Surfaces with $p_g = 0$, $K^2 = 1$

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The[1] purpose of this article is to present a uniform way of writing down the equations defining a minimal surface X having $p_g = 0$, $K^2 = 1$ and $\operatorname{Tors} X = \mathbb{Z}/5$, $\mathbb{Z}/4$ or $\mathbb{Z}/3$, where $\operatorname{Tors} X$ is the Severi torsion group of X; the method consists of writing down generators and relations for the canonical ring of the cover Y, where $\psi\colon Y \to X$ is the Abelian cover corresponding to $\operatorname{Tors} X$. Studying the canonical ring of Y, together with the action of $\operatorname{Gal} Y/X$, is equivalent to studying the $(\mathbb{Z} \oplus \operatorname{Tors} X)$-graded ring

$$R(X, K_X, \operatorname{Tors} X) = \bigoplus_{\substack{n\ge 0\\ \mathfrak{a}\in\operatorname{Tors} X}} H^0(X, nK_X + \mathfrak{a}).$$

As a corollary of this method I prove the existence of surfaces in each class, and the fact that each class forms an irreducible moduli space.

The surfaces with $\operatorname{Tors} X = \mathbb{Z}/5$ are due to Godeaux, and my treatment in §1 is intended to illustrate my method in a transparent case. A surface with $\operatorname{Tors} X = \mathbb{Z}/4$ has been constructed independently by Miyaoka [1], who also proved the irreducibility of the moduli space of Godeaux surfaces. The bulk of this paper (§3) is devoted to surfaces with $\operatorname{Tors} X = \mathbb{Z}/3$, for which the cover Y cannot be represented as a (weighted) complete intersection.

I have an argument based on writing down generators and relations of the canonical ring which I hope will prove that surfaces with $p_g = 0$, $K^2 = 1$ and $\operatorname{Tors} X = \mathbb{Z}/2$ form an irreducible moduli space. That such surfaces exist is shown by a recent example of a double plane due to Oort and Peters.[2]

The most interesting problem remaining is that of knowing whether there exist surfaces with $K^2 = 1$, $p_g = 0$ and no torsion.

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[2] See the end for some updates and corrections.

Conventions All varieties, morphisms, etc. in this article are defined over a fixed algebraically closed field of characteristic 0. By "surface" I mean a complete nonsingular algebraic surface X, and for such a surface, K_X denotes the divisor class associated to the invertible sheaf Ω^2_X of regular 2-forms, and $p_g = h^0(X, \mathcal{O}_X(K_X))$ is the geometric genus. The conditions $K^2_X > 0$ and $\operatorname{Tors} X \neq 0$ together imply that X is of general type.

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0 Some useful preliminaries

The following lemmas will be useful for describing the canonical system $|K_Y|$ of the cover $\psi\colon Y \to X$ of a given surface X with nontrivial torsion group. In particular I will need to show by arguing on X that $|K_Y|$ is without base points (or without fixed components).

Lemma 0.1 *Let X be a minimal surface of general type with $K^2 = 1$, and let D and D' be two distinct positive divisors numerically equivalent to K (so that $KD = KD' = D^2 = D'^2 = 1$); then D and D' are without common components, and hence intersect transversally in one point $P(D, D')$.*

Proof Write

$$D = C + \sum n_i C_i, \qquad D' = C' + \sum n'_j C'_j,$$

with C and C' irreducible such that $KC = KC' = 1$ and $KC_i = KC'_j = 0$.

If $C = C'$ then $D = D'$, since there is no nontrivial numerical relation between the C_i and C'_j (Bombieri [2], p. 451). The intersection pairing on the C_i and C'_j is even and negative definite, so that if the greatest common divisor E of D and D' is nonzero then $E^2 \leq -2$; hence $(D - E)(D' - E) = K^2 - 2KE + E^2 = 1 + E^2 < 0$, which contradicts the fact that $D - E$ and $D' - E$ are without common components.

Lemma 0.2 *Let X be a minimal surface of general type with $K^2 = 1$, and let D, D' and D'' be distinct positive divisors numerically equivalent to K, and such that $D' - D''$ is not linearly equivalent to 0.*

Then the two points of intersection $P(D, D')$ and $P(D, D'')$ are distinct.

