Infinitesimal view of extending a hyperplane section – deformation theory and computer algebra

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0.1 The extension problem

Given a variety $C \subset \mathbb{P}^{n-1}$, I want to study extensions of C as a hyperplane section of a variety in $\mathbb{P}^n$:

$$\begin{array}{ccc} C & \subset & \mathbb{P}^{n-1} \\ \bigcap & & \bigcap \\ X & \subset & \mathbb{P}^n \end{array} \quad \text{with} \quad C = \mathbb{P}^{n-1} \cap X;$$

that is, $C : (x_0 = 0) \subset X$, where x_0 is the new coordinate in $\mathbb{P}^n$. I will always take the intersection in the sense of homogeneous coordinate rings, which is a somewhat stronger condition than saying that C is the ideal-theoretical intersection $C = \mathbb{P}^{n-1} \cap X$.

0.2

Some cases of varieties not admitting any extension were known to the ancients: for example, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$ has no extensions other than cones because all varieties of degree 3 are classified ([Scorza1–2, XXX], compare [Swinnerton-Dyer]); and systematic obstructions of a topological nature to the existence of X were discovered from around 1976 by Sommese and others (see [Sommese1], [Fujita1], [Bădescu], [L'vovskii1–2]). More recent work of Sommese points to the conclusion that very few projective varieties C can be hyperplane sections; for example, Sommese

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[Sommese2–3] gives a detailed classification of the cases for which K_C is not ample when $C = \mathbb{P}^{n-1} \cap X$ is a smooth hypersection of a smooth 3-fold X; this amounts to numerical obstructions to the existence of a smooth extension of C in terms of the Mori cone of C.

0.3 The infinitesimal view

Here I'm interested in harder cases, for example the famous problem of which smooth curves C of genus g lie on a K3 surface $C \subset X$; the *infinitesimal view* of this problem is to study the schemes $C \subset 2C \subset 3C \subset \cdots$ that would be the Cartier divisors $kC : (x_0^k = 0) \subset X$ if X existed. Here each step is a linear problem in the solution to the previous one. For example, assuming that C is smooth, the first step is the vector space

$$\mathbb{H}^{(1)} = \bigl\{2C \subset \mathbb{P}^n \text{ extending } C\bigr\} = H^0(N_{\mathbb{P}^{n-1}|C}(-1))$$

or dividing out by coordinate changes,

$$\mathbb{T}^1_{-1} = \bigl\{2C \text{ extending } C\bigr\} = \operatorname{coker}\{H^0(T_{\mathbb{P}^{n-1}}(-1)) \to H^0(N_{\mathbb{P}^{n-1}|C}(-1))\bigr\}.$$

Singularity theorists know this as the graded piece of degree -1 of the deformation space $\mathbb{T}^1$ of the cone over C. However, the extension from $(k-1)C$ up to kC is only an affine linear problem for $k \geq 3$ (because there is no trivial or cone extension of $2C$); in particular 1st order deformations may be obstructed.

0.4

This paper aims to sketch some general theory surrounding the infinitesimal view, and to make the link with deformation theory as practised by singularity theorists. My main interest is to study concrete examples, where the extension-deformation theory can be reduced to explicit polynomial calculation, giving results on moduli spaces of surfaces; for this reason, I have not taken too much trouble to work in intrinsic terms. It could be said that the authors of the intrinsic theory have not exactly gone out of their way to make their methods and results accessible.

The indirect influence on the material of §1 of Grothendieck and Illusie's theory of the cotangent complex [Grothendieck, Illusie2] will be clear to the experts (despite my sarcasm concerning their presentation); §1 can be seen as an attempt to spell out a worthwhile special case of their theory in concrete terms (compare also [Artin]), and I have groped around for years for the translation given in (1.15, 1.18, 1.21) of the enigma [Illusie1, (1.5–7)]. Thus even a hazy understanding of the Grothendieck ideology can be an incisive weapon, which I fear may not pass on to the next generation.

0.5

Already considerations of 1st and 2nd order deformations lead even in reasonably simple cases to calculations that are too heavy to be moved by hand. An eventual aim of this work is to set up an algorithmic procedure to determine the irreducibility or otherwise of the moduli space of Godeaux surfaces with torsion $\mathbb{Z}/2$ or $\{0\}$, suitable for programming into computer algebra (although this paper falls short of accomplishing this); see §2 for this motivation and §6 for a 'pseudocode' description of a computer algebra algorithm that in principle calculates moduli spaces of deformations.

0.6 Acknowledgements

The ideas and calculations appearing here have been the subject of many discussions over several years with Duncan Dicks, and I must apologise to him for the overlap between some sections of this paper and his thesis [Dicks]. I have derived similar (if less obviously related) benefit from the work of Margarida Mendes Lopes [Mendes Lopes]. I am very grateful to David Epstein for encouragement.

I would like to thank Fabrizio Catanese for persuading me to go the extremely enjoyable conference at L'Aquila, and Prof. Laura Livorni and the conference organisers for their hospitality. This conference and the British SERC Math Committee have provided me, in entirely different ways, with a strong challenge to express myself at length on this subject.

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Coda to §3. Canonical map of an irreducible Gorenstein curve and 'general' divisors.

§4. Graded rings on hyperelliptic curves. Easy result that should be better known: if C is hyperelliptic and D a divisor made up of Weierstrass points then the ring $R(C, \mathcal{O}_C(D))$ has a very concrete description.

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§6. Six minuets for a mechanical clock. A 'pseudocode' description of a computer algebra algorithm to compute moduli spaces of deformations.

References

Part I
General theory

1 The Hilbert scheme of extensions

This *overture in the French style* is mainly formalism, and the reader should skip through it rapidly, perhaps taking in the main theme Definition 1.7 and its development in Theorem 1.15; Pinkham's example in §2 gives a quick and reasonably representative impression of what's going on.

1.1

Let $C, \mathcal{O}_C(1)$ be a polarised projective k-scheme (usually a variety), and

$$S = R(C, \mathcal{O}_C(1)) = \bigoplus_{i\geq 0} H^0(C, \mathcal{O}_C(i))$$

the corresponding graded ring. Suppose given a ring $\overline{R} \subset S$ of *finite colength*, that is, such that $S/\overline{R}$ is a finite dimensional vector space. Often $\overline{R} = S$, but

I do not assume this: for example, if $C \subset \mathbb{P}^{n-1}$ is a smooth curve that is not projectively normal, its homogeneous coordinate ring $\overline{R} = k[x_1, \dots, x_n]/I_C$ is of finite colength in $R(C, \mathcal{O}_C(1))$ (the normalisation of $\overline{R}$).

Throughout, a graded ring R is a graded k-algebra

$$R = \bigoplus_{i \geq 0} R_i$$

graded in positive degrees, with $R_0 = k$.

Main problem Main Problem. Given a graded ring $\overline{R}$ and $a_0 \in \mathbb{Z}$, $a_0 > 0$. Describe the set of pairs $x_0 \in R$, where R is a graded ring and $x_0 \in R_{a_0}$ a non-zerodivisor, homogeneous of degree a_0, such that

$$\overline{R} = R/(x_0).$$

Notice that since x_0 is a non-zerodivisor, the ideal $(x_0) = x_0 R \cong R$. If R is given, then I write

$$R^{(k)} = R/(x_0^{k+1}),$$

and call $R^{(k)}$ the *kth order infinitesimal neighbourhood* of $\overline{R} = R^{(0)}$ in R.

This notation and terminology will be generalised in (1.8).

1.2 The hyperplane section principle

Let R be a graded ring and $x_0 \in R$ a homogeneous non-zerodivisor of degree $\deg x_0 = a_0 > 0$; set $\overline{R} = R/(x_0)$. The hyperplane section principle says that quite generally, the generators, relations and syzygies of R reduce mod x_0 to those of $\overline{R} = R/(x_0)$, and in particular, occur in the same degrees. In more detail:

Proposition *(i) Generators. Quite generally, let $R = \bigoplus R_i$ be a graded ring, and $\overline{R} = R/(x_0)$, where $x_0 \in R_{a_0}$. Suppose that $\overline{R}$ is generated by homogeneous elements $x_1, \dots, x_n$ of degree $\deg x_i = a_i$; then R is generated by $x_0, x_1, \dots, x_n$. That is,*

$$\overline{R} = k[x_1, \dots, x_n]/\overline{I} \implies R = k[x_0, \dots, x_n]/I,$$

where $\overline{I} \subset k[x_1, \dots, x_n]$ and $I \subset k[x_0, \dots, x_n]$ are the ideals of relations holding in $\overline{R}$ and R. (See (1.3, (3)) for the several abuses of notation involved in the x_i.)

(ii) *Relations. Keep the notation and level of generality of (i). Suppose that* $f(x_1, \dots, x_n) \in \overline{I}$ *is a homogeneous relation of degree* d *holding in* $\overline{R}$*; then there is a homogeneous relation* $F(x_0, \dots, x_n) \in I$ *of degree* d *holding in* R *such that* $F(0, x_1, \dots, x_n) \equiv f(x_1, \dots, x_n)$.

Let $f_1, \dots, f_m \in \overline{I}$ *be a set of homogeneous relations holding in* $\overline{R}$ *that generates* $\overline{I}$*, and for each* i*, let* $F_i(x_0, \dots, x_n) \in I$ *be a homogeneous relation in* R *such that* $F_i(0, x_1, \dots, x_n) = f_i(x_1, \dots, x_n)$*. Now assume that* x_0 *is a non-zerodivisor. Then* $F_1, \dots, F_m$ *generate* I*; that is,*

$$\overline{I} = (f_1, \dots, f_n) \implies I = (F_1, \dots, F_n) \quad \textit{with} \quad F_i \mapsto f_i.$$

(iii) *Syzygies. Quite generally, let* $F_1, \dots, F_m \in k[x_0, \dots, x_n]$ *be homogeneous elements, and consider the ideal* $I = (F_1, \dots, F_m)$ *and the quotient graded ring* $R = k[x_0, \dots, x_n]/I$*. For each* i*, write* $f_i = F_i(0, x_1, \dots, x_n) \in k[x_1, \dots, x_n]$*, and set* $\overline{I} = (f_1, \dots, f_m)$ *and*

$$\overline{R} = R/(x_0) = k[x_1, \dots, x_n]/\overline{I}.$$

Then the following 3 *conditions are equivalent:*

(a) $x_0 \in R$ *is a non-zerodivisor in* R*;*

(b) $(x_0) \cap I = x_0 I \subset k[x_0, \dots, x_n]$*;*

(c) for every syzygy

$$\sigma : \sum_i \ell_i f_i \equiv 0 \in k[x_1, \dots, x_n]$$

between the f_i *there is a syzygy*

$$\Sigma : \sum_i L_i F_i \equiv 0 \in k[x_0, \dots, x_n]$$

between the F_i *with* $L_i(0, x_1, \dots, x_n) = \ell_i(x_1, \dots, x_n)$.

Remark 1.3 (1) This is standard Cohen–Macaulay formalism, see for example [Mumford1] or [Saint-Donat, (6.6) and (7.9)]; everything works just as well if the non-zerodivisor x_0 is replaced by a regular sequence $(\xi_1, \dots, \xi_k)$.

(2) Recall the general philosophy of commutative algebra that ‘graded is a particular case of local’. The assumption that R is graded and $a_0 > 0$

is used in every step of the argument to reduce the degree and make possible proofs by induction.

In the more general deformation situation $x_0 \in R$ or $x_0 \in H^0(\mathcal{O}_X)$, one must either assume that R or $\mathcal{O}_X$ is (x_0)-adically complete (for example (R, m) is a complete local ring and $x_0 \in m$); or honestly face the convergence problem of analytic approximation of formal structures. This is the real substance of Kodaira and Spencer's achievement in the global analytical context, and, in the algebraic setup, is one of the main themes of [Artin].

By (ii), R is determined by finitely many polynomials of given degree, so it depends *a priori* on a finite dimensional parameter space. Morally speaking, rather than graded and degree < 0, the right hypothesis for the material of this section (and for the algorithmic routines of §6) should be that $\mathbb{T}^1$ and $\mathbb{T}^2$ are finite dimensional.

(3) Abuse of notation. There are two separate abuses of notation in writing x_i: (a) the same x_i is used for the variables in the polynomial ring $k[x_1, \dots, x_n]$ and for the ring element $x_i = \operatorname{im} x_i \in R = k[x_1, \dots, x_n]/I$; there is no real ambiguity here, since I usually write $=$ for equality in R and $\equiv$ for identity of polynomials. (b) I identify the variables in the two polynomial rings $k[x_0, \dots, x_n]$ and $k[x_1, \dots, x_n]$; this means that there is a chosen lifting $k[x_1, \dots, x_n] \hookrightarrow k[x_0, \dots, x_n]$ of the quotient map $k[x_0, \dots, x_n] \to k[x_1, \dots, x_n] = k[x_0, \dots, x_n]/(x_0)$. Notice that from a highbrow point of view, I always work in a given trivial extension of a (smooth) ambient space (with a given retraction or 'face operator'), thus sidestepping the unspeakable if more intrinsic theory of the cotangent complex [Grothendieck, Illusie1–2, Lichtenbaum and Schlessinger].

(4) Higher syzygies for $\overline{R}$ extend to R in a similar way; in fact (1.2, ii–iii) can be lumped together as a more general statement on modules.

(5) The notation of (1.2) will be used throughout §1. I'll write $d_i = \deg f_i$ and $s_j = \deg \sigma_j$.

1.4 Proof of (1.2, i)

Easy: mod x_0, every homogeneous $g \in R$ can be written as a polynomial in $x_1, \dots, x_n$, so that

$$g = g_0(x_1, \dots, x_n) + x_0 g',$$

where $g' \in R$ is of degree $\deg g - a_0 < \deg g$, and induction.

1.5 Proof of (1.2, ii)

It's traditional at this point to draw the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & (x_0)\cap I & \to & I & \to & \overline{I} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (x_0) & \to & k[x_0,\dots,x_n] & \to & k[x_1,\dots,x_n] & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & x_0R & \to & R & \to & \overline{R} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Now $I \to \overline{I}$ is surjective by the Snake Lemma. Take any $f \in \overline{I}$ homogeneous of degree d and $F \in I$ with $F \mapsto f$. Then $f = F - x_0 g$ (this uses the lift $f \in k[x_1,\dots,x_n] \subset k[x_0,\dots,x_n]$). If I take only the homogeneous piece of F and g of degree d then $f = F - x_0 g$ still holds, so $F \mapsto f$.

Now suppose that $\{F_1,\dots,F_n\}$ are chosen to map to a generating set $\{f_1,\dots,f_n\}$ of $\overline{I}$, and let $G \in I$ be any homogeneous element. Then since $G \mapsto g \in \overline{I} = (f_1,\dots,f_n)$, I can write

$$g = \sum \ell_i f_i$$

with homogeneous $\ell_i \in k[x_1,\dots,x_n]$, so

$$H = G - \sum \ell_i F_i \in (x_0)\cap I.$$

Claim *If x_0 is a non-zerodivisor of R then $(x_0)\cap I = x_0 I$.* Because

$$H = x_0 H' \in I \implies x_0 H' = 0 \in R \implies H' = 0 \in R \implies H' \in I.$$

Thus $G = \sum \ell_i F_i + x_0 G'$ with $G' \in I$, so I'm home by induction.

1.6 Proof of (1.2, iii)

(a) $\implies$ (b) has just been proved, and $\Longleftarrow$ is just as elementary. I prove (b) $\implies$ (c). Write $F_i = f_i + x_0 g_i$, and suppose the syzygy of $\overline{R}$ is $\sigma : \sum \ell_i f_i = 0$. Then

$$I \ni \sum \ell_i F_i = x_0 \sum \ell_i g_i \in (x_0),$$

so that (b) implies that $\sum \ell_i g_i \in I$, and so $\sum \ell_i g_i \equiv \sum m_i F_i$. Then

$$\sum L_i F_i = 0, \quad \text{where } L_i = \ell_i - x_0 m_i.$$

Conversely, assume (c) and let $g \in k[x_0, \dots, x_n]$ be such that $x_0 g \equiv \sum \ell_i F_i$. Then $\sum \ell_i f_i \equiv 0$ so that by (c) there exist $L_i \mapsto \ell_i$ with $\sum L_i F_i \equiv 0$, and

$$x_0 g \equiv x_0 \sum (\ell_i - L_i) F_i,$$

so cancelling x_0 gives $g \in I$. Q.E.D.

1.7 The Hilbert scheme of extensions of $\overline{R}$

This solves Problem 1.1: the set of rings $R, x_0 \in R_{a_0}$ such that $\overline{R} = R/(x_0)$ can be given as the set of polynomials F_i extending the relations f_i of $\overline{R}$ such that the syzygies σ_j extend to Σ_j.

To discuss this in more detail, fix once and for all the ring $\overline{R}$, its generators $x_1, \dots, x_n$, relations f_i and syzygies σ_j.

I also fix the polynomial ring $k[x_0, \dots, x_n]$ overlying R and discuss the set of extension rings R together with the data $\{F_i, \Sigma_j\}$ of relations and syzygies as in (1.2). Then

$$\left\{ \forall R, \{F_i, \Sigma_j\} \;\middle|\; R/(x_0) \cong \overline{R} \right\} \xleftrightarrow{\text{bij}}$$
$$\text{BH} = \left\{ \begin{array}{l} F_i = f_i + x_0 g_i \\ \Sigma_i : L_{ij} = \ell_{ij} + x_0 m_{ij} \end{array} \;\middle|\; \sum L_{ij} F_i \equiv 0 \right\}.$$

The set on the right-hand side has a natural structure of an affine scheme $\text{BH} = \text{BH}(\overline{R}, a_0)$, the *big Hilbert scheme of extensions* of $\overline{R}$. For the $g_i \in k[x_0, \dots, x_n]$ and $m_{ij} \in k[x_0, \dots, x_n]$ are finitely many polynomials of given degrees, so their coefficients are finite in number, and can be taken as coordinates in an affine space; the conditions $\sum L_{ij} F_i \equiv 0$ are then a finite set of polynomial relations on these coefficients.

Remark The (small) Hilbert scheme

$$\mathbb{H}(\overline{R}, a_0) = \left\{ \forall R, x_0 \mid R/(x_0) \cong \overline{R} \right\}$$

is part of primeval creation, so can't be redefined: it parametrises ideals $I \subset k[x_0, \dots, x_n]$ such that $(x_0) \cap I = x_0 I$ and $I/x_0 I = \overline{I}$, and is a locally closed subscheme of the Grassmannian of $I_{\leq d} \subset k[x_0, \dots, x_n]_{\leq d}$ (for some large d), as usual in the philosophy of [Mumford2].

Throwing away the extra data $\{F_i, L_{ij}\}$ corresponds to dividing out BH by an equivalence relation. This is rather harmless, and mainly a matter of notation: (1) the equivalence relation is of the form

$$F_i \sim F_i + x_0\left(\sum m_j F_j\right), \quad \text{and similarly for the } \Sigma_j,$$

(because I'm only concerned with the ideal I generated by the F_i, and have fixed $f_i = F_i \bmod x_0$), and is therefore given by a nilpotent group action; (2) by methods of Macaulay and Gröbner, any vector space associated with $k[x_0, \dots, x_n]$ can be given a preferred ordered monomial basis, in terms of which, for a fixed k-valued point $R \in \mathbb{H}(\overline{R}, a_0)$, there is a 'first choice' for the extra data $\{F_i, L_{ij}\}$; I will refer to $\{F_i, L_{ij}\}$ as the *coordinates* of the corresponding $R \in \mathbb{H}(\overline{R}, a_0)$.

Definition 1.8 Suppose given a graded ring $\overline{R}$ and a degree a_0. A ring $R^{(k)}$ together with a homogeneous element $x_0 \in R^{(k)}_{a_0}$ such that $\overline{R} = R^{(k)}/(x_0)$ is a *kth order infinitesimal extension* of $\overline{R}$ if $x_0^{k+1} = 0$ and $R^{(k)}$ is flat over the subring $k[x_0]/(x_0^{k+1})$ generated by x_0.

Remark Write $\mu_i = \mu_{x_0^i}\colon R^{(k)} \to R^{(k)}$ for the map given by multiplication by x_0^i; flatness is of course equivalent to saying that

$$\operatorname{im} \mu_{k+1-i} = \ker \mu_i \cong R^{(i-1)} = R^{(k)}/(x_0^i)$$

for each $0 < i \le k$. This is the nearest a nilpotent element of order $k+1$ comes to being a non-zerodivisor: x_0 kills only multiples of x_0^k.

The material of (1.2–7) can be easily rewritten in this context: to give $R^{(k)}$ is the same thing as to give relations $F_i \in k[x_0, \dots, x_n]/(x_0^{k+1})$ extending f_i and such that the syzygies extend mod x_0^{k+1}.

Thus the *big Hilbert scheme* $\mathrm{BH}^{(k)} = \mathrm{BH}^{(k)}(\overline{R}, a_0)$ of *kth order infinitesimal extensions* of $\overline{R}$ is given as in (1.7) as the affine scheme parametrising power series

$$F_i = f_i + x_0 f_i' + x_0^2 f_i'' + \cdots + x_0^k f_i^{(k)}$$

and

$$L_{ij} = \ell_{ij} + x_0 \ell_{ij}' + \cdots + x_0^k \ell_{ij}^{(k)}$$

for which the syzygies are satisfied up to kth order:

$$\sum L_{ij} F_i \equiv 0 \mod x_0^{k+1}.$$

Similarly, the (small) Hilbert scheme $\mathbb{H}^{(k)} = \mathbb{H}^{(k)}(\overline{R}, a_0)$ parametrises ideals $I^{(k)} \subset k[x_0, \dots, x_n]/(x_0^{k+1})$ with $(x_0) \cap I^{(k)} = x_0 I^{(k)}$ and $I^{(k)}/x_0 I^{(k)} = \overline{I}$.

1.9 There is no convergence problem

In the previous sections I have defined a Hilbert scheme $\mathbb{H}(\overline{R})$ parametrising extension of $\overline{R}$, and kth order Hilbert schemes $\mathbb{H}^{(k)}(\overline{R})$ parametrising kth order infinitesimal extensions $R^{(k)}$, with morphisms $\varphi_k\colon \mathbb{H}^{(k)} \to \mathbb{H}^{(k-1)}$ truncating $R^{(k)}$ into $R^{(k-1)} = R^{(k)}/(x_0^k)$. Now it is fortunate and obvious that $\mathbb{H}(\overline{R}) = \mathbb{H}^{(k)}(\overline{R})$ for sufficiently large k; this follows for reasons already described in (1.3, (2)). More precisely, the syzygies $\sum \ell_{ij} f_i = 0$ are identities between homogeneous polynomials of given degrees, and F_i, L_{ij} have the same degrees as f_i and ℓ_{ij}; therefore, as soon as

$$(k+1)\cdot a_0 = \deg x_0^{k+1} > \max\bigl(\deg \textstyle\sum \ell_{ij} f_i\bigr),$$

I get the implication

$$\sum L_{ij}F_i \equiv 0 \bmod x_0^{k+1} \implies \sum L_{ij}F_i \equiv 0.$$

Notice that, for reasons mentioned in (1.3, (2)), the Hilbert schemes of (1.7–8) are finite dimensional with no restrictions on singularities. This contrasts with deformation theory, where for example a cone over a singular curve already has infinite dimensional versal deformation; of course, I'm only trading in the negatively graded portion of the deformation theory.

