

## Bogomolov's theorem $c_1^2 \leq 4c_2$

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This paper is an account of the following theorem of F. A. Bogomolov:

**Theorem 0.** *Let  $X$  be a non-singular projective surface defined over the complex numbers such that for some  $n \geq 1$   $H^0(K_X^{\otimes n}) \neq 0$ . Then we have the relation  $c_1^2 \leq 4c_2$  between the Chern numbers of  $X$ .*

The proof is based on the two following results, which are of the interest for their own sake. The first introduces a strong notion of *instability* of rank 2 vector bundles.

**Theorem 1.** *Let  $E$  be a vector bundle of rank 2 over a projective non-singular surface defined over the complex numbers. Then  $c_1(E)^2 > 4c_2(E)$  if and only if there exists an exact sequence*

$$(*) \quad 0 \rightarrow L \rightarrow E \rightarrow I_Z \cdot M \rightarrow 0,$$

with  $L$  and  $M$  line bundles, and  $Z$  a 0-dimensional subscheme of  $X$  with sheaf of ideals  $I_Z$ , such that, letting  $\Delta = L \otimes M^{-1}$  we have

$$\text{and} \quad \left. \begin{aligned} c_1(\Delta)^2 &> 4 \deg Z \\ c_1(\Delta) \cdot H &> 0. \end{aligned} \right\} \quad (0)$$

Here  $H$  denotes the class of the hyperplane section of  $X$ , and  $\deg Z$  denotes the length of  $O_Z$ , that is

$$\deg Z = \sum_{P \in \text{Supp } Z} \dim_k O_{Z,P}.$$

The two conditions  $c^2 > 0$  and  $c \cdot H > 0$  define the *numerically positive cone* in the Neron-Severi group  $NS(X)$ , and the conditions (0) imply that  $c_1(\Delta)$  belongs to this positive cone. The line bundle  $L$  occurring in Theorem 1 is *necessarily* unique, since any other line bundle  $L' \subset E$  has a non-zero morphism to  $M$ , and hence  $L' = M(-D)$ , with  $D \geq 0$  a positive divisor, and hence for the corresponding  $\Delta'$  we have  $c_1(\Delta') \cdot H < 0$ . This unicity, together with the fact that we have a purely numerical criterion for instability is in striking contrast with the case of vector bundles over

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curves.

A result closely related to Theorem 1 for vector bundles over  $P^2$  was obtained by Schwarzenberger [5], who only made use of the  $n=1$  stage of the argument given in § 1.

Note that both the conclusion and the hypotheses of Theorem 1 are invariant under the operations of tensoring  $E$  with a line bundle and of going over to the dual bundle  $E^*$ .

**Theorem 2.** *Let  $X$  be a non-singular projective surface over the complex numbers, and let  $L \subset \Omega_X^1$  be a line bundle; then*

(i) *if for some  $n \geq 1$   $h^0(L^{\otimes n}) > 1$  then there is a morphism  $f: X \rightarrow C$  of  $X$  onto a non-singular curve  $C$  of positive genus, and a divisor  $D$  on  $X$  made up of components of the fibres of  $f$  such that the two subsheaves  $L$  and  $f^*(\Omega_C^1)(D)$  of  $\Omega_X^1$  coincide.*

Hence

(ii) *there exists a constant  $k$  such that for all  $n$*

$$h^0(L^{\otimes n}) < nk;$$

and

(iii)  *$L$  is not in the positive cone of  $NS(X)$ .*

There is no loss of generality in assuming that the line bundle  $L \subset \Omega_X^1$  is a full subsheaf of  $\Omega_X^1$  (Definition 2, § 1, see also Lemma 3, § 2), in which case the divisor  $D$  of Theorem 2 is positive, and is computed in Lemma 4 of § 2.

The assertion (i) with  $n=1$  is an old lemma of Castelnuovo ([6], see also Proposition 2 of § 2); in this case we have the additional fact that  $H^0(X, L) = f^*H^0(C, \Omega_C^1)$ , so that  $C$  has genus at least 2. There are many examples of surfaces having a rational pencil  $X \rightarrow P^1$  with multiple fibres for which the corresponding  $L \subset \Omega_X^1$  has  $h^0(L^{\otimes n}) \geq 2$  for some  $n > 1$ . Nevertheless, Theorem 2 is proved by an easy reduction to the  $n=1$  case by means of ramified coverings corresponding to sections of  $L^{\otimes n}$ .

Theorem 0 follows at once from Theorems 1 and 2, since in the exact sequence (\*) we have  $\Delta = L \otimes M^{-1}$  and  $\det E = L \otimes M$ , so that  $L^{\otimes 2} = \Delta \otimes \det E$ ; note that the condition  $H^0(K_X^{\otimes 2}) \neq 0$  is essential: for minimal models of ruled surfaces of genus  $\geq 2$  we have  $c_1^2 > 4c_2$  (both sides being negative), and the tangent bundle is indeed unstable.

Sections 1 and 2 contain proofs of Theorems 1 and 2 respectively; I add a brief discussion of the significance of the invariants  $c_1^2$  and  $c_2$  and of the ratio  $\alpha = c_1^2/c_2$  for algebraic surfaces (§ 0) and of other possible applications of Bogomolov's methods (§ 3). Appendix 1 gives Mumford's proof of C. P. Ramanujam's strong form of the Kodaira vanishing theorem from Theorem 1 as an indication of the strength of this result.

Finally, the stronger inequality  $c_1^2 \leq 3c_2$  (Conjecture 1 in § 0) has been proved independently by Miyaoka [7] and Yau [44], and Yau has also proved the conjecture of Severi mentioned at the top of p. 460.

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### § 0. The invariants $c_1^2$ and $c_2$

In this paper all surfaces will be non-singular and projective, and defined over the complex field  $k = \mathbb{C}$ ; they will usually be denoted  $X$ . The Chern numbers  $c_1^2$  and  $c_2$  are the Chern numbers of the tangent (or cotangent) bundle of  $X$ ; apart from standard notation I will introduce in this section the letter  $\alpha$  for the ratio  $c_1^2/c_2$ .

$c_1^2$  and  $c_2$  provide two crude numerical invariants of algebraic surfaces; to the topologist they mean the Euler characteristic  $\chi(X, \mathbb{Z}) = c_2$  and the index  $\tau(X) = \frac{1}{3}(c_1^2 - 2c_2)$ . In algebraic geometry they occur as  $K^2 = c_1^2$  and  $(p_a + 1) = \frac{1}{2}(c_1^2 + c_2)$ .

