(\rho_k)$, $V''_{n-1}(\rho_{n-2})$ and $V_{n+1}(\rho_{n-2})$ for any $2 \leq k \leq n-3$. Similarly $\mathcal{I}(W)/\mathfrak{n}$ contains $V_n(\rho_{n-1}')$ and $V_n(\rho_n')$ as well as $W = V_2(\rho_1')$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k=1}^{\ell-1} S_k \bar{V}_2(\rho_1') = W + \sum_{k=1}^{\ell-2} S_k \bar{V}_2(\rho_1') + \overline{S}_{\ell+1}$$

In particular, $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}\, G} \deg(\rho)\rho$. Hence $\mathcal{I}(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$. We see that

$$V(\mathcal{I}(W)) := \mathcal{I}(W) / \{\mathfrak{m}\mathcal{I}(W) + \mathfrak{n}\} \simeq W \simeq \rho_1'.$$

It follows that $\mathcal{I}(W) \in E(\rho_1')$.

Next we assume $W \neq V_2(\rho_1'), V_\ell(\rho_1')$. Then we first see that $x^3y \in \mathcal{I}(W)$ because $x^3y - (x^3y - 2tx^{\ell+2}) = 2tx^{\ell+2} \in \mathfrak{n}$. It follows that $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{\ell+1}(\rho_2)$, $V_{k+1}(\rho_k)$, $V_{\ell-k+3}(\rho_k)$, $V''_{n-1}(\rho_{n-2})$, $V_{n+1}(\rho_{n-2})$, $V_n(\rho_{n-1}')$ and $V_n(\rho_n')$ where $3 \leq k \leq n-3$. Since $S_1 \cdot W + V_{\ell+1}(\rho_2) = V_3(\rho_2) + V_{\ell+1}(\rho_2) \simeq \rho_2^{\oplus 2}$, $\mathcal{I}(W)/\mathfrak{n}$ also contains $2\rho_2$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{m \geq 0}^{\ell-2} S_m \bar{V}_2(\rho_1') = W + \sum_{m=0}^{\ell-3} S_m \bar{V}_2(\rho_1') + \overline{S}_{\ell+1}.$$

Hence we have $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}\, G} \deg(\rho)\rho$. Therefore $\mathcal{I}(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$. By the above structure of $\mathcal{I}(W)/\mathfrak{n}$, $V(\mathcal{I}(W)) \simeq W \simeq \rho_1'$. It follows that $\mathcal{I}(W) \in E(\rho_1') \setminus P(\rho_1', \rho_2)$.

Case $\mathcal{I}(W) \in P(\rho_1', \rho_2)$ Let $W = W(\rho_1', \rho_2) := V_\ell(\rho_1') \oplus V_3(\rho_2)$. Now $\mathcal{I}(W)/\mathfrak{n}$ contains x^2y and xy^2, hence also $V_{i+1}(\rho_i)$, $V_{\ell-i+3}(\rho_i)$ for $3 \le i \le n-3$, $V_{\ell+1}(\rho_2)$, $V_{n+1}(\rho_{n-2})$, $V_n(\rho_{n-1}')$ and $V_n(\rho_n')$. Similarly, $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{n-1}''(\rho_{n-2})$. We note that $\{\mathcal{I}(W)/\mathfrak{n}\}[\rho_1'] = W = V_\ell(\rho_1') = \{S_1 \cdot V_{\ell-1}(\rho_2)\}[\rho_1']$ and $\{\mathcal{I}(W)/\mathfrak{n}\}[\rho_2] = V_3(\rho_2) \oplus V_{\ell+1}(\rho_2) = S_1 \cdot V_2(\rho_1') \oplus V_{\ell+1}(\rho_2)$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{m=0}^{\ell-2} S_m \bar{V}_3(\rho_2) = W + \sum_{m=0}^{\ell-3} S_m \bar{V}_3(\rho_2) + \overline{S}_{\ell+1}$$

Hence we have $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho\in\operatorname{Irr} G} \deg(\rho)\rho$. Therefore $\mathcal{I}(W) \in \operatorname{Hilb}^G(\mathbb{A}^2)$. We also see that $\mathcal{I}(W) \in P(\rho_1', \rho_2)$, because

$$\begin{aligned} V(\mathcal{I}(W)) &= V_\ell(\rho_1') \oplus \{S_1 \cdot V_2(\rho_1')\}[\rho_2] \\ &= \{S_1 \cdot V_{\ell-1}(\rho_2)\}[\rho_1'] \oplus V_3(\rho_2) \simeq \rho_1' \oplus \rho_2. \end{aligned}$$

Case $\mathcal{I}(W) \in E(\rho_k) \setminus P(\rho_{k\pm1}, \rho_k)$ **for** $2 \le k \le n-3$ We consider now $W \in C(\rho_k) = \mathbb{P}(\rho_k \subset V_{k+1}(\rho_k) \oplus V_{\ell-k+1}(\rho_k))$ with $W \ne V_{k+1}(\rho_k), V_{\ell-k+1}(\rho_k)$. Let $\mathcal{I}(W) = W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$.

Hence we may assume that $x^{k+1}y - ty^{\ell-k+1} \in W$ for a nonzero constant t. Since $x^{k+3}y = x^2(x^{k+1}y - ty^{\ell-k+1}) + tx^2y^{\ell-k+1}$, and $x^2y^2 \in \mathfrak{n}$, $\mathcal{I}(W)$ contains $x^{k+3}y$. Similarly, $ty^{\ell-k+2} = -y(x^{k+1}y - ty^{\ell-k+1}) + x^{k+1}y^2$ gives $y^{\ell-k+2} \in \mathcal{I}(W)$. Hence we see that $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{\ell-i+1}(\rho_i)$ for $2 \le i \le k-1$, $V_{i+1}(\rho_i)$ for $k+2 \le i \le n-3$, $V_{\ell-i+3}(\rho_i)$ for $2 \le i \le n-3$, $V_{n-1}'(\rho_{n-2})$, $V_{n-1}''(\rho_{n-2})$, $V_\ell(\rho_1')$, $V_n(\rho_{n-1}')$ and $V_n(\rho_n')$. Since $xy^{\ell-k+1} \in V_{\ell-k+2}(\rho_{k+1})$, we have $V_{\ell-k+3}(\rho_k) \subset \mathcal{I}(W)/\mathfrak{n}$ and $x^{k+2}y = x(x^{k+1}y - ty^{\ell-k+1}) + txy^{\ell-k+1} \in \mathcal{I}(W)/\mathfrak{n}$. Hence $V_{k+2}(\rho_{k+1}) \subset \mathcal{I}(W)/\mathfrak{n}$ if $k \le n-4$. It follows that

$$\begin{aligned} \mathcal{I}(W)/\mathfrak{n} &= W + \sum\nolimits_{m=1}^{\ell-k} S_m \bar{V}_{k+1}(\rho_k) + \sum\nolimits_{m=0}^{k-1} S_m \bar{V}_{\ell-k+2}(\rho_{k-1}) \\ &= W + \sum\nolimits_{m=1}^{\ell-2k} S_m \bar{V}_{k+1}(\rho_k) + \sum\nolimits_{m=\ell-k+2}^{\ell+1} \overline{S}_m. \end{aligned}$$

It follows from $W \simeq \rho_k$ that $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho\in\operatorname{Irr} G} \deg(\rho)\rho$. Therefore $\mathcal{I}(W) \in \operatorname{Hilb}^G(\mathbb{A}^2)$. It is easy to see that $V(\mathcal{I}(W)) \simeq W \simeq \rho_k$ so that $\mathcal{I}(W) \in E(\rho_k)$.

Case $\mathcal{I}(W) \in P(\rho_k, \rho_{k+1})$ Let $W = W(\rho_k, \rho_{k+1}) := V_{\ell-k+1}(\rho_k) \oplus V_{k+2}(\rho_{k+1})$ for $2 \le k \le n-4$. For $k = n-3$, set

$$W = W(\rho_{n-3}, \rho_{n-2}) := V_n(\rho_{n-3}) \oplus V_{n-1}''(\rho_{n-2}).$$

Now $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{\ell-i+1}(\rho_i)$ for $2 \le i \le k$, $V_{i+1}(\rho_i)$ for $k+1 \le i \le n-3$, $V_{\ell-i+3}(\rho_i)$ for $2 \le i \le n-2$, $V_{n-1}''(\rho_{n-2})$ and $V_n(\rho_i')$ for $i = n-1, n$. Similarly

$V_\ell(\rho_1') \subset \mathcal{I}(W)/\mathfrak{n}$. Hence $\mathcal{I}(W) \in P(\rho_k, \rho_{k+1}) \subset \mathrm{Hilb}^G(\mathbb{A}^2)$. We also see that

$$V(\mathcal{I}(W)) = \begin{cases} V_{\ell-k+1}(\rho_k) \oplus \{S_1 \cdot V_{k+1}(\rho_k)\}\,[\rho_{k+1}] & \text{for } 2 \le k \le n-4, \\ V_n(\rho_{n-3}) \oplus \{S_1 \cdot V_{n-2}(\rho_{n-3})\}\,[\rho_{n-2}] & \text{for } k = n-3 \end{cases}$$
$$= \begin{cases} \{S_1 \cdot V_{\ell-k}(\rho_{k+1})\}\,[\rho_k] \oplus V_{k+1}(\rho_k) \simeq \rho_k \oplus \rho_{k+1}, \\ \{S_1 \cdot V'_{n-1}(\rho_{n-2})\}\,[\rho_{n-3}] \oplus V''_{n-1}(\rho_{n-2}) \simeq \rho_{n-3} \oplus \rho_{n-2}. \end{cases}$$

Case $\mathcal{I}(W) \in E(\rho_{n-2}) \setminus \big(P(\rho_{n-2}, \rho_{n-3}) \cup P(\rho_{n-2}, \rho'_{n-1}) \cup P(\rho_{n-2}, \rho'_n)\big)$ Let $W \in C(\rho_{n-2}) = \mathbb{P}(V_{n-1}(\rho_{n-2}))$, and define $\mathcal{I}(W) := W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$. Set

$$W_0 = S_1 \cdot V_{n-2}(\rho'_{n-1}), \quad W_\infty = S_1 \cdot V_{n-2}(\rho'_n) \quad \text{and} \quad W_1 = V''_{n-1}(\rho_{n-2}).$$

Let $H = x^{n-2} - i^{n/2}y^{n-2}$ and $G = x^{n-2} + i^{n/2}y^{n-2}$. Then $W = \{xH - txG, yH + tyG\}$ for some t. Assume $t \neq 0, 1, \infty$, or equivalently, $W \neq W_\lambda$ for $\lambda = 0, 1, \infty$. Then $x^n \in \mathcal{I}(W)/\mathfrak{n}$, so that $V_\ell(\rho_1')$, $V_{\ell-i+1}(\rho_i)$ for $2 \le i \le n-3$ and $V_{\ell-i+3}(\rho_i)$ for $2 \le i \le n-2$ are contained in $\mathcal{I}(W)/\mathfrak{n}$. We also see that $xyH \in V_n(\rho'_{n-1}) \subset \mathcal{I}(W)/\mathfrak{n}$ and $xyG \in V_n(\rho'_n) \subset \mathcal{I}(W)/\mathfrak{n}$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{m=n}^{\ell+1} \overline{S}_m.$$

Since $W \simeq \rho_{n-2}$, we have $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}\, G} \deg(\rho)\rho$ with $V(\mathcal{I}(W)) \simeq W$. It follows that $\mathcal{I}(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$.

Case $\mathcal{I}(W) \in E(\rho'_{n-1}) \setminus P(\rho_{n-2}, \rho'_{n-1})$ Let $W \in C(\rho'_{n-1}) := \mathbb{P}(V_{n-2}(\rho'_{n-1}) \oplus V_n(\rho'_{n-1}))$. Assume $W \neq V_n(\rho'_{n-1})$. Then $\mathcal{I}(W)/\mathfrak{n}$ contains $x^n y$ and hence x^n. It follows that $\mathcal{I}(W)/\mathfrak{n}$ contains $V_{\ell-i+1}(\rho_i)$, $V_{\ell-i+3}(\rho_i)$ for $2 \le i \le n-3$, and $V_{n+1}(\rho_{n-2})$. We also see that $\mathcal{I}(W)/\mathfrak{n}$ contains $x^{n-1} - i^{n/2}xy^{n-2}$ so that $\{\mathcal{I}(W)/\mathfrak{n}\} \cap V_{n-1}(\rho_{n-2}) \simeq \rho_{n-2}$. Similarly we see easily that $V_\ell(\rho_1'), V_n(\rho_n') \subset \mathcal{I}(W)/\mathfrak{n}$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{m=1}^{2} S_m \bar{V}_{n-2}(\rho'_{n-1}) + \sum_{m=n+1}^{\ell+1} \overline{S}_m.$$

Since $W \simeq \rho'_{n-1}$, $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}\, G} \deg(\rho)\rho$. Therefore $\mathcal{I}(W) \in E(\rho'_{n-1}) \subset \mathrm{Hilb}^G(\mathbb{A}^2)$ with $V(\mathcal{I}(W)) \simeq W$.

Case $\mathcal{I}(W) \in P(\rho_{n-2}, \rho'_{n-1})$ We consider

$$W = W(\rho_{n-2}, \rho'_{n-1}) := S_1 \cdot V_{n-2}(\rho'_{n-1})[\rho_{n-2}] \oplus V_n(\rho'_{n-1}) = W_0 \oplus V_n(\rho'_{n-1}).$$

Then $\mathcal{I}(W)/\mathfrak{n}$ contains x^n, therefore $\mathcal{I}(W)/\mathfrak{n}$ contains $V_\ell(\rho_1')$, $V_{\ell-i+1}(\rho_i)$, $V_{\ell-i+3}(\rho_i)$ for $2 \le i \le n-3$, $V_{n+1}(\rho_{n-2})$ and $V_n(\rho_n')$. Since $W \subset \mathcal{I}(W)/\mathfrak{n}$, we see that $\mathcal{I}(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}\, G} \deg(\rho)\rho$. Hence $\mathcal{I}(W) \in P(\rho_{n-2}, \rho'_{n-1}) \subset \mathrm{Hilb}^G(\mathbb{A}^2)$ with $V(\mathcal{I}(W)) \simeq W$.