Actually, most of what I do in this paper will work with minor modifications for deformations in degree 0 (that is, considering algebras R or $R^{(k)}$ over $k[\![\lambda]\!]$ or $k[\lambda]/(\lambda^{k+1})$ that are flat deformations of $\overline{R}$ and are graded with $\deg\lambda = 0$). Then the convergence problem is nontrivial, but well understood: the Hilbert scheme of $\overline{R}$ is a (bounded) projective k-scheme H, and a formal deformation R over $k[\![\lambda]\!]$ is a formal curve in H, so can be analytically or algebraically approximated.

Deformation obstructions and iterated linear structure

In the remainder of this section I give a concrete description of the tower of schemes

$$\mathbb{H} \to \cdots \to \mathbb{H}^{(k)} \to \mathbb{H}^{(k-1)} \to \cdots \to \mathbb{H}^{(1)} \to \mathbb{H}^{(0)} = \mathrm{pt}.$$

in terms of $\overline{R}$ and the $\overline{R}$-modules $\overline{I}/\overline{I}^2$ (the 'conormal sheaf' to $\overline{R}$); see Theorem 1.15. It is worth treating the 1st order case separately, although logically it is covered by the higher order statement (1.15).

1.10 1st order theory

Theorem $\mathbb{H}^{(1)}(\overline{R}, a_0) = \mathrm{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-a_0}$.

Notes (a) The subscript refers of course to the degree $-a_0$ piece of the graded module Hom, consisting of $\overline{R}$-linear maps of degree $-a_0$.

(b) If it helps at all, think of the right-hand side as the group of splittings of

$$0 \to x_0\overline{R} \to \overline{I}/\overline{I}^2 \oplus x_0\overline{R} \to \overline{I}/\overline{I}^2 \to 0.$$

where the middle term is $(\overline{I}, x_0)/(\overline{I}, x_0)^2 \subset k[x_0, \dots, x_n]/(\overline{I}, x_0)^2$ split by the lifting (1.3, (3)).

(c) For the relation with $\mathbb{T}^1$ see (1.22) below.

Coordinate-free proof of the theorem

Write $A = k[x_1, \dots, x_n]$ for brevity. Starting from $\overline{I} \subset A$, look for vector subspaces

$$I^{(1)} \subset A \oplus x_0 A$$

satisfying

(a) $I^{(1)} + x_0 A = \overline{I} \oplus x_0 A$;

(b) $x_0 A \cap I^{(1)} = x_0 I^{(1)} = 0 \oplus x_0 \overline{I}$;

(c) $I^{(1)}$ is an ideal of $A \oplus x_0 A$.

Subgroups $I^{(1)}$ enjoying (a) and (b) are in bijection with maps $\varphi\colon \overline{I} \to \overline{R}$ by a standard graph argument: if $I^{(1)}$ is given, then by (a), for all $f \in \overline{I}$ there exists f' such that $f + x_0 f' \in I^{(1)}$, and by (b), f' is uniquely determined mod $\overline{I}$, so $f \mapsto f'$ mod $\overline{I}$ defines $\varphi\colon \overline{I} \to \overline{R}$. The inverse construction is obvious.

Finally, it's an easy calculation to see that $I^{(1)}$ is an ideal if and only if φ is A-linear. Graded ideals $I^{(1)}$ correspond to graded maps φ of degree $-a_0$.

1.11 Coordinate proof of (1.10)

It is useful to have a coordinate proof, both for algorithmic applications, and to make the link with the projective resolution of $\overline{I}/\overline{I}^2$ in (1.13–17). By construction, $\mathrm{BH}^{(1)}(\overline{R}, a_0) =$

$$\left\{(f_i + x_0 f_i'), (\ell_{ij} + x_0 \ell_{ij}') \;\middle|\; \forall j, \sum (\ell_{ij} + x_0\ell_{ij}')(f_i + x_0 f_i') \equiv 0 \bmod x_0^2\right\},$$

since $\sum \ell_{ij} f_i = 0$ is given, strip off this term and divide through by x_0 (thus reducing degrees by a_0); then the defining equation is

$$\sum \ell_{ij} f'_i + \sum \ell'_{ij} f_i \equiv 0 \in A_{s_j - a_0},$$

where $s_j = \deg \Sigma_j$. Since the second summand is an arbitrary element of $\overline{I}$, the condition on $\{f'_i\}$ is

$$\sum \ell_{ij} f'_i = 0 \in \overline{R}_{s_j - a_0} \quad \text{(for all } j).$$

Now $\overline{I}$ is the A-module generated by $\{f_i\}$ with relations $\sum \ell_{ij} f_i = 0$, so that $f_i \mapsto f'_i$ defines an A-linear map $\varphi\colon \overline{I} \to \overline{R}$, hence an element $\varphi \in \operatorname{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-a_0}$.

Clearly, φ only depends on $f'_i \in \overline{R}$; the equivalence relation $\sim$ on $\{(f'_i, \ell'_{ij})\} \in \mathrm{BH}^{(1)}(\overline{R}, a_0)$ defining $\mathbb{H}^{(1)}(\overline{R}, a_0)$ ignores the ℓ'_{ij} completely, and takes account only of the classes $f'_i \in \overline{R}$ of the $f'_i \bmod \overline{I}$ (as described in (1.7)). It is now easy to see that there is a bijection between the 3 sets:

(1) $\mathrm{BH}^{(1)}(\overline{R}, a_0)/\sim$;

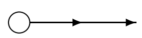
(2) ideals $I^{(1)} \subset k[x_0, x_1, \dots, x_n]/(x_0^2)$ generated by $\{f_i + x_0 f'_i\}$;

(3) $\operatorname{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-a_0}$. Q.E.D.

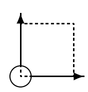
1.12 Preparations for the main theorem

Higher order deformation theory studies extensions of an individual ring $R^{(k-1)}$ to $R^{(k)}$, and on the level of the Hilbert schemes, the morphisms $\varphi_k\colon \mathbb{H}^{(k)} \to \mathbb{H}^{(k-1)}$ defined by the forgetful maps $R^{(k)} \mapsto R^{(k-1)}$.

I first indicate briefly why the higher order problem differs from the 1st. To extend the ring $R^{(1)}$ to $R^{(2)}$ one takes account of new terms in x_0^2; however, x_0 is already involved in $R^{(1)}$, so that x_0^2 is not just a coordinate in a transverse extension of the ambient space. A picture in the simplest case may help:



$k[x]/x^3$



$k[x,y]/(x,y)^2$

1.13 The extension problem in coordinates and the obstruction ψ

The forgetful maps $R^{(k)} \mapsto R^{(k-1)}$ define morphisms $\varphi_k\colon \mathrm{BH}^{(k)} \to \mathrm{BH}^{(k-1)}$ that take the kth order power series

$$f_i + x_0 f_i' + x_0^2 f_i'' + \cdots + x_0^k f_i^{(k)} \quad \text{and} \quad \ell_{ij} + x_0 \ell_{ij}' + \cdots + x_0^k \ell_{ij}^{(k)} \tag{1}$$

into their $(k-1)$st order truncations

$$F_i = f_i + x_0 f_i' + \cdots + x_0^{k-1} f_i^{(k-1)} \text{ and } L_{ij} = \ell_{ij} + x_0 \ell_{ij}' + \cdots + x_0^{k-1} \ell_{ij}^{(k-1)}. \tag{2}$$

Now suppose given $(k-1)$st order power series F_i and L_{ij} satisfying

$$\sum L_{ij} F_i \equiv 0 \bmod x_0^k; \tag{3}$$

extending these to kth order power series satisfying the same mod x_0^{k+1} is clearly equivalent to fixing up the new kth order terms $f_i^{(k)}$ and $\ell_{ij}^{(k)}$ to satisfy

$$\sum \ell_{ij} f_i^{(k)} + \sum_{a=1}^{k-1} \sum \ell_{ij}^{(a)} f_i^{(k-a)} + \sum \ell_{ij}^{(k)} f_i \equiv 0 \tag{4}$$

(this has involved using (3) to kill terms not divisible by x_0^k, then dividing through by x_0^k, thus lowering degrees by ka_0); that is,

$$\sum \ell_{ij} f_i^{(k)} + \sum \ell_{ij}^{(k)} f_i \equiv - \sum_{a=1}^{k-1} \sum \ell_{ij}^{(a)} f_i^{(k-a)}. \tag{5}$$

This is a set of inhomogeneous linear equations in the new unknowns $f_i^{(k)}$ and $\ell_{ij}^{(k)}$. As in (1.11), the second term on the left-hand side is just an arbitrary element of $\overline{I}_{s_j - ka_0}$, so working mod $\overline{I}$ allows me to forget it:

$$\sum \ell_{ij} f_i^{(k)} = - \sum_{a=1}^{k-1} \sum \ell_{ij}^{(a)} f_i^{(k-a)} = \psi_j \in \overline{R}_{s_j - ka_0}. \tag{6}$$

(5) are of course the equations of $\mathrm{BH}^{(k)}$ as a scheme over $\mathrm{BH}^{(k-1)}$. This is all one needs from the point of view of practical computation: the right-hand side of (6) is a given vector $\psi = \{\psi_j\} \in \bigoplus_j \overline{R}_{s_j - ka_0}$ (depending in a bilinear way on the given $(k-1)$st order power series F_i and L_{ij}). Maybe the left-hand side fails to hit ψ at all, so there's an obstruction to extending

$R^{(k-1)}$. But if it does hit ψ, the ambiguity in the choice of $f_i^{(k)}$ is the vector space $\{f_i^{(k)} \mid \sum \ell_{ij} f_i^{(k)} = 0 \in \overline{R}_{s_j - ka_0}\}$; this space depends only on $\overline{R}$ (not on $R^{(k-1)}$), and is the same space as in (1.11) with a_0 replaced by ka_0. This shows that $\mathrm{BH}^{(k)} \to \mathrm{BH}^{(k-1)}$ is an affine fibre bundle over its image.

Definition. Write

$$\psi \colon \mathrm{BH}^{(k-1)}(\overline{R}, a_0) \to \bigoplus \overline{R}(s_j)_{-ka_0}$$

for the polynomial map defined by $\psi_j = -\sum_{a=1}^{k-1} \sum_i \ell_{ij}^{(a)} f_i^{(k-a)} \mod \overline{I}$ (see (6)).

1.14 Notation

For brevity, write $A = k[x_1, \dots, x_n]$, and let

$$\cdots \xrightarrow{(m_{jn})} \bigoplus A(-s_j) \xrightarrow{(\ell_{ij})} \bigoplus A(-d_i) \xrightarrow{(f_i)} \overline{I} \to 0$$

be the projective resolution of $\overline{I}$ as A-module. (Recall $d_i = \deg f_i$ and $s_j = \deg \sigma_j$; the twistings $A(-d_i)$ etc. are a traditional device to make the homomorphisms graded of degree 0.)

Then $\mathrm{Ext}^i_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})$ is the homology of the dual complex

$$0 \to \bigoplus \overline{R}(d_i) \xrightarrow{\delta_0} \bigoplus \overline{R}(s_j) \xrightarrow{\delta_1} \cdots$$

(the *conormal complex* of $\overline{R} = A/\overline{I}$), and in particular there is an exact sequence

$$0 \to \mathrm{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R}) \to \bigoplus \overline{R}(d_i) \xrightarrow{\delta_0} \ker \delta_1 \xrightarrow{\pi} \mathrm{Ext}^1_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R}) \to 0.$$

Main theorem 1.15 *(i)* $\psi \colon \mathrm{BH}^{(k-1)} \to \bigoplus \overline{R}(s_j)_{-ka_0}$ *factors through a morphism of schemes* $\Psi \colon \mathbb{H}^{(k-1)}(\overline{R}, a_0) \to \bigoplus \overline{R}(s_j)_{-ka_0}$ *(the target is a finite dimensional vector space viewed as an affine variety).*

(ii) $\Psi \colon \mathbb{H}^{(k-1)}(\overline{R}, a_0) \to \ker \delta_1 \subset \bigoplus \overline{R}(s_j)_{-ka_0}$.
Therefore, the middle square in the diagram

$$\begin{array}{ccccccc}
 & & \mathbb{H}^{(k)}(\overline{R}, a_0) & \xrightarrow{\varphi_k} & \mathbb{H}^{(k-1)}(\overline{R}, a_0) & & \\
 & & \downarrow & & \downarrow \Psi & & \\
0 \to \mathrm{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0} & \to & \bigoplus \overline{R}(d_i)_{-ka_0} & \xrightarrow{\delta_0} & (\ker \delta_1)_{-ka_0} & & \\
 & & & & & \xrightarrow{\pi} & \mathrm{Ext}^1_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0} \to 0
\end{array}$$

is Cartesian.

In other words, the morphism $\varphi_k\colon \mathbb{H}^{(k)}(\overline{R}, a_0) \to \mathbb{H}^{(k-1)}(\overline{R}, a_0)$ *has the following structure: its image* $\operatorname{im}\varphi_k$ *is the scheme-theoretic fibre over* 0 *of the morphism* $\pi\circ\Psi\colon \mathbb{H}^{(k-1)}(\overline{R}, a_0) \to \operatorname{Ext}^1_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0}$*; so* $\pi\circ\Psi(R^{(k-1)})$ *is the* obstruction to extending $R^{(k-1)}$ to kth order*. And* $\varphi_k\colon \mathbb{H}^{(k)}(\overline{R}, a_0) \to \operatorname{im}\varphi_k \subset \mathbb{H}^{(k-1)}(\overline{R}, a_0)$ *is a fibre bundle (in the Zariski topology), with fibre an affine space over* $\operatorname{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0}$.

Given (i) and (ii), the Cartesian square just restates equations (6): a point $R^{(k)} \in \mathbb{H}^{(k)}(\overline{R}, a_0)$ over $R^{(k-1)}$ is $\{f_i^{(k)} \in \overline{R}(d_i - ka_0)\}$ which maps to $\Psi(R^{(k-1)})$ on taking $\sum \ell_{ij} f_i^{(k)}$, that is, under δ_0.

This can only exist if $\Psi(R^{(k-1)})$ is a boundary, so maps to 0 in Ext^1. Thus the image of $\mathbb{H}^{(k)}(\overline{R}, a_0) \to \mathbb{H}^{(k-1)}(\overline{R}, a_0)$ is the fibre over 0 of the composite morphism $\pi\circ\Psi\colon \mathbb{H}^{(k-1)}(\overline{R}, a_0) \to \ker\delta_1 \to \operatorname{Ext}^1_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})$ given by the homology class of ψ; its fibres (when nonempty) are affine spaces under

$$\left\{f_i^{(k)} \in \overline{R}(d_i - ka_0) \;\middle|\; \sum \ell_{ij} f_i^{(k)} \equiv 0\right\} = \ker\delta_0 = \operatorname{Hom}^1_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0}.$$

So the only points to prove are (i) and (ii). I first give set-theoretic proofs, working with a $(k-1)$st order infinitesimal extension ring $R^{(k-1)} \in \mathbb{H}^{(k-1)}(\overline{R}, a_0)$ defined over the base field k (sorry about the notation clash), and choosing a lift to $\{F_i, L_{ij}\} \in \mathrm{BH}^{(k-1)}(\overline{R}, a_0)$. In fact the same proof works scheme-theoretically; the necessary technicalities are not hard, but are left to (1.20) to allow the reader to skip them.

1.16 Proof of (i)

This is similar to the proof of (1.11). Suppose that (F_i, L_{ij}) and $('F_i, 'L_{ij})$ are two choices of coordinates for $R^{(k-1)}$, with $\left\{F_i, 'F_i \in I_{d_i}^{(k-1)}\right\}$ and

$$\sum L_{ij} F_i \text{ and } \sum ('L_{ij})('F_i) \equiv 0 \bmod x_0^k.$$

Then the difference

$$\sum L_{ij} F_i - \sum ('L_{ij})('F_i) \in x_0^k \cap I^{(k-1)} = x_0^k \overline{I}.$$

Since ψ_j is just minus the coefficient of x_0^k in $\sum L_{ij} F_i$ reduced mod $\overline{I}$, this proves that it depends only on $I^{(k-1)}$, or $R^{(k-1)} \in \mathbb{H}^{(k-1)}(\overline{R}, a_0)$, so Ψ is well defined.

1.17 Proof of (ii)

I introduce notation for the 2nd syzygies involved in δ_1. These form the next term in the projective resolution of $\overline{I}$ in (1.14), and are identities of the form

$$\mu : \sum_j m_j \ell_{ij} = 0$$

holding between the 1st syzygies $\sum_i \ell_{ij} f_i = 0$. Suppose that $\mu_n = (m_{jn})$ is a generating set. Then as mentioned in (1.3, (4)), these identities lift to identities

$$\sum_j M_{jn} L_{ij} = \sum_j (m_{jn} + x_0 m'_{jn} + \cdots + x_0^{k-1} m_{jn}^{(k-1)}) \\ \times (\ell_{ij} + \cdots + x_0^{k-1} \ell_{ij}^{(k-1)}) \equiv 0 \bmod x_0^k$$

between the 1st syzygies for $R^{(k-1)}$. Now hold on tight, please: write Δ_n for the x_0^k term in

$$\sum_j \Big(M_{jn} \sum_i L_{ij} F_i \Big) = \sum_i \Big(\sum_j M_{jn} L_{ij} F_i \Big).$$

I'm going to calculate Δ_n on the two sides. Working on the left, since $\sum_i L_{ij} F_i \equiv -\psi_j x_0^k \bmod x_0^{k+1}$ and $M_{jn} \equiv m_{jn} \bmod x_0$, the coefficient of x_0^k is $\Delta_n = -\sum_j m_{jn} \psi_j$. On the right, since $\sum_j M_{jn} L_{ij} \equiv 0 \bmod x_0^k$, I can pick out the leading term and write

$$\sum_j M_{jn} L_{ij} \equiv \theta_{ni} x_0^k \bmod x_0^{k+1};$$

then since $F_i \equiv f_i \bmod x_0$, it follows that $\Delta_n = \sum_i \theta_{ni} f_i \in \overline{I}$. Now I've won: δ_1 is the map given on $\bigoplus \overline{R}(s_j)_{-ka_0}$ by the matrix (m_{jn}), and I've just proved that each coefficient

$$\Delta_n = -\sum_j m_{jn} \psi_j = \sum_i \theta_{ni} f_i = 0 \in \overline{R}. \quad \text{Q.E.D.}$$

1.18 Normal structure to $R^{(k-1)}$ and obstruction calculus

It is interesting to give an alternative coordinate-free treatment of what Theorem 1.15 means for a given ring $R^{(k-1)}$ (that is, a given k-valued point of $\mathbb{H}^{(k-1)}$).

Same result kth order extensions $R^{(k)}$ of a given $(k-1)$st order extension ring $R^{(k-1)}$ are in bijection with splittings of a certain 'normal' sequence

$$0 \to x_0^k\overline{R} \to N \to \overline{I}/\overline{I}^2 \to 0 \qquad (*)$$

of graded $\overline{R}$-modules deduced from $R^{(k-1)}$.

Compare with (1.10, Note b). This gives the obstruction $\psi(R^{(k-1)}) \in \operatorname{Ext}^1_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0}$ to the existence of an extension, and if it vanishes, the set of extensions is an affine space under $\operatorname{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0}$.

1.19 Proof. Step 1 (notation and set-up)

I start from the ideal

$$J = I^{(k-1)} \subset A[x_0]/(x_0^k) = A \oplus x_0A \oplus \cdots \oplus x_0^{k-1}A$$

defining $R^{(k-1)}$ (so satisfying $J \cap x_0^{k-1}A = x_0^{k-1}J = x_0^{k-1}\overline{I}$). Introduce subgroups

$$L = (\overline{I} + (x_0)) \cdot_{\text{new}} J \subset M = J \oplus x_0^kA \subset A \oplus x_0A \oplus \cdots \oplus x_0^kA.$$

Here M is the inverse of J under $A[x_0]/(x_0^{k+1}) \to A[x_0]/(x_0^k)$, hence an ideal. For L, note that the old multiplication by x_0 in $A[x_0]/(x_0^k)$ does not correspond to multiplication in $A[x_0]/(x_0^{k+1})$; also, $J = J \oplus 0 \subset A[x_0]/(x_0^{k+1})$ is not an ideal (I'm paying here for the abuse of notation (1.3, (3))). So I'm defining L (*a priori* only a subgroup) by

$$L = \overline{I} \cdot J + x_0 \cdot_{\text{new}} J;$$

the first summand is $\overline{I} \cdot J \oplus 0$, and in the second, the operation $x_0 \cdot_{\text{new}}$ just shifts down the terms

$$f + x_0f' + \cdots + x_0^{k-1}f^{(k-1)} \in J \;\mapsto\; x_0f + x_0^2f' + \cdots + x_0^kf^{(k)}$$

without killing anything off.

Step 2 (definition of the $\overline{R}$-module N)

(i) L is an ideal of $A[x_0]/(x_0^{k+1})$; (ii) multiplication by x_0 and $\overline{I}$ both take M into L. Therefore the quotient $N = M/L$ is an $\overline{R}$-module.

Proof (i) Elements of A multiply each summand of L to itself, and x_0 multiplies the first summand into the second, so OK. (ii) Similarly, x_0 and $\overline{I}$ both multiply the first summand $J \oplus 0$ of M to L; the second summand $x_0^k A$ gets killed by x_0 and multiplied into $x_0 \cdot_{\text{new}} J$ by $\overline{I}$, because $\overline{I} = J$ mod x_0.

Step 3 (exact sequence $(*)$)

(i) $x_0^k A \cap L = x_0^k \overline{I}$; (ii) $x_0^k A + L = (\overline{I} J + x_0 J) \oplus x_0^k A \subset J \oplus x_0^k A$. Therefore the module N defined in Step 2 fits into the exact sequence

$$0 \to x_0^k \overline{R} \to N \to \overline{I}/\overline{I}^2 \to 0 \qquad (*)$$

of graded $\overline{R}$-modules, induced by the split sequence $x_0^k A \hookrightarrow M \twoheadrightarrow J$ in the definition of M.

Proof (i) Since $x_0^k A \cap \overline{I} J = \emptyset$, this is clear:

$$x_0^k A \cap x_0 \cdot_{\text{new}} J = x_0(x_0^{k-1} A \cap J) = x_0^k \overline{I}.$$

(ii) Adding $x_0^k A$ kills the difference between $\cdot_{\text{new}}$ and $\cdot_{\text{old}}$, so

$$L + x_0^k A = \overline{I} J + x_0 \cdot_{\text{new}} J + x_0^k A = (\overline{I} J + x_0 J) \oplus x_0^k A.$$

The terms of the exact sequence follow because $x_0^k A / x_0^k \overline{I} = x_0^k \overline{R}$ and

$$\{J \oplus x_0^k A\} / \{(\overline{I} J + x_0 J) \oplus x_0^k A\} = J / (\overline{I} J + x_0 J) = \overline{I}/\overline{I}^2.$$

Step 4 (graph argument as in (1.10))

Starting from

$$J = I^{(k-1)} \subset A[x_0]/(x_0^k) = A \oplus x_0 A \oplus \cdots \oplus x_0^{k-1} A,$$

look for subgroups

$$J' = I^{(k)} \subset M = J \oplus x_0^k A \subset A[x_0]/(x_0^{k+1}) = A \oplus x_0 A \oplus \cdots \oplus x_0^k A,$$

satisfying

(a) $J' + x_0^k A = J \oplus x_0^k A$;

(b) $x_0^k A \cap J' = x_0^k \overline{I}$;

(c) J' is an ideal of $A \oplus x_0 A \oplus \cdots \oplus x_0^k A$.