On blowing up a point  $P \in X$ ,  $c_1^2$  decreases by 1 and  $c_2$  increases by 1; the arithmetic genus is birationally invariant, and to get a second birational invariant we have to take  $c_1^2$  or  $c_2$  of the minimal model—which gives us the minimal value of  $c_2$  and the maximal value of  $c_1^2$  for the given birational class. The Euler characteristic of  $O_X(nK_X)$  is given in terms of  $c_1^2$  and  $c_2$ :

$$\chi(O_X(nK_X)) = (p_a + 1) + \binom{n}{2} K^2$$

which takes on the form

$$P_m(X) = H^0(O_X(mK_X)) = (p_a + 1) + \binom{m}{2} K^2 \quad (m \geq 2)$$

for  $n+m \geq 5$  we have  $c_1^2 < \frac{4}{3}c_2$ , the ratio  $\alpha$  approaching  $\frac{4}{3}$  if  $n$  and  $m$  tend together to infinity.

For the complete intersection of  $r$  hypersurfaces in  $P^{r+2}$  there is the general bound  $c_1^2 < (2r/(r+1))c_2$ , and again  $\alpha$  can be made to approach this limit by letting the degrees of the hypersurfaces tend uniformly to infinity. To get a family of surfaces with  $\alpha$  approaching 2 we have to let  $r$  tend to infinity as well.

These examples are of course all simply connected.

(ii) *Quotients of symmetric domains.* Let  $D$  be a 2-dimensional bounded symmetric domain, and let  $\Gamma$  be a discontinuous group of transformations of  $D$  acting without fixed points and with compact quotient  $D/\Gamma$ . According to Hirzebruch [14] the Chern numbers of  $D/\Gamma$  are proportional to those of the compact symmetric space  $D'$  dual to  $D$ . There are only two possible choices for  $D$ :

$$D_1^2 = \{(z_1, z_2); |z_i|^2 < 1\} \subset D_1'^2 = P^1 \times P^1$$

and

$$D_2 = \{(z_1, z_2); |z_1|^2 + |z_2|^2 < 1\} \subset D_2' = P^2.$$

The quotients  $D_1^2/\Gamma$  will thus have  $c_1^2 = 2c_2$ , and those of  $D_2$  will have  $c_1^2 = 3c_2$ . Included in the case  $D_1^2$  are the product of two curves of genus  $\geq 2$ , for which the group  $\Gamma$  splits as a product of two groups acting on the factors. However, if  $\Gamma$  satisfies a suitable irreducibility condition then every subgroup  $\Gamma' \subset \Gamma$  of finite index has finite commutator quotient  $\Gamma'^{ab}$ , so every finite covering  $Y \rightarrow X$  has  $q(Y) = 0$ .

Conjecture 2 above implies in particular that this does not happen for a group  $\Gamma$  of transformations of  $D_2$ .

(iii) *Kodaira surfaces.* Kodaira [15] constructs a class of surfaces  $X$  having a morphism  $X \rightarrow C$  with no singular fibres (see also [16]); he gives the values of  $c_1^2$  and  $c_2$  for these examples, and one sees that

$$2c_2 < c_1^2 < 2.35c_2.$$

(iv) *Surfaces with ordinary singularities.* Let  $X \subset P^3$  be a surface of degree  $n$  having only ordinary singularities (see [17], [18], [19]), that is a double curve  $\Delta$  of degree  $m$  and genus  $p_a(\Delta)$ , a finite number  $t$  of ordinary triple points and a finite number of pinch points. The following formulae for the invariants of the normalisation  $Y \rightarrow X$  are implicit in [17], and given explicitly in ([18], p. 36, see also [19]):

$$p_a(Y) = \binom{n-1}{3} - (n-4)m + p_a(\Delta) - 1,$$

$$c_1^2(Y) = n(n-4)^2 - (5n-24)m + 4p_a(\Delta) - 4 + t,$$

$$c_2(Y) = n(n^2 - 4n + 6) - (7n-24)m + 8p_a(\Delta) - 8 - t.$$

There are similar formulae for the invariants of the minimal resolution  $Y \rightarrow X$  of a surface in  $P^3$  with only isolated singularities in terms of local analytic invariants of the singular points given in [20].

Conjecture 2 must be regarded as a very difficult problem, since even the simplest possible case, that of proving that there do not exist surfaces of general type topologically equivalent to  $P^2$  has been open since Severi [21].

To get some sort of upper bound for  $\alpha$  we can write

$$\alpha + 1 = \frac{12(p_a + 1)}{2(p_a + 1) + h^{1,1} - 2q},$$

and note that if  $h^{1,1} > 2q$  we have  $\alpha < 5$ . In fact, as remarked by Van de Ven ([22], [23]), provided the Albanese map  $X \rightarrow \text{Alb } X$  maps onto a surface we always have  $h^{1,1} \geq 2q - 1$ , which gives the limit  $\alpha \leq 5 + 6/(2p_a + 1)$  (asymptotically  $\sim 5$ ) in this case. On the other hand, if  $X \rightarrow \text{Alb } X$  maps onto a curve  $C$  (of genus  $q$ ), with general fibre of genus  $\pi$ , say, then there doesn't seem to be a better estimate than the formula for the Euler characteristic

$$c_2 = \chi(X, \mathcal{Z}) \geq (2q - 2)(2\pi - 2),$$

which in the worst case  $\pi = 2$  lead Van de Ven to the estimate  $\alpha \leq 8$  ([22], [23]).

The special methods available for surfaces with a pencil of curves of genus 2 [24] do not seem to imply any trivial improvement of this estimate; it is interesting to speculate whether the asymptotic properties for such surfaces (in the sense of Conjectures 1 and 2) are in any way special.

### § 1. Instability of vector bundles

Let  $E$  be a rank 2 vector bundle over  $X$ ; the proof of Theorem 1 is based on a consideration of the auxiliary vector bundles

$$H_n = S^{2n} E \otimes (\det E)^{\otimes -n};$$

$H_n$  is a vector bundle of rank  $2n + 1$ , and one checks easily the following result.