Proof Let $\mathfrak{a} \in \operatorname{Pic} X$ be the class of $D' - D''$; by hypothesis $\mathfrak{a} \neq 0$. I have to show that the restriction $\mathfrak{a}_D$ is nontrivial. This follows at once from the cohomology exact sequence of

$$0 \to \mathcal{O}_X(\mathfrak{a} - D) \to \mathcal{O}_X(\mathfrak{a}) \to \mathfrak{a}_D \to 0,$$

and the fact that $H^1(\mathcal{O}_X(\mathfrak{a} - D))$ (by Ramanujam's form of Kodaira vanishing, see [5] or [6]).

Lemma 0.3 *Let X be a surface with $p_g = 0$, $K^2 = 1$ and let $\mathfrak{a} \in \operatorname{Tors} X$ be a nontrivial torsion element, of order n, say. Then*

$$h^0(X, \mathcal{O}_X(K + \mathfrak{a})) = 1, \quad h^1(X, \mathcal{O}_X(K + \mathfrak{a})) = 0.$$

Proof From the Riemann–Roch formula and the fact that $h^2(K + \mathfrak{a}) = h^0(\mathfrak{a}) = 0$, it follows that

$$h^0(K + \mathfrak{a}) - h^1(K + \mathfrak{a}) = 1.$$

Now if $H^1(X, \mathcal{O}_X(K + \mathfrak{a})) \neq 0$ it follows that the étale covering $Y \to X$ corresponding to $\mathfrak{a}$ (where $Y = \operatorname{Spec}_X(\mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{a}) \oplus \cdots \oplus \mathcal{O}_X((n-1)\mathfrak{a}))$ has $H^1(\mathcal{O}_Y) \neq 0$; thus Y has étale covers of large finite order, which gives a contradiction as in [2], p. 488.

I will use continuously the following fact:

Proposition 0.4 $h^0(nK + \mathfrak{a}) = 1 + \binom{n}{2}$ *for all $n \geq 1$ and $\mathfrak{a} \in \operatorname{Tors} X$ with the exception of $n = 1$, $\mathfrak{a} = 0$.*

1 The Godeaux surface with $p_g = 0$, $K^2 = 1$ and $\operatorname{Tors} X = \mathbb{Z}/5$

In this section I show how to write down generators and relations for the pluricanonical ring of a Godeaux surface, starting from sections of the line bundles $K + \mathfrak{a}$ with $\mathfrak{a} \in \operatorname{Tors} X = \mathbb{Z}/5$.

Let X be a surface as in the section heading. The elements of $\operatorname{Tors} X = \mathbb{Z}/5$ will be denoted $1, 2, 3, 4, 0 \pmod 5$. Let $x_i \in H^0(K + i)$ be nonzero elements for $i = 1, 2, 3, 4$.

Any monomial in the x_i is a section of some $NK + l$:

$$\prod x_i^{a_i} \in H^0(NK + l), \quad \text{where} \quad \sum a_i = N \quad \text{and} \quad \sum i a_i \equiv l \mod 5.$$

For example, $H^0(5K)$ contains the following 12 elements:

$$\begin{aligned} &x_1^5, x_2^5, x_4^5, x_3^5, \\ &x_1^3 x_3 x_4, x_2^3 x_1 x_4, x_4^3 x_1 x_2, x_3^3 x_2 x_4, \\ &x_1^2 x_2^2 x_4, x_2^2 x_4^2 x_3, x_4^2 x_3^2 x_1, x_3^2 x_1^2 x_2. \end{aligned} \tag{1}$$

Since $h^0(5K) = 11$ there is at least one notrivial relation g between these elements. It will turn out that these x_i and the relation g are in a certain sense a set of generators and relations of the canonical ring of X.