As in (1.10), subgroups J' for which (a) and (b) hold correspond bijectively with maps $\varphi\colon J \to \overline{R}$. For if J' is given, then by (a), for all $F = f + x_0 f' + \cdots + x_0^{k-1} f^{(k-1)} \in J$ there exists $f^{(k)}$ such that $F + x_0^k f^{(k)} \in J'$, and by (b), $f^{(k)}$ is uniquely determined mod $\overline{I}$, so $F \mapsto f^{(k)}$ mod $\overline{I}$ defines $\varphi\colon J \to \overline{R}$. The map φ is not quite what I want to express that J' is an ideal. Try instead the following:

$$\Phi\colon J \to (J \oplus x_0^k A)/L = N \quad \text{by} \quad F \mapsto \Phi(F) = F + x_0^k f^{(k)} \bmod L.$$

After passing to quotients as in Step 3, any map of this form (with first component F) obviously splits $(*)$ as a sequence of vector spaces; also, by Step 3, (i), knowing $f^{(k)}$ mod $\overline{I}$ is equivalent to $\Phi(F) = F + x_0^k f^{(k)}$ mod L, so that maps Φ also correspond bijectively with subspaces J' satisfying (a) and (b).

Step 5

J' is an ideal if and only if Φ is $A[x_0]/(x_0^k)$-linear.

It is trivial to check that Φ is A-linear if and only if $AJ' \subset J'$, so the point is to deal with the multiplications (old and new) by x_0. It's easy to check in turn that the following 3 conditions are equivalent:

(I) J' is closed under multiplication by x_0;

(II) $f + x_0 f' + x_0^2 f'' + \cdots + x_0^k f^{(k)} \in J' \implies x_0 f + x_0^2 f' + \cdots + x_0^k f^{(k-1)} \in J'$;

(III) $F = f + x_0 f' + \cdots + x_0^{k-1} f^{(k-1)} \in J \implies \Phi(x_0 F) = 0$.

Since $\Phi\colon J \to (M \oplus x_0^k A)/L$ is $A[x_0]/(x_0^k)$-linear if and only if the induced splitting $J/(\overline{I}J + x_0 J) = \overline{I}/\overline{I}^2 \to N$ is $\overline{R}$-linear, this completes the proof of (1.13). Q.E.D.

Technical appendix

1.20 Scheme-theoretic proof of (1.15)

Required to prove that all the maps in

$$\begin{tikzcd} \mathrm{BH}^{(k-1)} \arrow[d] \arrow[ddr] & \\ \mathbb{H}^{(k-1)} \arrow[d] & \\ \ker\delta_1 & \subset \quad \bigoplus \overline{R}(s_j)_{-ka_0} \end{tikzcd}$$

are morphisms of schemes. The proof given in (1.16–17) is inadequate because it deals only with k-valued points of $\mathrm{BH}^{(k-1)}$ and $\mathbb{H}^{(k-1)}$.

Thus the point is just to use the proper definition of the Hilbert scheme $\mathbb{H}^{(k-1)}$ as the functor of ideals extending $\overline{I}$. More precisely, this is the functor on k-algebras S

$$S \mapsto \left\{S\text{-submodules } J^{(k-1)} \subset S[x_0,\dots,x_n]/(x_0^k) \text{ s.t. (a-c) hold}\right\},$$

where (a) $J^{(k-1)}+(x_0) = \overline{I}\cdot S[x_1,\dots,x_n]\oplus(x_0)$; (b) $(x_0)\cap J^{(k-1)} = x_0J^{(k-1)}$; and (c) $J^{(k-1)}$ is an ideal, that is, $x_iJ^{(k-1)} \subset J^{(k-1)}$ for each i. Since these conditions are locally closed, $\mathbb{H}^{(k-1)}$ is a representable functor, naturally represented by a subscheme of the Grassmannian of linear subspaces of $\left(k[x_0,\dots,x_n]/(x_0^k)\right)_{\le d}$ for some large d.

There is a universal sheaf of ideals $\underline{J}^{(k-1)} \subset \mathcal{O}_{\mathbb{H}^{(k-1)}}[x_0,\dots,x_n]/(x_0^k)$ over $\mathbb{H}^{(k-1)}$ defined by tautological Grassmannian considerations, and the proof of (1.15) just consists of applying the arguments of (1.16–17) to the stalks of $\underline{J}^{(k-1)}$. That is, let $P \in \mathbb{H}^{(k-1)}$ be a scheme-theoretic point, $S = \mathcal{O}_{\mathbb{H}^{(k-1)},P}$ its local ring and $J^{(k-1)} \subset S[x_0,\dots,x_n]/(x_0^k)$ the stalk of $\underline{J}^{(k-1)}$; then, taking F_i to be elements of $S[x_0,\dots,x_n]/(x_0^k)$ generating $J^{(k-1)}$ and similarly for L_{ij}, M_{jn} etc., the arguments of (1.16–17) go through without modification.

1.21 The obstruction as a cup product

The fact that the obstruction to extending $R^{(k-1)}$ is bilinear in the coordinates F_i and L_{ij} (see (1.13)) has a highbrow interpretations (see [Illusie1, (1.7, iii)]). Consider the diagram

$$\begin{array}{ccc} R^{(k-1)} & \leftarrow & ? \\ \uparrow & & \uparrow \\ k[x_0]/(x_0^k) & \leftarrow & k[x_0]/(x_0^{k+1}) \\ \uparrow & & \\ k & & \end{array}$$

The $\{L_{ij}\}$ can be interpreted as a normal or 'Kodaira–Spencer' class of $R^{(k-1)}$ over $k[x_0]/(x_0^k)$. The kernel of $k[x_0]/(x_0^{k+1}) \to k[x_0]/(x_0^k)$ is the 1-dimensional vector space $V = \langle x_0^k \rangle$; since multiplication by x_0^k is nonzero, this extension has a nonzero class, which pulls back to $R^{(k-1)}$ as multiplication by the F_i. The obstruction can be thought of as a cup product of these two classes.

1.22 Coordinate changes, $\mathbb{T}^1$ and moduli space of extensions

The Hilbert schemes $\mathbb{H}(\overline{R}, a_0)$ and $\mathbb{H}^{(k)}(\overline{R}, a_0)$ contain every extension of $\overline{R}$, but in a redundant way. One can define the *moduli space* of extensions, and compute it as $\mathbb{H}$ or $\mathbb{H}^{(k)}$ divided out by an equivalence relation given by coordinate changes of the form

$$x_h \mapsto x_h + \alpha_h x_0 \quad \text{with} \quad \deg \alpha_h = a_h - a_0.$$

Thus the answer to the 1st order extension problem for smooth $C \subset \mathbb{P}^{n-1}$ is

$$\mathbb{T}^1_{-1} = \{2C \text{ extending } C\} = \operatorname{coker}\Big\{ H^0(T_{\mathbb{P}^{n-1}}(-1)) \to H^0(N_{\mathbb{P}^{n-1}|C}(-1)) \Big\}.$$

Here $H^0(N_{\mathbb{P}^{n-1}|C}(-1)) = \mathbb{H}^{(1)}$ is the set of subschemes $2C \subset \mathbb{P}^n$ extending C in a fixed coordinate system $(x_0, x_1, \dots, x_n)$. Passing to the cokernel consists exactly of dividing out by $H^0(T_{\mathbb{P}^{n-1}}(-1))$, corresponding to coordinate changes $x_h \mapsto x_h + \alpha_h x_0$ fixing $\mathbb{P}^{n-1}$.

Quite generally, for 1st order extensions, these coordinate changes have the effect

$$f_i' \mapsto f_i' + \sum_h \alpha_h \frac{\partial f_i}{\partial x_h},$$

where $\deg \alpha_h = a_h - a_0$; thus in the deformation theory of a hypersurface singularity $V(f)$, the f' can be chosen arbitrarily, so that the Hilbert scheme

$\mathbb{H}^{(1)}$ is just one graded piece of a polynomial ring, and $\mathbb{T}^1$ is one graded piece of the quotient by the Jacobian ideal $(\partial f/\partial x_h)$. The Jacobian matrix $J(\partial f_i/\partial x_h)$ plays a similar role for complete intersection singularities.

Dividing out by the coordinate changes is not essential for most theoretical purposes: the redundant information contained in the Hilbert scheme does not affect questions such as the connectedness, irreducibility or unirationality of the set of extensions, or obstructions, or forcing into determinantal form. That is why this section has mainly concentrated on Hilbert schemes.

However, in practical (hand or machine) calculations it's often essential to use coordinate changes of this form to cut down the large number of free parameters I have to carry around. Thus in calculations I usually work with $\mathbb{T}^1$ moduli schemes of deformations. See (2.4), (5.13) and (6.3, Step 1) for practical illustrations.

In intrinsic terms, $\mathbb{T}^1(\overline{R}) = \mathrm{Ext}^1_{\overline{R}}(\Omega^1_{\overline{R}}, \overline{R})$; the reader versed in these matters will know that in the derived category there is no essential difference between working with this group or with $\mathrm{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})$.

1.23 Abstract versus projective extensions

There is a corresponding 'abstract' extension problem, in which C is not thought of in an ambient projective space, but an ample normal bundle $\mathcal{O}(1)$ is fixed; for example, assuming that C is nonsingular, the set parametrising the abstract extensions $C \subset 2C$ such that $\ker\{\mathcal{O}_{2C} \to \mathcal{O}_C\} \cong \mathcal{O}_C(-1)$ is the whole of $H^1(T_C(-1))$, by analogy with Kodaira–Spencer deformation theory. In this context the graded piece $\mathbb{T}^1_{-1}$ is recovered as the set of abstract extensions $C \subset 2C$, together with a lift $\widetilde{x}_i \in H^0(\mathcal{O}_{2C}(1))$ of each $x_i \in H^0(\mathcal{O}_C(1))$. Thus $\mathbb{T}^1_{-1}$ maps to $H^1(T_C(-1))$, with image consisting of abstract extensions that satisfy an infinitesimal analogue of linear normality.

1.24 The unobstructed case

The ideal case of the extension problem is when every $R^{(k)}$ has an automatic extension to higher order; then

$$\mathbb{H}(\overline{R}, a_0) = \bigoplus_{k>0} \mathbb{H}^{(1)}(\overline{R}, ka_0).$$

The trivial case of this is a hypersurface: if $\overline{R}$ is $k[x_1, \dots, x_n]/(f_d)$ then its extensions are given by

$$F_d(x_0, \dots, x_n) = f_d + x_0 f_{d-1} + \cdots + x_0^k f_{d-k} + \cdots$$

where the bits $x_0^k f_{d-k}$ added in at the kth order are chosen freely from a vector space. A less trivial case is provided by the (as yet informal) notion of a *flexible format* (see (5.6)): let me say (vaguely) that a *format* Φ is a way of writing down relations for a ring depending on parameters, (the *entries* of Φ). The example to bear in mind is a generic determinantal. Make the following assumptions:

(i) Flexibility. $\overline{R}$ is given by relations in format Φ, and the format is *flexible*, in the sense that varying freely the entries of Φ (in a small neighbourhood) leads to flat deformations of $\overline{R}$; in determinantal cases this may come about because the syzygies are all consequences of the format (see for example (5.5) and (5.8)).

(ii) Completeness. Suppose that, in addition, every 1st order extension $R^{(1)} \in \mathbb{H}^{(1)}(\overline{R}, ka_0)$ of degree $-ka_0$ (for each $k > 0$) can be written in the same format Φ; this may have to be checked by explicit 1st order calculations (see for example the proof of Theorem 5.11 in (5.13–14)).

Then making this choice automatically lifts $R^{(1)}$, and so all the affine fibre bundles in (1.15) can be trivialised. Thus by induction, for each k, the relations for $R^{(k-1)}$ can be written in the format Φ, and by flexibility, they define an extension to all higher orders. The set of extensions $R^{(k)}$ over a fixed $R^{(k-1)}$ is then just the vector space $\mathbb{H}^{(1)}(\overline{R}, ka_0)$, and the completeness assumption is that every element of this can be obtained by varying the kth order terms of the entries of Φ. It follows in turn that $R^{(k-1)}$ can also be written in the format Φ, and so

$$\mathbb{H}(\overline{R}, a_0) = \bigoplus_{k>0} \mathbb{H}^{(1)}(\overline{R}, ka_0).$$

1.25 Relation with graded versal deformations

Suppose that $C, \mathcal{O}_C(1)$ is a polarised variety; let $P \in S = \operatorname{Spec} R(C, \mathcal{O}_C(1))$ be the affine cone over C. Since R is a graded ring, S enjoys an action of the multiplicative group $\mathbb{G}_m$. Assume for simplicity that C is nonsingular, so that $P \in S$ is an isolated singularity. Then by Schlessinger's theorem [Artin, Schlessinger] (plus Artin algebraic approximation or its analytic equivalent), there exists a versal deformation

$$\begin{array}{ccc} S & \subset & \Sigma \\ \downarrow & & \downarrow \\ 0 & \in & V; \end{array}$$

V is a finite dimensional formal scheme (or germ of a local analytic space) with a given deformation $\Sigma \to V$ such that every infinitesimal (or local analytic) deformation of $P \in S$ over a parameter scheme T is obtained as the pullback of Σ by a morphism $\varphi_\Sigma \colon T \to V$, with the 1st derivative of φ_Σ uniquely determined by Σ.

Pinkham [Pinkham, (2.3)] proved that V and the versal family $\Sigma \to V$ also have $\mathbb{G}_m$ actions; this incidentally allows me to be a little vague about the category in which V is defined: by the 'graded is local' principle described in (1.3, (2)), the scheme V is determined by its formal completion at the cone point $0 \in V$. The $\mathbb{G}_m$ action on V corresponds to a grading of the tangent space $\mathbb{T}^1 = T_0 V = \sum_i \mathbb{T}^1_i$. An extension of C corresponds to a graded 1-parameter deformation of $P \in S$, hence to a formal (or analytic) curve $(\operatorname{Spec} k[\![x_0]\!]) \to (0 \in V)$ with tangent vector in $\mathbb{T}^1_{-1}$, 2nd derivative in $\mathbb{T}^1_{-2}$, etc.

In the unobstructed case, V is local analytically isomorphic to an open ball in $\mathbb{T}^1$, so that the $\mathbb{G}_m$-action on V can be linearised, giving a decomposition of V similar to that in (1.23), and a graded analytic curve in V can be constructed with arbitrary derivatives of each order.

2 Examples, comments, propaganda

In this lightweight *scherzo*, a transparent example is followed by a brief description of the motivation and some speculative future applications of the infinitesimal view; the material is taken mainly from a recent (so far unsuccessful) research grant application.

2.1 Pinkham's example ([Pinkham, (8.6)])

Consider the normal rational curve $C_4 \subset \mathbb{P}^4$ given parametrically by $(s^4, s^3t, s^2t^2, st^3, t^4)$. The homogeneous graded ring of C_4 is of the form

$$k[s^4, s^3t, s^2t^2, st^3, t^4] = k[x_1, \dots, x_5]/I$$

where I is generated by the 6 relations

$$I\colon \operatorname{rank} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix} \le 1.$$

Geometrically, if $S \subset \mathbb{P}^5$ is a surface extending C and not a cone, then S is either a quartic scroll or the Veronese surface.

2.2 1st order deformation calculation

To find all 1st order extensions of C_4, I must (1) write out the 6 relations coming from the determinantal explicitly, then (2) write out all possible 1st order variation of the relations by adding arbitrary multiples of x_0, and then (3) take account of the syzygies mod x_0^2. This is tedious but wholly mechanical:

$$
\begin{array}{r|c|c|l}
x_3 & & x_4 & R_{12} : x_1x_3 = x_2^2 \;+ \\
-x_2 & & & R_{13} : x_1x_4 = x_2x_3 + \\
x_1 & x_5 & & R_{23} : x_2x_4 = x_3^2 \;+ \\
 & -x_4 & x_1 & R_{24} : x_2x_5 = x_3x_4 + \\
 & x_3 & & R_{34} : x_3x_5 = x_4^2 \;+ \\
 & & -x_2 & R_{14} : x_1x_5 = x_2x_4 +
\end{array}
\begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + a_{35}x_5 \\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + a_{45}x_5 \\
a_{51}x_1 + a_{52}x_2 + a_{53}x_3 + a_{54}x_4 + a_{55}x_5 \\
a_{61}x_1 + a_{62}x_2 + a_{63}x_3 + a_{64}x_4 + a_{65}x_5
\end{bmatrix} x_0
$$

2.3

The only syzygies I require are the 3 written out in columns on the left. I work out the effects of the first syzygy slowly before dumping the whole calculation before the reader.

Thus the first syzygy gives

$$
\begin{aligned}
0 &= x_3R_{12} - x_2R_{13} + x_1R_{23} \\
&= x_3(-x_1x_3 + x_2^2) - x_2(-x_1x_4 + x_2x_3) + x_1(-x_2x_4 + x_3^2) \\
&\quad + \Big[a_{11}x_1x_3 + a_{12}x_2x_3 + a_{13}x_3^2 + a_{14}x_3x_4 + a_{15}x_3x_5 \\
&\quad - a_{21}x_1x_2 - a_{22}x_2^2 - a_{23}x_2x_3 - a_{24}x_2x_4 - a_{25}x_2x_5 \\
&\quad + a_{31}x_1^2 + a_{32}x_1x_2 + a_{33}x_1x_3 + a_{34}x_1x_4 + a_{35}x_1x_5\Big]x_0.
\end{aligned}
$$

The terms on the first line all cancel out (the syzygy for $\overline{R}$), and cancelling x_0 and using the relations in $\overline{R}$ gives the following equality in $\overline{R}$:

$$
\begin{aligned}
0 = a_{31}x_1^2 &+ (-a_{21} + a_{32})x_1x_2 + (a_{11} - a_{22} + a_{33})x_1x_3 \\
&+ (a_{12} - a_{23} + a_{34})x_2x_3 + (a_{13} - a_{24} + a_{35})x_2x_4 \\
&+ (a_{14} - a_{25})x_3x_4 + a_{15}x_3x_5.
\end{aligned}
$$

Since the 9 monomials

$$
x_1^2,\ x_1x_2,\ x_1x_3,\ x_2x_3,\ x_2x_4,\ x_3x_4,\ x_3x_5,\ x_4x_5,\ x_5^2
$$

form a basis of $\overline{R}_2 = H^0(\mathbb{P}^1, \mathcal{O}(8))$, I can equate coefficients; thus the 3 syzygies imply

$$
\begin{array}{rllll}
I. & a_{31}=0, & a_{21}=a_{32}, & a_{11}-a_{22}+a_{33}=0, & \\
 & & & a_{12}-a_{23}+a_{34}=0, & \\
 & a_{15}=0, & a_{14}=a_{25}, & a_{13}-a_{24}+a_{35}=0; & \\
II. & a_{35}=0, & a_{34}=a_{45}, & a_{33}-a_{44}+a_{55}=0, & \\
 & & & a_{32}-a_{43}+a_{54}=0, & \\
 & a_{51}=0, & a_{41}=a_{52}, & a_{31}-a_{42}+a_{53}=0; & \\
III. & a_{41}=0, & a_{42}=a_{61}, & a_{43}=a_{62}, & a_{11}+a_{44}-a_{63}=0, \\
 & a_{14}=0, & a_{15}=0, & a_{13}=a_{65}, & a_{12}+a_{45}-a_{64}=0.
\end{array}
$$

2.4

At this stage I also want to make a normalisation as described in (1.22), using a transformation of the form $x_1 \mapsto x_1 + \lambda x_0$ to arrange $a_{13} = 0$, and similarly with $x_2, \dots, x_5$ to get $a_{34} = a_{33} = a_{32} = a_{53} = 0$.

Having done this, the result of the 1st order deformation calculation can be written in human-readable form (well, almost!). I write it down here together with the set-up for the 2nd order calculation:

$$
\begin{array}{rcl}
x_1x_3 &=& x_2^2 + \\
x_1x_4 &=& x_2x_3 + \\
x_1x_5 &=& x_2x_4 + \\
x_2x_5 &=& x_3x_4 + \\
x_3x_5 &=& x_4^2 + \\
x_2x_4 &=& x_3^2 +
\end{array}
\begin{bmatrix}
ax_1 + bx_2 \\
ax_2 + bx_3 \\
fx_2 + (a+g)x_3 + bx_4 \\
fx_3 + gx_4 \\
fx_4 + gx_5 \\
0
\end{bmatrix} x_0
\begin{array}{c} + \\ + \\ + \\ + \\ + \\ + \end{array}
\begin{bmatrix}
\alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6
\end{bmatrix} x_0^2
$$

Here (a, b, f, g) are free parameters, coordinates on the 4-dimensional vector space

$$\mathbb{T}^1_{-1} = \operatorname{coker}\bigl\{H^0(T_{\mathbb{P}^4}(-1)) \to H^0(N_{\mathbb{P}^4|C}(-1))\bigr\}.$$

2.5 The 2nd order calculation

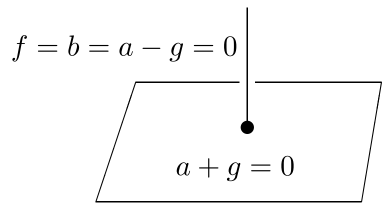
Using the same 3 syzygies gives

$$
\begin{array}{rllll}
I. & \alpha_1 = 0, & \alpha_2 = ab, & \alpha_6 = -a^2, & \\
II. & \alpha_4 = fg, & \alpha_5 = 0, & \alpha_6 = -g^2, & \\
III. & \alpha_4 = -af, & \alpha_1 = 0, & \alpha_3 = (a+g)a + fb, & (a+g)b = 0.
\end{array}
$$

The conclusion: the 1st order deformation with coordinates (a, b, f, g) admits an extension to 2nd order if and only if

$$(a+g)f = (a+g)b = (a+g)(a-g) = 0;$$

and the 2nd order extension is unique if it exists. That is, the locus of $\mathbb{T}^1(\overline{R})$ corresponding to genuine extensions of $\overline{R}$ is the union of the 3-plane $(a+g) = 0$ and the line $f = b = a - g = 0$:



2.6 Determinantal interpretations

It is not hard to see that if $a = -g$ then the 6 relations can be recast in the determinantal form

$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & x_3 + ax_0 & x_4 + fx_0 \\ x_2 + bx_0 & x_3 - ax_0 & x_4 & x_5 \end{pmatrix} \le 1.$$

That is, the original 2×4 matrix had special or nongeneric entries $a_{12} = a_{21} = x_2$, $a_{13} = a_{22} = x_3$ and $a_{14} = a_{23} = x_4$, and the extension is obtained by allowing these to become general linear forms in 6 variables. These are of course the equations defining a quartic scroll in $\mathbb{P}^5$.

On the other hand, if $b = f = 0$ and $a = g$ then the 6 relations can be written in the form

$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & x_3 + ax_0 \\ x_2 & x_3 - ax_0 & x_4 \\ x_3 + ax_0 & x_4 & x_5 \end{pmatrix} \le 1.$$

That is, the original 6 relations can be recast in the alternative symmetric determinantal form

$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_3 & x_4 & x_5 \end{pmatrix} \le 1.$$

As a symmetric determinantal, this is nongeneric only in the 'persymmetric' coincidence $a_{13} = a_{22} = x_3$, and the given extension is the least I can do to cure this. When $a \ne 0$, the determinantal equations are of course the equations of the Veronese surface in $\mathbb{P}^5$.