**Proposition 1.** (i) *There are canonical isomorphisms*

$$H_n^* = H_n \quad \text{and} \quad \det H_n = \mathcal{O}_X;$$

(ii) *the Riemann-Roch formula for  $H_n$  gives*

$$\chi(H_n) = (2n + 1)(p_a(X) + 1) + (c_1(E)^2 - 4c_2(E)) \left[ \sum_{i=1}^n i^2 \right].$$

Thus if  $c_1(E)^2 > 4c_2(E)$  we know that  $h^0(H_n) + h^2(H_n)$  grows with  $n$  like  $n^3$ ; more precisely

**Proposition 2.** *Let  $E$  be a rank 2 vector bundle on  $X$  such that  $c_1(E)^2 > 4c_2$ .*

Then there exists a constant  $k > 0$  such that for all sufficiently large  $n$   $h^0(H_n) > kn^3$ .

*Proof.* By Serre duality and (i) of Proposition 1 we have

$$h^2(H_n) = h^0(H_n \otimes K_X),$$

so that it will be sufficient to get a lower bound for the difference between  $h^0(H_n)$  and  $h^0(H_n \otimes K_X)$ .

**Lemma 3.** *Let  $L$  be any line bundle on  $X$ ; then there exists a constant  $K > 0$  such that for all  $n$*

$$h^0(H_n \otimes L) - h^0(H_n) < Kn^2.$$

*Proof.* Suppose that the divisor class of  $L$  has been expressed as the difference

$$[L] \sim C - D,$$

with  $C$  and  $D$  (say) smooth curves. Restricting the sections of  $H_n \otimes L$  down to  $C$  gives the exact sequence

$$0 \rightarrow H^0(H_n \otimes O_X(-D)) \rightarrow H^0(H_n \otimes L) \rightarrow H^0(C, (H_n \otimes L)|_C);$$

hence the lemma will follow from the bound

$$h^0(C, (H_n \otimes L)|_C) < Kn^2.$$

However, this is obvious, since  $(H_n \otimes L)|_C$  is a standard construction on  $C$  starting from  $E|_C$  and  $L$ ; thus if  $E|_C$  is expressed as an extension of two line bundles  $L_1$  and  $L_2$  we have

$$h^0(C, (H_n \otimes L)|_C) \leq \sum_{p=-n}^n h^0(L_1^p \otimes L_2^{-p} \otimes L).$$

**Remark.** This proves a slightly stronger form of Proposition 2, namely that for any curve  $D \subset X$ ,  $h^0(H_n \otimes O_X(-D))$  grows like  $n^3$ . We will however only need the fact that for some  $n$   $h^0(H_n) > 2n + 1$ .

Let me fix some notation; let  $P(E) \xrightarrow{\pi} X$  be the projectivisation of the bundle  $E$ , and let  $O_{P(E)}(1)$  be the Grothendieck tautological line bundle on  $P(E)$ . Let  $M_n$  be the line bundle

$$M_n = O_{P(E)}(2n) \otimes (\pi^*(\det E))^{\otimes -n};$$

we have

$$M_n = M_1^{\otimes n}, \quad \pi_* M_n = H_n, \quad \text{and} \quad H^0(X, H_n) = H^0(P(E), M_n).$$

**Definition 1.** A quasi-section of the fibration  $\pi: P(E) \rightarrow X$  is an irreducible

for minimal models of surfaces of general type ([8], [9]).

The invariants  $p_g$ ,  $q$  and  $h^{1,1}$  relate to  $c_1^2$  and  $c_2$  as follows :

$$p_g - q + 1 = p_a + 1 = \frac{1}{2}(c_1^2 + c_2)$$

and

$$c_2 = \chi(X, \mathcal{Z}) = 2 - 4q + 2p_g + h^{1,1}.$$

A supplementary discrete invariant of  $X$  is the fundamental group  $\pi_1$ ; this is related to  $q$  by the fact that the Abelianisation  $\pi_1^{ab} = H_1(X, \mathcal{Z})$  has free part of rank  $2q$ .

There is a general lower bound for  $c_1^2$  in terms of  $c_2$  for minimal models of surfaces of general type provided by M. Noether's inequality

$$p_g \leq \frac{1}{2}K^2 + 2 \quad (0)$$

([8], Theorem 9, p. 208)); this gives  $c_2 \leq 5c_1^2 + 36$ . The surfaces approaching the bound (0) are analogous to curves having non-trivial linear systems of very low degree (Castelnuovo [10], Horikawa [11], [12], [13]); there are very many such examples, so the bound (0) is a very satisfying one. We shall be interested in inequalities in the other direction.

The following two questions have played an important role in Bogomolov's thinking; the first is referred to in Bombieri's paper ([8], p. 219)) as a traditional conjecture.

**Conjecture 1.**  $c_1^2 \leq 3c_2$ .

**Conjecture 2.** All surfaces for which the index  $\tau(X) = \frac{1}{3}(c_1^2 - 2c_2)$  is positive have infinite fundamental group  $\pi_1(X)$ ; furthermore, if  $\tau(X) > 0$  then  $X$  has a finite unramified covering  $Y$  with  $q(Y) \neq 0$ .

Using the identities given above, the first can be rewritten

$$1'. \quad h^{1,1} \geq p_g + q + 1,$$

and the second implies :

$$2'. \quad \text{If } \pi_1(X) = 0 \text{ then } h^{1,1} > 2(p_g + 1).$$

The theorem proved in this article is equivalent to

$$h^{1,1} \geq 2q + \frac{2}{3}(p_a + 1).$$

**Some examples.** (i) *Complete intersections.* The non-singular hypersurface of degree  $n$  in  $P^3$  has

$$c_1^2 = n(n-4)^2, \quad c_2 = n(n^2 - 4n + 6);$$

for  $n \geq 4$  we have  $c_1^2 < c_2$ , and the ratio  $\alpha$  tends to 1 as  $n$  tends to infinity.

The complete intersection of two hypersurfaces of degree  $n$  and  $m$  in  $P^4$  has

$$c_1^2 = (n+m-5)^2 nm, \quad c_2 = (n^2 + nm + m^2 - 5(n+m) + 10)nm;$$

subvariety  $S \subset P(E)$  such that, equivalently  $S \cdot (\text{fibre}) = 1$ , or the restriction of the projection,  $\pi: S \rightarrow X$  is birational.

Note that  $\pi: S \rightarrow X$  is an isomorphism outside the subset

$$Y = \{x \in X \mid \pi^{-1}(x) \subset S\} \subset X,$$

which by the irreducibility of  $S$  has codimension at least 2, that is  $Y$  is a finite set of points.

**Proposition 4.** *There is a natural correspondence between quasi-sections  $S \subset P(E)$  and exact sequences*

$$(*) \quad 0 \rightarrow L \rightarrow E \rightarrow I_Z \cdot M \rightarrow 0,$$

with  $L$  and  $M$  line bundles and  $Z$  a zero-dimensional subscheme of  $X$  having sheaf of ideals  $I_Z$ .