Case $\mathcal{I}(W) \in E(\rho'_n) \setminus P(\rho_{n-2}, \rho'_n)$ **or** $\mathcal{I}(W) \in P(\rho_{n-2}, \rho'_n)$ This is similar to the above, and we omit the details. $\square$

Lemma 13.8 *For ρ' adjacent to ρ, the limit of $\mathcal{I}(W)$ as $\mathcal{I}(W) \in E(\rho)$ approaches $P(\rho, \rho')$ is $\mathcal{I}(W(\rho, \rho'))$.*

Proof We first consider $W \in C(\rho'_1)$ with $W \neq V_\ell(\rho'_1)$. Then by 13.7 we see that $\mathcal{I}(W) = W + V_3(\rho_2) + \sum_{m\geq 1} S_m V_3(\rho_2)$. Hence we have

$$\begin{aligned}\lim_{\substack{W\to V_\ell(\rho'_1)\\ W\in C(\rho'_1)}} \mathcal{I}(W) &= V_\ell(\rho'_1) + V_3(\rho_2) + \sum_{m\geq 1} S_m V_3(\rho_2)\\ &= \mathcal{I}(V_\ell(\rho'_1) \oplus V_3(\rho_2)) = \mathcal{I}(W(\rho'_1, \rho_2)).\end{aligned}$$

Next we consider $W \in C(\rho_2)$ with $W \neq V_3(\rho_2), V_{\ell-1}(\rho_2)$. Then by 13.7 we have $\mathcal{I}(W) = W + V_\ell(\rho'_1) + \sum_{m\geq 0} S_m V_4(\rho_3)$. Since $V_4(\rho_3) \subset S_1 V_3(\rho_2)$, we have

$$\begin{aligned}\lim_{\substack{W\to V_3(\rho_2)\\ W\in C(\rho_2)}} \mathcal{I}(W) &= V_\ell(\rho'_1) + V_3(\rho_2) + \sum_{m\geq 1} S_m V_3(\rho_2)\\ &= \mathcal{I}(W(\rho'_1, \rho_2)) = \lim_{\substack{W\to V_\ell(\rho'_1)\\ W\in C(\rho'_1)}} \mathcal{I}(W).\end{aligned}$$

Suppose that $W \in C(\rho_k) = \mathbb{P}(V_{\ell-k+1}(\rho_k) \oplus V_{k+1}(\rho_k))$ with $W \neq V_{k+1}(\rho_k)$, $V_{\ell-k+1}(\rho_k)$. By 13.7 we see

$$\mathcal{I}(W) = W + \sum_{m\geq 0} S_m V_{k+2}(\rho_{k+1}) + \sum_{m\geq 0} S_m V_{\ell-k+2}(\rho_{k-1}).$$

Thus for $2 \leq k \leq n-4$ we see that

$$\lim_{W\to V_{\ell-k+1}(\rho_k)} \mathcal{I}(W) = \mathcal{I}(W(\rho_k, \rho_{k+1})) = \lim_{W\to V_{k+2}(\rho_{k+1})} \mathcal{I}(W).$$

Similarly for $W \in C(\rho_{n-2})$ with $W \neq W_\lambda$ for $\lambda = 0, 1, \infty$ we have

$$\begin{aligned}\mathcal{I}(W) &= W + \sum_{m\geq 0} S_m V_n(\rho_{n-3}) + \sum_{\substack{m\geq 0\\ j=n-1,n}} S_m V_n(\rho'_j) = W + \sum_{m\geq n} S_m,\\ \lim_{W\to W_1} \mathcal{I}(W) &= \sum_{m\geq 0} S_m V_n(\rho_{n-3}) + \sum_{m\geq 0} S_m W_1 = \mathcal{I}(W_1 \oplus V_n(\rho_{n-3})),\end{aligned}$$

because $V_n(\rho_{n-3}) \subset S_1 W_0 + \mathfrak{n}$. Consequently

$$\begin{aligned}\lim_{W'\to V_n(\rho_{n-3})} \mathcal{I}(W') &= V_n(\rho_{n-3}) + \sum_{m\geq 0} S_m V_{n+1}(\rho_{n-4}) + \sum_{m\geq 0} S_m V''_{n-1}(\rho_{n-2})\\ &= \sum_{m\geq 0} S_m V_n(\rho_{n-3}) + \sum_{m\geq 0} S_m W_1 = \lim_{W''\to W_1} \mathcal{I}(W''),\end{aligned}$$

where $W' \in C(\rho_{n-3})$, $W'' \in C(\rho_{n-2})$. The limit when W approaches W_0 or W_∞ is similar. □

To complete the proofs of Theorem 13.6, we also need to prove:

Lemma 13.9 *$E(\rho)$ and $E(\rho')$ intersects at $P(\rho,\rho')$ transversally if ρ and ρ' are adjacent.*

Proof By the proof of Theorem 9.3, $X_G = \mathrm{Hilb}^G(\mathbb{A}^2)$ is smooth, with tangent space $T_{[I]}(X_G)$ at $[I]$ the G-invariant subspace $\mathrm{Hom}_{\mathcal{O}_{\mathbb{A}^2}}(I, \mathcal{O}_{\mathbb{A}^2}/I)^G$ of $T_{[I]}(\mathrm{Hilb}^n(\mathbb{A}^2))$, which is isomorphic to $\mathrm{Hom}_{\mathcal{O}_{\mathbb{A}^2}}(I, \mathcal{O}_{\mathbb{A}^2}/I)$, where $n = |G|$. Assume that ρ and ρ' are adjacent with $d(\rho') = d(\rho) + 1$. Let $W(\rho,\rho') = V_{\frac{h}{2}-d(\rho)}(\rho) \oplus V_{\frac{h}{2}-d(\rho')}(\rho')$. Then $\mathcal{I}(W(\rho,\rho')) \in P(\rho,\rho')$. We prove the following formula

$$
\begin{aligned}
T_{[I]}(X_G) &\simeq \mathrm{Hom}_{\mathcal{O}_{\mathbb{A}^2}}(I, \mathcal{O}_{\mathbb{A}^2}/I)^G \simeq \\
&\mathrm{Hom}_G(V_{\frac{h}{2}-d(\rho)}(\rho), V_{\frac{h}{2}+d(\rho)}(\rho)) \oplus \mathrm{Hom}_G(V_{\frac{h}{2}+d(\rho')}(\rho'), V_{\frac{h}{2}-d(\rho')}(\rho')),
\end{aligned}
$$

where $I = \mathcal{I}(W(\rho,\rho'))$. First assume $\rho = \rho_2$ and $\rho' = \rho'_1$. Then

$$
\begin{aligned}
&\mathrm{Hom}_{\mathcal{O}_{\mathbb{A}^2}}(I, \mathcal{O}_{\mathbb{A}^2}/I)^G \subset \\
&\qquad \mathrm{Hom}_G(V_\ell(\rho'_1), V_2(\rho'_1)) \oplus \mathrm{Hom}_G(V_3(\rho_2), V_1(\rho_2) \oplus V_{\ell-1}(\rho_2))
\end{aligned}
$$

Let φ be any element of $\mathrm{Hom}_{\mathcal{O}_{\mathbb{A}^2}}(I, \mathcal{O}_{\mathbb{A}^2}/I)^G$. A nontrivial G-isomorphism φ_0 of $V_3(\rho_2)$ onto $V_1(\rho_2)$ is given by $\varphi_0(x^2y) = x$, $\varphi_0(xy^2) = -y$. Therefore we may assume $\varphi = c\varphi_0 \mod V_{\ell-1}(\rho_2)$ for some constant c. Since φ defines an $\mathcal{O}_{\mathbb{A}^2}$-homomorphism, we have $y\varphi(x^2y) = x\varphi(xy^2)$, so that $2cxy = 0$ in $\mathcal{O}_{\mathbb{A}^2}/I$. It follows that $c = 0$, and $\varphi(V_3(\rho_2)) \subset V_{\ell-1}(\rho_2)$. Thus the formula for $I = \mathcal{I}(W(\rho'_1, \rho_2))$ is proved.

Now we consider the general case. By 13.7 we see that $\{\mathfrak{m}/I\}[\rho]$ contains $V_{\bar{h}+d(\rho)}(\rho)$ as a nontrivial factor, while $\{\mathfrak{m}/I\}[\rho']$ contains $V_{\bar{h}-d(\rho')}(\rho')$ similarly. Moreover by the proof in 13.7 we see that either of the linear subspaces $\mathrm{Hom}_G(V_{\frac{h}{2}-d(\rho)}(\rho), V_{\frac{h}{2}+d(\rho)}(\rho))$ and $\mathrm{Hom}_G(V_{\frac{h}{2}+d(\rho')}(\rho'), V_{\frac{h}{2}-d(\rho')}(\rho'))$ yield nontrivial deformations of the ideal I inside the exceptional set E. Since $\dim T_{[I]}(X_G) = 2$ by Theorem 9.3, these linear subspaces span $T_{[I]}(X_G)$. Hence we have

$$
\begin{aligned}
T_{[I]}(X_G) &\simeq \\
&\mathrm{Hom}_G(V_{\frac{h}{2}-d(\rho)}(\rho), V_{\frac{h}{2}+d(\rho)}(\rho)) \oplus \mathrm{Hom}_G(V_{\frac{h}{2}+d(\rho')}(\rho'), V_{\frac{h}{2}-d(\rho')}(\rho')),
\end{aligned}
$$

with

$$
\begin{aligned}
T_{[I]}(E(\rho)) &\simeq \mathrm{Hom}_G(V_{\frac{h}{2}-d(\rho)}(\rho), V_{\frac{h}{2}+d(\rho)}(\rho)), \\
T_{[I]}(E(\rho')) &\simeq \mathrm{Hom}_G(V_{\frac{h}{2}+d(\rho')}(\rho'), V_{\frac{h}{2}-d(\rho')}(\rho')).
\end{aligned}
$$

This completes the proof of Lemma 13.9 for $\rho, \rho' \neq \rho_{n-2}$. The cases $\rho = \rho_{n-2}$ are proved similarly. $\square$

Lemma 13.10 *Let $E^*(\rho)$ be the closure in E of the set*

$$\{\mathcal{I}(W); W \in C(\rho), W \neq V_{\frac{h}{2}\pm d(\rho)}\}.$$

Then $E^*(\rho)$ is a smooth rational curve.

Proof By Lemma 13.9, $E^*(\rho)$ is smooth at $\mathcal{I}(W(\rho,\rho'))$ for ρ' adjacent to ρ. It remains to prove the assertion elsewhere on $E^*(\rho)$.

Let $C^0(\rho) := \{W \in C(\rho); W \neq V_{\frac{h}{2}\pm d(\rho)}\}$ and $I := \mathcal{I}(W)$ for $W \in C^0(\rho)$. Since we have a flat family of ideals $\mathcal{I}(W)$ for $W \in C^0(\rho)$, we have a natural morphism $\iota\colon C^0(\rho) \to \mathrm{Hilb}^G(\mathbb{A}^2)$, and a natural homomorphism $(d\iota)_*\colon T_{[W]}(C(\rho)) \to T_{[I]}(\mathrm{Hilb}^G(\mathbb{A}^2))$. Equivalently there is a homomorphism

$$(d\iota)_*\colon \mathrm{Hom}(W, V_{\frac{h}{2}-d(\rho)}(\rho) + V_{\frac{h}{2}+d(\rho)}(\rho)/W) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{A}^2}}(I, \mathcal{O}_{\mathbb{A}^2}/I)^G$$

Let $\varphi \in T_{[W]}(C(\rho))$. Then $(d\iota)_*(\varphi)(I) \subset \mathfrak{m}/I$ because $C(\rho) \subset E$. Recall that $\{\mathfrak{m}/I\}[\rho_0] = 0$ by Corollary 9.6. Hence $(d\iota)_*(\varphi)(\mathfrak{n}) = 0$. Since $I/\mathfrak{n}$ is generated by W by 13.7, $(d\iota)_*(\varphi)$ is induced from φ by extending it to $\bigoplus S_k W$ as an $\mathcal{O}_{\mathbb{A}^2}$-homomorphism. Note that we have

$$V_{\frac{h}{2}-d(\rho)}(\rho) + V_{\frac{h}{2}+d(\rho)}(\rho)/W \subset \mathfrak{m}/I.$$

It follows that $(d\iota)_*$ is injective and that $C^0(\rho)$ is immersed at $\mathcal{I}(W)$. The same argument applies as well when $W = V_{\frac{h}{2}+d(\rho)}$ if there is no adjacent ρ' with $d(\rho') > d(\rho)$. Hence $E^*(\rho)$ is a smooth rational curve. $\square$

We will see $E(\rho) = E^*(\rho)$ soon in 13.11.