2.7 The chicken or the egg?

Since the answer here (and also in more substantial cases, for example §5) is expressed in terms of determinantal formats, it's tempting to think that the calculations can also. I believe that this view is mistaken, and that the deformation calculation is more fundamental: starting from one of the determinantal formats (as in (2.1)), my experience is that it's not possible to predict the other without carrying out what is in effect a deformation calculation (maybe only in part or in guessed form). In fact, my guess is that in some sense the determinantal format can be seen as arising from an elimination of deformation variables. A ring in determinantal format somehow managing to squeeze out of its straight-jacket under a deformation is an extremely delicate and interesting phenomenon; a similar example provides the main theme of §5, and I believe that there are many other substantial cases in this area of commutative algebra and algebraic geometry.

2.8

The rest of this section discusses briefly some of the motivation behind my study of the extension problem. As a method, the infinitesimal view has so far only led to convincing results in a small number of cases: the case of numerical quintics treated by Ed Griffin [Griffin2] (worked out in detail in §5), and Dicks' work [Dicks] on canonical rings of surfaces with $p_g = 3$, $K^2 = 4$ (another class studied by Horikawa). The idea in either case is that both the geometry and algebra of an individual surface and properties of its moduli space can be treated by viewing the surface as an extension of a general curve $C \in |K|$. Up to now, the computations have been considered to be very hard and long; one reason for the material of §1 is to express these (in so far as possible) as mechanical algorithms.

When we have these computations mechanised, I believe that there will turn out to be dozens of examples where the infinitesimal view can be used to study curves, surfaces, 3-folds and singularities and their moduli spaces. In many cases the defining relations of the graded ring fit into specific formats (for example, involving determinantals), so they give rise to interesting examples in commutative algebra.

2.9 Mumford's dream

A program advanced by D. Mumford (his problem to the Montreal NATO summer school in 1980) says that questions on the existence of surfaces or the number of components of their moduli space are 'in principle' solvable by computer; roughly, the idea is to define a graded ring by writing down relations (in terms of some chosen generators) as polynomials with indeterminate coefficients, and then to find the subvariety of the set of indeterminates where the graded ring has the required properties. Mumford actually proposed to use this to study surfaces with $p_g = 0$, $K^2 = 9$. However, I doubt if Mumford's program can be implemented on computers in the foreseeable future (even in simple key cases).

My point is that when detailed information about an ample divisor of the variety under study is available, it is reasonable to study it as an extension problem. This may lead to less intractable problems; the situation is analogous to replacing a general algebraic group by a nilpotent one – it's not far-fetched to see the iterated linear structure of (1.13) as a kind of nilpotent phenomenon.

2.10 Godeaux surfaces

Surfaces of general type have been the starting point for all of my work in the classification of varieties over the last 13 or 14 years. There is a paradox here, because although the subject as a whole has seen spectacular progress in many directions, some of the basic problems from which I started off seem just as hard now as ever.

A *Godeaux surface* is a minimal surface X of general type with $p_g = 0$, $K^2 = 1$. These are the smallest possible values of the numerical invariants; the torsion subgroup $(\operatorname{Pic} X)_{\text{Tors}} = \operatorname{Torsion} X = \pi_1^{\text{alg}}(X)$ of a Godeaux surface is one of $\mathbb{Z}/5$, $\mathbb{Z}/4$, $\mathbb{Z}/3$, $\mathbb{Z}/2$ or 0, and the surfaces in the first 3 cases are well understood (at least by me) [Reid1]. Godeaux surfaces with Torsion $= \mathbb{Z}/2$ has been the single most important motivating case for me, and I have put an enormous amount of effort into computing their canonical rings since 1977, before I knew of the link with deformation theory; the ring restricted to the unique curve $C \in |K + \sigma|$ has a nice hyperelliptic description as in Theorem 4.6. The calculation of the ring of the surface itself by the infinitesimal view is a key test of my ideas (see (6.5)), and was originally intended as §6 (*grosse Fuge*) of this paper, but I am reluctantly obliged to leave it to a future occasion.

A Godeaux surface with Torsion $= 0$ was constructed by R. Barlow in her 1982 Ph.D. thesis (see [Barlow1–2]), and is at present the only known simply connected surface of general type with $p_g = 0$; this has recently been the subject of interest on the part of differential geometers [Kotschik]. The following question is a distant aim.

Question. Is the moduli space of Godeaux surfaces with Torsion = 0 irreducible?

Under certain genericity assumptions, I have a concrete (but fairly complicated) geometric description of the graded ring $R(C) = R(C, K_{X|C})$ associated with a general 2-canonical curve $C \subset X$ in terms of liaisons, involving an elliptic curve $E \subset \mathbb{P}^3$ and a 3-torsion point of E. The ring $R(C)$ can then be written out in terms of generators, relations and syzygies, and the canonical ring $R(X)$ studied by feeding $R(C)$ into the extension methods of §1. Already $R(C)$ is a very big calculation.

2.11 Remark on 4-manifolds

The current view in 4-dimensional topology is that on the one hand, homotopy theory, surgery and Freedman's work reduces the classification of simply connected 4-manifolds up to homeomorphism to the intersection form on $H^2(M, \mathbb{Z})$; on the other hand, by Donaldson's work, their classification up to diffeomorphism is intimately related to the complex geometry of algebraic surfaces. So for example, although the chain of reasoning is long, and depends on some wild-looking conjectures arising out of Donaldson's work (see [Friedman and Morgan]), one now thinks of the problem of diffeomorphism type of simply connected 4-manifolds with intersection form $(+1, -8)$, as being closely related to my question (2.10). It is quite amazing that there is such a long chain of reasoning, starting at one end with topology and differential geometry, through algebraic geometry and the commutative algebra of complicated graded rings to computer algebra.

2.12 Speculative applications

There are many of these; to mention only rather substantial ones:

(a) The relative canonical algebra of a fibre space $f\colon X \to B$ of curves of genus g over a base curve. The ultimate aim here is to decide a conjecture of Xiao Gang on 'Morsification': the germ of f around a *degenerate* fibre can be deformed to a neighbouring fibration having only Morse critical points or nonsingular multiple fibres. The rings (and the calculations) arising here are like those for canonical rings of surfaces. For example, the genus 3 case has been studied in detail by Mendes Lopes [Mendes Lopes]: computing the canonical ring $R(F, K_F)$ of a nonreduced fibre of F is a nilpotent extension problem similar to §1, and the rings arising are in some cases similar to the numerical quintics of §5.

(b) Construct new surfaces embedded in $\mathbb{P}^4$, and hence new vector bundles on $\mathbb{P}^4$, starting from a cleverly set up curve $C \subset \mathbb{P}^3$; the point is that the construction and embedding of C can perhaps be done intrinsically and geometrically, even though the commutative algebra of $C \subset \mathbb{P}^3$ (the monad defining the corresponding vector bundle) is certain to be very complicated.

(c) Du Val singularities and 3-fold flip singularities. Another long term aim is an attack on the flip singularities that play a crucial role in Mori's theory of minimal models of 3-folds. A flip singularity $P \in X$ usually contains a Du Val surface singularity $P \in S \in |-K_X|$ as anticanonical divisor (the 'general elephant'). Technically, the problem is that already the 1st order normal data of X around S is quite awkward to specify: since S is not a Cartier divisor, the normal sheaf may be nontrivial on $S \setminus P$, and moreover, it may not be S_2 at the singularity.

(d) Permanence. A familiar phenomenon in projective geometry is that features of a hyperplane section $C \subset X$ often extend to X. For example, if an ample divisor $C \subset X$ of a surface X is a hyperelliptic curve, one may entertain certain expectations concerning the rational map $\varphi_C \colon X \to \mathbb{P}^N$; or if C is contained in a scroll $C \subset F \subset \mathbb{P}^{n-1}$, then one may hope to find a bigger scroll in $\mathbb{P}^n$ containing X, etc. (see for example [Serrano]). Similar remarks apply to permanence of features under small deformation. In some cases the infinitesimal view allows these questions to be treated together, and to be explained in terms of 1st order infinitesimal extensions.

Part II
Halfcanonical curves and the canonical ring of a regular surface

3 The canonical ring of a regular surface

Theorem 3.1 *Let X be a canonical surface (that is, the canonical model of a surface of general type). Suppose that*

(i) $p_g \geq 2$, $K^2 \geq 3$;

(ii) $q = 0$; *and*

(iii) X *has an irreducible canonical curve* $C \in |K_X|$.

Then the canonical ring $R = R(X, K_X)$ *is generated in degrees* ≤ 3 *and related in degrees* ≤ 6.

Standard convention

When R is generated in degrees ≤ 3, I write $R = k[x_i, y_j, z_k]/I$, with $\deg(x_i, y_j, z_k) = (1, 2, 3)$.

3.2 Counterexample (P. Francia and C. Ciliberto [Ciliberto, §4])

On a minimal surface of general type S, define a *Francia cycle* to be a 2-connected divisor E such that either $KE = 1$ and $E^2 = -1$, or $KE = 2$ and $E^2 = 0$ (think of E as a smooth elliptic or genus 2 curve). A famous theorem of Francia [Francia] says that, with finitely many exceptional families, $2K_X$ is very ample on the canonical model if and only if the minimal model S does not contain a Francia cycle.

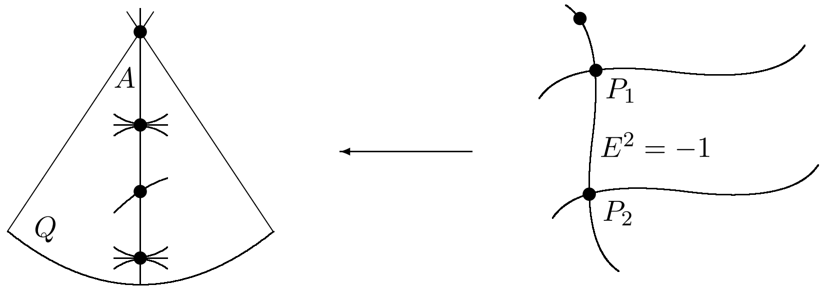
Without the assumption (iii), the conclusion of Theorem 3.1 fails infinitely often. The point is just that if the fixed part of $|K_X|$ contains a Francia cycle then the multiplication map

$$S^2 R_2 \oplus R_1 \cdot R_3 \to R_4$$

cannot be surjective. In fact φ_{4K} is very ample on E; however, φ_{2K} cannot be very ample on E for reasons of low degree, and the elements of $R_1 \cdot R_3$ vanish along E (by the assumption that E is fixed in $|K_X|$).

Consider the double cover of the quadric cone $Q \subset \mathbb{P}^3$ ramified in the vertex and in a curve $R \in |\mathcal{O}_Q(2m + 2n + 3)|$ that meets a given generator A in a $(2m + 1)$-tuple tacnode and a $(2n + 1)$-tuple tacnode (see figure): it is not hard to see that making a minimal resolution X leads to a elliptic curve E on X with $E^2 = -1$; E is fixed in $|K_X|$ because $K_X E = 1$ and E passes through 2 points P_i on the exceptional curves C_i that are base points of $K_X|C_i$.

Remark 3.3 (a) The statement of the theorem in terms of the canonical model X allows the possibility that the canonical system $|K_S|$ of the minimal nonsingular model S has -2-curves as fixed part; it would be quite unpleasant to have to do the proof in this context.



(b) Two relative versions of the problem are also of interest. If $f\colon X \to Y$ is a resolution of an isolated Gorenstein surface singularity, then the relative canonical algebra $\bigoplus f_*\omega_X^{\otimes k}$ is generated in degree ≤ 3 (see [Laufer2, (5.2)] and compare [Laufer1, (3.2) and (3.5)]) and I conjecture that it is related in degrees ≤ 6. If $f\colon X \to B$ is a fibre space of curves of genus $g = 2$ or 3 over a base curve then the relative canonical algebra $\bigoplus f_*\omega_X^{\otimes k}$ is generated in degree ≤ 3 and related in degrees ≤ 6 (a result due to Mendes Lopes [Mendes Lopes]); it would be interesting to know if Laufer's argument can be modified to prove this for all g.

(c) Theorem 3.1 may hold even without assumption (ii): that is, although $H^0(X, 2K_X) \to H^0(C, 2K_{X|C})$ is not surjective, it might be possible to show that it maps 'onto the bits that matter', as with other questions concerning the 2-canonical map [Francia, Reider]. Also, it seems reasonable to ask about weakening the irreducibility assumption (iii) to $C \in |K_X|$ 3-connected. Thus I conjecture that, with a finite number of exceptional families, the canonical ring of a surface $R(X, K_X)$ is generated in degrees ≤ 3 and related in degrees ≤ 6 if and only if the fixed part of $|K_S|$ on the minimal model does not contain a Francia cycle.

(d) Exercise. Prove that under the assumptions of (3.1), the 3-canonical model $X^{[3]}$ is projectively normal. [Hint: by (3.1), R_{3d} is spanned by monomials in x_i, y_j, z_k of degree $3d$; if such a monomial is a product of two monomials of degree divisible by 3 then OK. Thus $x_0 \cdot R_5$ is in the image of $S^2(R_3) \to R_6$. But if x_0 defines an irreducible curve C, the surjectivity of $S^2(R_3) \to R_6$ modulo x_0 follows by standard use of the free pencil trick on C.]

3.4

By the hyperplane section principle (1.2, i–ii), Theorem 3.1 will follow from the following more precise result for curves, applied to the curve $C \in |K_X|$ and the divisor $D = K_{X|C}$.

Theorem *Let C be an irreducible Gorenstein curve of genus $g \geq 2$, and D a Cartier divisor on C such that $2D \sim K_C$; assume that C and D are not in the 4 exceptional cases (i–iv) below. Then the graded ring $R(C, D)$ is generated in degrees ≤ 3 and related in degrees ≤ 6.*

Exceptional cases:

(i) C is hyperelliptic of genus $g \neq 2$ and $h^0(\mathcal{O}_C(D)) = 0$; in this case $R(C, D)$ is generated in degrees ≤ 4 and related in degrees ≤ 8.

(ii) $g = 2$, $D = P$ where P is a Weierstrass point; in this case

$$R(C, D) = k[x, y, z]/F, \quad \textit{with} \quad \deg(x, y, z, F) = 1, 2, 5, 10.$$

(iii) $g = 3$, $D = g^1_2$; in this case

$$R(C, D) = k[x_1, x_2, y]/F, \quad \textit{with} \quad \deg(x_1, x_2, y, F) = 1, 1, 4, 8.$$

(iv) $g = 3$, C is nonhyperelliptic and $h^0(\mathcal{O}_C(D)) = 0$; then $R(C, D)$ is generated in degrees ≤ 3, but requires one relation in degree 8 ($2D = K_C$ is very ample, mapping C to a plane quartic $C = C_4 \subset \mathbb{P}^2$, and the relation in degree 8 is the defining equation of C_4).

History

Cases (ii–iv) go back in effect to Enriques; for example, Case (iv) is treated in detail in [Catanese and Debarre].

The proof of Theorem 3.4 is a cheap adaptation of the early stages of the famous *Petri analysis* [4 authors, Chap. III, §2] for the canonical ring $R(C, 2D) = R(C, K_C)$ of a nonhyperelliptic curve C. More closely related to the point of view of [Mumford1] and [Fujita2], similar arguments show that $R(C, D)$ is generated in degrees ≤ 3 and related in degrees ≤ 6 for any divisor D of degree $\geq g + 1$ on an irreducible curve C.

3.5 Set-up for the proof of (3.4)

This section *waltzes* through the major case of a nonhyperelliptic curve C; the *trio* section §4 covers in much more detail the relative minor case when C is hyperelliptic; (see (3.11) if you don't know what it means for an irreducible Gorenstein curve C to be hyperelliptic). When $g=3$ and C is nonhyperelliptic then either (iv) holds, or $h^0(C,\mathcal{O}_C(D))=1$; I offer the reader the lovely exercise of seeing that in this case, which corresponds to a plane quartic with a bitangent line, $R(C,\mathcal{O}_C(D))$ is a complete intersection ring

$$R(C,D)=k[x,y_1,y_2,z]/(f,g) \quad \text{with} \quad \deg(f,g)=(4,6).$$

Thus I suppose throughout this section that C is nonhyperelliptic and $g\ge 4$. Introduce vector space bases as follows:

$$\begin{aligned}
&x_1,\dots,x_a\in H^0(D);\\
&y_1,\dots,y_g\in H^0(2D)=H^0(K_C);\\
&z_1,\dots,z_{2g-2}\in H^0(3D).
\end{aligned}$$

Write $I(m,n)$ for the kernel of the natural map

$$\varphi_{m,n}\colon H^0(mD)\otimes H^0(nD)\to H^0((m+n)D),$$

and

$$\psi_{\ell;m,n}\colon H^0(\ell D)\otimes I(m,n)\to I(\ell+m,n)$$

for the natural map.

Main lemma 3.6 *(I) $\varphi_{m,2}$ is surjective for every $m\ge 2$;*

(II) $I(m+2,2)=\operatorname{im}\psi_{2;m,2}+\operatorname{im}\psi_{m;2,2}$.

This result is similar to [Fujita2, Lemma 1.8]; the proof occupies (3.8–10) together with a technical appendix.

3.7 Lemma 3.6 $\implies$ Theorem 3.4

(I) implies by induction that if $m=2\ell\ge 2$ is even, then $H^0(mD)$ is spanned as a vector space by the set $S^\ell(y)$ of monomials of degree ℓ in the y_i; and if $m=2\ell+1\ge 3$ is odd then $H^0(mD)$ is spanned as a vector space by the set $z\otimes S^{\ell-1}(y)$ of monomials of the form z_j times a monomial of degree $\ell-1$ in the y_i. This obviously implies that $R(C,D)$ is generated in degrees ≤ 3.

The relations in low degrees can be written

$$\begin{array}{lll} \deg 2 & x_i x_j = L_{ij}(y) & \text{(linear forms)} \\ \deg 3 & x_i y_j = M_{ij}(z) & \text{(linear forms)} \\ \deg 4 & x_i z_j = N_{ij}(y) & \text{(quadratic forms)} \\ \deg 6 & z_i z_j = P_{ij}(y) & \text{(cubic forms).} \end{array}$$

These relations clearly allow any monomial of degree m in the x_i, y_j, z_k to be expressed as a linear combination of $S^\ell(y)$ if $m = 2\ell$ or of $z \otimes S^{\ell-1}(y)$ if $m = 2\ell + 1$.

For the relations, suppose that $F_m : f_m(x, y, z) = 0 \in R_m$ is a polynomial relation of degree m between the generators x, y, z of $R(C, D)$. I must show that F_m is a linear combination of products

$$\text{(monomial)} \times \text{(relation in degree} \leq 6).$$

Any term occuring in F_m can be expressed as a linear combination of monomials $S^\ell(y)$ or $z \otimes S^{\ell-1}(y)$ by using products of the relations just tabulated. Therefore I need only deal with linear dependence relations between these monomials in R_m (for $m \geq 7$).

By just separating off one y_i in each monomial in an arbitrary way, a linear combination of these monomials in R_m can be written as the image of an element $\xi \in R_{m-2} \otimes R_2$; to say that it vanishes in R_m means that $\xi \in I(m-2, 2)$. But then Lemma 3.6, (II) says that

$$\xi \in \operatorname{im} \psi_{2;m-4,2} + \operatorname{im} \psi_{m-4;2,2}.$$

This means that the relation in degree m corresponding to ξ is a sum of relations in degrees $m-2$ and 4 multiplied up into degree m. By induction, the result follows. Q.E.D.

3.8 Proof of (3.6, I), and notation

Let $A = P_3 + \cdots + P_g$ be a divisor on C made up of $g - 2$ general points. Since C is nonhyperelliptic, K_C is birational, so that $|2D - A| = |K_C - A|$ is a free pencil (by general position [4 authors, p. 109]); hence the free pencil trick gives the exact sequence

$$\begin{array}{ccccccc} & 0 \to H^0((m-2)D + A) & \to & H^0(2D - A) \otimes H^0(mD) & \to & H^0((m+2)D - A) \\ m = 2 & 1 & & 2 \times g & & 2g - 1 \\ m \geq 3 & (m-2)(g-1) - 1 & & 2 \times (m-1)(g-1) & & m(g-1) + 1; \end{array}$$

the indicated dimension count shows that the right-hand arrow

$$H^0(2D - A) \otimes H^0(mD) \to H^0((m+2)D - A) \to 0$$

is surjective.

Let $t_m \in H^0(mD)$ be an element not vanishing at any of $P_3, \dots, P_g$, and, as in the Petri analysis, choose the basis $y_1, \dots, y_g$ of $H^0(K_C)$ such that

$$y_1, y_2 \text{ bases } H^0(2D - A), \quad \text{and} \quad y_i(P_j) = \delta_{ij} \text{ for } i, j = 3, \dots, g$$

(Kronecker delta). Then by the free pencil trick,

$$H^0((m+2)D - A) = H^0(mD)y_1 \oplus H^0(mD)y_2,$$

and, obviously, $t_m y_i$ for $i = 3, \dots, g$ form a complementary basis of $H^0((m+2)D)$. This proves (3.6, I).

Similarly,

$$H^0((m+4)D - A) = H^0((m+2)D)y_1 + H^0((m+2)D)y_2$$

and $t_{m+2}y_i$ for $i = 3, \dots, g$ is a complementary basis of $H^0((m+4)D)$.

3.9

As u runs through $H^0(mD + A)$, the relations

$$\rho(u) = uy_1 \otimes y_2 - uy_2 \otimes y_1 \in I(m+2, 2)$$

express the fact that

$$H^0((m+2)D)y_1 \cap H^0((m+2)D)y_2 = H^0(mD+A)y_1y_2 \subset H^0((m+4)D-A),$$

which is part of the free pencil trick. The key to (3.6, II) is to prove that for $m \geq 3$,

$$\rho(u) \in \operatorname{im} \psi_{2;m,2} \quad \text{for all } u \in H^0(mD + A);$$

since $\xi \otimes \rho(v) = \rho(\xi v)$ for $\xi \in H^0(2D)$ and $v \in H^0((m-2)D + A)$, this follows trivially from the claim:

Claim.

$$H^0(2D) \otimes H^0((m-2)D + A) \to H^0(mD + A) \to 0$$

is surjective.

3.10 Proof of (3.6, II)

Claim 3.9 is proved in (3.15), and I first polish off (3.6, II) assuming it. Suppose that $m \geq 3$.

Step 1 The subspace $\{\rho(u)\} = \rho(H^0(mD + A))$ is the kernel of

$$H^0((m+2)D) \otimes H^0(2D - A) \to H^0((m+4)D - A) \subset H^0((m+4)D),$$

and the $t_m y_i \otimes y_i$ map to a complementary basis. Therefore, a subset

$$S \subset I(m+2, 2) = \ker\{H^0((m+2)D) \otimes H^0(2D) \to H^0((m+4)D)\}$$

will span $I(m+2, 2)$ as a k-vector space provided that

(i) S contains the $\rho(u)$; and

(ii) S spans a subspace complementary to

$$H^0((m+2)D) \otimes H^0(2D - A) \oplus \sum k \cdot t_m y_i \otimes y_i,$$

in other words, any $\eta \in H^0((m+2)D) \otimes H^0(2D)$ can be written

$$\eta = \eta_S + \eta_{2D-A} + \eta_t \qquad (*)$$

where $\eta_{2D-A} \in H^0((m+2)D) \otimes H^0(2D - A)$ and η_S, η_t are linear combinations of S and of the $t_m y_i \otimes y_i$ respectively.