Let $\psi \colon Y \to X$ be the étale cover corresponding to $\operatorname{Tors} X$; $\psi^* \operatorname{Tors} X = 0$, so that each $\psi^* x_i$ is a section of $\psi^*(K + i) = K_Y$. I will continue to denote these sections $x_i \in H^0(K_Y)$; each x_i defines a divisor D_i on X and a divisor $\psi^* D_i$ on Y which is invariant under the group action. In view of Lemma 0.1, the 3 divisors D_1, D_2, D_3 are disjoint on X, so that $\psi^* D_1$, $\psi^* D_2$ and $\psi^* D_3$, are disjoint on Y. In particular K_Y is without base points, and (since $K_Y^2 = 5$, $p_g = 4$), φ_{K_Y} is therefore a birational morphism onto a quintic $\overline{Y}$ of $\mathbb{P}^3$.

By definition, the cover $\psi \colon Y \to X$ is Spec of the $\mathcal{O}_X$-algebra

$$\bigoplus_{i=0}^{4} \mathcal{O}_X(i),$$

and has Galois group $\mathbb{Z}/5$ acting by multiplying the ith factor by ε^i, where ε is a primitive 5th root of 1. The $x_i \in H^0(K_Y)$ are just the eigenvectors of this action.

Theorem 1.1 *(i) The $x_i \in H^0(K_Y)$ generate the canonical ring of Y, and there is just on relation g between them, g being a linear combination of the elements (1) above.*

(ii) The pluricanonical ring $R(X)$ is the ring of invariants of the $\mathbb{Z}/5$ action on $R(Y)$; thus every $H^0(NK)$ is spanned by the monomials in the x_i belonging to it, and the only relations between these are multiples of g by some element of $H^0((N-5)K)$.

An obvious necessary and sufficient condition that a surface Y defined by a quintic g does not meet the 4 fixed points $(0,\dots,1,\dots,0)$ of the group action on $\mathbb{P}^3$ is that the coefficients of the x_i^5 in g do not vanish. One therefore obtains a description of the moduli space of Godeaux surfaces as in Miyaoka [1].

2 Surfaces with $K^2 = 1$, $p_g = 0$ and $|\operatorname{Tors}| = 4$

Theorem 2.1 *There are no surfaces with $p_g = 0$, $K^2 = 1$ and* $\operatorname{Tors} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proof Suppose that X is such a surface; let 01, 10, 11 be the nontrivial torsion elements of $\operatorname{Pic} X$, and x_{01}, x_{10} and x_{11} nonzero sections of the bundles $K+01$, $K+10$, $K+11$. The square x_{01}^2, x_{10}^2, x_{11}^2 all belong to $H^0(2K)$, so that there is a linear dependence relation between them.

Now let Y be the cover of X corresponding to $\operatorname{Tors} X$; the 3 sections x_{01}, x_{10}, x_{11} are linearly independent sections of K_Y and thus form a basis of $H^0(K_Y)$; but there is a nontrivial quadratic relation between them, which implies that the canonical linear system $|K_Y|$ is composed with a pencil: $|K_Y| = 2|D| + F$, where $|D|$ is a pencil without fixed part, and F the fixed part. Then since $K^2 = 4$, $F \neq 0$ – for otherwise $D^2 = 1$ and $KD = 2$, contradicting $D^2 \equiv KD \bmod 2$. But if $F \neq 0$ then the curves defined by x_{01} and x_{10} have a common component on Y, hence also on X, contradicting Lemma 0.1; this establishes the result.

Now let X be a surface with $K^2 = 1$, $p_g = 0$ and $\operatorname{Tors} = \mathbb{Z}/4$. As for the Godeaux surface, I will denote the elements of $\operatorname{Tors} X$ by $1, 2, 3, 0 \bmod 4$. For $i = 1, 2, 3$, let $x_i \in H^0(K+i)$ be nonzero. Then the $H^0(2K+i)$ contain the following elements:

$$\begin{aligned} x_1x_3, x_2^2 &\in H^0(2K), \\ x_2x_3, y_1 &\in H^0(2K+1), \\ x_1^2, x_3^2 &\in H^0(2K+2), \\ x_1x_2, y_3 &\in H^0(2K+3); \end{aligned} \tag{2}$$

the two elements occurring in $H^0(2K)$ and $H^0(2K+2)$ are linear independents – for otherwises, say $x_1x_3 + ax_2^2 = 0$, which contradicts Lemma 0.1. The generators $y_i \in H^0(2K+i)$ (for $i = 1$ or 3) are chosen to be linearly independent from the monomial in the x_i. Thus the elements of (2) provide bases for $H^0(2K+i)$.