13.11 Proof of Theorem 13.6 – Conclusion

Let E be the exceptional set of π, and E^* the union of all $E^*(\rho)$ for $\rho \in \mathrm{Irr}\, G$. Since $E^*(\rho) \subset E(\rho)$ by 13.7, E^* is a subset of E. Since π is a birational morphism, E is connected and it is set theoretically the total fiber $\pi^{-1}(0)$ over the singular point $0 \in S_G$. Hence in particular $P(\rho,\rho') \subset E$ for any ρ, ρ'. By Lemma 13.9, the dual graph of E^* is the same as the Dynkin diagram $\Gamma(\mathrm{Irr}\, G)$ of $\mathrm{Irr}\, G$. Hence E^* is connected because $\Gamma(\mathrm{Irr}\, G)$ is connected. By Lemma 13.10 E^* is smooth except at $\mathcal{I}(W(\rho,\rho'))$, while E^* has two smooth irreducible components $E^*(\rho)$ and $E^*(\rho')$ meeting transversally at $\mathcal{I}(W(\rho,\rho'))$ by Lemma 13.9. It follows that E^* is a connected component of E. Hence $E^* = E$. It follows that $E(\rho) = E^*(\rho)$ for all $\rho \in \mathrm{Irr}\, G$, $P(\rho,\rho') = \{\mathcal{I}(W(\rho,\rho'))\}$ for ρ, ρ' adjacent, and $P(\rho,\rho') = \emptyset$ otherwise. Similarly $Q(\rho,\rho',\rho'') = \emptyset$. Thus Theorem 13.6 is proved.

13.12 Conclusion

The proof of Theorem 13.6 proves also Theorems 10.4 and 10.7 automatically. Theorems 10.5–10.6 are clear from Tables 7–8. Since any subscheme in $\mathrm{Hilb}^G(\mathbb{A}^2)$ with support outside the exceptional set E is a G-orbit of $|G|$ distinct points in $\mathbb{A}^2 \setminus \{0\}$, the defining ideal I of it is given by using G-invariant functions as follows

$$I = \big(F(x,y) - F(a,b), G(x,y) - G(a,b), H(x,y) - H(a,b)\big),$$

where $F(x,y) = x^\ell + y^\ell$, $G(x,y) = xy(x^\ell - y^\ell)$, $H(x,y) = x^2y^2$ and $(a,b) \neq (0,0)$. Thus we obtain a complete description of the ideals in $\mathrm{Hilb}^G(\mathbb{A}^2)$.

14 The binary tetrahedral group E_6

14.1 Character table

The binary tetrahedral group $G = \mathbb{T}$ is defined as the subgroup of $\mathrm{SL}(2,\mathbb{C})$ of order 24 generated by $\mathbb{D}_2 = \langle\sigma,\tau\rangle$ and μ:

$$\sigma = \begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, \quad \mu = \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon^7, & \varepsilon^7 \\ \varepsilon^5, & \varepsilon \end{pmatrix},$$

where $\varepsilon = e^{2\pi i/8}$ [Slodowy80], p. 74. G acts on $\mathbb{A}^2$ from the right by $(x,y) \mapsto$

ρ	1	2	3	4	5	6	7	d	$(\frac{h}{2} \pm d)$
	1	-1	τ	μ	μ^2	μ^4	μ^5		
$(\sharp)$	1	1	6	4	4	4	4		
ρ_0	1	1	1	1	1	1	1	(2)	–
ρ_2	2	-2	0	1	-1	-1	1	1	$(5,7)$
ρ_3	3	3	-1	0	0	0	0	0	$(6,6)$
ρ_2'	2	-2	0	ω^2	$-\omega$	$-\omega^2$	ω	1	$(5,7)$
ρ_1'	1	1	1	ω^2	ω	ω^2	ω	2	$(4,8)$
ρ_2''	2	-2	0	ω	$-\omega^2$	$-\omega$	ω^2	1	$(5,7)$
ρ_1''	1	1	1	ω	ω^2	ω	ω^2	2	$(4,8)$

Table 9: Character table of E_6

$(x,y)g$ for $g \in G$. $\mathbb{D}_2$ is a normal subgroup of G and the following is exact:

$$1 \to \mathbb{D}_2 \to G \to \mathbb{Z}/3\mathbb{Z} \to 1.$$

See Table 9 for the character table of G [Schur07] and the other relevant invariants. The Coxeter number h of E_6 is equal to 12. Let $\omega = (-1+\sqrt{3}i)/2$.

14.2 Symmetric tensors modulo $\mathfrak{n}$

Let S_m be the space of homogeneous polynomials in x and y of degree m. The G-modules S_m and $\overline{S}_m := S_m(\mathfrak{m}/\mathfrak{n})$ by ρ_2 decompose into irreducible G-modules. We define a G-submodule of $\mathfrak{m}/\mathfrak{n}$ by $\bar{V}_i(\rho_j) := S_i(\mathfrak{m}/\mathfrak{n})[\rho_j]$ the sum of all copies of ρ in $S_i(\mathfrak{m}/\mathfrak{n})$, and define $V_i(\rho_j)$ to be a G-submodule of S_i such that $V_i(\rho_j) \simeq \bar{V}_i(\rho_j)$, $V_i(\rho_j) \equiv \bar{V}_i(\rho_j) \mod \mathfrak{n}$. We use $V_i(\rho_j)$ and $\bar{V}_i(\rho_j)$ interchangeably whenever this is harmless. For a G-module W we define $W[\rho]$ to be the sum of all the copies of ρ in W.

It is known by [Klein], p. 51 that there are G-invariant polynomials A_6, A_8, A_6^2 and A_{12} respectively of homogeneous degrees 6, 8, 12 and 12. In his notation, we may assume that $A_6 = T$, $A_8 = W$ and $A_{12} = \varphi^3$. See 14.3.

The decomposition of S_m and $\overline{S}_m$ for small values of m are given in Table 10. The factors of $\overline{S}_m$ in brackets are those in S_{McKay}. We see by Table 10 that $V_{6\pm d(\rho)}(\rho) \simeq \rho^{\oplus 2}$ if $d(\rho) = 0$, or ρ if $d(\rho) \geq 1$. We also see that $\overline{S}_{6-k} \simeq \overline{S}_{6+k}$ for any k. Thus Theorems 10.5–10.6 for E_6 follows from Table 10 immediately.

14.3 Generators of $V_j(\rho)$

We prepare some notation for Table 11. Let

$$\begin{aligned}
p_1 &= x^2 - y^2, \quad p_2 = x^2 + y^2, \quad p_3 = xy \\
q_1 &= x^3 + (2\omega + 1)xy^2, \quad q_2 = y^3 + (2\omega + 1)x^2y, \\
s_1 &= x^3 + (2\omega^2 + 1)xy^2, \quad s_2 = y^3 + (2\omega^2 + 1)x^2y \\
\gamma_1 &= x^5 - 5xy^4, \quad \gamma_2 = y^5 - 5x^4y, \quad T = p_1p_2p_3, \\
\varphi &= p_2^2 + 4\omega p_3^2, \quad \psi = p_2^2 + 4\omega^2 p_3^2, \quad W = \varphi\psi.
\end{aligned}$$

We note that $\mathfrak{n}$ is generated by T, W and φ^3 (or ψ^3) by [Klein], p. 51.

Computations give Table 11. We note the relations

$$\begin{aligned}
\rho_2' &= \rho_1' \cdot \rho_2 = \rho_1'' \cdot \rho_2'', \qquad & \rho_2'' &= \rho_1' \cdot \rho_2' = \rho_1'' \cdot \rho_2, \\
\rho_2 &= \rho_1' \cdot \rho_2'' = \rho_1'' \cdot \rho_2', \qquad & \rho_3 &= \rho_1' \cdot \rho_3 = \rho_1'' \cdot \rho_3.
\end{aligned}$$

In view of Table 10, each irreducible G factor appears in $\overline{S}_m$ with multiplicity at most one except when $m = 6$, $\rho = \rho_3$. Therefore the following congruence of G-modules modulo $\mathfrak{n}$ are clear from the fact that these G-modules are nontrivial modulo $\mathfrak{n}$.

$$\begin{aligned}
V_3(\rho_2'')\varphi &\equiv V_3(\rho_2')\psi, \qquad & V_4(\rho_3)\varphi &\equiv V_4(\rho_3)\psi, \\
V_1(\rho_2)\varphi^2 &\equiv V_5(\rho_2)\psi, \qquad & V_5(\rho_2)\varphi &\equiv V_1(\rho_2)\psi^2, \\
V_2(\rho_3)\varphi^2 &\equiv V_2(\rho_3)\psi^2, \qquad & V_3(\rho_2')\varphi^2 &\equiv V_3(\rho_2'')\psi^2.
\end{aligned}$$

m	S_m	$\overline{S}_m$
0	ρ_0	0
1	ρ_2	ρ_2
2	ρ_3	ρ_3
3	$\rho_2' + \rho_2''$	$\rho_2' + \rho_2''$
4	$\rho_1' + \rho_1'' + \rho_3$	$(\rho_1' + \rho_1'') + \rho_3$
5	$\rho_2 + \rho_2' + \rho_2''$	$(\rho_2 + \rho_2' + \rho_2'')$
6	$\rho_0 + 2\rho_3$	$(2\rho_3)$
7	$2\rho_2 + \rho_2' + \rho_2''$	$(\rho_2 + \rho_2' + \rho_2'')$
8	$\rho_0 + \rho_1' + \rho_1'' + 2\rho_3$	$(\rho_1' + \rho_1'') + \rho_3$
9	$\rho_2 + 2\rho_2' + 2\rho_2''$	$\rho_2' + \rho_2''$
10	$\rho_1' + \rho_1'' + 3\rho_3$	ρ_3
11	$2\rho_2 + 2\rho_2' + 2\rho_2''$	ρ_2
12	$2\rho_0 + \rho_1' + \rho_1'' + 3\rho_3$	0

Table 10: Irreducible decompositions of $\overline{S}_m(E_6)$

m	ρ	$V_m(\rho)$	m	ρ	$V_m(\rho)$
1	ρ_2	x, y	7	ρ_2	$s_1\varphi, s_2\varphi$
2	ρ_3	x^2, xy, y^2	7	ρ_2'	$s_1\psi, s_2\psi$
3	ρ_2'	q_1, q_2	7	ρ_2''	$q_1\varphi, q_2\varphi$
3	ρ_2''	s_1, s_2	8	ρ_1'	ψ^2
4	ρ_1'	φ	8	ρ_1''	φ^2
4	ρ_1''	ψ	8	ρ_3	$p_1p_2\varphi, p_2p_3\varphi, p_3p_1\varphi$
4	ρ_3	p_1p_2, p_2p_3, p_3p_1	9	ρ_2'	$x\psi^2, y\psi^2$
5	ρ_2	γ_1, γ_2	9	ρ_2''	$x\varphi^2, y\varphi^2$
5	ρ_2'	$x\varphi, y\varphi$	10	ρ_3	$x^2\varphi^2, xy\varphi^2, y^2\varphi^2$
5	ρ_2''	$x\psi, y\psi$	11	ρ_2	$q_1\varphi^2, q_2\varphi^2$
6	ρ_3	$V_2(\rho_3)\varphi \oplus V_2(\rho_3)\psi$			

Table 11: $V_m(\rho)(E_6)$

For instance, $s_i\varphi - q_i\psi \equiv 0 \mod T$, so that $V_3(\rho_2'')\varphi \equiv V_3(\rho_2')\psi$. Since $p_1p_2(\varphi-\psi) \equiv 0 \mod T$, $p_2p_3(\varphi - \omega\psi) \equiv 0 \mod T$ and $p_3p_1(\varphi - \omega^2\psi) \equiv 0 \mod T$ so that $V_4(\rho_3)\varphi \equiv V_4(\rho_3)\psi$.

Lemma 14.4 *1.*

$$S_m\bar{V}_4(\rho_1') = \begin{cases} \rho_2' & \text{for } m=1, \\ \rho_3 & \text{for } m=2, \\ \rho_2+\rho_2'' & \text{for } m=3, \text{ and} \\ \rho_1''+\rho_3 & \text{for } m=4. \end{cases}$$

2. $S_m\bar{V}_4(\rho_1') = \overline{S}_{m+4}$ *for* $m \geq 5$, *and* $S_m\bar{V}_5(\rho_2') = S_{m+1}\bar{V}_4(\rho_1')$ *for* $m \geq 1$.

3. $S_m\bar{V}_5(\rho_2) = \rho_3$ *for* $m = 1$, $\rho_2' + \rho_2''$ *for* $m = 2$, *and* $\overline{S}_{k+5}$ *for* $m \geq 3$.

4. $S_1\bar{V}_7(\rho_2') = \rho_1' + \rho_3$.

Proof (1) is clear for $k = 1, 2$. Next we consider $S_3V_4(\rho_1')$. By Table 10 $S_3V_4(\rho_1') \simeq S_3 \otimes V_4(\rho_1') \simeq \rho_2'' + \rho_2$. We prove $S_1 \cdot A_6 \neq \{S_3V_4(\rho_1')\}[\rho_2] = V_3(\rho_2'')V_4(\rho_1')$. For otherwise, A_6 is divisible by $\varphi \in V_4(\rho_1')$, whence $A_6/\varphi \in V_2(\rho_1'') = \{0\}$, a contradiction. Hence we have $S_3\bar{V}_4(\rho_1') = \rho_2 + \rho_2''$. Similarly $S_4V_4(\rho_1') = \rho_0 + \rho_1'' + \rho_3$ where $\{S_4V_4(\rho_1')\}[\rho_0] = S_0 \cdot A_8$. The factors ρ_1'' and ρ_3 in $S_4V_4(\rho_1')$ are not divisible by A_6. In fact, otherwise $\{S_4V_4(\rho_1')\}[\rho_3] = S_2 \cdot A_6$ because $S_2 \simeq \rho_3$. It follows that A_6 is divisible by φ, which is a contradiction. Therefore $S_4\bar{V}_4(\rho_1') = \rho_1'' + \rho_3$. Finally we see $S_5V_4(\rho_1') = \rho_2 + \rho_2' + \rho_2''$ where $\{S_5V_4(\rho_1')\}[\rho_2] = S_1 \cdot A_8$. The factors ρ_2' and ρ_2'' in $S_5V_4(\rho_1')$ are not divisible by A_6. For instance if $\{S_5V_4(\rho_1')\}[\rho_2'] = V_3(\rho_2') \cdot A_6$, then since the generators of $V_3(\rho_2')$ are coprime, A_6 is divisible by φ, a contradiction. It follows that $S_5\bar{V}_4(\rho_1') = \rho_2' + \rho_2'' = \overline{S}_9$. The rest of (1) is clear. (2) is clear from (1).