*Proof.* The correspondence is given as follows: let  $N$  denote the restriction to  $S$  of the line bundle  $O_{P(E)}(1)$ . The kernel of the surjection  $O_{P(E)}(1) \rightarrow N$  is the line bundle  $O_{P(E)}(1) \otimes O_{P(E)}(-S)$ , which has trivial restriction to each fibre  $\pi^{-1}(x)$ , and is hence of the form  $\pi^*L$  for some line bundle  $L$  on  $X$ . Applying the sheaf-theoretic direct image functor to the exact sequence of sheaves on  $P(E)$

$$0 \rightarrow \pi^*L \rightarrow O_{P(E)}(1) \rightarrow N \rightarrow 0$$

gives the exact sequence

$$0 \rightarrow L \rightarrow E \rightarrow \pi_*N \rightarrow 0.$$

Since  $\pi: S \rightarrow X$  is an isomorphism outside the finite set  $Y \subset X$ ,  $\pi_*N$  is locally free outside  $Y$ . To see that  $N$  is necessarily of the form  $I_Z \cdot M$ , with  $M$  locally free, consider an open  $U \subset X$  over which  $L$  and  $E$  are trivialisable:  $L|_U = O_U \rightarrow E|_U = O_U \oplus O_U$  is given by two sections  $f$  and  $g$  of  $O_U$ , and the divisors of zeroes of  $f$  and  $g$  have no common components—since otherwise the cokernel  $\pi_*N$  would have sections supported on a divisor. Hence  $f$  and  $g$  define a 0-dimensional subscheme  $Z$ . From the exact sequence

$$0 \rightarrow O_U \xrightarrow{(f,g)} O_U \oplus O_U \xrightarrow{\begin{pmatrix} -f \\ g \end{pmatrix}} I_Z \cdot O_U \rightarrow 0$$

one sees that the cokernel  $\pi_*N$  is of the required form.

Note that  $\text{Supp } Z \subset Y$ .

The following definition is implicit in the arguments of Proposition 4, and we will find it useful later:

**Definition 2.** Let  $\mathcal{E}$  be a torsion-free sheaf on an integral scheme  $S$ . A subsheaf  $\mathcal{F} \subset \mathcal{E}$  is said to be *full* if equivalently:

(i) the quotient  $\mathcal{E}/\mathcal{F}$  is torsion free ;

or

(ii) For any sheaf  $\mathcal{G}$  with  $\mathcal{F} \subset \mathcal{G} \subset \mathcal{E}$  and  $\mathcal{F} = \mathcal{G}$  at the generic point of  $S$ ,  $\mathcal{F} = \mathcal{G}$ .

**Lemma 5.** *Suppose that we have an exact sequence (\*) as in Proposition 4 ; then*

(i)  $\det E = L \otimes M$ , so that  $c_1(E) = c_1(L) + c_1(M)$  ;

and

(ii)  $c_2(E) = c_1(L) \cdot c_1(M) + \deg Z$ ,

with  $\deg Z$  the length of  $O_Z$ , that is

$$\deg Z = \sum_{P \in \text{Supp } Z} l(O_{Z,P})$$

*Proof.* (i) follows from the fact that both sides are line bundles, isomorphic outside a set of codimension 2.

For (ii), note that  $L \rightarrow E$  is given by a section  $s \in \Gamma(E \otimes L^{-1})$ , which by hypothesis has  $Z$  as its zero locus ; hence  $c_2(E \otimes L^{-1}) = \deg Z$ , and (ii) follows formally.

With this lemma the "if" part of Theorem 1 is of course a triviality ; I now want to give Bogomolov's method of constructing a quasi-section  $S \subset P(E)$  for which the corresponding  $L$  is positive, under the hypothesis that  $c_1^2 > 4c_2$ .

For  $x \in X$  and any  $n$ , let  $U(n, x)$  denote the subspace of sections of  $H_n$  vanishing at  $x$  ;  $U(n, x)$  has codimension at most  $\text{rk } H_n = 2n + 1$  in  $H^0(H_n)$ . Any  $s \in H^0(H_n)$  can be considered as a section of the line bundle  $M_n$  on  $P(E)$ , and  $(s)$  denotes the corresponding divisor ;  $M_n$  has degree  $2n$  on the fibres of  $P(E)$ .

**Proposition 6.** *There exists a quasi-section  $S \subset P(E)$  such that for all  $n, x$  and  $s \in U(n, x)$ ,  $s$  vanishes to order at least  $n + 1$  on  $S$ , that is*

$$(s) > (n + 1)S.$$

Before giving the proof, let us derive Theorem 1 as an easy corollary. For this, let  $S$  be the section provided by Proposition 6, and let  $L, M, Z$  and the exact sequence (\*) correspond to  $S$  as in Proposition 4. Letting  $\Delta = L \otimes M^{-1} = L^{\otimes 2} \otimes \det E^{-1}$ , Lemma 5 gives  $c_1(\Delta)^2 > 4 \deg Z$ , so that it will be sufficient to prove that for some  $i > 0$   $H^0(X, \Delta^{\otimes i}) \neq 0$ .

From Propositions 4 and 6

$$H^0(P(E), M_n(-(n + 1)S)) \neq 0 ;$$

since obviously  $H^0(P(E), M_n(-(2n + 1)S)) = 0$ , the exact sequences

$$0 \rightarrow H^0(M_n(-(k + 1)S)) \rightarrow H^0(M_n(-kS)) \rightarrow H^0(S, (M_n(-kS))|_S)$$

for  $n + 1 \leq k \leq 2n$  give

$$\sum_{k=n+1}^{2n} h^0(S, (M_n(-kS))|_S) \geq h^0(P(E), M_n(-(n+1)S)) > 0,$$

so that for some  $i > 0$  we have  $H^0(S, (M_n(-(n+i)S))|_S) \neq 0$ .