Step 2 Now set $S = \operatorname{im} \psi_{2;m,2} + \operatorname{im} \psi_{m;2,2}$. By (3.9), $\operatorname{im} \psi_{2;m,2}$ contains $\rho(u)$ for $u \in H^0(mD + A)$. Therefore, it is enough to verify $(*)$ for any $\eta \in H^0((m+2)D) \otimes H^0(2D)$.

Break up $H^0((m+2)D) \otimes H^0(2D)$ as a direct sum of the following 4 pieces:

$$V_1 = H^0((m+2)D) \otimes H^0(2D - A);$$
$$V_2 = H^0((m+2)D - A) \otimes \sum k \cdot y_i;$$
$$V_3 = \sum k \cdot t_{m+2} y_i \otimes y_j \text{ summed over } i, j = 3, \dots, g \text{ with } i \neq j;$$
$$V_4 = \sum k \cdot t_{m+2} y_i \otimes y_i \text{ for } i = 3, \dots, g.$$

For V_1 and V_4 there's not much to prove. Also since

$$H^0((m+2)D - A) = H^0(mD)y_1 + H^0(mD)y_2$$

and $R(2,2)$ contains $y_1 \otimes y_i - y_i \otimes y_1$ and $y_2 \otimes y_i - y_i \otimes y_2$ for $i = 3, \dots, g$, it follows that $V_2 \subset V_1 + \operatorname{im} \psi_{m;2,2}$.

Finally, for the summand V_3, note that for $i, j = 3, \dots, g$ and $i \neq j$,

$$y_i y_j \in H^0(4D - A) = H^0(2D)y_1 + H^0(2D)y_2,$$

so that I(2, 2) contains the Petri relation

$$y_i \otimes y_j - a_{ij} \otimes y_1 - b_{ij} \otimes y_2 \text{ with } a_{ij}, b_{ij} \in H^0(K_C).$$

Therefore also

$$t_{m+2} y_i \otimes y_j \in V_1 + \operatorname{im} \psi_{m;2,2}.$$

This completes the proof of (3.6, II), modulo Claim 3.9.

Coda to §3. 'General' divisors and the proof of (3.9)

Lemma 3.11 (hyperelliptic dichotomy) *Let C be an irreducible Gorenstein curve of genus $g = p_a C \geq 2$.*

- *(i) The canonical linear system $|K_C|$ is free;*
- *(ii) K_C is very ample unless φ_K is a 2-to-1 flat morphism to a normal rational curve.*

Proof (See [Catanese, 3] for a discussion of a more general problem; however, my proof of (ii) seems to be new even in the nonsingular case!).

(i) Suppose $P \in C$ is a base point of $|K_C|$; then $h^0(m_P \cdot \mathcal{O}_C(K_C)) = g$ and by RR $h^1(m_P \cdot \mathcal{O}_C(K_C)) = 2$, so by Serre duality the inclusion

$$\operatorname{Hom}(\mathcal{O}_C, \mathcal{O}_C) = k \subset \operatorname{Hom}(m_P, \mathcal{O}_C)$$

is strict. A nonconstant element of $\operatorname{Hom}(m_P, \mathcal{O}_C)$ is a rational function $h \in k(C)$ such that $h \cdot m_P \subset \mathcal{O}_C$. Since $\deg h \cdot m_P = \deg m_P = -1$, it is easy to see that $h \cdot m_P = m_Q$ for some $P \neq Q \in C$, and it follows that P and Q are Cartier divisors on C, hence nonsingular points, and as usual h defines a birational morphism $C \to \mathbb{P}^1$, necessarily an isomorphism.

(ii) If $\varphi_K \colon C \to \mathbb{P}^{g-1}$ is not birational then it is clearly 2-to-1 to a normal rational curve. Suppose it is birational to a curve of degree $2g-2$. If $A = P_3 + \cdots + P_g$ is a divisor on C made up of $g-2$ general points then $|K_C - A|$ is a free pencil by general position, and arguing as in (3.8), $S^d(H^0(K_C)) \to H^0(dK_C)$ is surjective; thus the ring $R(C, K_C)$ is generated by $H^0(K_C)$. Therefore the ample divisor K_C is very ample. Q.E.D.

3.12

Claim 3.9 will also follow from the free pencil trick, once I prove that the divisor $A = P_1 + \cdots + P_{g-2}$ made up of $g-2$ general points is 'general enough' for $|D+A|$ to be free and birational.

Proposition *Let C be an irreducible Gorenstein curve of genus g, and D a divisor class such that $2D \sim K_C$. Let $A = P_3 + \cdots + P_g$ be a divisor on C made up of $g-2$ general points. Then*

(i) Suppose that $g \geq 3$, and that C is nonhyperelliptic if $g = 3$; then $h^0(\mathcal{O}_C(D)) \leq g-2$, so that $H^0(C, \mathcal{O}_C(D-A)) = 0$ and $h^0(C, \mathcal{O}_C(D+A)) = g-2$.

(ii) Suppose that $g \geq 4$, and that C is nonhyperelliptic if $g = 4$; then $|D+A|$ is free; it's a free pencil if g=4.

(iii) Suppose that $g \geq 5$, and that C is nonhyperelliptic if $g = 5$; then φ_{D+A} is birational.

3.13 Proof of (3.12, ii)

It's enough to prove $\mathrm{Hom}(m_P, \mathcal{O}_C(D-A))$=0 for every $P \in C$, since then by duality and RR,

$$h^0(m_P \cdot \mathcal{O}_C(D+A)) = g-3 < h^0(\mathcal{O}_C(D+A)) = g-2,$$

and $|D+A|$ is free.

Case $H^0(D) = 0$ Then $h^0(m_P \cdot \mathcal{O}_C(D)) = 0$ for every $P \in C$, so by RR, $h^1(m_P \cdot \mathcal{O}_C(D)) = 1$. By Serre duality,

$$\dim \mathrm{Hom}(m_P, \mathcal{O}_C(D)) = 1$$

for every $P \in C$; hence there is just a 1-dimensional family of effective divisors A (of any degree) with $\mathrm{Hom}(m_P, \mathcal{O}_C(D-A)) = 0$ for any P. Since A varies in a family of dimension $g-2 \geq 2$, it can be chosen to avoid this set.

Case $H^0(D) \neq 0$ By RR and duality, the inclusion

$$H^0(\mathcal{O}_C(D)) \subset \mathrm{Hom}(m_P, \mathcal{O}_C(D))$$

is strict only for P in the base locus of $|D|$; therefore I can assume that the general divisor A imposes linearly independent conditions on each of the vector spaces $\mathrm{Hom}(m_P, \mathcal{O}_C(D))$ (there are in effect only finitely many of them). So if $\mathrm{Hom}(m_P, \mathcal{O}_C(D - A)) \neq 0$ for $P \in C$ then

$$\dim \mathrm{Hom}(m_P, \mathcal{O}_C(D)) \geq g - 1.$$

Using RR and duality as usual, this is the same as

$$h^0(m_P \cdot \mathcal{O}_C(D)) \geq g - 2.$$

This contradicts (a singular analogue of) Clifford's theorem: by the linear-bilinear trick, the map

$$S^2 H^0(m_P \cdot \mathcal{O}_C(D)) \to H^0(m_P^2 \cdot \mathcal{O}_C(K_C))$$

has rank $\geq 2h^0 - 1$ (with equality if and only if the image of C under the rational map defined by $H^0(m_P \cdot \mathcal{O}_C(D))$ is a normal rational curve), so

$$g \geq h^0(m_P^2 \cdot \mathcal{O}_C(K_C)) + 1 \geq 2h^0(m_P \cdot \mathcal{O}_C(D)) \geq 2(g - 2),$$

that is, $g \leq 4$ and C is hyperelliptic in case of equality. This contradiction proves (ii). The reader can do (i) as an exercise in the same vein.

3.14 Proof of (3.12, iii)

This is very similar: I prove that there exists a nonsingular point Q such that $\mathrm{Hom}(m_P, \mathcal{O}_C(D + Q - A)) = 0$ for every $P \in C$; as before, RR and duality imply that

$$h^0(m_P \cdot \mathcal{O}_C(D + A - Q)) = h^0(\mathcal{O}_C(D + A)) - 2,$$

so that φ_{D+A} is an isomorphism near Q.

Case $h^0(D) \leq 1$ Then $h^0(D + Q) = 1$ for a general point Q, and fixing such a point, RR and duality imply that

$$\dim \mathrm{Hom}(m_P, \mathcal{O}_C(D + Q)) = 2$$

for every $P \in C$; therefore the family of effective divisors A such that $\mathrm{Hom}(m_P, \mathcal{O}_C(D + Q - A)) \neq 0$ for any P has dimension 2, and as A varies in a family of dimension $g - 2 \geq 3$, I can choose it to avoid this.

Case $h^0(D) \geq 2$ I pick a general Q, so that $h^0(D+Q) = h^0(D)$; then, as before, the inclusion

$$H^0(\mathcal{O}_C(D+Q)) \subset \operatorname{Hom}(m_P, \mathcal{O}_C(D+Q))$$

is strict only for P a base point of $|D+Q|$; so that there are only finitely many distinct vector spaces $\operatorname{Hom}(m_P, \mathcal{O}_C(D+Q))$, and I can assume that the general divisor A imposes linearly independent conditions on each of them. Thus $\operatorname{Hom}(m_P, \mathcal{O}_C(D+Q-A)) \neq 0$ implies

$$\dim \operatorname{Hom}(m_P, \mathcal{O}_C(D+Q)) \geq g-1, \quad \text{that is,} \quad h^0(m_P \cdot \mathcal{O}_C(D+Q)) \geq g-2.$$

As before, the linear-bilinear trick gives

$$\begin{aligned} \operatorname{rank}\bigl\{S^2 H^0(m_P \cdot \mathcal{O}_C(D+Q)) \to H^0(m_P^2 \cdot \mathcal{O}_C(K_C+2Q))\bigr\} \\ \geq 2h^0(m_P \cdot \mathcal{O}_C(D+Q)) - 1. \end{aligned}$$

Now $|K_C + 2Q|$ is free and $H^0(\mathcal{O}_C(K_C+2Q)) = g+1$, so

$$g+1 \geq h^0(m_P^2 \cdot \mathcal{O}_C(K_C+2Q)) + 1 \geq 2h^0(m_P \cdot \mathcal{O}_C(D+Q)) \geq 2(g-2);$$

that is, $g \leq 5$ and C is hyperelliptic in case of equality. Q.E.D.

3.15 Proof of Claim 3.9

$h^0(D+A) = g-2$. If $g \geq 5$ then φ_{D+A} is birational, so that I can choose a divisor $B = Q_1 + \cdots + Q_{g-4}$ made up of general points, and sections $s_i \in H^0(D+A)$ such that $s_i(Q_j) = \delta_{ij}$. Then using the free pencil trick in the usual way shows that

$$H^0(2D) \otimes H^0(D+A-B) \to H^0(3D+A-B)$$

is surjective; if $t \in H^0(2D)$ doesn't vanish at $Q_1, \dots, Q_4$ then $s_i t$ for $i = 1, \dots, g-4$ is a complementary basis of $H^0(3D+A)$. The statement for $m \geq 4$ is an easy exercise using the same method. Q.E.D.

4 Graded rings on hyperelliptic curves

4.1 Notation, introduction

A nonsingular hyperelliptic curve of genus g is a 2-to-1 cover $\pi\colon C \to \mathbb{P}^1$ branched in $2g+2$ distinct points

$$\{Q_1, \dots, Q_{2g+2}\} \subset \mathbb{P}^1,$$

lifting to points $\{P_1, \dots, P_{2g+2}\} \subset C$ (see the picture below); the $P_i \in C$ are the *Weierstrass points*, the points of C for which $2P_i \in g^1_2$. If $D = \sum d_i P_i$ is a divisor on C made up of Weierstrass points, or equivalently, invariant under the hyperelliptic involution $\iota\colon C \to C$, I am going to describe an automatic and painless way of writing down a vector space basis of $H^0(C, \mathcal{O}_C(D))$, and a presentation of the ring $R(C, \mathcal{O}_C(D))$ by generators and relations.

In a nutshell, the method is the following. Fix homogeneous coordinates on $\mathbb{P}^1$, or equivalently, a basis $(t_1, t_2) \in H^0(\mathbb{P}^1, \mathcal{O}(1)) = H^0(C, g^1_2)$. For each $i = 1, \dots, 2g+2$, let

$$u_i\colon C \hookrightarrow \mathcal{O}_C(P_i)$$

be the constant section. Since $2P_i \in g^1_2$, it follows that $u_i^2 \in H^0(C, g^1_2)$, so that I can write

$$u_i^2 = \ell_i(t_1, t_2), \tag{$*$}$$

where ℓ_i is the linear form in t_1 and t_2 defining the branch point $Q_i \in \mathbb{P}^1$. Now it is more-or-less clear that any vector space of the form $H^0(C, \mathcal{O}_C(D))$ has a basis consisting of monomials in the u_i, and that the only relations between these are either of a trivial monomial kind or are derived from $(*)$.

4.2 Easy preliminaries

(i) The decomposition of $\pi_*\mathcal{O}_C$ into the ± 1-eigensheaves of ι is

$$\pi_*\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g-1),$$

and the algebra structure on $\pi_*\mathcal{O}_C$ is given by a multiplication map

$$f\colon S^2(\mathcal{O}_{\mathbb{P}^1}(-g-1)) = \mathcal{O}_{\mathbb{P}^1}(-2g-2) \to \mathcal{O}_{\mathbb{P}^1},$$

which is a polynomial $f_{2g+2}(t_1, t_2)$ vanishing at the $2g+2$ branch points Q_i;

(ii) the Weierstrass points add up to a divisor in $|(g+1)g^1_2|$, that is

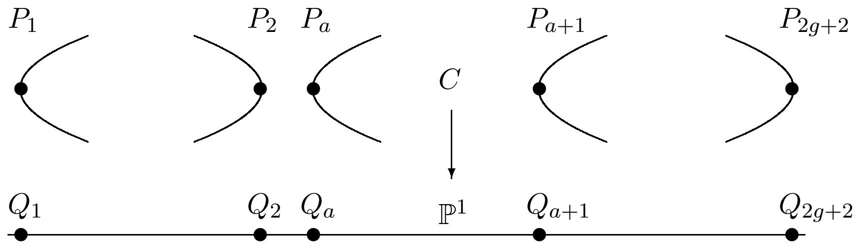
$$P_1 + \cdots + P_{2g+2} \sim (g+1)g^1_2;$$

(iii) locally near a branch point, $\pi_*\mathcal{O}_C(P_i) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(Q_i) \cdot u_i$.

Remark 4.3 For any partition $\{P_1, \dots, P_a\} \cup \{P_{a+1}, \dots, P_{2g+2}\}$ of the Weierstrass points into two sets,

$$P_1 + \cdots + P_a + (g+1-a)g^1_2 \sim P_a + 1 + \cdots + P_{2g+2},$$

as follows from (ii) and $2P_i \sim g^1_2$.



This is important in what follows (see (4.5)); it corresponds to passing between the ± 1-eigensheaves of

$$\pi_* \mathcal{O}_C(P_1 + \cdots + P_a + k g^1_2).$$

Proof (i) is standard; one affine piece of C is

$$C : (y^2 = f_{2g+2}(t)).$$

It's easy to see that y/t^{g+1} is a rational function on C with

$$\operatorname{div}(y/t^{g+1}) = P_1 + \cdots + P_{2g+2} - (g+1) \cdot g^1_2;$$

this proves (ii). For (iii), if t is a local parameter on $\mathbb{P}^1$ at a branch point $Q \in \mathbb{P}^1$ and $u^2 = t \cdot (\text{unit})$, then u is a local parameter at $P \in C$, so $1/t$ has a simple pole at Q and the -1-eigensheaf of $\pi_* \mathcal{O}_C(P)$ is $\mathcal{O}_{\mathbb{P}^1} \cdot u/t = \mathcal{O}_{\mathbb{P}^1}(Q) \cdot u$. Q.E.D.

4.4 Simplest examples of graded rings

(a) Let $D = g^1_2$; then $H^0(\mathcal{O}_C(D)) = (t_1, t_2)$, and

$$H^0(\mathcal{O}_C(kD)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k - g - 1));$$

thus for $k \leq g$ all the sections of $\mathcal{O}_C(kD)$ are in the $+1$-eigenspace, so no new generators are needed, and I get the final generator

$$w \in H^0(\mathcal{O}_C((g+1)D))$$

in degree $g+1$ satisfying $w^2 = f_{2g+2}(t_1, t_2)$. So

$$R(C, g^1_2) = k[t_1, t_2, w]/F, \quad \text{with} \quad \deg(t_1, t_2, w, F) = 1, 1, g+1, 2g+2,$$

and $C = C_{2g+2} \subset \mathbb{P}(1, 1, g+1)$.

(b) Let $D = P$ with $P \in C$ a Weierstrass point; write $P = P_{2g+2}$ and $P_1, \dots, P_{2g+1}$ for the remaining Weierstrass points, and

$$u\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P) \quad \text{and} \quad v\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_1 + \cdots + P_{2g+1})$$

for the two constant sections. Since $u^2\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(2P) = \mathcal{O}_C(g^1_2)$ is the constant section, I can choose the coordinates (t_1, t_2) so that $u^2 = t_1$, and $t_2 \in H^0(C, \mathcal{O}_C(2P))$ is a complementary basis element. Now

$$\pi_*\mathcal{O}_C(2kP) = \pi_*\mathcal{O}_C(kg^1_2) = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-g-1),$$

and by (4.2, iii),

$$\pi_*\mathcal{O}_C((2k+1)P) = \pi_*\mathcal{O}_C(P) \otimes \mathcal{O}_{\mathbb{P}^1}(k) = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-g),$$

so that monomials $u^\ell, u^{\ell-2}t_2, \dots$ base $H^0(\mathcal{O}_C(\ell P))$ for $\ell \le 2g$; but in degree $2g+1$ there is a new section z in the -1-eigenspace. Under the linear equivalence

$$(2g+1)P \sim P_{2g+2} + g \cdot g^1_2 \sim P_1 + \cdots + P_{2g+1},$$

z is the constant section $v\colon C \hookrightarrow \mathcal{O}_C(P_1 + \cdots + P_{2g+1})$; in more detail, if y is chosen as in (4.2, ii) then

$$\begin{aligned} \operatorname{div}(t_1^{g+1}/y) &= (g+1)(2P) - (P_1 + \cdots + P_{2g+2}) \\ &= (2g+1)P - (P_1 + \cdots + P_{2g+1}) \end{aligned}$$

so that $z = vt_1g + 1/y$. If $f = f_{2g+1}(t_1, t_2)$ is the form defining the $2g+1$ branch points in $\mathbb{P}^1$, then $z^2 = f(u^2, t_2)$, so

$$R(C, P) = k[u, t_2, z]/F, \quad \text{with} \quad \deg(u, t_2, z, F) = 1, 2, 2g+1, 4g+2,$$

and $C = C_{4g+2} \subset \mathbb{P}(1, 2, 2g+1)$.

Remark The ring $R(C, g^1_2)$ of (i) can be obtained by eliminating the elements of $R(C, P)$ of odd degree; that is, $R(C, 2P) = R(C, P)^{(2)}$. This means replacing u by $t_1 = u^2$, z by $w = uz$, and

$$F\colon z^2 = f_{4g+2} \quad \text{by} \quad F'\colon w^2 = u^2 f_{4g+2}(u, t_2) = f'_{2g+2}(t_1, t_2).$$

Lemma 4.5 *Let D be a divisor on C. Equivalent conditions:*

(i) the divisor class of D is invariant under ι, that is $D \sim \iota^* D$;

(ii) $D \sim D'$ with $D' = \iota^* D'$;

(iii) D is made up of Weierstrass points, that is (after a possible renumbering),

$$D \sim P_1 + \cdots + P_a + bg^1_2.$$

Proof The implications (ii) $\implies$ (iii) $\implies$ (i) are trivial, so assume (i). By adding on a large multiple of g_2^1 if necessary, I assume that D is effective. If $\iota^*D \sim D$ but $\iota^*D \neq D$ then $|D|$ is a nontrivial linear system. I pick one Weierstrass point, say P_1; then the divisor class $D - P_1$ is invariant under ι, and $|D - P_1|$ contains an effective divisor, so that induction on $\deg D$ proves (ii). Q.E.D.

Remark Since $D + \iota^*D \sim (\deg D) \cdot g_2^1$ for any divisor D on a hyperelliptic curve, a 4th equivalent condition on D is

(iv) $2D \sim (\deg D) \cdot g_2^1$.

This set of divisors includes of course all divisor classes with $2D \sim 0$ or $2D \sim K_C$, etc.

Useful fact: each 2-torsion divisor on a hyperelliptic curve is (up to renumbering) of the form

$$P_1 + \cdots + P_{2a} - a \cdot g_2^1 \sim P_{2a+1} + \cdots + P_{2g+2} - (g + 1 - a) \cdot g_2^1.$$

Go on, check for yourself that there are 2^{2g} of these!

Theorem 4.6 *(I) For an invariant divisor $D = P_1 + \cdots + P_a + bg_2^1$, set*

$$D' = P_{a+1} + \cdots + P_{2g+2} + (a + b - g - 1)g_2^1,$$

so that $D \sim D'$ by Remark 4.3. Write $u\colon C \hookrightarrow C(P_1 + \cdots + P_a)$ and $v\colon C \hookrightarrow C(P_{a+1} + \cdots + P_{2g+2})$ for the constant sections. Then

$$\pi_*\mathcal{O}_C(D) = \mathcal{O}_{\mathbb{P}^1}(b) \cdot u \oplus \mathcal{O}_{\mathbb{P}^1}(a + b - g - 1) \cdot v$$

and

$$H^0(\mathcal{O}_C(D)) = H^0(\mathcal{O}_{\mathbb{P}^1}(b)) \cdot u \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(a + b - g - 1)) \cdot v.$$

In other words, if I write $S^k(t_1, t_2) = \{t_1^k, t_1^{k-1}t_2, \ldots, t_2^k\}$ for the set of $k+1$ monomials of degree k (or $\emptyset$ if $k < 0$) then $H^0(\mathcal{O}_C(D))$ has basis

$$S^b(t_1, t_2) \cdot u, \quad S^{a+b-g-1}(t_1, t_2) \cdot v.$$

(II) Write $f_a(t_1, t_2)$ and $g_{2g+2-a}(t_1, t_2)$ for the forms defining $Q_1 + \cdots + Q_a$ and $Q_{a+1} + \cdots + Q_{2g+2}$ in $\mathbb{P}^1$. Then the graded ring $R(C, \mathcal{O}_C(D))$ is generated by monomials in $R(C, \mathcal{O}_C(kD))$ for suitable initial values of k, and related by monomial relations together with relations deduced from

$$u^2 = f_a(t_1, t_2), \quad v^2 = g_{2g+2-a}(t_1, t_2).$$

Proof of (I) $\pi_*\mathcal{O}_C(D)$ has a uniquely determined $\mathbb{Z}/2$ action compatible with the inclusion $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_1+\cdots+P_a)$, and the $+1$-eigensheaf is clearly $\mathcal{O}_{\mathbb{P}^1}(b)\cdot u$. Multiplication by the rational function $y/t_1^{g+1} \in k(C)$ described in the proof of (4.2, ii) induces an isomorphism

$$\mathcal{O}_C(D) \cong \mathcal{O}_C(D'),$$

and since y/t_1^{g+1} is in the -1-eigenspace, the isomorphism interchanges the ± 1-eigensheaves. This proves (I).