Next, we prove that $S_1\bar{V}_5(\rho_2) = \rho_3$. First, Table 11 gives $\dim S_1V_5(\rho_2) = 4$. Thus $S_1V_5(\rho_2) \simeq \rho_2 \otimes \rho_2 \simeq \rho_0 + \rho_3$. Hence $\{S_1V_5(\rho_2)\}[\rho_0] = S_0 \cdot A_6$. It follows that $S_1\bar{V}_5(\rho_2) = \rho_3$. Now consider $S_2V_5(\rho_2)$. Since $\dim S_1 \otimes V_5(\rho_2) = 4$, we have $\dim S_2 \otimes V_5(\rho_2) \geq 5$. We see that $S_2V_5(\rho_2) = S_2 \otimes V_5(\rho_2) = \rho_2 + \rho_2' + \rho_2''$, and that $\rho_2 \simeq S_1 \cdot A_6 \subset S_2V_5(\rho_2)$, $V_3(\rho_2'')V_4(\rho_1'') = V_7(\rho_2') \simeq \rho_2'$ and $V_3(\rho_2')V_4(\rho_1') = V_7(\rho_2'') \simeq \rho_2''$. Hence $S_2\bar{V}_5(\rho_2) = \rho_2' + \rho_2''$.

On the other hand, $S_1V_3(\rho_2'') = S_1 \otimes V_3(\rho_2'') = \rho_1'' + \rho_3$, so that $S_1V_7(\rho_2') = S_1V_3(\rho_2'')V_4(\rho_1'') = \rho_1' + \rho_3$. We prove that $S_1\bar{V}_7(\rho_2') = \rho_1' + \rho_3$. For otherwise, by Table 10, we have $\{S_1\bar{V}_7(\rho_2')\}[\rho_3] = 0$ so that $\{S_1V_7(\rho_2')\}[\rho_3] = S_2A_6$. $V_7(\rho_2')$ is divisible by ψ, so that A_6 is divisible by ψ. Hence $A_6/\psi \in V_2(\rho_1')$, which contradicts $S_2 = \rho_3$. Therefore $\{S_1V_7(\rho_2')\}[\rho_3] = \rho_3$ and $S_1\bar{V}_7(\rho_2') = \rho_1' + \rho_3$. Similarly $S_1\bar{V}_7(\rho_2'') = \rho_1'' + \rho_3$. This proves (4). Moreover $S_3\bar{V}_5(\rho_2) = S_1S_2\bar{V}_5(\rho_2) = S_1(\bar{V}_7(\rho_2') + \bar{V}_7(\rho_2''))$ so that $S_3\bar{V}_5(\rho_2) \supset \rho_1' + \rho_1'' + \rho_3 = \overline{S}_8$. This proves (3). $\square$

Lemma 14.5 *Let $W_k = S_1 \cdot \bar{V}_5(\rho_2^{(k)})$ $(\simeq \rho_3)$ for any $k = 0, 1, 2$, where $\rho_2^{(k)} = \rho_2, \rho_2', \rho_2''$. Let $W \in \mathbb{P}(V_6(\rho_3))$. Then $S_1 W = \rho_2 + \rho_2' + \rho_2''$ if and only if $W \neq W_k$ for $k = 1, 2, 3$.*

Proof We see $S_1 \cdot W_1 = S_2 \cdot \bar{V}_5(\rho_2') = S_3 \cdot \bar{V}_4(\rho_1') = \rho_2 + \rho_2''$ by Lemma 14.4. Similarly $S_1 \cdot W_2 = S_3 \cdot \bar{V}_4(\rho_1'') = \rho_2 + \rho_2'$. Also by Lemma 14.4, (3) we have $S_1 \cdot W_0 = \rho_2' + \rho_2''$.

Conversely assume $W \neq W_k$ for any k. Choose and fix a G-module isomorphism $h\colon W_1 \to W_2$. For instance, $h(p_k\varphi) = \omega^{-k} p_k \psi$. Then h induces a natural isomorphism $\{S_1 \otimes h\}[\rho_2]\colon \{S_1 \otimes W_1\}[\rho_2] \to \{S_1 \otimes W_2\}[\rho_2]$, which induces an isomorphism $\{S_1 \cdot h\}[\rho_2]\colon \{S_1 \cdot W_1\}[\rho_2] \to \{S_1 \cdot W_2\}[\rho_2]$. Since $\overline{S}_7$ contains a single ρ_2, we have $\{S_1 \cdot W_1\}[\rho_2] \simeq \{S_1 \cdot W_2\}[\rho_2]$ $(\simeq \rho_2)$ by $\{S_1 \cdot h\}[\rho_2]$. It follows that $\{S_1 \cdot h\}[\rho_2]$ is a nonzero constant multiple of the identity. Since $V_6(\rho_3) = W_1 \oplus W_2$, this proves uniqueness of the G-submodule $W \simeq \rho_3$ of $V_6(\rho_3)$ such that $\{S_1 \cdot W\}[\rho_2] = 0$. Since $\{S_1 \cdot W_0\}[\rho_2] = 0$, we have $\{S_1 \cdot W\}[\rho_2] \neq 0$ by the assumption $W \neq W_0$. Similarly there exists a unique proper G-submodule $W \in V_6(\rho_3)$ such that $\{S_1 \cdot W\}[\rho_2'] = 0$ or $\{S_1 \cdot W\}[\rho_2''] = 0$. As we saw above, $\{S_1 \cdot W_1\}[\rho_2'] = 0$ and $\{S_1 \cdot W_2\}[\rho_2''] = 0$. Therefore $S_1 \cdot W = \rho_2 + \rho_2' + \rho_2''$ if $W \neq W_k$ for $k = 0, 1, 2$. $\square$

14.6 Proof of Theorem 10.7 in the E_6 case

Consider $I \in X_G$ in the exceptional set E, or equivalently, $I \in X_G$ with $I \subset \mathfrak{m}$. For a finite submodule W of $\mathfrak{m}$ we define $\mathcal{I}(W) = W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$ and $V(\mathcal{I}(W)) := \mathcal{I}(W)/\mathfrak{m}\mathcal{I}(W) + \mathfrak{n}$. We write $\equiv$ for congruence modulo $\mathfrak{n}$.

Case $\mathcal{I}(W) \in E^0(\rho_1')$ Let $W \in \mathbb{P}(V_4(\rho_1') \oplus V_8(\rho_1'))$, so that $W \simeq \rho_1'$. Suppose that $W \neq V_8(\rho_1')$ and set $\mathcal{I}(W) = W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$. Since $\overline{S}_{12} = 0$, by Lemma 14.4 we have $S_k \cdot W \equiv S_k \cdot \bar{V}_4(\rho_1')$ for $k \geq 4$. Also by Lemma 14.4 $S_k \cdot \bar{V}_4(\rho_1') = \overline{S}_{k+4}$ for $k \geq 5$. Hence $\overline{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 9$. Since $S_k \cdot W = S_k \cdot \bar{V}_4(\rho_1') \mod \overline{S}_9$ for $k \geq 1$, we deduce that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k \geq 1} S_k \cdot \bar{V}_4(\rho_1') = W + \sum_{k=1}^{4} S_k \cdot \bar{V}_4(\rho_1') + \sum_{k=9}^{11} \overline{S}_k.$$

We see by Lemma 14.4

$$W + S_4\bar{V}_4(\rho_1') = \rho_1' + \rho_1'' + \rho_3 = \frac{1}{2}(\overline{S}_4 + \overline{S}_8),$$

$$S_1\bar{V}_4(\rho_1') + S_3\bar{V}_4(\rho_1') = \rho_2 + \rho_2' + \rho_2'' = \frac{1}{2}(\overline{S}_5 + \overline{S}_7),$$

$$S_2\bar{V}_4(\rho_1') = \rho_3 = \frac{1}{2}\overline{S}_6.$$

By duality $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}\,G}\deg(\rho)\rho$. Thus $\mathcal{I}(W) \in X_G$ and $V(\mathcal{I}(W)) \simeq W$.

Case $\mathcal{I}(W) \in E^0(\rho_2')$ Let $W \in \mathbb{P}(V_5(\rho_2') \oplus V_7(\rho_2'))$ with $W \simeq \rho_2'$. Suppose $W \neq V_5(\rho_2'), V_7(\rho_2')$. Since $\overline{S}_{12} = 0$, we have $S_k \cdot W \equiv S_k \cdot \bar{V}_5(\rho_2') = \overline{S}_{k+5}$ for $k \geq 5$ by the condition $W \neq V_7((\rho_2')$. We also see that $S_4 \cdot W = S_4 \cdot \bar{V}_5(\rho_2')$ mod $\overline{S}_{11} = \overline{S}_9$. Therefore $\overline{S}_9 \subset \mathcal{I}(W)/\mathfrak{n}$. Hence $S_k \cdot W = S_k \cdot \bar{V}_5(\rho_2')$ mod $\overline{S}_9$ for $k \geq 2$. Since $S_1 \cdot \bar{V}_5(\rho_2') = \rho_3$ and $S_1 \cdot \bar{V}_7(\rho_2') = \rho_1' + \rho_3$, we have $S_1 \cdot W \equiv \rho_1' + \rho_3$ and $\{S_1 \cdot W\}[\rho_1'] \equiv \bar{V}_8(\rho_1') \subset \mathcal{I}(W)/\mathfrak{n}$ by the assumption $W \neq V_5(\rho_2')$. Since $S_3\bar{V}_5(\rho_2') = \rho_1'' + \rho_3$, we have $\overline{S}_8 = \bar{V}_8(\rho_1') \oplus S_3\bar{V}_5(\rho_2') \subset \mathcal{I}(W)/\mathfrak{n}$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k\geq 1} S_k \cdot \bar{V}_5(\rho_2') = W + \sum_{k=1}^{2} S_k \cdot \bar{V}_5(\rho_2') + \sum_{k=8}^{11} \overline{S}_k \quad \text{and}$$

$$W + S_1\bar{V}_5(\rho_2') + S_2\bar{V}_5(\rho_2') = \rho_2 + \rho_2' + \rho_2'' + \rho_3 = \frac{1}{2}(\overline{S}_5 + \overline{S}_6 + \overline{S}_7).$$

Hence $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}\,G}\deg(\rho)\rho$. Thus $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

Case $\mathcal{I}(W) \in E^0(\rho_1'')$ **or** $\mathcal{I}(W) \in E^0(\rho_2'')$ These cases are similar.

Case $\mathcal{I}(W) \in E^0(\rho_2)$ Let $W \in \mathbb{P}(V_5(\rho_2)\oplus V_7(\rho_2))$, so that $W \simeq \rho_2$. Suppose that $W \neq V_7(\rho_2)$. As above, we see that $\overline{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 10$. It follows that $S_3 \cdot W = S_3 \cdot \bar{V}_5(\rho_2)$ mod $\overline{S}_{10} = \overline{S}_8$. Therefore $\overline{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 8$. Similarly $S_2 \cdot W \equiv S_2 \cdot \bar{V}_5(\rho_2) = \rho_2' + \rho_2''$ mod $\overline{S}_8$ and $S_1 \cdot W \equiv S_1 \cdot \bar{V}_5(\rho_2) = \rho_3$ mod $\overline{S}_8$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k\geq 1} S_k \cdot \bar{V}_5(\rho_2) = W + \sum_{k=1}^{2} S_k \cdot \bar{V}_5(\rho_2) + \sum_{k=8}^{11} \overline{S}_k, \quad \text{and}$$

$$W + S_1\bar{V}_5(\rho_2) + S_2\bar{V}_5(\rho_2) = \rho_2 + \rho_2' + \rho_2'' + \rho_3 = \frac{1}{2}(\overline{S}_5 + \overline{S}_6 + \overline{S}_7).$$

Hence $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}\,G}\deg(\rho)\rho$. Thus $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

Case $\mathcal{I}(W) \in E^0(\rho_3)$ Let $W \in \mathbb{P}(V_6(\rho_3))$. Let $W_k = S_1 \cdot V_5(\rho_2^{(k)})$ for any $k = 0, 1, 2$ where $\rho_2^{(k)} = \rho_2, \rho_2', \rho_2''$. Now we suppose that $W \neq W_k$. Then $S_1 \cdot W \equiv \overline{S}_7$ by Lemma 14.5 so that $\mathcal{I}(W)$ contains $\overline{S}_k$ for any $k \geq 7$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k\geq 1} S_k W = W + \sum_{k=7}^{11} \overline{S}_k.$$

Hence $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}\,G}\deg(\rho)\rho$, and so $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

Case $\mathcal{I}(W) \in P(\rho_1', \rho_2')$ Let $W = W(\rho_1', \rho_2') := V_8(\rho_1') \oplus V_5(\rho_2')$. Recall that $W = \{S_1 \cdot V_7(\rho_2')\}[\rho_1'] \oplus V_5(\rho_2') = V_8(\rho_1') \oplus S_1 \cdot V_4(\rho_1')$. By Lemma 14.4, we see that $S_1 \cdot \bar{V}_5(\rho_2') = \rho_3$, $S_2 \cdot \bar{V}_5(\rho_2') = \rho_2 + \rho_2''$, $S_3 \cdot \bar{V}_5(\rho_2') = \rho_1'' + \rho_3$ and $\overline{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 8$. It follows that $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho \in \operatorname{Irr} G} \deg(\rho)\rho$ by Table 10. Therefore $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

Case $\mathcal{I}(W) \in P(\rho_2', \rho_3)$ Let $W = W(\rho_2', \rho_3) := V_7(\rho_2') \oplus S_1 V_5(\rho_2') = V_7(\rho_2') \oplus W_1$. We recall that $S_1 \cdot W_1 = \rho_2 + \rho_2''$, so that $\overline{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 7$. Since $W_1 = \rho_3$ we have $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho \in \operatorname{Irr} G} \deg(\rho)\rho$ by Table 10. Therefore $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

Cases $\mathcal{I}(W) \in P(\rho_2, \rho_3)$ or $\mathcal{I}(W) \in P(\rho_2'', \rho_3)$ Similar.