Using the definitions

$$O_{P(E)}(1)|_S = N, \quad O_{P(E)}(1) \otimes O_{P(E)}(-S) = \pi^*L$$

and

$$M_n = O_{P(E)}(2n) \otimes (\pi^* \det E)^{\otimes -n}$$

occurring in Proposition 4, we can rewrite

$$M_n(-(n+i)S)|_S = (N \otimes \pi^*L \otimes \pi^*(\det E)^{-1})^{\otimes n} \otimes (\pi^*L \otimes N^{-1})^{\otimes i}.$$

Since  $\pi_*N \subset M$ , we have  $N \subset \pi^*M$ , and  $N^{\otimes(n-i)} \subset \pi^*M^{\otimes(n-i)}$ , so that using (i) of Lemma 5 we have

$$M_n(-(n+i)S)|_S \subset (\pi^*L \otimes \pi^*M^{-1})^{\otimes i} = \pi^*(\Delta)^{\otimes i};$$

hence  $H^0(S, \pi^*(\Delta)^{\otimes i}) = H^0(X, \Delta^{\otimes i}) \neq 0$  for some  $i > 0$ . This proves Theorem 1.

Note that we have only made use of the existence of a single non-zero section  $s \in H^0(H_n)$  whose divisor  $(s)$  on  $P(E)$  has a quasi-section  $S$  with multiplicity  $\geq n+1$ .

*Proof of Proposition 6.* We consider a non-zero element  $s \in U(n, x)$ , and show that the divisor of  $s$  contains some quasi-section  $S$  with multiplicity at least  $n+1$ . The fact that  $S$  is then independent of  $n$  and  $x$  is essentially trivial, but as remarked above has not been used in the proof of Theorem 1.

The formal proof of this fact is a little abstract, using the simplest case of Mumford's theory of stability of points under the action of a reductive group [28]; in order to explain why we should expect a rabbit from this particular hat, I would like to illustrate this argument by the simplest example. A section  $s$  of  $H_n = S^{2n}E \otimes (\det E)^{\otimes -n}$  gives us, for every point  $x \in X$  an element  $s_x \in S^{2n}E_x \otimes (\det E_x)^{\otimes -n}$ ; choosing a basis  $(e_1, e_2)$  of  $E_x$  this can be represented, up to a factor, as a homogeneous form of degree  $2n$  in 2 variables:

$$s_x = f(x_1, x_2) = \sum a_i x_1^i x_2^{2n-i} = \prod (x_1 - \alpha_i x_2).$$

The roots  $\alpha_i$  of this form will of course vary on changing the basis of  $E_x$ ; however, the discriminant of  $f$ ,

$$\text{disc}(f) = \prod_{i \neq j} (\alpha_i - \alpha_j),$$

which is a polynomial in the coefficients  $a_i$  of  $f$ , will only change by scalar factor under such a base change. Taking account of these scalar factors, we see that to

$s$  we can associate a section disc ( $s$ ) of the line bundle  $(\det H_n)^{\otimes N}$  in such a way that disc ( $s$ ) vanishes at  $x \in X$  if and only if the form associated to  $s_x$  has a repeated root, or in other words the divisor ( $s$ ) on  $P(E)$  meets the fibre over  $x$  in a divisor having a point of multiplicity at least 2. But  $\det H_n \stackrel{\Delta}{=} O_X$ , so that any section is a constant, and if  $s$  was chosen to vanish at some given point of  $X$  disc ( $s$ ) vanishes identically, and ( $s$ ) has some horizontal double component.

The formal argument is a generalisation of this, and goes in two steps:

**Step A.** Let  $W$  be the standard 2-dimensional representation of  $GL(2)$ , and let  $W_n = S^{2n}W \otimes (\det W)^{\otimes -n}$ .

The diagonal scalars of  $GL(2)$  act trivially on the representation  $W_n$ , so that  $W_n$  is a representation of the reductive group  $PGL(2) = G$ . We have the notion of an *unstable point* of  $W_n$  under  $G$  ([28], p. 36)); the following conditions are equivalent:

(i) letting  $G \cdot f \subset W_n$  denote the orbit of  $f$  under  $G$ ,

$$O \in \overline{G \cdot f},$$

— denoting the closure in the Zariski topology.

(ii) For every homogeneous polynomial function  $P$  of degree  $m \geq 1$  on  $W_n$  invariant under  $G$ ,  $P(f) = 0$ .

In our special case,  $f \in W_n$  can be regarded as a homogeneous polynomial in 2 variables, and it is well known (and goes back to Hilbert) that we have a third equivalent condition ([28], pp. 77–78)):

(iii)  $f$  has a root of multiplicity  $n+1$ , that is for some basis  $e_1, e_2$  of  $W$ ,  $f = e_1^{n+1} f' / (e_1 \wedge e_2)^n$ , with  $f'$  of degree  $n-1$ .

The assertion we need is (iii), and we prove it by checking (ii).

**Step B.** The invariant functions on  $W_n$  of degree  $m$  define a canonical direct summand  $I_{n,m}$  in  $S^m(W_n^*)$ :

$$(0) \quad S^m(W_n^*) = I_{n,m} \oplus S'.$$

The vector bundle  $H_n$  on  $X$  can be described as the vector bundle associated to the  $GL(2)$  bundle  $E$  corresponding to the representation  $W_n$ ; the structure group of  $H_n$  is  $PGL(2)$ . Associated to the decomposition (0) above we have a decomposition

$$S^m(H_n^*) = I_{n,m} \oplus S',$$

with  $I_{n,m}$  a trivial vector bundle.

Thus every invariant polynomial  $p \in S^m(W_n^*)$  corresponds to a global section  $P$  of  $I_{n,m}$  on  $X$ . Under the canonical pairing

$$S^m(H_n^*) \times H_n \rightarrow O_X,$$

$P$  takes a global section  $s$  of  $H_n$  into a global function  $P(s) \in H^0(X, \mathcal{O}_X)$ .

If  $s \in U(n, x)$  then  $P(s)$  vanishes at  $x$ , and hence vanishes identically.

This completes the proof of Proposition 6, and hence of Theorem 1.

## § 2. The stability of the tangent bundle

We begin with the following fundamental fact:

**Proposition 1.** *Let  $X$  be a compact complex surface, and let  $\omega \in H^0(X, \Omega_X^1)$  be a global holomorphic 1-form; then  $d\omega = 0$ .*

*Proof.* For compact Kähler manifolds there is the more general fact that  $d\omega = 0$  for any global holomorphic  $p$ -form ([30], § 4.7), ([31], § IV, 4); the proof involves the rudiments of the theory of harmonic forms, so I give an easy alternative proof.