A monomial in the x_i and y_i is as before an element of $H^0(NK+l)$:

$$\prod x_i^{a_i} y_i^{b_j} \in H^0(NK+l),$$

where $\sum(a_i+2b_i)=N$ and $\sum(ia_i+ib_i)\equiv l \bmod 4$.

In particular, one observes that $H^0(4K)$ and $H^0(4K+2)$ each contains 8 monomials in the x_i and y_i, so that there is a linear dependence relation between each of these sets, q_0 and q_2 respectively:

$$\begin{array}{rcll} x_1^4, x_2^4, x_3^4, x_1^2x_2^2, x_1x_2^2x_3, x_1x_2y_1, x_2x_3y_3 & \in & H^0(4K) & : q_0; \\ x_1^2x_2^2, x_2^2x_3^2, x_1^3x_2, x_1x_3^3, x_1x_2y_3, x_2x_3y_1, y_1^2, y_3^2 & \in & H^0(4K+2) & : q_2. \end{array}$$

As in §1 it will turn out that the x_i and y_i provide a set of generators for the canonical ring of X, and the q_0 and q_2 the only relations.

Let $\psi\colon Y\to X$ be as before the étale cover corresponding to $\operatorname{Tors} X$. As before, let $x_i\in H^0(K_Y)$ and $y_i\in H^0(2K_Y)$ be the elements ψ^*x_i and ψ^*y_i. By Lemma 0.1, the x_i have no common zero on X, so that $|K_Y|$, and *a fortiori* $|2K_Y|$ is without base points. A basis for $|2K_Y|$ is given by the elements (2), so that φ_{2K_Y} is contained in the cone on the Veronese surface $\overline{F}$, the projective variety corresponding to the graded ring $k[x_1,x_2,x_3,y_1,y_2]$. φ_{2K_Y} is actually the complete intersection inside this variety of the two hypersurfaces defined by q_0 and q_2 above.

Theorem 2.2 *(i) The x_i and y_i generate the canonical ring of Y, and the q_0 and q_2 above are the only relations.*

(ii) The canonical ring $R(X)$ is the invariant subring of the action of $\mathbb{Z}/4$ on $R(Y)$.

One verifies easily that the fixed loci of the action of $\mathbb{Z}/4$ on the cone on the Veronese are contained in the union of the following 3 linear varieties:

$$\begin{array}{l} V_1: x_1=x_2=x_3=0, \\ V_2: x_2=y_1=y_3=0, \\ V_3: x_1=x_3=y_1=y_3=0; \end{array} \tag{3}$$

it is easy to write down necessary and sufficient conditions on the coefficients of q_0 and q_2 so that the locus $\overline{Y} : q_0 = q_2 = 0$ does not meet V_1, V_2, V_3; for example

$$q_0 = x_1^4 + x_2^4 + x_3^4 + y_1 y_3 \quad \text{and} \quad q_2 = x_1^3 x_2 + x_1 x_3^3 + y_1^2 + y_3^2$$

satisfy this condition, and also define a nonsingular Y.

The canonical class of the surface Y so constructed is calculated as follows: let $F = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)) \xrightarrow{\pi} \mathbb{P}^2$ be the standard scroll; F has two line bundles $L = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ and the tautological bundle M of the Proj (so that $\pi_* M = \mathcal{O} \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)$). The map φ_M has image $\varphi_M(F) = \overline{F}$, the cone on the Veronese. The inverse image $B = \varphi_M^{-1}(\text{vertex})$ is the unique divisor in the linear system $|M - 2L|$.

One sees that the the canonical class K_F is given by $K_K \sim -3M - L$. Since Y is the complete intersection in F of two divisors in $|2M|$ we have $K_Y \sim M_{|Y} - L_{|Y}$ (by the adjunction formula), so that $K_Y \sim B_{|Y} + L_{|Y}$. But Y and B are disjoint, so that $B_{|Y} = 0$ and $K_Y \sim L_{|Y}$. Thus the linear system $|K_Y|$ is just the restriction to Y of the linear system L, so that Y satisfies $K^2 = 4$, $p_g = 3$ and $\varphi_M(Y) \subset \mathbb{P}^7$ is bicanonic.