4.7

I will regard (II) as a principle, and not go into the long-winded general proof, which involves introducing notation $k_0^{\pm}$, $k_1^{\pm}$ for the smallest even and odd values of k for which each eigensheaf of $\pi_*\mathcal{O}_C(kD)$ has sections, and a division into cases according to which of these is smaller.

I now give a much more precise statement and proof of (II) in the main case of interest. Suppose that, in the notation of Theorem 4.5,

$$b \geq 0 \quad \text{and} \quad a+2b < g+1 \leq 2a+3b.$$

Note that $2D = (a+2b)\cdot g_2^1$, so that

$$\pi_*\mathcal{O}_C(2D) = \mathcal{O}(a+2b) \oplus \mathcal{O}(a+2b-g-1)\cdot uv,$$

where $uv\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_1+\cdots+P_{2g+2})$ is the constant section. Write $V^{\pm}$ to denote the ± 1-eigenspaces of a vector space on which ι acts; the point of these inequalities is just to ensure that

$$\begin{aligned}
&H^0(D)^+ = H^0(\mathcal{O}(b))\cdot u \neq 0,\\
&H^0(2D)^- = H^0(\mathcal{O}(a+2b-g-1))\cdot uv = 0, \quad \text{(so also } H^0(D)^- = 0\text{)}\\
&H^0(3D)^- = H^0(\mathcal{O}(2a+3b-g-1))\cdot v \neq 0.
\end{aligned}$$

Notice that this case covers all effective halfcanonical divisors on a hyperelliptic curve of genus $g \geq 4$, for which $a+2b = g-1$.

Theorem *The graded ring $R(C,D)$ is generated by the following bases:*

$$\begin{aligned}
&(x_0,x_1,\dots,x_b) = S^b(t_1,t_2)\cdot u = t_1^b u, t_1^{b-1}t_2 u,\dots,t_2^b u \in H^0(D)^+;\\
&(y_0,y_1,\dots,y_d) = S^d(t_1,t_2) = t_1^d, t_1^{d-1}t_2,\dots,t_2^d \in H^0(2D)^+;\\
&(z_0,z_1,\dots,z_c) = S^c(t_1,t_2)\cdot v = t_1^c v, t_1^{c-1}t_2 v,\dots,t_2^c v \in H^0(3D)^-.
\end{aligned}$$

where I set $d = a + 2b = \deg D$ and $c = 2a + 3b - g - 1$ for brevity. The relations are given as follows:

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & \dots & x_{b-1} & y_0 & y_1 & \dots & y_{d-1} & z_0 & z_1 & \dots & z_{c-1} \\ x_1 & x_2 & \dots & x_b & y_1 & y_2 & \dots & y_d & z_1 & z_2 & \dots & z_c \end{pmatrix} \le 1$$

(the x or z columns are omitted if $b = 0$ or $c = 0$). And

$$\begin{aligned} x_i x_j &= F_{i+j}(y_0, \dots, y_d) \quad \text{for all } 0 \le i, j \le b, \\ z_i z_j &= G_{i+j}(y_0, \dots, y_d) \quad \text{for all } 0 \le i, j \le c, \end{aligned}$$

where

$$\begin{aligned} F_{i+j} &= t_1^{2b-i-j} t_2^{i+j} f_a(t_1, t_2) \quad \text{rendered as a linear form in } y_0, \dots, y_d; \\ G_{i+j} &= t_1^{2c-i-j} t_2^{i+j} g_{2g+2-a}(t_1, t_2) \quad \text{rendered as a cubic form in } y_0, \dots, y_d; \end{aligned}$$

Remark 4.8 (a) Notice that, as promised, the first set consists of monomial relations, and the second of relations deduced from

$$u^2 = f_a(t_1, t_2), \quad v^2 = g_{2g+2-a}(t_1, t_2).$$

(b) The first set of determinantal relations $\operatorname{rank} A \le 1$ says simply that the ratio $(t_1 : t_2)$ defining $\pi\colon C \to \mathbb{P}^1$ is well defined. In fact the projective toric variety defined by $\operatorname{rank} A \le 1$ is a *weighted scroll*, that is, a fibre bundle $\varphi\colon F \to \mathbb{P}^1$ with fibre the weighted projective space $\mathbb{P}(1, 2, 3)$: in more detail, $F = \operatorname{Proj}_{\mathbb{P}}^1(\mathcal{A})$, where $\mathcal{A}$ is the graded $\mathbb{P}^1$-algebra

$$\mathcal{A} = \bigoplus \varphi^* \mathcal{O}_F(k) = \operatorname{Sym}\Big\{\mathcal{O}_{\mathbb{P}^1}(b)_1 \oplus \mathcal{O}_{\mathbb{P}^1}(d)_2 \oplus \mathcal{O}_{\mathbb{P}^1}(c)_3\Big\}.$$

This means that the x_i, y_i and z_i can be written simply as

$$\begin{aligned} (x_0, \dots, x_b) &= S^b(t_1, t_2) \cdot u, \\ (y_0, \dots, y_d) &= S^d(t_1, t_2) \cdot w, \\ (z_0, \dots, z_c) &= S^c(t_1, t_2) \cdot v, \end{aligned}$$

where $u \in H^0(F, \mathcal{O}_F(1) \otimes \varphi^* \mathcal{O}_{\mathbb{P}^1}(-b))$ is a global basis of the summand $\mathcal{A}_1(-b)$ over $\mathbb{P}^1$, and similarly for w and v. In these terms, $C \subset F$ is the codimension 2 complete intersection of type $(2, 6)$ defined by

$$u^2 = f_a(t_1, t_2) w, \quad v^2 = g_{2g+2-a}(t_1, t_2) w^3.$$

4.9 Proof of the theorem

By (4.5, I), I have an explicit monomial basis of each $H^0(nD)$ in terms of t_1, t_2, u and v; clearly, each monomial in $H^0(2nD)$ equals

$$\text{either} \quad y^{\underline{m}} \quad \text{with } \deg y^{\underline{m}} = 2n \quad \text{or} \quad y^{\underline{m}'} x_i z_j \quad \text{with } \deg y^{\underline{m}'} = 2n-4,$$

and similarly each monomial in $H^0((2n+1)D)$ equals

$$\text{either} \quad y^{\underline{m}} x_i \quad \text{with } \deg y^{\underline{m}} = 2n \quad \text{or} \quad y^{\underline{m}'} z_j \quad \text{with } \deg y^{\underline{m}'} = 2n-2,$$

Moreover, an expression in x_i, y_j, z_k containing a quadratic term in x_i or z_k can obviously be translated to these using the second set of relations in (4.6). Now make a *first choice* of monomial representative of each such monomial element: for example, ordinary alphanumeric order picks $x_0y_1y_d$ ahead of $x_1y_0y_d$ or $x_0y_2y_{d-1}$ etc., which are equal to it in $R(C, D)$; and it is trivial to go from any $y^{\underline{m}}$ or $y^{\underline{m}'}x_iz_j$ or $y^{\underline{m}}x_i$ or $y^{\underline{m}'}z_j$ to its first choice representative using the monomial relations of the first set of (4.6). Q.E.D.

4.10 Singular curves and Theorem 3.1, hyperelliptic case

For a nonsingular curve, the hyperelliptic case of Theorem 3.1 is included in (4.6). This analysis extends without difficulty to the case of a singular irreducible hyperelliptic curve C; for brevity I restrict myself to the main point, which is to describe the Cartier divisors on C playing the role of the sums of Weierstrass points $P_1 + \cdots + P_a$ in (4.4, iii). The reader may wish to fill in the details as an extended exercise.

The branch points of the cover $C \to \mathbb{P}^1$ divide into two types:

Cusp-like points In local analytic coordinates, $y^2 = x^{2k+1}$. At such a point $P \in C$ there is a unique Cartier divisor $P(1) = \mathrm{div}_P(y/x^k)$ of degree 1 such that $\iota^*P(1) = P(1)$. This has the property that

$$2P(1) = \mathrm{div}_P(x) \sim g^1_2,$$

If $k = 0$, then $P(1) = P$ is just an ordinary Weierstrass point of C. In general, the pullback of P(1) is just the Weierstrass point of the normalised curve, with multiplicity 1 (since y/x^k is a local parameter there), but

$$P(1) + ig^1_2 \text{ is effective } \iff i \geq k.$$

The divisor $A_P = P(1) + k \cdot g^1_2$ is a Cartier divisor of degree of degree $2k+1$ on C, and plays the role of $2k+1$ coincident Weierstrass points.

Node-like points In local analytic coordinates, $y^2 = x^{2k}$. At such a point $P \in C$ there is a unique nonzero Cartier divisor $P(0) = \mathrm{div}_P(y/x^k)$ of degree 0 such that $\iota^* P(0) = P(0)$. This satisfies $2P(0) = 0$, and the pullback of $P(0)$ to the normalisation is 0 (since $y/x^k = \pm 1$ is invertible at the two points), but

$$P(0) + ig^1_2 \text{ is effective } \iff i \geq k.$$

The divisor $A_P = P(0) + k \cdot g^1_2$ is a Cartier divisor of degree of degree $2k$ on C, and plays the role of $2k$ coincident Weierstrass points.

The divisors $\sum A_P$ summed over distinct branch points P and of degree $a \leq g-1$ are characterised as the Cartier divisors on C invariant under ι and with $h^0 = 1$, in complete analogy with sums of distinct Weierstrass points. Now by analogy with Lemma 4.4, it can be seen that any Cartier divisor (or divisor class) on C invariant under ι is a sum of divisors of the form $P(1)$ for cusp-like P, of divisors of the form $P(0)$ for node-like P, and of a multiple of g^1_2. Any effective Cartier divisor D invariant under ι is of the form

$$D = \sum A_P + bg^1_2, \quad \text{with } b \geq 0,$$

summed over a subset of the branch points P, and as in Theorem 4.5, if I set $a = \deg \sum A_P$ and write $\sum' A_P$ for the complementary sum, then $D \sim D'$ where

$$D' = \sum{}' A_P + (a + b - g - 1)g^1_2.$$

The statement and proof of Theorems 4.5–6 now go through with only minor changes.

Part III
Applications

5 Numerical quintics and other stories

5.0 Preview

In this *toccata* section I work out in detail the deformation theory in degree ≤ 0 of the ring $\overline{R} = R(C, \mathcal{O}_C(D))$, where C is a nonsingular hyperelliptic curve of genus 6 and $D = 2g^1_2 + P \in \frac{1}{2}K_C$; in substance, the results are due to Horikawa [Horikawa] and Griffin [Griffin], although my treatment is novel and quite fun.

It turns out that the ring $\overline{R}$ admits two quite different representations in determinantal format. Each of these is *flexible*, in the sense of (1.24) so that changing freely the coefficients of the given matrix format preserves flatness, and thus gives rise to a large family of unobstructed deformations; in fact, I prove that in degrees ≤ 0, deformations from either family have codimension 1 in all 1st order deformations. These two families intersect transversally and their union gives exactly the deformations that extend to 2nd order. (There is an amazingly close analogy with Pinkham's example (2.1–6).) The main results are (5.11) and (5.16).

I conclude the section (5.17) by showing how to apply this to the classification and deformation theory of numerical quintics of dimension ≥ 2; for surfaces, these are of course fundamental classical results of Horikawa.

5.1 The ring for C

Let C be a nonsingular hyperelliptic curve of genus 6, and $D = 2g^1_2 + P$ with $P = P_0$ a Weierstrass point. By the standard hyperelliptic stuff (4.6), $R(C, \mathcal{O}_C(D))$ has generators

$$\begin{aligned} x_1, x_2, x_3 &= t_1^2 u, t_1 t_2 u, t_2^2 u; \\ y &= t_2^5; \\ z_1, z_2 &= t_1 v, t_2 v, \end{aligned}$$

where (t_1, t_2) is a basis of $H^0(g^1_2) = H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, and $u \colon \mathcal{O} \hookrightarrow \mathcal{O}(P_0)$, $v \colon \mathcal{O} \hookrightarrow \mathcal{O}(P_1 + \cdots + P_{13})$, so

$$u^2 = t_1 \quad \text{and} \quad v^2 = f_{13}(t_1, t_2).$$

By (4.6), the ideal of relations is generated by 9 relations that can be written down as two groups: the 6 relations from

$$\operatorname{rank} A \le 1 \quad \text{where} \quad A = \begin{pmatrix} x_1 & x_2 & x_3^2 & z_1 \\ x_2 & x_3 & y & z_2 \end{pmatrix},$$

(the vestigial symmetry $a_{12} = a_{21} = x_2$ of A is one crucial ingredient in what follows) and the 3 arising from $v^2 = f_{13}$:

$$\begin{aligned} z_1^2 &= t_1^2 f_{13} = g_0(x_1, \ldots, y), \\ z_1 z_2 &= t_1 t_2 f_{13} = g_1(x_1, \ldots, y), \\ z_2^2 &= t_2^2 f_{13} = g_2(x_1, \ldots, y), \end{aligned}$$

where $\deg g_i = 6$. Write

$$f_{13}(t_1, t_2) = a_0 t_1^{13} + a_1 t_1^{12} t_2 + \cdots + a_{10} t_1^3 t_2^{10} + 2b_1 t_1^2 t_2^{11} + b_2 t_1 t_2^{12} + b_3 t_2^{13}$$

(note the names of last 3 coefficients $2b_1, b_2, b_3$); it is easy to see that the nonsingularity of C implies $b_3 \neq 0$. Then

$$\begin{array}{rclcccccc}
g_0 & = & x_1^2 \cdot h & + & 2x_1x_2 \cdot b_1y^2 & + & x_2^2 \cdot b_2y^2 & + & (x_3^2)^2 \cdot b_3y, \\
g_1 & = & x_1x_2 \cdot h & + & (x_1x_3 + x_2^2) \cdot b_1y^2 & + & x_2x_3 \cdot b_2y^2 & + & x_3^2y \cdot b_3y, \\
g_2 & = & x_2^2 \cdot h & + & 2x_2x_3 \cdot b_1y^2 & + & x_3^2 \cdot b_2y^2 & + & y^2 \cdot b_3y,
\end{array}$$

where

$$\begin{aligned} h = {} & a_0x_1^4 + a_1x_1^3x_2 + a_2x_1^3x_3 + a_3x_1^2x_2x_3 + a_4x_1^2x_3^2 + a_5x_1^2y \\ & + a_6x_1x_2y + a_7x_1x_3y + a_8x_2x_3 + a_9x_3^2y + a_{10}y^2. \end{aligned}$$

More explicitly, I have the following 9 relations.

5.2 Table of all relations for $\overline{R} = R(C, \mathcal{O}_C(D))$

$$\begin{array}{rrcl}
S: & x_1x_3 & = & x_2^2; \\
T_1: & x_1y & = & x_2x_3^2; \\
T_2: & x_2y & = & x_3^3; \\
U_1: & x_1z_2 & = & x_2z_1; \\
U_2: & x_2z_2 & = & x_3z_1; \\
V: & x_3^2z_2 & = & yz_1; \\
-W_0: & z_1^2 & = & hx_1^2 + 2b_1y^2 \cdot x_1x_2 + b_2y^2 \cdot x_2^2 + b_3y \cdot x_3^4; \\
-W_1: & z_1z_2 & = & hx_1x_2 + b_1y^2 \cdot (x_1x_3 + x_2^2) + b_2y^2 \cdot x_2x_3 + b_3y \cdot x_3^2y; \\
-W_2: & z_2^2 & = & hx_2^2 + 2b_1y^2 \cdot x_2x_3 + b_2y^2 \cdot x_3^2 + b_3y \cdot y^2.
\end{array}$$

The syzygies holding between these relations are written out explicitly in Table 5.4.

5.3 First determinantal format

The above notation has been massaged slightly to make the g_i into explicit quadratic expressions in the rows of A; that is,

$$\begin{pmatrix} g_1 & g_2 \\ g_2 & g_3 \end{pmatrix} = A_0M_0{}^tA_0,$$

where

$$A_0 = \begin{pmatrix} x_1 & x_2 & x_3^2 \\ x_2 & x_3 & y \end{pmatrix} \quad \text{and} \quad M_0 = \begin{pmatrix} h & b_1y^2 & 0 \\ b_1y^2 & b_2y^2 & 0 \\ 0 & 0 & b_3y \end{pmatrix}.$$

In other words, the final 3 relations for the z_iz_j can be written in the form

$$AM({}^tA) = 0,$$

where A is as above and M is the symmetric matrix with homogeneous entries

$$M = \begin{pmatrix} M_0 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{of degrees} \quad \begin{pmatrix} 4 & 4 & 3 & 2 \\ 4 & 4 & 3 & 2 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$$

5.4

The following key observation is due (in a slightly harder context) to Duncan Dicks: the syzygies holding between the 9 relations of Table 5.2 are all consequences of the determinantal format.

Proposition *The following are 16 syzygies between the relations $S, T_1, \dots,$ V of Table 5.2; they generate the module of all syzygies. Moreover, they can be deduced from the determinantal format.*

Table of syzygies for $\overline{R} = R(C, \mathcal{O}_C(D))$

First set:

$$\begin{aligned} x_1T_2 - x_2T_1 + x_3^2S &\equiv 0 & \qquad x_2T_2 - x_3T_1 + yS &\equiv 0 \\ x_1U_2 - x_2U_1 + z_1S &\equiv 0 & x_2U_2 - x_3U_1 + z_2S &\equiv 0 \\ x_1V - x_3^2U_1 + z_1T_1 &\equiv 0 & x_2V - yU_1 + z_2T_1 &\equiv 0 \\ x_2V - x_3^2U_2 + z_1T_2 &\equiv 0 & x_3V - yU_2 + z_2T_2 &\equiv 0 \end{aligned}$$

Second set:

$$\begin{aligned} x_2W_0 - x_1W_1 &\equiv -(b_1x_1 + b_2x_2)y_2S - b_3x_3^2yT_1 + z_1U_1; \\ x_3W_0 - x_2W_1 &\equiv (x_1h + b_1x_2y_2)S - b_3x_3^2yT_2 + z_1U_2; \\ yW_0 - x_3^2W_1 &\equiv (x_1h + b_1x_2y_2)T_1 + (b_1x_1 + b_2x_2)y_2T_2 + z_1V; \\ z_2W_0 - z_1W_1 &\equiv (x_1h + b_1x_2y_2)U_1 + (b_1x_1 + b_2x_2)y_2U_2 + b_3x_3^2yV; \\ x_2W_1 - x_1W_2 &\equiv -(b_1x_2 + b_2x_3)y_2S - b_3y_2T_1 + z_2U_1; \\ x_3W_1 - x_2W_2 &\equiv (x_2h + b_1x_3y_2)S - b_3y_2T_2 + z_2U_2; \\ yW_1 - x_3^2W_2 &\equiv (x_2h + b_1x_3y_2)T_1 + (b_1x_2 + b_2x_3)y_2T_2 + z_2V; \\ z_2W_1 - z_1W_2 &\equiv (x_2h + b_1x_3y_2)U_1 + (b_1x_2 + b_2x_3)y_2U_2 + b_3y_2V. \end{aligned}$$

5.5 Proof

I first show how to derive the 16 syzygies from the determinantal format (which proves they are in fact syzygies). Let $A = (a_{ij})$ and $M = (m_{ij})$ be 2×4 and 4×4 matrixes whose entries are weighted homogeneous polynomials in a polynomial ring of degrees

$$\deg a_{ij} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \deg m_{ij} = \begin{pmatrix} 4 & 4 & 3 & 2 \\ 4 & 4 & 3 & 2 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$$

and I the ideal generated by the 9 polynomial relations

$$\operatorname{rank} A \le 1, \quad AM(tA) = 0.$$

There are two ways of deducing syzygies from the format of these relations: first, an obvious determinantal trick is to double a row of A, so that any 3×3 minor vanishes identically. This leads to the first 8 syzygies of Table 5.4. Next, write

$$A^* = \begin{pmatrix} a_{12} & -a_{11} \\ a_{22} & -a_{21} \\ a_{32} & -a_{31} \\ a_{42} & -a_{41} \end{pmatrix}$$

so that $(A^*)A$ is a 4×4 skew matrix with entries the 2×2 minors of A. Then the expression $(A^*)AM({}^tA)$ can be parsed in two different ways: A^* times $AM({}^tA)$ is a linear combination of the second set of relations with coefficients from A^*; whereas $(A^*)A$ times $M({}^tA)$ is a linear combination of the first set of relations (the 2×2 minors of A) with coefficients from $M({}^tA)$. Equating these leads to the second set of 8 syzygies in Table 5.4.

Finally, why are these all the syzygies? The assertion is that any identity between the 9 relations is a linear combination of the 16 given ones. I sketch a proof by a calculation similar to that of (4.6), but more unpleasant. I can write the identity $\sum \ell_{ij} f_i = 0$, where the ℓ_{ij} are monomials (and the f_i are the 9 relations (5.2), possibly repeated). First of all, since z_1S, $z_1T_1, \dots, z_1V$ all appear on the right-hand side of one of the 16 given syzygies, I can subtract off multiples of them and assume that none of the ℓ_{ij} are divisible by z_1, except possibly if $f_i = W_0$, W_1 or W_2. But an easy argument on the highest power of z_1 then shows that none of the ℓ_{ij} can be divisible by z_1. Similarly for z_2.

Now assuming that none of the ℓ_{ij} are divisible by z_1 or z_2, it's not hard to see that none of W_0, W_1, W_2 can appear in any syzygies at all, and in turn, the same for U_1, U_2, V. Finally, it's not hard to see that the first two syzygies of the table are the only ones between S, T_1, T_2.

I apologise for the above proof *by intimidation*. Here is a proof *by appeal to authority*. The statement is of a kind covered by monomial bases algorithms of Macaulay and Gröbner (so the skeleton of a proof just given is part of an algorithm); in particular, Bayer and Stillman's computer program Macaulay [Bayer and Stillman] can calculate the entire projective resolution of the ring defined by the 9 relations (5.2) in a few seconds (on an obsolete home microcomputer), and confirms the 16 syzygies. (Macaulay assumes working over a prime finite field, preferably $k = \mathbb{Z}/(31,991)$, and that constants in k are chosen for the coefficients of h, b_1, b_2 and b_3; however, this proof 'after specialisation' obviously implies the statement I need.) Q.E.D.

Remark 5.6 (a) The power of Proposition 5.4, and of the analogous result (5.8) for the second determinantal format, is that varying the entries of the matrixes A and M leads to flat deformations of the ring $\overline{R}$ (since the syzygies only depend on the determinantal format of the equations, this corresponds to varying the relations together with the syzygies). In this case, I say that the format of the equations is *flexible*: since the coefficients of the entries can vary freely in an open set of k^N, leading to large unobstructed families of deformations. It is not clear how to formalise this as a definition, since the expression 'format of the equations' is vague. However, it includes well-known and very useful formats such as generic determinantals.