Consider the  $C^\infty$ -differential  $d\omega \wedge \bar{d}\bar{\omega}$ , which is of type  $(2, 2)$ ; wherever  $d\omega \neq 0$ ,  $d\omega \wedge \bar{d}\bar{\omega} > 0$ . On the other hand  $d\omega \wedge \bar{d}\bar{\omega}$  is the total differential of a  $C^\infty$  3-form:

$$(d + \bar{d})(d\omega \wedge \bar{\omega}) = d^2\omega \wedge \bar{\omega} + d\omega \wedge d\bar{\omega} + \bar{d}d\omega \wedge \bar{\omega} + d\omega \wedge \bar{d}\bar{\omega} = d\omega \wedge \bar{d}\bar{\omega};$$

hence by Stokes' Theorem the integral of  $d\omega \wedge \bar{d}\bar{\omega}$  over the whole of  $X$  is zero:

$$\int d\omega \wedge \bar{d}\bar{\omega} = \int d\omega \wedge \bar{\omega} = 0,$$

and hence  $d\omega = 0$ .

**Proposition 2** (Castelnuovo's lemma, [6], see also [22]). *Let  $\omega_1$  and  $\omega_2 \in H^0(\Omega_X^1)$  be two linearly independent global holomorphic 1-forms such that*

$$\omega_1 \wedge \omega_2 = 0;$$

*then there exists a morphism  $X \xrightarrow{f} C$ , and global holomorphic 1-forms  $s_1$  and  $s_2 \in H^0(C, \Omega_C^1)$  such that  $\omega_i = f^*s_i$ .*

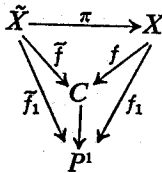
*Proof.* We will use the following obvious fact:

**Lemma 3.** *Let  $E$  be a rank 2 vector bundle over a factorial variety  $V$ , and let  $\mathcal{F} \subset E$  be a subsheaf of rank 1; then there is a unique full subsheaf (see Def. 2, § 1)  $L \subset E$  which coincides with  $\mathcal{F}$  at the generic point of  $V$ , and  $L$  is a line bundle. Furthermore,  $\mathcal{F} \subset L$ .*

$L$  is just the subsheaf of  $E$  consisting of rational sections of  $\mathcal{F}$ ; the argument used at the end of the proof of Proposition 4 of § 1 shows that  $L$  is locally free and that  $E/L$  is torsion-free.

The condition  $\omega_1 \wedge \omega_2 = 0$  implies that  $\omega_1$  and  $\omega_2$  belong to the same line bundle  $L \subset \Omega_X^1$ ; these two sections define a rational map  $f_1: X \rightarrow P^1$ . The sequence of

blowings-up  $\tilde{X} \xrightarrow{\pi} X$  which reduce the singularities of  $f_1$ , and the Stein factorisation of  $f_1$  fit together into the diagram



When we have shown that  $C$  has genus  $\geq 1$  it will follow by a traditional argument in the theory of surfaces (see Zariski [32], p. 25 or Zariski and Schilling [33]) that  $f$  is a morphism, and hence  $\tilde{X}=X$ . Setting  $\tilde{\omega}_i = \pi^* \omega_i$ , we can assume  $\tilde{X}=X$  (the full bundle  $\tilde{L}$  spanned by  $\tilde{\omega}_i$  in the sense of Lemma 3 is not necessarily  $\pi^*L$ ).

The map  $f_1$  is given by taking the ratio  $\omega_1 : \omega_2$  as a function on  $X$ ; thus  $\omega_1 = f_1^*(x)\omega_2$ ,  $x$  being a coordinate (rational) function on  $P^1$ .

Now use Proposition 1 to give us

$$0 = d\omega_1 = d(f_1^*(x)\omega_2) = f_1^*(dx) \wedge \omega_2;$$

that is:  $f_1^*(dx)$  is a rational section of the line bundle  $L \subset \Omega_X^1$  corresponding to  $\omega_1$  and  $\omega_2$ . Hence  $f_1^*\Omega_{P^1}^1 \subset L$ ; since obviously  $f_1^*(\Omega_{P^1}^1)$  and  $f^*(\Omega_C^1)$  coincide on an open set of  $X$  (in fact outside the ramification of  $C \rightarrow P^1$ ), we also have  $f^*(\Omega_C^1) \subset L$ .

Now for an arbitrary morphism  $f: X \rightarrow C$  of a non-singular surface  $X$  onto a non-singular curve  $C$  we have the notion of *ramification divisor*  $D(f)$  on  $X$ : for  $P \in C$ , decompose the (scheme-theoretic) fibre  $f^*P = \sum_i n_i C_i$  as a sum of irreducible components, and set  $D(f)_P = f^*P - (f^*P)_{\text{red}} = \sum_i (n_i - 1)C_i$ , and define  $D(f) = \sum_{P \in C} D(f)_P$ . The idea behind the definition is the following fact:

**Lemma 4.** *We have an inclusion  $f^*(\Omega_C^1)(D(f)) \subset \Omega_X^1$  as a full subbundle.*

An easy computation in local coordinates at a non-singular point of  $C_i$  shows that the inclusion  $df: f^*(\Omega_C^1) \subset \Omega_X^1$  has divisor of zeroes  $(n_i - 1)C_i$ , proving the lemma.

Returning to the proof of Proposition 2, we have

**Corollary 5.**  $L = f^*(\Omega_C^1)(D(f))$ .

This is just the unicity assertion in Lemma 3.

Now I claim that  $f_*L = \Omega_C^1$ ; this is clear outside those points  $P \in C$  for which the fibre  $f^*C$  is non-reduced. But for such a point, since  $f^*\Omega_C^1 \subset L \subset f^*(\Omega_C^1(P))$  we have  $\Omega_C^1 \subset f_*L \subset \Omega_C^1(P)$ ; however, any section of  $\Omega_C^1(P)$  around  $P$  which does not belong to  $\Omega_C^1$  must generate  $\Omega_C^1(P)$  at  $P$ , so that it corresponds to a (rational) section of  $f^*\Omega_C^1$  over  $X$  having pole exactly  $f^*P$  around  $f^{-1}P$ , and hence not belonging to  $L$ .

This proves the proposition, since  $f_*\omega_i$  provide the required sections of  $\Omega_C^1$ , ensuring that  $C$  has genus  $\geq 2$ .

*Proof of Theorem 2.* For a line bundle  $L \subset \Omega_X^1$  and a rational map  $f: X \rightarrow C$  of  $X$  onto a curve  $C$ , the condition  $f^*\Omega_C^1 \subset L$  is equivalent to the condition that the fibres of  $f$  are tangent at the general point of  $X$  to the foliation corresponding to  $L$ . The function field  $k(C)$  of  $C$  is then determined intrinsically on  $X$  as the field of functions on  $X$  whose differential belongs to  $L \subset \Omega_X^1$ ; thus the map  $X \xrightarrow{f} C$  is unique if it exists if we assume that  $k(C)$  is algebraically closed in  $k(X)$ .