Proof of Theorem 2.2 We have seen that $|2K_Y|$ is without base points. By counting degrees one sees that φ_{2K_Y} is either birational or is a double cover onto a surface of degree 8. Let me eliminate the second possibility. The restriction of φ_{2K_Y} to the general curve $C \in |K_Y|$ is the complete canonical system of C; in order that φ_{2K_Y} be 2–1, C must be hyperelliptic. Now the canonical map $\varphi_{K_Y} : Y \to \mathbb{P}^2$ can be described as the composite of φ_{2K_Y} with the projection from the vertex of the cone on the Veronese. Thus this 4-fold cover splits as a composite $Y \to \overline{Y} \to \mathbb{P}^2$ of two double covers; the first factor in this composite is composed with the hyperelliptic involution on C, but this implies that $|K_Y|$ cuts out on C a g_4^1 which is composed with the hyperelliptic g_2^1 of C; such a linear system is not complete, and this contradicts $H^1(\mathcal{O}_Y) = 0$. Essentially the same argument is that above every line $l \subset \mathbb{P}^2$ one has on $\overline{Y}$ a curve which is the canonical image of a hyperelliptic curve, which is hence rational; but then the ramification of $\overline{Y} \to \mathbb{P}^2$ must be in a conic, which contradicts the completeness of $|K_Y|$.

The following Lemma 2.3 implies at once that the image $\varphi_{2K_Y}(Y) = \overline{Y}$ of the birational map φ_{2K_Y} is the complete intersection in $\overline{F}$ of the hypersurfaces defined by q_0 and q_2. Comparing $K_{\overline{Y}}$ as computed by the adjunction formula with K_Y shows that $\overline{Y}$ has only rational double points; the fact that the x_i and y_i then span the canonical ring of Y is then a standard verifiction that certain linear systems on F cut out complete linear systems on $\overline{Y}$.

Lemma 2.3 *$\overline{Y} \subset \overline{F}$ is not contained in any reducible divisor $Q \in |\mathcal{O}_{\overline{F}}(2)|$.*

Proof Equivalently, the inverse image $\varphi_M^{-1}(\overline{Y}) \subset F$ is not contained in any divisor $Q \in |2M|$ which splits as $Q = Q_1 + Q_2$, with the possible exception $Q_1 = B$, $Q_2 \in |2M - B| = |B + 4L|$. Since the image $\varphi_M(Q_2)$ of $Q_2 \in |B+2L|$ does not span $\mathbb{P}^7$, it is enough to check that $\overline{Y}$ is not contained in any divisor $Q_2 \in |B + 3L|$. But $H^0(F, \mathcal{O}_F(B + 3L))$ is spanned by the monomials

$$\begin{gathered} x_1^2x_2, x_3^2x_2, x_1y_3, x_3y_1, \\ x_1^2x_3, x_1x_2^2, x_3^3, x_2y_3, \\ x_1x_2x_3, x_2^3, x_1y_1, x_3y_2, \\ x_1^3, x_1x_2^2, x_2^2x_3, x_2y_1. \end{gathered}$$

These monomials are linearly independent as elements of $H^0(Y, 3K_Y)$, as follows easily from the splitting into eigenspaces of the $\mathbb{Z}/4$ action, and Lemmas 0.2 and 0.3. For example, a nontrivial relation between the elements in the eigenspace of 1 can be written

$$x_1(ax_1x_2 + by_3) = x_3(cx_2x_3 + dy_1),$$

which contradicts the choice of y_i. The other cases are similar, and the lemma is proved.

3 $K^2 = 1$, $p_g = 0$ and $\operatorname{Tors} X = \mathbb{Z}/3$

In this section I show how to write down the equations defining a surface X with the invariants $p_g = 0$, $K^2 = 1$ and $\operatorname{Tors} X = \mathbb{Z}/3$.