The following Lemma is proved in the same manner as before. It allows us to complete the proof of Theorem 10.7 by the same argument as in Section 13.

Lemma 14.7 *Each $E(\rho)$ is a smooth rational curve. Moreover, if ρ and ρ' are adjacent then*

1. *as $\mathcal{I}(W) \in E(\rho)$ approaches the point $P(\rho, \rho')$, the limit of $\mathcal{I}(W)$ is $\mathcal{I}(W(\rho, \rho'))$;*
2. *$E(\rho)$ and $E(\rho')$ intersect transversally at $P(\rho, \rho')$.*

14.8 Conclusion

Theorem 10.4 also follows from the lemma. Theorem 10.7, (3) follows from Tables 10–11 and Lemma 14.5.

Let $I \in X_G$. If $\operatorname{Supp}(\mathcal{O}_{\mathbb{A}^2}/I)$ is not the origin, then

$$I = (T(x, y) - T(a, b), \varphi^3(x, y) - \varphi^3(a, b), W(x, y) - W(a, b))$$

where $(a, b) \neq (0, 0)$.

By the same argument as in Section 13 we thus obtain a complete description of the G-invariant ideals in X_G.

15 The binary octahedral group E_7

15.1 Character table

The binary octahedral group $\mathbb{O}$ is defined as the subgroup of $\mathrm{SL}(2, \mathbb{C})$ of order 48 generated by $\mathbb{T} = \langle \sigma, \tau, \mu \rangle$ and κ:

$$\sigma = \begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}, \ \tau = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, \ \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7, & \varepsilon^7 \\ \varepsilon^5, & \varepsilon \end{pmatrix}, \ \kappa = \begin{pmatrix} \varepsilon, & 0 \\ 0, & \varepsilon^7 \end{pmatrix},$$

where $\varepsilon = e^{2\pi i/8}$ [Slodowy80], p. 73. G acts on $\mathbb{A}^2$ from the right by $(x,y) \mapsto (x,y)g$ for $g \in G$. $\mathbb{D}_2$ and $\mathbb{T}$ are normal subgroups of G and the following sequences are exact:

$$1 \to \mathbb{T} \to G \to \mathbb{Z}/2\mathbb{Z} \to 1$$

and

$$1 \to \mathbb{D}_2 \to G \to S_3 \to 1,$$

where S_3 is the symmetric group on 3 letters.

See Table 12 for the character table of G and other relevant invariants. E_7 has Coxeter number $h = 18$.

15.2 Symmetric tensors modulo $\mathfrak{n}$

The G-modules S_m and $\overline{S}_m := S_m(\mathfrak{m}/\mathfrak{n})$ by $\rho_{\mathrm{nat}} := \rho_2$ for small values of m split into irreducible G-modules as in Table 13. The factors of $\overline{S}_m$ in brackets are those in S_{McKay}. We use the same notation $\bar{V}_m(\rho)$ and $V_m(\rho)$ for $\rho \in \operatorname{Irr} G$ as before. Let $\varphi = p_2^2 + 4\omega p_3^2$, $\psi = p_2^2 + 4\omega^2 p_3^2$, $T(x,y) = (x^4 - y^4)xy$. In Table 14 we denote by $W_j^{(i)} \simeq \rho_4$ the G-submodules of $V_9(\rho_4) \simeq \rho_4^{\oplus 2}$; $W_2'' := S_1 \cdot V_8(\rho_2'')$, $W_3 := S_1 \cdot V_8(\rho_3)$, $W_3' := S_1 \cdot V_8(\rho_3')$,

Lemma 15.3 *The G-module $S_m\bar{V}_k(\rho)$ splits into irreducible G-submodules as in Table 15. We read the table as $S_2\bar{V}_6(\rho_1') = \rho_3'$, $S_2\bar{V}_8(\rho_2'') = \rho_3 + \rho_3'$ and so on.*

Proof The assertions for $(m,k) = (1,6), (2,6), (3,6)$ are clear. There are three generators A_8, A_{12} and A_{18} of respective degrees 8, 12 and 18 for the ring of G-invariant polynomials. We know that $A_8 = \varphi\psi$, $A_{12} = T^2$ by [Klein], p. 54.

Note first that $S_m = S_{m-8} \cdot A_8 \oplus \overline{S}_m$ for $m = 10, 11$ and

$$S_4V_6(\rho_1') = (\rho_2'' + \rho_3') \otimes \rho_1' = \rho_2'' + \rho_3, \quad S_5V_6(\rho_1') = (\rho_2' + \rho_4) \otimes \rho_1' = \rho_2 + \rho_4.$$

If $\{S_4\bar{V}_6(\rho_1')\}[\rho_3] = 0$ in $\overline{S}_{10}$, then $\{S_4V_6(\rho_1')\}[\rho_3] = S_2 \cdot A_8$. A_8 would be divisible by T, a generator of $V_6(\rho_1')$. However, this is impossible. Hence $\{S_4\bar{V}_6(\rho_1')\}[\rho_3] = \rho_3$ so that $S_4\bar{V}_6(\rho_1') = \rho_2'' + \rho_3$. $S_5\bar{V}_6(\rho_1') = \rho_2 + \rho_4$ is proved similarly.

Since $S_6V_6(\rho_1') = (\rho_1')^2 + \rho_3 + \rho_3' = \rho_0 + \rho_3 + \rho_3'$, $S_6\bar{V}_6(\rho_1') = \rho_3 + \rho_3'$ or ρ_3. If $S_6\bar{V}_6(\rho_1') = \rho_3$, then $S_6[\rho_3] \cdot V_6(\rho_1')$ is divisible by T^2, so that $S_6[\rho_3]$ is divisible by T. Since $\deg T = 6$, this is impossible. Hence $S_6\bar{V}_6(\rho_1') = \rho_3 + \rho_3'$.

Next we have $S_7V_6(\rho_1') = \rho_2' + \rho_2 + \rho_4$ and $\{S_7V_6(\rho_1')\}[\rho_2] = \rho_2 \cdot A_{12}$. If $\{S_7\bar{V}_6(\rho_1')\}[\rho_4] = 0$, then $\{S_7V_6(\rho_1')\}[\rho_4] = V_7[\rho_4]V_6(\rho_1') = \rho_4 \cdot A_{12}$ or $\rho_4 \cdot A_8$. In the first case, $V_7[\rho_4]$ is divisible by T, which is impossible because $\deg T = 6$ and $\dim S_1 = 2 < \deg \rho_4 = 4$. In the second case, $V_7[\rho_4]$ is divisible by A_8,

ρ	1	2	3	4	5	6	7	8	d	$(\frac{h}{2}\pm d)$
	1	-1	μ	μ^2	τ	κ	$\tau\kappa$	κ^3		
$\sharp$	1	1	8	8	6	6	12	6		
ρ_0	1	1	1	1	1	1	1	1	(3)	–
ρ_2	2	-2	1	-1	0	$\sqrt{2}$	0	$-\sqrt{2}$	2	$(7,11)$
ρ_3	3	3	0	0	-1	1	-1	1	1	$(8,10)$
ρ_4	4	-4	-1	1	0	0	0	0	0	$(9,9)$
ρ'_3	3	3	0	0	-1	-1	1	-1	1	$(8,10)$
ρ'_2	2	-2	1	-1	0	$-\sqrt{2}$	0	$\sqrt{2}$	2	$(7,11)$
ρ'_1	1	1	1	1	1	-1	-1	-1	3	$(6,12)$
ρ''_2	2	2	-1	-1	2	0	0	0	1	$(8,10)$

Table 12: Character table of E_7

m	S_m	$\overline{S}_m$
1	ρ_2	ρ_2
2	ρ_3	ρ_3
3	ρ_4	ρ_4
4	$\rho''_2+\rho'_3$	$\rho''_2+\rho'_3$
5	$\rho'_2+\rho_4$	$\rho'_2+\rho_4$
6	$\rho'_1+\rho_3+\rho'_3$	$(\rho'_1)+\rho_3+\rho'_3$
7	$\rho_2+\rho'_2+\rho_4$	$(\rho_2+\rho'_2)+\rho_4$
8	$\rho_0+\rho''_2+\rho_3+\rho'_3$	$(\rho''_2+\rho_3+\rho'_3)$
9	$\rho_2+2\rho_4$	$(2\rho_4)$
10	$\rho''_2+2\rho_3+\rho'_3$	$(\rho''_2+\rho_3+\rho'_3)$
11	$\rho_2+\rho'_2+2\rho_4$	$(\rho_2+\rho'_2)+\rho_4$
12	$\rho_0+\rho'_1+\rho''_2+\rho_3+2\rho'_3$	$(\rho'_1)+\rho_3+\rho'_3$
13	$\rho_2+2\rho'_2+2\rho_4$	$\rho'_2+\rho_4$
14	$\rho'_1+\rho''_2+2\rho_3+2\rho'_3$	$\rho''_2+\rho'_3$
15	$\rho_2+\rho'_2+3\rho_4$	ρ_4
16	$\rho_0+2\rho''_2+2\rho_3+2\rho'_3$	ρ_3
17	$2\rho_2+\rho'_2+3\rho_4$	ρ_2
18	$\rho_0+\rho'_1+\rho''_2+3\rho_3+2\rho'_3$	0

Table 13: Irreducible decompositions of $S_m(E_7)$ and $\overline{S}_m(E_7)$

m	ρ	$V_m(\rho)$
7	ρ_2	$7x^4y^3+y^7, -x^7-7x^3y^4$
11	ρ_2	$x^{10}y-6x^6y^5+5x^2y^9, -xy^{10}+6x^5y^6-5x^9y^2$
8	ρ_3	$-2xy^7-14x^5y^3, x^8-y^8, 2x^7y+14x^3y^5$
10	ρ_3	$4x^{10}+60x^6y^4, 5x^9y+54x^5y^5+5xy^9$
		$60x^4y^6+4y^{10}$
9	ρ_4	$W_2''+W_3=W_3+W_3'=W_3'+W_2''\simeq\rho_4^{\oplus 2}$
9	W_2''	$12x^6y^3+12x^2y^7, x^9-10x^5y^4+xy^8$
		$-x^8y+10x^4y^5-y^9, 12x^7y^2+12x^3y^6$
9	W_3	$21x^6y^3+3x^2y^7, -x^9+7x^5y^4+2xy^8$
		$-2x^8y-7x^4y^5+y^9, -3x^7y^2-21x^3y^6$
9	W_3'	x^3T, x^2yT, xy^2T, y^3T
8	ρ_3'	x^2T, xyT, y^2T
10	ρ_3'	$-3x^8y^2-14x^4y^6+y^{10}, 8x^7y^3+8x^3y^7$
		$x^{10}-14x^6y^4-3x^2y^8$
7	ρ_2'	xT, yT
11	ρ_2'	$-11x^8y^3-22x^4y^7+y^{11}, 11x^3y^8+22x^7y^4-x^{11}$
6	ρ_1'	T
12	ρ_1'	$x^{12}-33x^8y^4-33x^4y^8+y^{12}$
8	ρ_2''	$\psi^2, -\varphi^2$
10	ρ_2''	$x^5y\psi-xy^5\varphi, -x^5y\varphi+xy^5\psi$

Table 14: $V_m(\rho)(E_7)$

which is impossible. It follows that $\{S_7\bar{V}_6(\rho_1')\}[\rho_4]=\rho_4$. If $\{S_7\bar{V}_6(\rho_1')\}[\rho_2']=0$, then $V_7[\rho_2]V_6(\rho_1')=\rho_2'\cdot A_{12}$ or $\rho_2'\cdot A_8$. In the first case $V_7[\rho_2]$ is divisible by T, which contradicts Table 14. In the second case $V_7[\rho_2]$ is divisible by A_8, absurd. Hence $\{S_7\bar{V}_6(\rho_1')\}[\rho_2']=\rho_2'$. It follows that $S_7\bar{V}_6(\rho_1')=\rho_2'+\rho_4=\overline{S}_{13}$.

We note next $\dim S_1V_{11}(\rho_2')\geq 3$. If $\dim S_1V_{11}(\rho_2')=3$, then there exists a $f\in S_{10}$ such that $V_{11}(\rho_2')=S_1\cdot f$. Hence $f\in S_{10}[\rho_1']=\{0\}$, a contradiction. Hence $\dim S_1V_{11}(\rho_2')=4$. so that $S_1V_{11}(\rho_2')=\rho_1'+\rho_3'$. If $\{S_1\bar{V}_{11}(\rho_2')\}[\rho_3']=0$, we have $\{S_1V_{11}(\rho_2')\}[\rho_3']=V_4[\rho_3']\cdot A_8$ by Table 13. Since $\dim S_1<\deg\rho_3'=3$, there exists a nontrivial element of $\{S_1V_{11}(\rho_2')\}[\rho_3']$ divisible by both x and A_8. Hence $V_{11}(\rho_2')$ contains a nontrivial element divisible by A_8. This implies that $V_{11}(\rho_2')$ is divisible by A_8. Then $V_3(\rho_2')=V_{11}(\rho_2')A_8^{-1}=\rho_2'$, which contradicts $S_3=\rho_4$. Hence $S_1\bar{V}_{11}(\rho_2')=\rho_1'+\rho_3'$.