Theorem 2 is a consequence of the next assertion:

**Proposition 6.** *Let  $L \subset \Omega_X^1$  be a (full) line bundle, and suppose that for some  $n \geq 1$  there are two linearly independent sections  $s_1$  and  $s_2 \in H^0(L^{\otimes n})$ . Let  $X \xrightarrow{f_1} P^1$  be the rational map defined by  $s_1$  and  $s_2$ , and  $X \xrightarrow{f} C$  be its Stein factorisation.*

*Then  $f^*\Omega_C^1 \subset L \subset \Omega_X^1$ .*

*Furthermore,  $f$  is a morphism.*

*Proof.* Note that for a rational map  $f: X \rightarrow C$  and a line bundle  $M$  on  $C$ ,  $f^*M$  is a well-defined line bundle on  $X$ .

There exists a canonical finite covering  $Y_1 \xrightarrow{\pi_1} X$  having Galois group  $Z/n \oplus Z/n$ , and sections  $t_i \in H^0(Y_1, \pi_1^*L)$  such that  $\pi_1^*s_i = t_i^{\otimes n}$  ( $i=1, 2$ ). Desingularising  $Y_1$  if necessary we get a covering  $Y \xrightarrow{\pi} X$  and sections  $t_i \in H^0(Y, \pi^*L)$  such that  $\pi^*s_i = t_i^{\otimes n}$ . The embedding  $L \subset \Omega_X^1$  lifts to an embedding  $\pi^*L \subset \pi^*\Omega_X^1$ , and composing with the standard map  $d\pi: \pi^*\Omega_X^1 \rightarrow \Omega_Y^1$  gives us a line bundle (not necessarily full)  $\pi^*L \subset \Omega_Y^1$ , and two linearly independent sections  $t_1$  and  $t_2 \in H^0(Y, \pi^*L)$ .

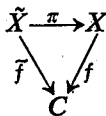
Now consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ g \downarrow & & \downarrow f \\ D & & C \\ \downarrow & & \downarrow \\ P^1 & \longrightarrow & P^1 \end{array}$$

$$(x, y) \longmapsto (x^n, y^n),$$

where the left-hand column is given by Proposition 2 applied to the sections  $t_1$  and  $t_2$  of  $\pi^*L$  on  $Y$ , and the right-hand side is the Stein factorisation of the map defined by  $s_1$  and  $s_2$ . By Proposition 2 we know that  $g^*\Omega_D^1$  coincides at the general point of  $Y$  with  $\pi^*L$ , so that the fibres of  $g$  are tangent to the foliation defined by  $\pi^*L$ ; since the above diagram commutes it follows that the fibres of  $f$  are tangent to the foliation defined by  $L$  at a general point of  $X$ .

This proves the first part of the proposition. For the last assertion we let as before  $\tilde{X} \xrightarrow{\pi} X$  be a sequence of blowings-up which reduces the indeterminacy of  $f$ :



Then if we let  $\tilde{L} \subset \Omega_{\tilde{X}}$  be the full line bundle corresponding to  $\pi^*L$ , then we have both that  $\tilde{L} = \tilde{f}^* \Omega_C^1(D(\tilde{f}))$ , and that the morphism  $\tilde{f}$  is defined by two sections  $\pi^*s_i$  of  $\pi^*L \subset \tilde{L}$ . But if  $f$  was not regular at some point  $P \in X$  then some component of  $\pi^{-1}(P)$  maps surjectively to  $C$  under  $\tilde{f}$ , and the  $\pi^*s_i$  vanish on it. This contradicts  $\tilde{L} = \tilde{f}^* \Omega_C^1(D(\tilde{f}))$ , and so  $f$  is regular.

Theorem 2 is proved.

### § 3. Possible extensions

I would like to mention the following problems concerning the possible generalisations of Theorems 0, 1 and 2; some of these will be treated in F. A. Bogomolov's forthcoming article [0].

(i) *Vector bundles of higher rank on a surface.*

Let  $E$  be a vector bundle of rank  $n$  on a surface. For a full subsheaf  $F \subset E$  of rank  $r$ , let  $F'$  denote the quotient  $E/F$ ;  $F'$  is then torsion free of rank  $r' = n - r$  and locally free outside a finite set  $Z = Z(F)$ . Set

$$\Delta = \Delta(F) = (\det F)^{\otimes r'} \otimes (\det F')^{\otimes -r},$$

where  $\det F'$  is the uniquely defined line bundle extending the usual  $\det F'$  outside  $Z$ .

**Definition 1.**  $E$  is said to be *f-unstable* if there exists a flag

$$F_1 \subset F_2 \subset \dots \subset F_k \subset E, \tag{0}$$

with the  $F_i$  full subsheaves of  $E$ , and integers  $d_i \geq 0$ , not all zero, such that the line bundle

$$L = \bigotimes_i (\Delta_i)^{\otimes d_i}$$

has  $H^0(L) \neq 0$ .

By analogy with the  $n=2$  case one could hope for the following two steps to work:

(a)  $c_1^2 > (2n/(n-1))c_2$  implies that either  $H^0(H_m)$  or  $H^0(H_m^*)$  grows with  $m$ , where  $H_m$  is the auxiliary bundle

$$H_m = S^{m,n}E \otimes (\det E)^{-m};$$

(one checks easily that  $\det H_m = O_X$ ).

(b) Let  $s \in H^0(H_m)$  be a section chosen to vanish at some point  $x \in X$ ; then the general fibre  $s_P$  of  $s$  defines an unstable orbit of  $PGL(n)$  in the representation

$W_m$  to which  $H_m$  is associated. The stabiliser of this orbit provides a parabolic subgroup  $P_s \subset PGL(n)$ , and  $s$  provides a reduction of the structural group of  $E$  to  $P_s$ , which in turn defines a flag (0).

(a) and (b), together with the analog of Lemma 5 of § 1 would give the following result:

$$c_1(E)^2 > \frac{2n}{n-1} c_2(E)$$

implies that  $E$  is  $f$ -unstable, with the line bundle  $L$  in the numerically positive cone.