I start off with a description of the final construction, intended to clarify the rather complicated arguments that follow; the fact that this construction actually provides surfaces S with the required invariants follows from this description, from the form of the equations f and g given below, and from Bertini's theorem and a simple nonsingularity computation. Most of the work of this section is to show that *any* surface X with the above invariants is given by my construction.

The scroll F Consider the projective variety $\overline{F} = \operatorname{Proj} R$ associated to the graded ring $R = k[x_1, x_2, y_0, y_1, y_2]$ with $\deg x_i = 1$, $\deg y_i = 2$. The elements of degree 2

$$x_1^2, x_2^2, x_1x_2, y_0, y_1, y_2 \tag{4}$$

define an embedding of $\overline{F}$ in $\mathbb{P}^5$ as a quadric of rank 3, the cone with vertex $\mathbb{P}^2$ (coordinates y_0, y_1, y_2) over the plane conic (with coordinates x_1^2, x_1x_2, x_2^2.

The natural desingularisation F of $\overline{F}$ is the rational scroll

$$F = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}), \qquad \pi\colon F \to \mathbb{P}^1,$$

where $\mathcal{E}$ is the rank 4 vector bundle $\mathcal{E} = \mathcal{O} \oplus \bigoplus_{i=0}^2 \mathcal{O}(-2)$. Write A for a fibre of $\pi\colon F \to \mathbb{P}^1$, so that $\mathcal{O}_F(A) = \pi^*\mathcal{O}(1)$, and B for the divisor of F corresponding to the unique section of $\mathcal{E}$, so that $\mathcal{O}_F(B)$ is the tautological bundle of F.

The linear system $|2A + B|$ defines a morphism

$$\varphi\colon F \to \overline{F} \subset \mathbb{P}^5;$$

and $B = \varphi^{-1}(\text{vertex})$. Obviously $B = \mathbb{P}^1 \times \mathbb{P}^2$, with φ projecting B to the second factor.

On F take the bihomogeneous coordinates $((x_1, x_2), (t, y_0, y_1, y_2))$, where (x_1, x_2) are coordinates on the base $\mathbb{P}^1$, and $t \in H^0(\mathbb{P}^1, \mathcal{E})$ and the $x_i \in H^0(\mathcal{O}_F(2A+B)) = H^0(\mathbb{P}^1, \mathcal{E} \otimes \mathcal{O}(2))$ are natural coordinates in the bundle $\mathcal{E}$. The bihomogeneity is expressed by the fact that the linear system $|nA+mB|$ on F is based by monomials

$$t^c x_1^{a_1} x_2^{a_2} y_0^{b_0} y_1^{b_1} y_2^{b_2} \quad \text{with } \textstyle\sum a_i + 2\sum b_i = n \text{ and } c + \sum b_i = m.$$

If we omit t, these are just the elements of degree n in the graded ring R having degree $\leq m$ in the y_i.

To construct the surface X with the specified invariants, I must construct its cyclic 3-fold cover Y, which is a surface having $p_g = 2$, $K^2 = 3$. The construction will be done as follows: take two irreducible divisor Q and C in F with $Q \in |6A + 2B|$ and $C \in |6A + 3B|$ such that

(i) Q contains 3 lines $\mathbb{P}^1 \times p_i \subset B$, with p_i noncollinear points, and is nonsingular along them;

(ii) C contains the 3 lines $\mathbb{P}^1 \times p_i$, and contains 3 fibres Q_i of $Q \to \mathbb{P}^1$;

(iii) $Q \cap C = \widetilde{Y} + \sum_3 Q_i$, with $\widetilde{Y}$ a surface which is nonsingular along $\mathbb{P}^1 \times p_i$, and has only rational double points elsewhere;

(iv) B touches $\widetilde{Y}$ along the 3 lines $\mathbb{P}^1 \times p_i$.

(These conditions are not independent, and in particular (iv) is a consequence of (i), (ii) and (iii), as one sees by an argument similar to those given below.)