It is clear from $\rho_2\otimes\rho_2''=\rho_4$ and Table 13 that $S_1\bar{V}_8(\rho_2'')=\rho_4$.

m	k	ρ	$S_m\bar{V}_k(\rho)$	m	k	ρ	$S_m\bar{V}_k(\rho)$
1	6	ρ'_1	ρ'_2	2	8	ρ''_2	$\rho_3+\rho'_3$
2	6		ρ'_3	3	8		$\rho_2+\rho'_2+\rho_4$
3	6		ρ_4	1	7	ρ_2	ρ_3
4	6		$\rho''_2+\rho_3$	2	7		ρ_4
5	6		$\rho_2+\rho_4$	3	7		$\rho''_2+\rho'_3$
6	6		$\rho_3+\rho'_3$	4	7		$\rho'_2+\rho_4$
7	6		$\rho'_2+\rho_4$	5	7		$\rho'_1+\rho_3+\rho'_3$
1	11	ρ'_2	$\rho'_1+\rho'_3$	1	10	ρ_3	$\rho_2+\rho_4$
1	8	ρ''_2	ρ_4	1	10	ρ'_3	$\rho'_2+\rho_4$

Table 15: Decomposition of $S_m\bar{V}_k(\rho)$

Next $S_2\otimes V_8(\rho''_2)=\rho_3+\rho'_3$ by Table 12. Since $\dim S_2V_8(\rho''_2)\geq 4$, we have $S_2V_8(\rho''_2)=\rho_3+\rho'_3$. If $\{S_2\bar{V}_8(\rho''_2)\}[\rho_3]=0$, then $\{S_2V_8(\rho''_2)\}[\rho_3]=S_2\cdot A_8$. Since $\deg\rho''_2<\deg\rho_3$ and $V_8(\rho''_2)$ is generated by φ^2 and ψ^2, there exists a nontrivial element of $\{S_2V_8(\rho''_2)\}[\rho_3]$ divisible by both φ^2 and A_8. Since φ and ψ are coprime, S_{10} contains a nontrivial element divisible by $\varphi^2\psi$, a contradiction. If $\{S_2\bar{V}_8(\rho''_2)\}[\rho'_3]=0$, then $\{S_2V_8(\rho''_2)\}[\rho'_3]=S_2\cdot A_8=\rho_3$, a contradiction. Hence $S_2\bar{V}_8(\rho''_2)=\rho_3+\rho'_3$.

Next we consider $S_3\bar{V}_8(\rho''_2)$. Since $\dim S_2V_8(\rho''_2)=6$ by the above proof, we have $\dim S_3V_8(\rho''_2)\geq 7$. By Table 12 $S_3\otimes V_8(\rho''_2)=\rho_2+\rho'_2+\rho_4$ so that $S_3V_8(\rho''_2)=\rho_2+\rho'_2+\rho_4$. Assume $S_3\bar{V}_8(\rho''_2)\neq\rho_2+\rho'_2+\rho_4$. Then by Table 13 the only possibility is that $\{S_3\bar{V}_8(\rho''_2)\}[\rho_4]=0$. Assume $\{S_3V_8(\rho''_2)\}[\rho_4]=S_3\cdot A_8$ so that there exists an element of $\{S_3V_8(\rho''_2)\}[\rho_4]$ divisible by both φ^2 and A_8. Therefore there exists a nontrivial element of S_3 divisible by ψ, which is a contradiction. Hence $S_3\bar{V}_8(\rho''_2)=\rho_2+\rho'_2+\rho_4$.

Clearly $S_1V_7(\rho_2)=\rho_0+\rho_3$, $S_2V_7(\rho_2)=\rho_2+\rho_4$. Hence $S_1\bar{V}_7(\rho_2)=\rho_3$ and $S_2\bar{V}_7(\rho_2)=\rho_4$.

Next $S_3\otimes V_7(\rho_2)=\rho_4\otimes\rho_2=\rho''_2+\rho_3+\rho'_3$ by Table 12. Since $\dim S_2V_7(\rho_2)=6$, we have $\dim S_3V_7(\rho_2)\geq 7$ so that $S_3\otimes V_7(\rho_2)=\rho''_2+\rho_3+\rho'_3$. It is clear that $\{S_1V_7(\rho_2)\}[\rho_0]=S_0\cdot A_8$, $\{S_2V_7(\rho_2)\}[\rho_2]=S_1\cdot A_8$. Hence $\{S_3V_7(\rho_2)\}[\rho_3]=S_2\cdot A_8$. It is clear that $\{S_3V_7(\rho_2)\}[\rho'_3]\neq S_2\cdot A_8$ and $\{S_3V_7(\rho_2)\}[\rho''_2]\neq S_2\cdot A_8$. Hence $S_3\bar{V}_7(\rho_2)=\rho''_2+\rho'_3$.

Next we see $\dim S_4V_7(\rho_2)=10$, $S_4V_7(\rho_2)\simeq S_4\otimes V_7(\rho_2)=\rho'_2+2\rho_4$. Hence $S_4\bar{V}_7(\rho_2)=\rho'_2+\rho_4$ by Table 13. It is easy to see that $\dim S_5V_7(\rho_2)=12$. Hence $S_5V_7(\rho_2)=S_5\otimes V_7(\rho_2)=\rho'_1+\rho''_2+\rho_3+2\rho'_3$ so that $S_5\bar{V}_7(\rho_2)=\rho'_1+\rho_3+\rho'_3=\overline{S}_{12}$ by Table 13.

Similarly we see easily that $\dim S_1V_{10}(\rho_3)=\dim S_1V_{10}(\rho'_3)=6$. Hence

$S_1V_{10}(\rho_3) = \rho_2 + \rho_4$, $S_1V_{10}(\rho_3') = \rho_2' + \rho_4$. If $\{S_1\bar{V}_{10}(\rho_3)\}[\rho_4] = 0$, then $\{S_1V_{10}(\rho_3)\}[\rho_4] = S_3 \cdot A_8$. Therefore there exists a nontrivial element of $V_{10}(\rho_3)$ divisible by A_8 so that $V_{10}(\rho_3)$ is divisible by A_8. This implies that $\bar{V}_{10}(\rho_3) = 0$. But by the choice of it, $V_{10}(\rho_3) \simeq \bar{V}_{10}(\rho_3)$, a contradiction. This completes the proof. $\square$

Corollary 15.4 *1.* $S_1\bar{V}_6(\rho_1') = \bar{V}_7(\rho_2')$, $S_2\bar{V}_6(\rho_1') = \bar{V}_8(\rho_3')$, $S_1\bar{V}_7(\rho_2) = \bar{V}_8(\rho_3)$.

2. $S_3\bar{V}_8(\rho_2'') = \overline{S}_{11}$, $S_5\bar{V}_7(\rho_2) = \overline{S}_{12}$, $S_7\bar{V}_6(\rho_1') = \overline{S}_{13}$.

3. $S_2\bar{V}_8(\rho_3') = \rho_2'' + \rho_3$, $S_2\bar{V}_8(\rho_2'') = \rho_3 + \rho_3'$, $S_2\bar{V}_8(\rho_3) = \rho_2'' + \rho_3'$.

Proof Clear. $\square$

We omit the proof of Theorem 10.7 because we need only to follow the proof in the E_6 case verbatim.

15.5 Conclusion

We also can give a complete description of G-invariant ideals in X_G. Let

$$\chi = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}, \quad F(x,y) = \chi T, \quad W(x,y) = \varphi\psi.$$

Let $I \in X_G$. If $\mathrm{Supp}(\mathcal{O}_{\mathbb{A}^2}/I)$ is not the origin, then we know that

$$I = \big(W(x,y) - W(a,b), T^2(x,y) - T^2(a,b), F(x,y) - F(a,b)\big).$$

where $(a,b) \neq (0,0)$.

16 The binary icosahedral group E_8

16.1 Character table

The binary icosahedral group $\mathbb{I}$ is defined as the subgroup of $\mathrm{SL}(2,\mathbb{C})$ of order 120 generated by σ and τ:

$$\sigma = -\begin{pmatrix} \varepsilon^3, & 0 \\ 0, & \varepsilon^2 \end{pmatrix}, \quad \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} -(\varepsilon - \varepsilon^4), & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3, & \varepsilon - \varepsilon^4 \end{pmatrix}$$

where $\varepsilon = e^{2\pi i/5}$. We note $\sigma^5 = \tau^2 = -1$. G acts on $\mathbb{A}^2$ from the right by $(x,y) \mapsto (x,y)g$ for $g \in G$. G is isomorphic to $\mathrm{SL}(2,\mathbb{F}_5)$. An isomorphism of G with $\mathrm{SL}(2,\mathbb{F}_5)$ is given by $\sigma \mapsto \left(\begin{smallmatrix} 3 & 3 \\ 3 & 0 \end{smallmatrix}\right)$, $\tau \mapsto \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. Let $\eta = \varepsilon^2 = e^{4\pi i/5}$. In

ρ	1	2	3	4	5	6	7	8	9	d	$(\frac{h}{2} \pm d)$
	1	-1	σ	σ^2	σ^3	σ^4	τ	$\sigma^2\tau$	$\sigma^7\tau$		
$\sharp$	1	1	12	12	12	12	30	20	20		
ρ_0	1	1	1	1	1	1	1	1	1	(5)	$-$
ρ_2	2	-2	μ^+	$-\mu^-$	μ^-	$-\mu^+$	0	-1	1	4	$(11, 19)$
ρ_3	3	3	μ^+	μ^-	μ^-	μ^+	-1	0	0	3	$(12, 18)$
ρ_4	4	-4	1	-1	1	-1	0	1	-1	2	$(13, 17)$
ρ_5	5	5	0	0	0	0	1	-1	-1	1	$(14, 16)$
ρ_6	6	-6	-1	1	-1	1	0	0	0	0	$(15, 15)$
ρ_4'	4	4	-1	-1	-1	-1	0	1	1	1	$(14, 16)$
ρ_2'	2	-2	μ^-	$-\mu^+$	μ^+	$-\mu^-$	0	-1	1	2	$(13, 17)$
ρ_3''	3	3	μ^-	μ^+	μ^+	μ^-	-1	0	0	1	$(14, 16)$

Table 16: Character table of E_8

m	$\overline{S}_m$	m	$\overline{S}_m$
0	0	30	0
1	ρ_2	29	ρ_2
2	ρ_3	28	ρ_3
3	ρ_4	27	ρ_4
4	ρ_5	26	ρ_5
5	ρ_6	25	ρ_6
6	$\rho_3'' + \rho_4'$	24	$\rho_3'' + \rho_4'$
7	$\rho_2' + \rho_6$	23	$\rho_2' + \rho_6$
8	$\rho_4' + \rho_5$	22	$\rho_4' + \rho_5$
9	$\rho_4 + \rho_6$	21	$\rho_4 + \rho_6$
10	$\rho_3 + \rho_3'' + \rho_5$	20	$\rho_3 + \rho_3'' + \rho_5$
11	$(\rho_2) + \rho_4 + \rho_6$	19	$(\rho_2) + \rho_4 + \rho_6$
12	$(\rho_3) + \rho_4' + \rho_5$	18	$(\rho_3) + \rho_4' + \rho_5$
13	$(\rho_2' + \rho_4) + \rho_6$	17	$(\rho_2' + \rho_4) + \rho_6$
14	$(\rho_3'' + \rho_4' + \rho_5)$	16	$(\rho_3'' + \rho_4' + \rho_5)$
15	$(2\rho_6)$		

Table 17: Irreducible decompositions of $\overline{S}_m(E_8)$

Slodowy's notation [Slodowy80], p. 74

$$\tau = \frac{1}{\eta^2 - \eta^3}\begin{pmatrix} \eta + \eta^4, & 1 \\ -1, & -\eta - \eta^4 \end{pmatrix}.$$

See Table 16 for the character table of G [Schur07] and the other relevant invariants. The Coxeter number h of E_8 is equal to 30. Let $\mu^{\pm} = \frac{1 \pm \sqrt{5}}{2}$.

m	k	ρ	$S_m\bar{V}_k(\rho)$	m	k	ρ	$S_m\bar{V}_k(\rho)$
1	11	ρ_2	ρ_3	1	16	ρ_5	$\rho_4 + \rho_6$
2	11		ρ_4	1	13	ρ_2'	ρ_4'
3	11		ρ_5	2	13		ρ_6
4	11		ρ_6	3	13		$\rho_3'' + \rho_5$
5	11		$\rho_3'' + \rho_4'$	4	13		$\rho_4 + \rho_6$
6	11		$\rho_2' + \rho_6$	5	13		$\rho_3 + \rho_4' + \rho_5$
7	11		$\rho_4' + \rho_5$	1	16	ρ_4'	$\rho_2' + \rho_6$
8	11		$\rho_4 + \rho_6$	1	14	ρ_3''	ρ_6
9	11		$\rho_3 + \rho_3'' + \rho_5$	2	14		$\rho_4' + \rho_5$
1	18	ρ_3	$\rho_2 + \rho_4$	3	14		$\rho_2' + \rho_4 + \rho_6$
1	17	ρ_4	$\rho_3 + \rho_5$				

Table 18: Irreducible decompositions of $S_m\bar{V}_k(\rho)$

16.2 Symmetric tensors modulo $\mathfrak{n}$

The G-modules $\overline{S}_m := S_m(\mathfrak{m}/\mathfrak{n})$ by $\rho_{\mathrm{nat}} := \rho_2$ for small values of m split into irreducible G-modules as in Table 17. The factors of $\overline{S}_m$ in brackets are those in S_{McKay}. We use the same notation $\bar{V}_m(\rho)$ and $V_m(\rho)$ for $\rho \in \mathrm{Irr}\, G$ as before.