(ii) *Vector bundles on higher-dimensional varieties.*

Let  $E$  be a rank 2 vector bundle on some variety  $V \subset P^N$  of dimension  $n$  such that

$$c_1(E)^2 \cdot H^{n-2} > 4c_2(E) \cdot H^{n-2};$$

then the restriction of  $E$  down to the general surface section  $X = H^{n-2} \subset V$  satisfies the hypotheses of Theorem 1, so that  $E|_X$  has a positive subbundle  $L_X$ . Do the  $L_X$  come from restricting down a positive subbundle  $L \subset E$  over the whole of  $V$ ?

(iii) *The stability of the tangent bundle to a higher-dimensional variety.*

Let  $V$  be a complex projective manifold, and let  $F \subset \Omega_V^p$  be a rank  $k$  subbundle. Can we limit the Iitaka Kodaira dimension  $\kappa(\det F, V)$  [34] of the line bundle  $\det F$ ?

As Bogomolov points out, the method of proof of the Castelnuovo-Bogomolov Theorem 2 shows easily that in the (essentially equivalent) cases  $p=1$  and  $k=1$  we have

$$(p=1) \quad \kappa(\det F, V) \leq k$$

$$(k=1) \quad \kappa(F, V) \leq p.$$

In view of the poverty of methods available for dealing with varieties of Kodaira dimension 0 [35], the following result is of interest:

**Theorem 3.** *Let  $V$  be a projective non-singular variety of dimension  $n$  over the complex numbers such that  $\text{Pic } V = \mathbf{Z}$  and the anticanonical class  $-K_V$  is a positive generator of  $\text{Pic } V$ . Then the tangent bundle  $T_V$  is stable.*

For otherwise there is a subbundle  $F \subset \Omega_V^1$  of rank  $k < n$  for which the corresponding

$$\Delta(F) = (\det F)^{\otimes n-k} \otimes (\det F')^{\otimes -k} = (\det F)^{\otimes n} \otimes K_V^{\otimes -k}$$

has a section; it follows that  $\det F = K_V^{\otimes m}$  with  $m < 0$ , contradicting the above Castelnuovo-Bogomolov theorem.

Assuming positive answers to the problems mentioned in (i) and (ii) above, we get restrictions on the Chern numbers of  $V$ ; for example, under the additional con-

dition that  $-K$  is ample, we get the bound  $(-K)^3 \leq 72$  for Fano 3-folds. This bound, going back to Fano ([36], [37], p. 83-93), has been recently reestablished by Iskovskikh ([38], see also [39]) by fairly difficult arguments, and under further conditions.

(iv) *Extensions to characteristic  $p$ .*

Shafarevich and Mumford have both pointed out that the proof of Theorem 1 breaks down in characteristic  $p$ , since the part of the divisor ( $s$ ) which meets each fibre in a single point of large multiplicity will in general be a multiple of a surface  $T \subset P(E)$  for which the projection  $T \rightarrow X$  is purely inseparable; the same proof then gives merely the instability of  $\pi^*E$  for a suitable purely inseparable cover  $\pi: Y \rightarrow X$ .

Theorem 2 however would seem to be hopelessly false in characteristic  $p$ ; the proof breaks down since (a) Proposition 1 of § 2 is false, so that we do not know that  $\omega_1 \wedge \omega_2 = 0$  implies that setting  $f = \omega_1 / \omega_2$ ,  $df \wedge d\omega_1 = 0$ ; and (b) even if we knew this, it would still leave the possibility that the map defined by the linear system  $|nL|$  (with  $L \subset \Omega_X^1$ ) is composed of an inseparable morphism of surfaces. The problem suggests itself of knowing whether any positive  $L \subset \Omega_X^1$  corresponds to an inseparable morphism  $f: X \rightarrow Y$  of degree  $p$ , in the sense that  $L$  coincides at the generic point with the image of  $df: f^*\Omega_Y^1 \rightarrow \Omega_X^1$  (see [40]).

There seem to be examples, due to Parshin (unpublished) of surfaces for which the ratio  $\alpha = c_1^2 / c_2$  is arbitrarily large.

**Appendix. The Kodaira vanishing theorem**

**Theorem** (Kodaira [41], C. P. Ramanujam [42], [43]). *Let  $X$  be a non-singular projective surface over the complex numbers, and let  $L \in \text{Pic } X$  be a line bundle such that  $c_1(L)^2 > 0$  and  $c_1(L) \cdot C \geq 0$  for every curve  $C$  on  $X$ .*

*Then  $H^1(L^{-1}) = 0$ .*

*Proof* (D. Mumford). To  $\alpha \in H^1(L^{-1})$  there corresponds in a canonical way an extension

$$(\alpha) \quad 0 \rightarrow O_X \rightarrow E \rightarrow L \rightarrow 0$$

of vector bundles on  $X$ , which is split if and only if  $\alpha = 0$ .

The bundle  $E$  has Chern classes  $c_1(E) = c_1(L)$  and  $c_2(E) = 0$ , so that it satisfies  $c_1(E)^2 > 4c_2(E)$ ; applying Theorem 1 gives us another exact sequence involving  $E$ :

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & O_X & & & \\ & & & \downarrow & & & \\ 0 & \rightarrow & L(-D) & \rightarrow & E & \rightarrow & I_Z \cdot O_X(D) \rightarrow 0 \\ & & & & \downarrow & & \\ & & & & L & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

The positive sub-bundle provided by Theorem 1 has a non-trivial morphism to  $L$ , and is therefore of the form  $L(-D)$ , with  $D \geq 0$  a positive divisor; and from Theorem 1 we have that  $\Delta = L(-2D)$  is in the numerically positive cone of  $NS(X)$ . The numerical conditions we have now contradict the index theorem unless  $D=0$ , in which case the extension  $(\alpha)$  is split and  $\alpha=0$ .

For by Lemma 5 of § 1 we have

$$c_2(E) = 0 = (c_1(L) - D) \cdot D + \deg Z,$$

so that  $c_1(L) \cdot D \leq D^2$ ; and  $c_1(L)^2 \geq 2c_1(L) \cdot D$  follows from the fact that  $L(-2D)$  is in the positive cone and the hypothesis made on  $L$ . Thus either

$$\det \begin{vmatrix} c_1(L)^2 & c_1(L) \cdot D \\ c_1(L) \cdot D & D^2 \end{vmatrix} = c_1(L)^2 D^2 - (c_1(L) \cdot D)^2 > 0,$$

which contradicts the index theorem, or  $D^2 = c_1(L) \cdot D = 0$ , which implies  $D=0$  (by the index theorem once more).

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