(b) The relations $\operatorname{rank} A \le 1$, $AM({}^tA) = 0$ for generic matrixes A and M (with M symmetric) are analogous to the defining equations of a Schubert cell. If the weights were all 1, it is easy to see that the corresponding projective variety is just

$$\mathbb{P}^1 \times (\text{universal quadric of } \mathbb{P}^3).$$

Here M gives the quadric $Q \subset \mathbb{P}^3$, the rows of A a point of Q, and the columns a point of $\mathbb{P}^1$.

5.7 Second determinantal form

Given a 6×6 skew matrix $N = \{n_{ij}\}$, the condition $\operatorname{rank} N \le 2$ is expressed by the vanishing of the 15 (diagonal) Pfaffians of the 4×4 skew matrixes obtained by picking 4 rows and the corresponding columns. More concretely, for $i < j < k < \ell$,

$$ij.k\ell = \mathrm{Pf}_{ij.k\ell}(N) = n_{ij}n_{k\ell} - n_{ik}n_{j\ell} + n_{i\ell}n_{jk}.$$

Let me while away a happy half-hour by evaluating the 4×4 Pfaffians of the following beauty:

$$M = \begin{pmatrix} 0 & \beta & y_1 & z_1 & x_2 & x_1 \\ & 0 & y_2 & z_2 & x_3 & x_2 \\ & & 0 & q & z_2 & z_1 \\ & & & 0 & py_2 & py_1 \\ & & & & 0 & p\beta \\ & & & & & 0 \end{pmatrix} \quad \text{of degrees } \deg n_{ij} = \begin{pmatrix} 0 & 0 & 2 & 3 & 1 & 1 \\ 0 & 0 & 2 & 3 & 1 & 1 \\ 2 & 2 & 4 & 5 & 3 & 3 \\ 3 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 3 & 4 & 2 & 2 \\ 1 & 1 & 3 & 4 & 2 & 2 \end{pmatrix}$$

(here β, p and q are homogeneous elements of degrees $0, 2$ and 5 to be filled in subsequently; the antidiagonal symmetry is part of the format). The answer, Oh delight! is the following 3 groups:

$$\begin{array}{lll} I. & 12.56 & x_1x_3 - x_2^2 + p\beta^2 \\ & 12.36 & x_1y_2 - x_2y_1 + \beta z_1 \\ & 12.35 & x_2y_2 - x_3y_1 + \beta z_2 \\ & 12.46 & x_1z_2 - x_2z_1 + \beta py_1 \\ & 12.45 & x_2z_2 - x_3z_1 + \beta py_2 \\ & 12.34 & y_2z_1 - y_1z_2 + \beta q \end{array}$$

$$\begin{array}{lll} II. & 13.46 & x_1q - z_1^2 + py_1^2 \\ & 13.45 & x_2q - z_1z_2 + py_1y_2 \\ & 23.45 & x_3q - z_2^2 + py_2^2 \end{array}$$

$$\begin{array}{ll} III. & 23.46 = 13.45 \\ & 13.56 = 12.46 \\ & 23.56 = 12.45 \\ & 14.56 = p \times 12.36 \\ & 24.56 = p \times 12.35 \\ & 34.56 = p \times 12.34 \end{array}$$

Remark (i) If $\beta = 0$, these relations can be put back in the first determinantal form: you just have to express x_1q, x_2q, x_3q as quadratics in the rows of the 2×4 matrix in the top-right.

(ii) On the other hand, if $\beta \neq 0$ then the relations give z_1 and z_2 as polynomials in the other variables. If there is only 1 variable of degree 2, and p, y_1, y_2 are near their values for $\overline{R}$ (given in (5.8) below) then y_1 and y_2 are also functions of x_1, x_2, x_3, and z_1, z_2, y_1 and y_2 can be eliminated to give the single quintic relation 12.34. This discovery is essentially due to Griffin [Griffin1–2].

(iii) Conversely, a set of relations in the first determinantal form given by matrixes A and M as in (5.5) can be put in the Pfaffian form if and only if the first 2×2 minor of

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

can be made symmetric (by row and column operations), or equivalently, the single degree 2 relation $S : (a_{11}a_{22} - a_{12}a_{21})$ is a quadratic form of rank 3.

Proposition 5.8 *(a) If I set*

$$y_1 = x_3^2 \text{ and } y_2 = y, \quad \beta = 0, \quad p = b_3 y, \quad q = hx_1 + 2b_1x_2y_2 + b_2x_3y_2$$

then the Pfaffian relations of (5.7) generate the same ideal as the 9 relations of (5.2).

(b) All the syzygies holding between the Pfaffian relations of (5.7) can be deduced from the determinantal format. Thus here too, arbitrary (small) variations of y_1, y_2, β, p, q give rise to flat deformations of the ring $\overline{R}$.

Proof (a) is a trivial substitution: for example, 23.45 becomes

$$-z_2^2 + x_3q + py^2 = -z_2^2 + x_1x_3h + 2b_1x_2x_3y^2 + b_2x_3^2y^2 + b_3y^3 = W_2 + hS.$$

Thus under this specialisation, one can read off

$$\begin{pmatrix} 34.56 & 24.56 & 23.56 & 23.46 & 23.45 \\ & 14.56 & 13.56 & 13.46 & 13.45 \\ & & 12.56 & 12.46 & 12.45 \\ & & & 12.36 & 12.35 \\ & & & & 12.34 \end{pmatrix} = \begin{pmatrix} -b_3yV & b_3yT_2 & U_2 & W_1 - b_1y^2S & W_2 + hS \\ & b_3yT_1 & U_1 & W_0 + b_2y^2S & W_1 - b_1y^2S \\ & & S_2 & U_1 & U_2 \\ & & & T_1 & T_2 \\ & & & & -V \end{pmatrix}$$

5.9 Syzygies between Pfaffians

Suppose that $B = (b_{ij})$ is a $(2k+1)\times(2k+1)$ skew matrix, and $P = (\mathrm{Pf}_{ii})$ the column formed by the $(2k+1)$ diagonal $2k\times 2k$ Pfaffians of B; then $BP = 0$ (or by symmetry $({}^tP)B = 0$). It's useful to know also that the adjugate matrix (maximal minors) of B is $\operatorname{ajd} B = P({}^tP)$; and of course, $B(\operatorname{ajd} B) = (\det B)\cdot \mathrm{id} = 0$.

This applies to every 5×5 diagonal block of N; thus if I make the skew 6×6 matrix $P = (P_{ij})$ with entries the 4×4 Pfaffians of N,

$$\begin{pmatrix} 0 & 34.56 & -24.56 & 23.56 & -23.46 & 23.45 \\ -34.56 & 0 & 14.56 & -13.56 & 13.46 & -13.45 \\ 24.56 & -14.56 & 0 & 12.56 & -12.46 & 12.45 \\ -23.56 & 13.56 & -23.56 & 0 & 12.36 & -12.35 \\ 23.46 & -13.46 & 12.46 & -12.36 & 0 & 12.34 \\ -23.45 & 13.45 & -12.45 & 12.35 & -12.34 & 0 \end{pmatrix}$$

(that is, $P_{ij} = \pm \mathrm{Pf}_{kl.mn}$ with $\pm = \operatorname{sign}(ijklmn)$, the *Pfaffian adjugate* of a $(2k+2)\times(2k+2)$ skewsymmetric matrix N), then the off-diagonal elements of NP are identically zero; it's not hard to check that the diagonal elements are all equal to Pf, the 6×6 Pfaffian of N, so that

$$\Sigma = NP = \mathrm{Pf}\cdot\mathrm{id}\,.$$

Since NP is 6×6, this provides *a priori* 35 identities between the relations of (5.7).

5.10 Proof of (b)

It is clearly enough to prove that after making the specialisation of (5.8, a), the determinantal syzygies just described generate the same module as the 16 syzygies of (5.4). This is a delicious calculation: write $\Sigma = NP = (\sigma_{ij})$, where

$$N = \begin{pmatrix} 0 & 0 & x_3^2 & z_1 & x_2 & x_1 \\ 0 & 0 & y & z_2 & x_3 & x_2 \\ -x_3^2 & -y & 0 & hx_1 + 2b_1x_2y^2 + b_2x_3y^2 & z_2 & z_1 \\ -z_1 & -z_2 & -hx_1 - 2b_1x_2y^2 - b_2x_3y^2 & 0 & b_3y^2 & b_3x_3^2y \\ -x_2 & -x_3 & -z_2 & -b_3y^2 & 0 & 0 \\ -x_1 & -x_2 & -z_1 & -b_3x_3^2y & 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0 & -b_3yV & -b_3yT_2 & U_2 & -W_1+b_1y^2S & W_2+hS \\ b_3yV & 0 & b_3yT_1 & -U_1 & W_0+b_2y^2S & -W_1+b_1y^2S \\ b_3yT_2 & -b_3yT_1 & 0 & S & -U_1 & U_2 \\ -U_2 & U_1 & -S & 0 & T_1 & -T_2 \\ W_1-b_1y^2S & -W_0-b_2y^2S & U_1 & -T_1 & 0 & -V \\ -W_2-hS & W_1-b_1y^2S & -U_2 & T_2 & V & 0 \end{pmatrix}$$

Then $\sigma_{14}, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{24}, \sigma_{23}, \sigma_{25}$ and σ_{26} are identically the 8 syzygies of the first set of (5.4); and $\sigma_{12}, \sigma_{22}, \sigma_{33}, \sigma_{35}, \sigma_{32}, \sigma_{11}, \sigma_{33}, \sigma_{21}, \sigma_{36}, \sigma_{31}$ are the 8 syzygies of the second set plus some multiples of the first. Go on, have a go! Q.E.D.

Theorem 5.11 *(I) In degree 0, every 1st order deformation $R^{(1)}$ of $\overline{R}$ can be put in the second determinantal form (5.7–8).*

(II) In degree < 0, every 1st order extension $R^{(1)}$ of $\overline{R}$ can be put in the first determinantal form (5.3–4). (In degree ≤ -2, $R^{(1)}$ can be put in either form.)

Remark 5.12 (a) Fixing a degree a ≤ 0 and making a 1-parameter extension (or deformation), 1st order deformations are thus unobstructed. What happens when extensions (in degree < 0) get mixed up with deformations (in degree 0) is more exciting, and is discussed in (5.15–16) below.

(b) The results here are exactly what one should expect. Plane quintics depend on 12 moduli, whereas hyperelliptic curves of genus 6 depend on 11, and (I) says that the latter can be seen as a smooth codimension 1 degeneration of the former; the only surprise is how complicated the algebra underlying this simple geometry turns out to be. Horikawa's geometric considerations show that a numerical quintic surface having a hyperelliptic canonical curve $C \in |K_X|$ has a genus 2 pencil, and can therefore be written in determinantal form; the result (II) says that this also holds for every 1st order extension $2C$ of C in degree -1.

(c) To understand the difference between the two cases, note that the deformation as a Pfaffian with $\beta = 0$ is impossible in deg < 0 (since β would have to be a polynomial of degree < 0. On the other hand, if you don't add an element x_0 of degree 1 to the polynomial ring, an arbitrary deformation of $S : x_1x_3 = x_2^2$ remains a quadric of rank 3; so in the first determinantal format, A must start with the symmetric 2×2 block

$$\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$$

5.13 Setting up the 1st order deformation calculation

By (1.10–11), the Hilbert scheme of 1st order extensions of $\overline{R}$ in degree $-a < 0$ (or deformations in degree $a = 0$) is the vector space

$$\mathbb{H}^{(1)}(\overline{R}, a) = \mathrm{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-a} = \Big\{ (f_i' \in \overline{R}_{d_i - a}) \;\Big|\; \forall j, \sum \ell_{ij} f_i' = 0 \in \overline{R}_{s_j - a} \Big\}.$$

A useful observation is that although the expression on the right says to use all 16 of the syzygies of (5.4) for $\overline{R}$, they are all implied by the following very convenient subset:

First few syzygies

$$\begin{array}{rrcl}
\sigma_1 : & x_1T_2 & \equiv & x_2T_1 - x_3^2S; \\
\sigma_2 : & x_1U_2 & \equiv & x_2U_1 - z_1S; \\
\sigma_3 : & x_1V & \equiv & x_3^2U_1 - z_1T_1; \\
\sigma_4 : & x_1W_1 & \equiv & x_2W_0 + (b_1x_1 + b_2x_2)y^2S + b_3x_3^2yT_1 - z_1U_1; \\
\sigma_5 : & x_1W_2 & \equiv & x_2W_1 + (b_1x_2 + b_2x_3)y^2S + b_3y^2T_1 - z_2U_1.
\end{array}$$

This is true because every syzygy Σ of (5.4) has a monomial multiple which is a linear combination of these 5, as can be checked by an elementary calculation; for example,

$$x_1(x_2V - x_3^2U_2 + z_1T_2) = x_2\sigma_3 - x_3^2\sigma_2 + z_1\sigma_1.$$

Thus, since each of $x_1, x_2, x_3, y, z_1, z_2$ is a non-zerodivisor of $\overline{R}$, I need only verify the condition $\sum \ell_{ij} f_i' = 0 \in \overline{R}_{s_j - a}$ for these 5 values of j.

For the 1st order calculation, I've got to write down all S', T_1', U_1' and $W_0' \in \overline{R}$ of degrees $2-a$, $3-a$, $4-a$ and $6-a$ such that in turn T_2', U_2', V', W_1' and W_2' can be found to satisfy

$$\begin{array}{rrcl}
\sigma_1' : & x_1T_2' & = & x_2T_1' - x_3^2S'; \\
\sigma_2' : & x_1U_2' & = & x_2U_1' - z_1S'; \\
\sigma_3' : & x_1V' & = & x_3^2U_1' - z_1T_1'; \\
\sigma_4' : & x_1W_1' & = & x_2W_0' + (b_1x_1 + b_2x_2)y^2S' + b_3x_3^2yT_1' - z_1U_1'; \\
\sigma_5' : & x_1W_2' & = & x_2W_1' + (b_1x_2 + b_2x_3)y^2S' + b_3y^2T_1' - z_2U_1'.
\end{array}$$

Each of these equalities in $\overline{R}$ is written as a condition of divisibility by x_1; this is a very concrete linear condition on S', T'_1, U'_1 and W'_0, especially since by (4.5, I), it is natural to write down a monomial basis of each $\overline{R}_d$ in alphanumeric order, with x_1 first:

Table of bases of $R_d = H^0(C, \mathcal{O}_C(dD))$:

$H^0(D)$	x_1, x_2, x_3
$H^0(2D)$	$x_1^2, x_1x_2, x_1x_3, x_2x_3, x_3^2, y$
$H^0(3D)$	$x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2x_3, x_1x_3^2, x_1y, x_2y, x_3y; z_1, z_2$
$H^0(4D)$	$x_1^4, x_1^3x_2, x_1^3x_3, x_1^2x_2x_3, x_1^2x_3^2, x_1^2y, x_1x_2y, x_1x_3y, x_2x_3y,$ $x_3^2y, y^2; x_1z_1, x_1z_2, x_2z_2, x_3z_2$
etc.	

5.14 1st order deformation calculation in degree 0

The proof of (5.11) is similar to that of (2.2–6), and I omit some details. *A priori*,

$$S' = \varepsilon_1 x_1^2 + \varepsilon_2 x_1x_2 + \varepsilon_3 x_1x_3 + \varepsilon_4 x_2x_3 + \varepsilon_5 x_3^2 + \varepsilon_6 y;$$

However, since the relation is

$$x_1x_3 = x_2^2 + \lambda S',$$

(with $\deg\lambda = 0$), I can make coordinate changes of the form $x_i \mapsto x_i + \lambda \sum a_{ij}x_j$ to kill all the terms except the last, so assume $S' = \varepsilon_6 y$. Similarly, I can reduce T'_1 to

$$T'_1 = \alpha_7 x_2y + \alpha_8 x_3y + \beta_1 z_1 + \beta_2 z_2.$$

by changes in y. Now plugging in σ'_1 gives

$$x_1T'_2 = x_2T'_1 - x_3^2S' \in H^0(4D);$$

since x_2x_3y, x_3^2y, x_2z_2 are linearly independent modulo multiples of x_1, I conclude $\varepsilon_6 = \lambda_8 = \beta_2 = 0$, and

$$S' = 0; T'_1 = \alpha x_2y + \beta z_1; \quad \text{and} \quad T'_2 = \alpha x_3y + \beta z_2.$$

Similarly, U'_1 can be reduced to $U'_1 = \gamma_{10}x_3^2y + \gamma_{11}y^2 + \delta_4x_3z_2$ by changes in z_1 and z_2, and plugging into σ'_2: $x_1U'_2 = x_2U'_1 - z_1S'$ gives $\gamma_{11} = \delta_4 = 0$, so that

$$U'_1 = \gamma x_3^2y \quad \text{and} \quad U'_2 = \gamma y^2.$$

As an arbitrary element of $H^0(6D)$, W_0' can be written as

$$\begin{aligned}W_0' = x_1^2h' &+ \delta_1x_1x_2y^2 + \delta_2x_1x_3y^2 + \delta_3x_2x_3y^2 + \delta_4x_3^2y^2 + \delta_5y^3 \\ &+ \varepsilon_1x_1x_2x_3z_2 + \varepsilon_2x_1x_3^2z_2 + \varepsilon_3x_1yz_2 + \varepsilon_4x_2yz_2 + \varepsilon_5x_3yz_2;\end{aligned}$$

(the ε_i are new, but they too will all die). A coordinate change of the form $z_1 \mapsto z_1 + \delta x_1Q$, $z_2 \mapsto z_2 + \delta x_2Q$ can be used to fix up $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ (same Q, so as not to alter U_1' and U_2'). Now the syzygy $\sigma_4'\colon x_1W_1' = x_2W_0' + b_3x_3^2y(\alpha x_2y + \beta z_1) - \gamma x_3^2yz_1$ gives

$$\delta_5 = 0 \quad\text{and}\quad \varepsilon_5 = -b_3\beta + \gamma,$$

so that

$$W_1' = x_1x_2h' + \delta_1x_1x_3y^2 + \delta_2x_2x_3y^2 + \delta_3x_3^2y^2 + (\delta_4 + b_3\alpha)y^3 + \varepsilon_4x_3yz_2;$$

in turn, plugging into $\sigma_5'\colon x_1W_2' = x_2W_1' + b_3y^2(\alpha x_2y + \beta z_1) - \gamma x_3^2yz_2$ implies $\delta_4 = -2b_3\alpha$, $\varepsilon_4 = 0$ and $\gamma = b_3\beta$. So finally

$$\begin{aligned}&S' = 0,\ \ T_1' = x_2y + \beta z_1,\ \ T_2' = x_3y + \beta z_2,\ \ U_1' = b_3\beta x_3^2y,\ \ U_2' = b_3\beta y^2,\\ &V' = -\beta(hx_1 + 2b_1x_2y^2 + b_2x_3y^2) - yz_2,\\ &W_0' = x_1^2h' + \delta_1x_1x_2y^2 + \delta_2x_1x_3y^2 + \delta_3x_2x_3y^2 - 2b_3x_3^2y^2,\\ &W_1' = x_1x_2h' + \delta_1x_1x_3y^2 + \delta_2x_2x_3y^2 + \delta_3x_3^2y^2 - b_3y^3,\\ &W_2' = x_1x_3h' + \delta_1x_2x_3y^2 + \delta_2x_3^2y^2 + \delta_3y^3.\end{aligned}$$

It is now easy to assemble the relations to 1st order into the Pfaffian determinantal format rank $N^{(1)} \le 2$, where

$$N^{(1)} = \begin{pmatrix} 0 & \lambda\beta & x_3^2 - \lambda\alpha y & z_1 & x_2 & x_1 \\ & 0 & y & z_2 & x_3 & x_2 \\ & & 0 & q + \lambda q' & z_2 & z_1 \\ & & & 0 & (p+\lambda p')y & (p+\lambda p')(x_3^2 - \lambda\alpha y) \\ & -\mathrm{sym} & & & 0 & \lambda(p+\lambda p')\beta \\ & & & & & 0 \end{pmatrix}$$

with $q' = x_1h' + \delta_1x_2y^2 + \delta_2x_3y^2$ and $p' = \delta_3y$.

5.15 1st order deformation calculation in degree < 0

The computation in degree $-2, -3, \ldots,$ is a straightforward exercise for the reader. (But beware: this kind of conclusion is not at all automatic: Dicks has an example of a ring having obstructed 1st order extensions in degree -4 only.)

I give the computation in degree -1 in skeleton form, since step by step it is almost identical to that in degree 0; I use Latin letters r, s, t, u, v instead of Greeks so that the notations of (5.13) and (5.14) can be added together in the contrapuntal climax (5.15–16).

Set $S' = 0$ (use $x_i \mapsto x_i + *x_0$) and $T'_1 = r_3x_2x_3 + r_4x_3^2 + r_5y$ (use $y \mapsto y + x_0(*x_1 + *x_2 + *x_3)$). Then

$$\sigma'_1\colon x_1T'_2 = x_2T'_1 - x_3^2S' \implies r_5 = 0 \text{ and } T'_2 = r_3x_3^2 + r_4y.$$

Set $U'_1 = t_7x_3y + s_1z_1 + s_2z_2$ (use $z_1 \mapsto z_1 + *x_0q$ and same for z_2). So

$$\sigma'_2\colon x_1U'_2 = x_2U'_1 - z_1S' \implies s_2 = t_7 = 0 \text{ and } U'_2 = s_1z_2.$$
$$\sigma'_3\colon x_1V' = x_3^2U'_1 - z_1T'_1 \implies r_4 = s_1 \text{ and } V' = -r_3x_3z_2.$$

So writing $r_3 = r$, $r_4 = s_1 = s$, this summarises as

$$S' = 0,\ T'_1 = rx_2x_3 + sx_3^2,\ T'_2 = rx_3^2 + sy,\ U'_1 = sz_1,\ U'_2 = sz_2,\ V' = -rx_3z_2,$$

all of which fits together as the x_0 terms in rank $A^{(1)} \le 1$, where

$$A^{(1)} = \begin{pmatrix} x_1 & x_2 - sx_0 & x_3^2 + rx_0x_3 & z_1 \\ x + 2 + sx_0 & x_3 & y & z_2 \end{pmatrix}.$$

Now set

$$\begin{aligned} -W'_0 = x_1^2h' &+ t_9x_1x_2x_3y + t_{10}x_1x_3^2y + t_{11}x_1y^2 + t_{12}x_2y^2 + t_{13}x_3y^2 \\ &+ u_6x_3^2z_2 + u_7yz_2 \end{aligned}$$

(using $z_1 \mapsto z_1 + x_0x_1m$, $z_2 \mapsto z_2 + x_0x_2m$ as before, and also $z_1 \mapsto z_1 + *x_0x_3^2$, $z_2 \mapsto z_2 + *x_0y$ to kill the term $u_5x_3^2z_1$). Then

$$\sigma'_4\colon x_1W'_1 = x_2W'_0 + b_3x_3^2y(rx_2x_3 + sx_3^2) - sz_1^2$$

implies $t_{13} = u_7 = 0$ and

$$\begin{aligned} -W'_1 = -sx_1h &+ x_1x_2h' + t_9x_1x_3^2y + t_{10}x_2x_3^2y \\ &+ (t_{11} - 2b_1s)x_2y^2 + (t_{12} - b_2s + b_3r)x_3y^2 + u_6yz_2. \end{aligned}$$

Then

$$\sigma'_5\colon x_1W'_2 = x_2W'_1 + b_3y^2(rx_2x_3 + sx_3^2) - sz_1z_2$$

implies $t_{12} = 2b_2s - 2b_3r$, $u_6 = 0$, and

$$-W_2' = -2sx_2h + x_1x_3h' + t_9x_1y^2 + t_{10}x_2y^2 + (t_{11} - 4b_1s)x_3y^2.$$

Thus, in conclusion, after the same massaging as in (5.2),

$$\begin{aligned}
-W_0' &= x_1^2h' + 2v_1x_1x_2x_3y + v_2x_2^2x_3y + v_3x_3^5 \\
&\qquad + 2b_1sx_1y^2 + 2b_2sx_2y^2 - 2b_3rx_3^3y, \\
-W_1' &= -sx_1h + x_1x_2h' + v_1(x_1x_3 + x_2^2)x_3y + v_2x_2x_3^2y + v_3x_3^3y \\
&\qquad + (b_2s - b_3r)x_3y^2, \\
-W_2' &= -2sx_2h + x_2^2h' + 2v_1x_2x_3^2y + v_2x_3^3y + (v_3 - 2b_1s)x_3y^2.
\end{aligned}$$

The reader familiar with the rules for matrix multiplication will see that to 1st order, these are the x_0 terms of the 3 relations $A^{(1)}M^{(1)}({}^tA^{(1)}) = 0$, where $A^{(1)}$ is given above and

$$M^{(1)} = \begin{pmatrix} h & b_1y^2 & & \\ b_1y^2 & b_2y^2 & & \\ & & b_3y & \\ & & & -1 \end{pmatrix} + x_0 \begin{pmatrix} h' & v_1x_3y & & \\ v_1x_3y & v_2x_3y & & \\ & & & \end{pmatrix}.$$

5.16 Mixing extensions and deformations

I now consider 'extension-deformations' of $\overline{R} = R(C, \mathcal{O}_C(D))$, depending on two variables, λ, x_0 of degrees 0 and 1. For example, this situation occurs if I want to study deformations of a given numerical quintic surface S extending C; or equally, if I have a given flat deformation $\overline{R}_\lambda$ of $\overline{R}$ and I want to study simultaneous extensions of the $\overline{R}_\lambda$.