We define irreducible G-submodules of $V_{15}(\rho_6)$ ($\simeq \rho_6^{\oplus 2}$) and σ_i, τ_j by

$$\begin{aligned} W_3'' &:= S_1V_{14}(\rho_3''), \quad W_4' := S_1V_{14}(\rho_4'), \quad W_5 := S_1V_{14}(\rho_5), \\ \sigma_1 &:= x^{10} + 66x^5y^5 - 11y^{10}, \quad \sigma_2 := -11x^{10} - 66x^5y^5 + y^{10} \\ \tau_1 &:= x^{10} - 39x^5y^5 - 26y^{10}, \quad \tau_2 := -26x^{10} + 39x^5y^5 + y^{10} \end{aligned}$$

Lemma 16.3 *The G-modules $S_m\bar{V}_k(\rho)$ split into irreducible G-submodules as in Table 18.*

Proof We give a brief proof of the lemma. Recall that the ring of G-invariant polynomials is generated by three elements A_{12}, A_{20} and A_{30} of degree 12, 20, 30 respectively. See [Klein], p. 55 or Table 4. Note that $S_1 \otimes V_{11}(\rho_2) = \rho_2 \otimes \rho_2 = \rho_0 + \rho_3$. Hence $S_1 \otimes V_{11}(\rho_2) = \rho_0 A_{12} + \rho_3$. In fact $A_{12} = xy(x^{10} + 11x^5y^5 - y^{10})$ by [Klein], p. 56. It follows that $S_1\bar{V}_{11}(\rho_2) = \rho_3$. Similarly $S_k \otimes V_{11}(\rho_2) \supset S_{k-1}A_{12}$. Therefore $S_2 \otimes V_{11}(\rho_2) = \rho_2 + \rho_4$, $S_2\bar{V}_{11}(\rho_2) = \rho_4$, $S_3 \otimes V_{11}(\rho_2) = \rho_3 + \rho_5$, $S_3\bar{V}_{11}(\rho_2) = \rho_5$, $S_4 \otimes V_{11}(\rho_2) = \rho_4 + \rho_6$, $S_4\bar{V}_{11}(\rho_2) = \rho_6$, $S_5 \otimes V_{11}(\rho_2) = \rho_3'' + \rho_4' + \rho_5$, $S_5\bar{V}_{11}(\rho_2) = \rho_3'' + \rho_4'$. All of these are proved as in Lemma 15.3. In fact, for instance $\dim S_5V_{11}(\rho_2) = 7$ by Table 19, and $\rho_6 \otimes \rho_2 = \rho_3'' + \rho_4' + \rho_5$ so that $S_5\bar{V}_{11}(\rho_2) = \rho_3'' + \rho_4'$.

We see $S_6\bar{V}_{11}(\rho_2) = \rho_2' + \rho_6$ because $\overline{S}_{17} = \rho_2' + \rho_4 + \rho_6$ and $\rho_2 \otimes S_5\bar{V}_{11}(\rho_2) = \rho_2 \otimes (\rho_3'' + \rho_4') = \rho_2' + 2\rho_6$ contains no ρ_4. $\overline{S}_{18} = \rho_3 + \rho_4' + \rho_5$ and $\rho_2 \otimes S_6\bar{V}_{11}(\rho_2) = \rho_2 \otimes (\rho_2' + \rho_6)$ contains no ρ_3, whence $S_7\bar{V}_{11}(\rho_2) = \rho_4' + \rho_5$. Similarly $S_8\bar{V}_{11}(\rho_2) = \rho_4 + \rho_6$ because $\overline{S}_{19} = \rho_2 + \rho_4 + \rho_6$, $\rho_2 \otimes S_7\bar{V}_{11}(\rho_2) = \rho_2' + \rho_4 + 2\rho_6$. By Table 17 $\overline{S}_{20} = \rho_3 + \rho_3'' + \rho_5$. $\rho_2 \otimes S_8\bar{V}_{11}(\rho_2) = \rho_3 + 2\rho_5 + \rho_3'' + \rho_4'$. Hence $S_9\bar{V}_{11}(\rho_2) = \rho_3 + \rho_3'' + \rho_5 = \overline{S}_{20}$.

$S_1\bar{V}_{18}(\rho_3) = \rho_2 + \rho_4$ follows from comparison of $S_1 \otimes \bar{V}_{18}(\rho_3)$ and S_{19} and the fact that any polynomial in $V_{18}(\rho_3)$ is not divisible by A_{12}.

Similarly $S_1\bar{V}_{17}(\rho_4) = \rho_3 + \rho_5$, $S_1\bar{V}_{16}(\rho_5) = \rho_4 + \rho_6$ and $S_1\bar{V}_{13}(\rho_2') = \rho_4'$. Since $\rho_3 \otimes \rho_2' = \rho_6$, we see $S_2\bar{V}_{13}(\rho_2') = \rho_6$. One checks $\dim S_3V_{13}(\rho_2') = \dim S_1W_4' = 8$ by using Table 19. It follows from this that $S_3\bar{V}_{13}(\rho_2') = \rho_3'' + \rho_5$. Similarly it is clear that $S_4V_{13}(\rho_2') = S_4 \otimes V_{13}(\rho_2') = \rho_4 + \rho_6$ and $S_5\bar{V}_{13}(\rho_2') = S_5 \otimes \bar{V}_{13}(\rho_2') = \overline{S}_{18}$. Note $\dim S_kV_{14}(\rho_3'') = 3(k+1)$ for $k = 1, 2, 3$ so that $S_kV_{14}(\rho_3'') = S_k \otimes V_{14}(\rho_3'')$. It follows from it that $S_k\bar{V}_{14}(\rho_3'') = S_k \otimes \rho_3''$ for $k = 1, 2, 3$. In particular, $S_2\bar{V}_{14}(\rho_3'') = \rho_3 \otimes \rho_3' = \rho_4' + \rho_5$, $S_3\bar{V}_{14}(\rho_3'') = \rho_2' + \rho_4 + \rho_6 = \overline{S}_{17}$. □

Corollary 16.4 *1.* $S_k\bar{V}_{11}(\rho_2) = \bar{V}_{11+k}(\rho_{k+2})$ *for* $1 \le k \le 3$*;* $S_1\bar{V}_{13}(\rho_2') = \bar{V}_{14}(\rho_4')$.

2. $S_9\bar{V}_{11}(\rho_2) = \overline{S}_{20}$, $S_5\bar{V}_{13}(\rho_2') = \overline{S}_{18}$, $S_3\bar{V}_{14}(\rho_3'') = \overline{S}_{17}$.

3. $S_2\bar{V}_{14}(\rho_5) = \rho_3'' + \rho_4'$, $S_2\bar{V}_{14}(\rho_4') = \rho_3'' + \rho_5$, $S_2\bar{V}_{14}(\rho_3'') = \rho_4' + \rho_5$.

Proof By Table 19, $\dim S_1W_3'' = 9$, $\dim S_1W_4' = 8$, $\dim S_1W_5 = 7$. Hence $S_2V_{14}(\rho_3'') = S_1W_3'' = \rho_4 + \rho_5$, $S_2\bar{V}_{14}(\rho_4') = S_2V_{14}(\rho_4') = S_1W_4' = \rho_3'' + \rho_5$, $S_5\bar{V}_{11}(\rho_2) = S_2\bar{V}_{14}(\rho_5) = S_1W_5 = \rho_3'' + \rho_4'$. □

In order to prove Theorem 10.7 in the E_8 case we have only to follow the proof of Theorem 10.7 in the D_n or E_6 case verbatim. We omit the details.

m	ρ	$V_m(\rho)$
11	ρ_2	$x\sigma_1, -y\sigma_2$
19	ρ_2	$-57x^{15}y^4 + 247x^{10}y^9 + 171x^5y^{14} + y^{19}$
		$-x^{19} + 171x^{14}y^5 - 247x^9y^{10} - 57x^4y^{15}$
12	ρ_3	$x^2\sigma_1, -5x^{11}y - 5xy^{11}, y^2\sigma_2$
18	ρ_3	$-12x^{15}y^3 + 117x^{10}y^8 + 126x^5y^{13} + y^{18}$
		$45x^{14}y^4 - 130x^9y^9 - 45x^4y^{14}$
		$x^{18} - 126x^{13}y^5 + 117x^8y^{10} + 12x^3y^{15}$
13	ρ_4	$x^3\sigma_1, -3x^{12}y + 22x^7y^6 - 7x^2y^{11}$
		$-7x^{11}y^2 - 22x^6y^7 - 3xy^{12}, y^3\sigma_2$
17	ρ_4	$-2x^{15}y^2 + 52x^{10}y^7 + 91x^5y^{12} + y^{17}$
		$10x^{14}y^3 - 65x^9y^8 - 35x^4y^{13}$
		$-35x^{13}y^4 + 65x^8y^9 + 10x^3y^{14}$
		$-x^{17} + 91x^{12}y^5 - 52x^7y^{10} - 2x^2y^{15}$
14	ρ_5	$x^4\sigma_1, -2x^{13}y + 33x^8y^6 - 8x^3y^{11}$
		$-5x^{12}y^2 - 5x^2y^{12}$
		$-8x^{11}y^3 - 33x^6y^8 - 2xy^{13}, -y^4\sigma_2$
16	ρ_5	$64x^{15}y + 728x^{10}y^6 + y^{16}$
		$66x^{14}y^2 + 676x^9y^7 - 91x^4y^{12}$
		$56x^{13}y^3 + 741x^8y^8 - 56x^3y^{13}$
		$91x^{12}y^4 + 676x^7y^9 - 66x^2y^{14}$
		$x^{16} + 728x^6y^{10} - 64xy^{15}$
13	ρ_2'	$y^3\tau_2, -x^3\tau_1$
17	ρ_2'	$x^{17} + 119x^{12}y^5 + 187x^7y^{10} + 17x^2y^{15}$
		$-17x^{15}y^2 + 187x^{10}y^7 - 119x^5y^{12} + y^{17}$
14	ρ_3''	$x^{14} - 14x^9y^5 + 49x^4y^{10}$
		$7x^{12}y^2 - 48x^7y^7 - 7x^2y^{12}$
		$49x^{10}y^4 + 14x^5y^9 + y^{14}$
16	ρ_3''	$3x^{15}y - 143x^{10}y^6 - 39x^5y^{11} + y^{16}$
		$-25x^{13}y^3 - 25x^3y^{13}$
		$x^{16} + 39x^{11}y^5 - 143x^6y^{10} - 3xy^{15}$

Table 19: $V_m(\rho)(E_8)$

m	ρ	$V_m(\rho)$
14	ρ_4'	$xy^3\tau_2, -x^4\tau_1, y^4\tau_2, -x^3y\tau_1$
16	ρ_4'	$-2x^{15}y + 77x^{10}y^6 - 84x^5y^{11} + y^{16}$
		$3x^{12}y^4 + 110x^7y^9 + 15x^2y^{14}$
		$15x^{14}y^2 - 110x^9y^7 + 35x^4y^{12}$
		$-x^{16} - 84x^{11}y^5 - 77x^6y^{10} - 2xy^{15}$
15	ρ_6	$W_3'' + W_4' = W_4' + W_5 = W_5 + W_3'' \simeq \rho_6^{\oplus 2}$
15	W_3''	$:= S_1V_{14}(\rho_3'')\ (\simeq \rho_6)$
		$x^{15} + 84x^{10}y^5 + 77x^5y^{10} + 2y^{15}$
		$-x^{14}y + 14x^9y^6 - 49x^4y^{11}$
		$-7x^{13}y^2 + 48x^8y^7 + 7x^3y^{12}$
		$7x^{12}y^3 - 48x^7y^8 - 7x^2y^{13}$
		$-49x^{11}y^4 - 14x^6y^9 - xy^{14}$
		$-2x^{15} + 77x^{10}y^5 - 84x^5y^{10} + y^{15}$
15	W_4'	$:= S_1V_{14}(\rho_4')\ (\simeq \rho_6)$
		$x^{15} + 39x^{10}y^5 - 143x^5y^{10} - 3y^{15}$
		$-2x^{14}y + 78x^9y^6 + 52x^4y^{11}$
		$x^{13}y^2 - 39x^8y^7 - 26x^3y^{12}$
		$-26x^{12}y^3 + 39x^7y^8 + x^2y^{13}$
		$52x^{11}y^4 - 78x^6y^9 - 2xy^{14}$
		$3x^{15} - 143x^{10}y^5 - 39x^5y^{10} + y^{15}$
15	W_5	$:= S_1V_{14}(\rho_5)\ (\simeq \rho_6)$
		$5x^{15} + 330x^{10}y^5 - 55x^5y^{10}$
		$-7x^{14}y + 198x^9y^6 - 43x^4y^{11}$
		$-19x^{13}y^2 + 66x^8y^7 - 31x^3y^{12}$
		$-31x^{12}y^3 - 66x^7y^8 - 19x^2y^{13}$
		$-43x^{11}y^4 - 198x^6y^9 - 7xy^{14}$
		$-55x^{10}y^5 - 330x^5y^{10} + 5y^{15}$

Table 19: $V_m(\rho)(E_8)$, continued

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We would like to mention some related problems that are unsolved or the subject of current research.

Conjecture 17.1 *Let G be any finite subgroup of* $\mathrm{SL}(3, \mathbb{C})$. *Then* $\mathrm{Hilb}^G(\mathbb{A}^3)$ *is a crepant smooth resolution of* $\mathbb{A}^3/G$.