Let A be a graded local Artinian k-algebra (whose degree 0 piece may be bigger than just k). A graded deformation of $\overline{R}$ over A is an A-algebra R_A that is both a flat deformation of $\overline{R}$ over A and graded as k-algebra. If A is generated by elements of different degrees, the ideal of relations defining R_A will be homogeneous in all the variables.

Write $B = k[\lambda, x_0]/(\lambda, x_0)^2$, where $\deg\lambda = 0$, as in (5.13) and $\deg x_0 = 1$. The set of R_B is strictly a 1st order problem, whose solution is just the direct sum of the two vector spaces studied in (5.13) and (5.14): every R_B can be written (after suitable coordinate changes) in the form $B[x_1, \dots, z_2]/I_B$, where I_B is generated by 9 relations

$${}_BF = F + \lambda F'_{(\lambda)} + x_0F'_{(x_0)},$$

where F is a relation for $\overline{R}$ as in (5.2), $F'_(\lambda)$ the bit added on in (5.13), and $F'(x_0)$ the bit added on in (5.14) (examples are given in the proof of (5.16)). I continue to use the notation of (5.13) and (5.14) without further mention.

5.17 The 2nd order obstruction

Now let $C = k[\lambda, x_0]/(\lambda^2, x_0^2)$; despite appearances, deformations of $\overline{R}$ over C is no longer a 1st order problem, since the maximal ideal m_C has $m_C^2 \neq 0$.

Theorem *Let R_B be a graded deformation of $\overline{R}$ over B; then*

$$R_B \text{ lifts to a deformation } R_C \iff \beta s = 0.$$

Proof If $\beta = 0$ then R_B can be written in the first determinantal format, and so is unobstructed; if $s = 0$ it can be written in the Pfaffian format, and is likewise unobstructed. So the point is to prove $\implies$.

By (5.13) and (5.14), the first 4 relations for R_B are

$$\begin{array}{rrcl} {}_BS: & x_1x_3 & = & x_2^2, \\ {}_BT_1: & x_1y & = & x_2x_3^2 + \lambda(\alpha x_2y + \beta z_1) + x_0(rx_2x_3 + sx_3^2), \\ {}_BT_2: & x_2y & = & x_33 + \lambda(\alpha x_3y + \beta z_2) + x_0(rx_3^2 + sy), \\ {}_BU_1: & x_1z_2 & = & x_2z_1 + \lambda(b_3\beta x_3^2y) + x_0(sz_1). \end{array}$$

The first syzygy $x_1T_2 - x_2T_1 + yS$ for $\overline{R}$ upgrades to one for R_B as follows:

$$x_1({}_BT_2) - x_2({}_BT_1) + y({}_BS) = \lambda(\alpha y({}_BS) + \beta({}_BU_1)) + x_0(rx_3({}_BS) + s({}_BT_1)).$$

A lift of R_B to R_C involves patching the relations ${}_BS$, ${}_BT_1$, ${}_BT_2$, etc. to

$${}_CS = {}_BS + x_0S'', \quad {}_CT_1 = {}_BT_1 + \lambda x_0 T_1'', \quad {}_CT_2 = {}_BT_2 + \lambda x_0 T_2'', \quad \text{etc.}$$

in such a way that all the syzygies can be extended (this is exactly the argument of (1.13, (5)). For this it is necessary that

$$x_1({}_CT_2) - x_2({}_CT_1) + y({}_CS) = \lambda(\alpha y({}_CS) + \beta(CU_1)) + x_0(rx_3({}_CS) + s({}_CT_1));$$

this is supposed to be an equality in $\overline{R}$ between the λx_0 terms, since the constant and 1st order terms have already been fixed up to vanish. Now I claim that if $\beta s \neq 0$ this inequality cannot hold for any choice of S'', T_1'', T_2''. In fact the λx_0 term of the right-hand side is already determined by the relations for R_B, and is

$$\lambda\beta x_0(sz_1) + x_0 s\lambda(\alpha x_2 y + \beta z_1)$$

However, the right-hand side consists of assorted multiples of x_1, x_2 and y, so can't possibly hit z_1. Q.E.D.

5.18 Numerical quintics

A *numerical quintic* is a polarised n-fold X, D such that

(i) D is ample;

(ii) $K_X = (3-n)D$,

(iii) $h^0(\mathcal{O}_X(D)) = n+2$; and

(iv) $D^n = 5$.

One hopes that under suitable nonsingularity conditions, the linear system $|D|$ contains $n-1$ elements whose intersection is a nonsingular curve C. This has been proved by Horikawa if X is a nonsingular surface or 3-fold. (If $|D|$ defines a generically finite map, then easy numerical considerations show that $|D|$ is free and birational, or has a single reduced point as its base locus and is 2-to-1.)

Corollary *Assuming this, the ring $R(X, \mathcal{O}_X(D))$, is either a quintic hypersurface, or of the first determinantal format (5.3); in the latter case, $\varphi_D(X)$ is a quadric of rank* 3 *or* 4*; if the rank is* 4 *then all small deformations of X are given by varying the coefficients in the determinantal format, and 1st order deformations are unobstructed. If the rank is* 3 *then deformations of X form two components, one of which consists of quintic hypersurfaces.*

6 Six minuets for a mechanical clock

6.1 Main algorithm

This section outlines routines to mechanise the ideas of §1, intended as a 'pseudocode' computer program to calculate the moduli space of deformations of a ring $\overline{R}$ as the subscheme in $\mathbb{T}^1_{<0}$ defined by the vanishing of an obstruction morphism

$$\texttt{obs}\colon \mathbb{T}^1_{<0} \to \mathbb{T}^2_{<0}.$$

More precisely, I describe an algorithm having the following input and output:

Data: A specification of a ring $\overline{R}$ in terms of generators, syzygies and second syzygies, that is, the ring $A = k[x_1, \dots, x_n]$, the generators (f_i) of $\overline{I}$ and the resolution (1.14)

$$\cdots \xrightarrow{(m_{jn})} \bigoplus A(-s_j) \xrightarrow{(\ell_{ij})} \bigoplus A(-di) \xrightarrow{(f_i)} \overline{R} \to 0.$$

Result: The graded vector spaces $\mathbb{T}^1_{<0}$, $\mathbb{T}^2_{<0}$ and the polynomial map $\texttt{obs}\colon \mathbb{T}^1_{<0} \to \mathbb{T}^2_{<0}$ between them.

The input and output are objects of the same type: the f_i are homogeneous polynomials, and so are the components $\texttt{obs}_\iota$ of $\texttt{obs}$. If the coefficients of the f_i are symbolic (expressions involving indeterminates), then so are the $\texttt{obs}_\iota$, whereas if the f_i are numerical (rational numbers or elements of a finite field), then so are the $\texttt{obs}_\iota$.

Some practical considerations on implementing this algorithm are discussed in (6.5) below.

6.2 Monomial basis routine

First of all, it's clear that the whole computation will only involve $\overline{R}_{\le d}$ for some d that is readily determined *a priori* from the data (in fact the minimum degree of the second syzygies $\min t_k$ will do). A monomial basis for $\overline{R}$ is a set of monomials $x^{\underline{m}_\lambda}$ in the variables $\{x_1, \dots, x_n\}$ whose images in $\overline{R}$ form a basis; because of what I just said, I will only need to deal with a finite set $\{x^{\underline{m}_\lambda}\}_{\lambda\in\Lambda}$ forming a basis of $\overline{R}_{\le d}$. I need the following:

(i) at the outset, a choice of a monomial basis $\{x^{\underline{m}_\lambda}\}$ of $\overline{R}$;

(ii) a general-purpose procedure `general_polynomial(name, deg)` that writes out a general polynomial of given degree

$$f = \sum c_{(\texttt{name})\lambda} \cdot x^{\underline{m}_\lambda},$$

with indeterminate coefficients, where $c_{(\texttt{name})}$ is a suitable name for the coefficients, and the sum takes place over all basic monomials of given degree.

(iii) a reduction procedure that takes any polynomial h into its normal form `hh` mod $\overline{I}$, with `hh` a linear combination of the $x^{\underline{m}_\lambda}$;

(iv) the *toll*, a vector $\texttt{toll} \equiv \{\texttt{toll}_i\}$ such that

$$h = \texttt{hh} + \sum \texttt{toll}_i \cdot f_i.$$

Here the normal form `hh` is the final outcome of a reduction process, and is sufficient for 1st order purposes; `toll` is a record of how each relation f_i is used in the reduction process. It's used to replace equalities in $\overline{R}$ by identities in $k[x_1, \dots, x_n]$; in particular, to determine $\ell^{(k)}_{ij}$ when going back from (1.12, (6)) to (1.12, (5)) for higher order work.

In concrete cases, the required monomial basis may be fixed up in an *ad hoc* way; for example, in §5 it's not hard to see how to use the relations (5.2) to pass from any element of the ring $\overline{R}$ to a combination of basic monomials as in (5.12).

6.3 Computation of $\mathbb{T}^1$ and $\mathbb{T}^2$

Since $H^{(1)}$, $\mathbb{T}^1$ and $\mathbb{T}^2$ are graded vector spaces, the 1st order computation breaks up into independent routines for the graded pieces of each degree $-k$.

The two vector spaces $H^{(1)}$ and $\mathbb{T}^2$ are the homology of the conormal complex

$$L_0 = \bigoplus \overline{R}(d_i) \xrightarrow{\delta_0} L_1 = \bigoplus \overline{R}(s_j) \xrightarrow{\delta_1} \cdots$$

where δ_0 and δ_1 are matrixes with polynomial entries, the transpose of (ℓ_{ij}), (m_{jk}) given in the data. An element of L_0 of degree $-k$ is a vector $\varphi = \{f_i'\}$ with $f_i \in \overline{R}_{d_i-k}$. First write out $f_i = \sum \mathtt{cf}_{*i\lambda} x^{\underline{m}_\lambda}$ using the procedure (6.2, ii) (putting enough information into the coefficient names $\mathtt{cf}_{*i\lambda}$ to distinguish them from all previous names).

To calculate $\mathbb{T}^1$ in any degree $-k < 0$:

Step 0 Generate $\{f_i'\}$ for each i, and make a list of their coefficients.

Step 1 Divide by the group of coordinate transformations

$$x_\ell \mapsto x_\ell + c_\ell x_0, \quad \text{where } c_\ell \in \overline{R}_{a_i-k}.$$

The effect is to replace f_i' by $f_i' + \sum c_\ell \partial f_i / \partial x_\ell$, so that the quotient by a group action is in this case just a quotient vector space. As in (2.4) and (5.13), this simply means using the coordinate changes to assign values (usually 0) to as many of the coefficients $\mathtt{cf}_{*i\lambda}$ of the f_i as possible. (To calculate $H^{(1)}$ this step would be omitted.)

Start a loop on j, ranging from 1 up to the number of syzygies: for each of the syzygies σ_j, carry out the following 3 steps.

Step 2 Evaluate the jth entry of $\delta_0(\{f_i'\})$; that is, calculate $\sum \ell_{ij} f_i'$ in the polynomial ring and carry out the reduction

$$\mathtt{to\_kill}_j := \text{normal form of } \sum \ell_{ij} f_i'$$

using the procedure (6.2, iii). The kernel of δ_0 is of course obtained by setting $\mathtt{to\_kill}_j = 0$ for each j.

Step 3 To equate coefficients in $\texttt{to\_kill}_j = 0$, start an internal loop through each monomial $x^{\underline{m}_\mu}$ of degree $s_j - k$. Set to zero the coefficient of $x^{\underline{m}_\mu}$ in $\texttt{to\_kill}_j$ to get a homogeneous linear equation

$$\texttt{eqh}(j, x^{\underline{m}_\mu}) := \big((\text{coefficient of } x^{\underline{m}_\mu} \text{ in } \texttt{to\_kill}_j) = 0\big)$$

in the list of indeterminate coefficients $\{\texttt{cf}_{*i\lambda}\}$. Solve this, to obtain a new relation of the form

$$\text{dependent coefficient} = \text{combination of others},$$

for one coefficient $\texttt{cf}_{*i\lambda}$, and assign this value.

Keep side-effect The subscript $*i\lambda$ is *a priori* not know; for the higher order computation, it is crucial to remember the point at which the coefficients $\texttt{cf}_{*i\lambda}$ are solved for; so at this point, make a table $\texttt{Table}_1$ relating $*i\lambda$ to $(j, x^{\underline{m}_\mu})$.

(end of $x^{\underline{m}_\mu}$ loop, end of j loop).

After all this, some of the coefficients $\texttt{cf}_{*i\lambda}$ have values assigned. Define $\texttt{defvar}[k]$ to be the set of remaining independent (unassigned) coefficients; these are coordinates on $\mathbb{T}^1_{-k}$.

The calculation of $\mathbb{T}^2_{-k}$ is similar, and I only sketch it: start from the vector space

$$\big\{g'_j = \sum \texttt{cg}_{*j\lambda} x^{\underline{m}_\lambda} \in \overline{R}_{s_j-k}\big\},$$

with a monomial basis, divide by $\operatorname{im}\delta_0$ (assigning values to some of the coefficients $\texttt{cg}_{*j\lambda}$), then use the linear equations defining $\ker\delta_1$ to assign values to more of the $\texttt{cg}_{*j\lambda}$. Here I again make a table $\texttt{Table}_2$ to remember when a coefficient $\texttt{cg}_{*j\lambda}$ is solved for (that is, assigned a value) by the linear equations defining $\ker\delta_1$. The meaning of this table is that δ_1 (the second syzygies) expresses the $x^{\underline{m}_\mu}$ term of the syzygy s_j (corresponding to $\texttt{cg}_{*j\lambda}$) as a linear combination of terms of the other syzygies, and does not give rise to new independent obstructions. At the end, the unassigned coefficients $\texttt{cg}_{*j\lambda}$ are coordinates on $\mathbb{T}^2_{-k}$.

Short-cuts

(1) It may not be necessary to write out all the $\{f'_i\}$ as general polynomials: if one entry ℓ_{ij} of the matrix δ_0 is a non-zerodivisor of $\overline{R}$ then the jth syzygy is equivalent to

$$\ell_{ij} \text{ divides } \textstyle\sum_{\iota\neq i} \ell_{\iota j} f'_\iota \quad \text{and} \quad f'_i := \Big\{\displaystyle\sum_{\iota\neq i} \ell_{\iota j} f'_\iota\Big\}/\ell_{ij}.$$

For example, in (5.12) all 5 of the syzygies used were treated in this way.

(2) The 1st order calculation of (6.3) can be done as a self-contained, purely linear routine. However, to avoid repeating all the 1st order work when proceeding to 2nd order, it is desirable to get hold of the `toll` in passing to the normal form in Step 2, which gives ℓ'_{ij}.

6.4 Higher order theory

The higher order theory is done by induction, and I start off the induction assuming that the calculation of $\mathbb{T}^1_{-\kappa}$ and $\mathbb{T}^2_{-\kappa}$ have been carried out for all $\kappa < k$, resulting in an array `defvar[1 .. k-1]`, where each `defvar`$[\kappa]$ consists of coordinates on $\mathbb{T}^1_{-\kappa}$. Write `defvars` for the union of these. The obstructions will be kept in a set `obs` of relations between the `defvars` (more usefully, as an ideal in k[`defvars`]), and I initialise `obs` := `{ }` to be the empty set (or the zero ideal).

I fix $\deg x_0 = 1$. Now I work by induction on k, starting with $k = 2$ because the 1st order part has already been done in (6.3). Consider kth order deformations of the form

$$f_i + x_0 f'_i + \cdots + x_0^k f_i^{(k)} \quad \text{and} \quad \ell_{ij} + x_0 \ell'_{ij} + \cdots + x_0^k \ell_{ij}^{(k)}$$

as in (1.13). Assuming that everything is known up to order $k-1$, the new unknowns $f_i^{(k)}$ must satisfy

$$\sum \ell_{ij} f_i^{(k)} = \psi_j \in \overline{R}_{s_j - ka_0}, \qquad (*)$$

where $\psi_j = -\sum_{a=1}^{k-1} \sum \ell(a)_{ij} f_i^{(k-a)}$ is as in (1.12, (6)).

I calculate all possibilities for the kth order terms at the same time as writing out the new obstructions.

Step 1 Write out the general form of $\varphi = \{f_i^{(k)}\}$ with $f_i^{(k)} \in \overline{R}_{d_i - k}$, and divide by the coordinate change $x_\ell \mapsto x_\ell + c_\ell x_0$, where $c_\ell \in \overline{R}_{a_i - k}$. This is exactly the same calculation as (6.3, Steps 0–1).

Now for each syzygy j, carry out the following steps, parallel to those in (6.3):

Step 2 Evaluate $\sum \ell_{ij} f_i^{(k)} - \psi_j$ where ψ_j is as in $(*)$ above, and reduce

$$\texttt{to\_kill}_j := \text{normal form of } \sum \ell_{ij} f_i^{(k)} - \psi_j$$

using the reduction procedure (6.2, iii); remember to set $\ell_{ij}^{(k)} := -\mathtt{toll}_i$. This is similar to (6.3, Step 2), except that carrying forward the ψ_j adds nonlinear terms from the order $< k$ calculations to the purely linear kth order part $\sum \ell_{ij} f_i^{(k)}$.

Step 3 Equate coefficients of each monomial $x^{\underline{m}_\mu}$ in $\mathtt{to\_kill}_j$, to get an equation $\mathtt{eqin}(j, x^{\underline{m}_\mu})$ in the kth order indeterminate coefficients $\{\mathtt{cf}_{*i\lambda}\}$ which is inhomogeneous linear in these, but with coefficients involving the $\mathtt{defvar}[\kappa]$ with $\kappa < k$.

Step 4 In deciding how to handle each equation $\mathtt{eqin}(j, x^{\underline{m}_\mu})$ there is a division into 3 cases, depending on the information remembered in the 'side-effect' tables $\mathtt{Table}_1$, $\mathtt{Table}_2$ constructed in (6.3, Step 3).

Case A If $\mathtt{Table}_1$ remembers that $\mathtt{eqh}(j, x^{\underline{m}_\mu})$ was used in (6.3, Step 3) of the 1st order problem to assign a value to $\mathtt{cf}_{*i\lambda}$ in the 1st order problem, then $\mathtt{eqin}(j, x^{\underline{m}_\mu})$ can also be used to assign a value to $\mathtt{cf}_{*i\lambda}$ (not the same value). This is true because the kth order indeterminates $\mathtt{cf}_{*i\lambda}$ appear with the same coefficients in the two equations.

Case B If $\mathtt{Table}_2$ remembers that in the 1st order problem the $x^{\underline{m}_\mu}$ term of the syzygy s_j was a linear combination of terms of the other syzygies, then just ignore $\mathtt{eqin}(j, x^{\underline{m}_\mu})$.

Case C If neither of Cases A–B hold then $\mathtt{eqin}(j, x^{\underline{m}_\mu})$ is a new obstructions, a component of $\mathtt{obs}\colon \mathbb{T}^1_{<0} \to \mathbb{T}^2_{-k}$.

(end of $x^{\underline{m}_\mu}$ loop, end of j loop).

6.5 Considerations of space and time

Concerning the feasibility of implementing this algorithm, I have a version of it running (written in Maple, [Maple]) to compute the deformation theory of a very specific example related to Godeaux surfaces with torsion $\mathbb{Z}/2$; in this example the ring needs 8 generators, 20 relations, 64 syzygies and 90 second syzygies, and the relations f_i have 10 indeterminates among their coefficients. My program works with symbolic coefficients, and polishes off easily the computation of $\mathbb{T}^1_{<0}$, which is 17-dimensional; at present I don't have the computation of $\mathbb{T}^2_{<0}$ implemented, although this does not present special difficulties (it should be approximately 10-dimensional). My computation of obstructions eventually grinds to a halt, growing too large for computer memory and the tolerance of my fellow-users (typically taking 80% of 16Mb memory, about 30 hours of CPU time, for an unfinished calculation). The obstructions are polynomials in 17 variables, with hundreds of nonzero symbolic coefficients (the first two take half a page each to print out, and after that 3 or 4 pages each), so that for example using the standard Gröbner basis package provided in Maple is not feasible to control them. I nevertheless expect that a modified version of this program will eventually run to completion, and decide the irreducibility of the moduli space of Godeaux surfaces with torsion $\mathbb{Z}/2$.

As described in (2.10), my preliminary notes on this calculation formed §6 (17 pages) of the preprint of this paper, and pending a more definitive composition these are still available on request, together with some version of my Maple routines.

On the other hand, there is the alternative approach in the spirit of Macaulay [Bayer and Stillman] in which all the rings are defined by polynomials with one word coefficients; I guess the advantage is that polynomials can be viewed as a much simpler data type (the array of its coefficients) instead of the recursive structure of symbolic computation packages such as Maple, thus saving on all the overheads of simplification. It seems clear to me that in these terms my algorithm will involve the same order of magnitude of complexity as the existing Macaulay package, so that rings of the size of my example (but defined by polynomials with coefficients in a prime field $\mathbb{Z}/(p)$) can be dealt with easily.

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