The conjecture is solved affirmatively in the Abelian case [Nakamura98], where for any finite Abelian subgroup G of $\mathrm{GL}(n, \mathbb{C})$ the Hilbert scheme $\mathrm{Hilb}^G(\mathbb{A}^n)$ is described as a (possibly nonnormal) toric variety. There is a McKay correspondence [Reid97], [INkjm98] similar to [GSV83]. See also [Nakamura98]. In general the normalization of $\mathrm{Hilb}^G(\mathbb{A}^n)$ is a torus embedding associated with a certain fan $\mathrm{Fan}(G)$ given explicitly by using some combinatorial data arising from the given group G. However in general it is not known whether $\mathrm{Hilb}^G(\mathbb{A}^n)$ is normal. There are various examples of $\mathrm{Hilb}^G(\mathbb{A}^n)$. Reid gave some examples of singular Hilb^G for finite Abelian subgroups G in $\mathrm{GL}(3, \mathbb{C})$ in private correspondence.

If G is the cyclic subgroup of $\mathrm{SL}(4, \mathbb{C})$ of order two generated by minus the identity then $\mathrm{Hilb}^G(\mathbb{A}^4)$ is nonsingular; however, it is not a crepant resolution of $\mathbb{A}^3/G$. There are also some examples of Abelian subgroups of $\mathrm{SL}(4, \mathbb{C})$ for which $\mathrm{Hilb}^G(\mathbb{A}^4)$ is singular, although a crepant resolution does exist. The simplest example is the Abelian subgroup of order eight consisting of diagonal 4×4 matrices with diagonal coefficients ± 1. [Kidoh98] gave a concrete description of $\mathrm{Hilb}^G(\mathbb{A}^2)$ for a finite Abelian subgroup G of $GL(2, \mathbb{C})$ by using two kinds of continued fractions.

We will treat the non-Abelian cases of Conjecture 17.1 elsewhere [GNS98]; in almost all the non-Abelian case, a certain beautiful duality in $\mathfrak{m}/\mathfrak{n}$ is observed [GNS98]. See also Section 7.

The following question would be important for future applications:

Problem 17.2 *Let G be a finite subgroup of* $\mathrm{SL}(n, \mathbb{C})$, *$N$ a normal subgroup of G. When is* $\mathrm{Hilb}^G(\mathbb{A}^n) \simeq \mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbb{A}^n))$*?*

Unfortunately the answer is negative in general in dimension three. This will appear in [GNS98].

References

[Arnold74] V.I. Arnold, *Critical points of smooth functions*, in Proc. Intern. Cong. of Math., Vancouver 1974, pp. 19–39.

[Beauville83] A. Beauville, *Variétés Kählériennes dont la première classe de Chern est nulle*, J. Diff. Geom. **18** (1983) 787–829.

[Blichfeldt05] H.F. Blichfeldt, *The finite discontinuous primitive groups of collineations in three variables* Math. Ann. **63** (1905) 552–572.

[Blichfeldt17] H.F. Blichfeldt, *Finite collineation groups*, The Univ. Chicago Press, Chicago, 1917.

[BR95] K. Behnke and O. Riemenschneider, *Quotient surface singularities and their deformations*, in Singularity Theory, World Scientific, 1995, pp. 1–54.

[Briançon77] J. Briançon, *Description de* $\mathrm{Hilb}^n \mathbb{C}\{x, y\}$, Invent. Math. **41** (1977) 45–89.

[Bourbaki] N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres 4,5 et 6 Masson, Paris–New York,1981.

[BV92] B. Blok, A. Varchenko, *Topological conformal field theories and the flat coordinates*, Intern. J. Mod. Phys. **7** (1992) 1467–1490.

[Cardy86] J.L. Cardy, *Operator content of two-dimensional conformally invariant theories*, Nucl. Phys. B **270** (1986) 186–204.

[Cardy88] J.L. Cardy, *Conformal invariance and statistical mechanics*, Champs, cordes et phenomenes Les Houches, 1988, pp. 169–245.

[Chevalley55] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77** (1955) 778–782.

[CIZ87] A. Capelli, C. Itzykson, J.B. Zuber, *Modular invariant partition functions in two dimensions*, Nucl. Phys. B **280** (1987) 445–465.

[Durfee79] A. Durfee, *Fifteen characterizations of rational double points and simple singularities*, L'Enseign. Math. **25** (1979) 131–163.

[EY89] T. Eguchi, S. K. Yang, *Virasoro algebras and critical phenomena* (in Japanese), Jour. Japan Phys. Soc. **44** (1988) 894–901.

[FGA] *Fondements de la Géométrie Algébrique*, Séminaire Bournaki,1957-62, Secrétariat Math., Paris (1962).

[EK97] D. E. Evans, Y. Kawahigashi, *Quantum symmetries on operator algebras*, Cambridge University Press, to appear.

[ES87] G. Ellingsrud and S.A. Strømme, *On the homology of the Hilbert schemes of points in the plane*, Invent. Math. **87** (1987) 343–352.

[Fujiki83] A. Fujiki, *On primitively symplectic compact Kähler V-mani folds*, Progress in Mathematics, Birkhäuser 39 (1983) 71–250.

[Fogarty68] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968) 511–521.

[Gabriel72] P. Gabriel, *Unzerlegbare Darstellungen I*, Manusc. Math. **6** (1972) 71–103.

[Gawedzki89] K. Gawedzki, *Conformal field theory*, Séminaire Bourbaki 1988/89, Asterisque (1989) No. 177–178, Exp. No. 704, 95–126.

[GHJ89] F. M. Goodman, P. de la Harpe, V. F. R. Jones, *Coxeter graphs and towers of algebras*, Mathematical sciences research institute publications **14**, Springer Verlag, 1989.

[GNS98] Y. Gomi, I. Nakamura, K. Shinoda, (in preparation)

[Göttsche91] L. Göttsche, *Hilbert schemes of zero-dimensional subschemes of smooth varieities*, Lecture Notes in Math., **1572**, Springer-Verlag, 1991.

[Grojnowski96] I. Grojnowski, *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, Math. Res. Letters, **3** (1996), 275–291.

[GSV83] G .Gonzalez-Sprinberg, J. Verdier, *Construction géométrique de la correspondence de McKay*, Ann. scient. Éc. Norm. Sup. **16** (1983) 409–449.

[GW86] D. Gepner, E. Witten, *String theory on group manifolds*, Nucl. Phys. B **278** (1986) 493–549.

[Hartshorne77] R. Hartshorne, *Algebraic geometry*, Garduate texts in Math. **52**, Springer, 1977.

[HHSV77] M. Hazewinkel, W. Hesselink, D. Siersma, F. D. Veldkamp, *The ubiquity of Coxeter–Dynkin diagrammes (an introduction to the A-D-E classification)*, Nieuw Archief voor Wiskunde **25** (1977) 257–307.

[Humphreys90] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge studies in advanced mathematics **29**, Cambridge University Press, 1990.

[Iarrobino72] A. Iarrobino, *Reducibility of the families of 0-dimensional schemes on a variety*, Invent. Math. **15** (1972) 72-77.

[Iarrobino77] A. Iarrobino, *Punctual Hilbert schemes*, Memoirs Amer. Math. Soc. **188**, Amer. Math. Soc. Providence, 1977.

[Ito95a] Y. Ito, *Crepant resolution of trihedral singularities and the orbifold Euler characteristic*, Intern. Jour. Math., **6** (1995) 33–43.

[Ito95b] Y. Ito, *Gorenstein quotient singularities of monomial type in dimension three*, J. Math. Sci. Univ. of Tokyo, **2** (1995) 419–440.

[INkjm98] Y. Ito and I. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, preprint (1998).

[IN96] Y. Ito and I. Nakamura, *McKay correspondence and Hilbert Schemes*, Proc. Japan Acad. **72** (1996) 135–138.

[IR96] Y. Ito, M. Reid, *The McKay correspondence for finite groups of* $\mathrm{SL}(3, C)$, Higher Dimensional Complex Varieties Proc. Internat. Conference, Trento (1996) 221–240.

[Izumi91] M. Izumi, *Application of fusion rules to classification of subfactors*, Publ. RIMS, Kyoto Univ. **27** (1991) 953–994.

[Jones91] V. F. R. Jones, *Subfactors and knots*, Regional conference series in mathematics **80**, Amer. Math. Soc. Providende,1991.

[Kato87] A. Kato, *Classification of modular invariant partition functions in two dimensions*, Modern Physics Letters A, **2** (1987) 585–600.

[Kac90] V. G. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge–New York, 1990.

[Khinich76] V. A. Khinich, *On the Gorenstein property of the ring of invariants of a Gorenstein ring*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976) 50–56, English transl. Math. USSR-Izv. **10** (1976) 47–53.

[Kidoh98] R. Kidoh, *Hilbert schemes and cyclic quotient surface singularities*, preprint (1998)

[Klein] F. Klein, *Vorlesungen über das Ikosaeder*, Birkhäuser B.G. Teubner, 1993.

[Knop87] F. Knop, *Ein neuer Zusammenhang zwischen einfachen Gruppen und einfachen Singularitäten*, Invent. Math. **90** (1987) 579–604.

[Knörrer85] H. Knörrer, *Group representations and the resolution of rational double points*, in Finite groups – Coming of Age, Contem. Math. **45**, 1985, pp. 175–222.

[Kronheimer89] P. B. Kronheimer, *The construction of ALE spaces as hyperkähler quotients*, Jour. Diff. Geometry **29** (1989) 665–683.

[KW88] V. G. Kac, M. Wakimoto, *Modular invariant representations of infinite dimensional Lie algebras and superalgebras*, Proc. Nat. Acad. Sci. U.S.A. **85** (1988) 4956–4960.

[Markushevich92] D. Markushevich, *Resolution of* $\mathbb{C}^3/H_{168}$, preprint (1992).

[McKay80] J. McKay, *Graphs, singularities, and finite group*, in Santa Cruz, conference on finite groups (Santa Cruz, 1979), Proc. Symp. Pure Math., AMS **37**, 1980, pp. 183–186.

[Mukai84] S. Mukai, *Symplectic structure of the moduli of bundles on abelian or K3 surface*, Inv. Math. **77** (1984) 101–116.

[MBD16] G. A. Miller, H. F. Blichfeldt, L. E. Dickson, *Theory and applications of finite groups*, Dover, New York, 1916.

[Nakajima96a] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on surfaces*, to appear in Ann. Math.

[Nakajima96b] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, 1996.

[Nakamura98] I. Nakamura, *Hilbert schemes of abelian group orbits*, preprint (1998).

[Ocneanu88] A. Ocneanu, *Quantized group string algebras and Galois theory for algebras*, in "Operator algebras and applications, Vol. 2 (Warwick, 1987)", London Math. Soc. Lecture Note Series **136**, Cambridge University Press, 1988, pp. 119–172.

[OSS80] C. Okonek, M. Schenider and H. Spindler, *Vector bundles on complex projective spaces*, Progress in Math., Birkhäuser **3**, 1980.

[OT92] P. Orlik, H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, 1992.

[Pasquier87a] V. Pasquier, *Two dimensional critical systems labelled by Dynkin diagrams*, Nucl. Phys. B **285** (1987) 162–172.

[Pasquier87b] V. Pasquier, *Operator contents of the ADE lattice models*, J. Phys. A: Math. Gen **20** (1987) 5707–5717.

[Pinkham80] H. Pinkham, *Singularités de Klein, I*, Séminaire sur les singularités de surfaces, Lecture Note in Math. **777**, Springer-Verlag, 1980, pp. 1–9.

[PZ96] V. B. Petkova, J. -B. Zuber, *From CFT to graphs*, Nucl. Phys. B **463** (1996) 161–193.

[Reid97] M. Reid, *McKay correspondence*, in Proc. of algebraic geometry symposium (Kinosaki, Nov 1996), T. Katsura (Ed.), 14–41, Duke file server alg-geom 9702016, 30 pp.

[Roan89] S-S. Roan, *On the generalization of Kummer surfaces*, J. Diff.Geom., **30** (1989) 523–537.

[Roan94] S-S. Roan, *On* $c_1 = 0$ *resolution of quotient singularities*, Intern. J. Math., **5** (1994) 523–536.

[Roan96] S-S. Roan, *Minimal Resolution of Gorenstein Orbifolds in Dimension Three* Topology **35** (1996) 489–508.

[Schur07] I. Schur, *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **132** (1907) 85–137.

[Slodowy80] P. Slodowy, *Simple singulalrities and simple algebraic groups*, Lecture Note in Math. **815**, Springer-Verlag, 1980.

[Slodowy90] P. Slodowy, *A new A-D-E classification*, Bayreut. Math. Schr. **33** (1990) 197–213.

[Slodowy95] P. Slodowy, *Groups and special singularities*, in Singularity Theory, Worldscientific, 1995, pp. 731–799.

[Steinberg64] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. Amer. Math. Soc. **112** (1964). 392–400

[YY93] Stephen S.-T. Yau and Y. Yu, *Gorenstein quotient singularities in dimension three*, Memoirs Amer. Math. Soc., **105**, Amer. Math. Soc. 1993.

[Watanabe74] K. Watanabe, *Certain invariant subrings are Gorenstein* I, II, Osaka J. Math., **11** (1974) 1–8, 379–388.

[Wenzl87] H. Wenzl, *On sequences of projections*, C. R. Math. Rep. Acad. Sci. Canada., **9** (1987) 5–9.

[Zuber90] J.-B. Zuber, *Graphs, algebras, conformal field theories and integrable lattice models*, Nucl. Phys. B (Proc. Suppl.) **18B** (1990) 313–326.

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