Then the 3 lines l_i are exceptional curves of the first kind on $\widetilde{Y}$ and contracting them gives the required surface Y. For the proof, note that the canonical class of $\widetilde{Y}$ (a divisor inside a divisor of F) is given by the adjunction formula:

$$K_F = -8A - 4B; \quad K_Q = \mathcal{O}_Q(-2A - 2B); \quad \text{and} \quad K_{\widetilde{Y}} = \mathcal{O}_Y(A + B).$$

But $\mathcal{O}_{\widetilde{Y}}(B) = 2\sum l_i$; since each $l_i^2 < 0$ (since it contracts under $\varphi\colon F \to \overline{F}$) it follows that $l_i^2 = -1$. The invariants $p_g = 2$ and $K_Y^2 = 3$ are easy.

The fact that conditions (i–iv) can be verified, and indeed that Y can be chosen invariant under a fixed-point free action of $\mathbb{Z}/3$ will be checked at the end of this section.

Now let X be any surface having $K^2 = 1$, $p_g = 0$ and $\operatorname{Tors} X = \mathbb{Z}/3$; as in the preceding sections, let 0, 1, 2 denote the elements of $\mathbb{Z}/3 \subset \operatorname{Pic} X$. In this case one can choose elements

$$\begin{aligned} & x_i \in H^0(K + i) \quad \text{for } i = 1, 2; \\ & y_i \in H^0(2K + i) \quad \text{for } i = 0, 1, 2; \\ \text{and} \qquad & z_i \in H^0(3K + i) \quad \text{for } i = 1, 2 \end{aligned}$$

so that the monomials in the x_i, y_i and z_i span the spaces $H^0(nK + i)$ for $n \leq 3$ as indicated in Table 1.

Let $\psi\colon Y \to X$ be the 3-fold cover of X corresponding to $\operatorname{Tors} X$. I want to study the 2-canonical map φ_{2K_Y} of Y in terms of X. Firstly, the two divisors $D_i \subset X$ defined by $x_i = 0$ meet transversally in a point P; since

$$H^0(\mathcal{O}_{D_1}(K_X + i)) = \begin{cases} 1 & \text{if } i = 2 \\ 0 & \text{if } i \neq 2 \end{cases}$$

and

$$H^0(\mathcal{O}_{D_1}(2K_X + i)) = \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i \neq 2, \end{cases}$$

none of the y_i can vanish at P.

Table 1:

sheaf	sections				relations
$K+1$	x_1				
$K+2$	x_2				
$2K$	x_1x_2	y_0			
$2K+1$	x_2^2	y_1			
$2K+2$	x_1^2	y_2			
$3K$	x_1^3, x_2^3	x_1y_2, x_2y_1			
$3K+1$	$x_1^2x_2$	x_1y_0, x_2y_2	z_1		
$3K+2$	$x_1x_2^2$	x_1y_1, x_2y_0	z_2		
$4K$	$x_1^2x_2^2,$	$x_1x_2y_0, x_1^2y_1, x_2^2y_2,$	$y_0^2, y_1y_2,$	x_1z_2, x_2z_1	R_0
$4K+1$	$x_1^4, x_1x_2^3,$	$x_1^2y_2, x_1x_2y_1, x_2^2y_0,$	$y_0y_1, y_2^2,$	x_2z_2	R_1
$4K+2$	$x_1^3x_2, x_2^4,$	$x_1^2y_0, x_1x_2y_2, x_2^2y_1,$	$y_0y_2, y_1^2,$	$x_1z_1,$	R_2
$5K$	$x_1^4x_2, x_1x_2^4,$	$x_1^3y_0, x_1^2x_2y_2, x_1x_2^2y_1, x_2^3y_0$			
	$x_1y_1^2, x_1y_0y_2, x_2y_0y_1, x_2y_2^2$		$x_1^2z_1, x_2^2z_2,$		$x_1R_2, x_2R_1,$
	y_1z_2, y_2z_1				S_0
$5K+1$	$x_1^3x_2^2, x_2^5,$	$x_1^3y_1, x_1^2x_2y_0, x_1x_2^2y_2, x_2^3y_1$			
	$x_1y_0^2, x_1y_1y_2, x_2y_0y_2, x_2y_1^2$		$x_1^2z_2, x_1x_2z_1,$		$x_1R_0, x_2R_2,$
	y_0z_1, y_2z_2				S_1
$5K+2$	$x_1^5, x_1^2x_2^3,$	$x_1^3y_2, x_1^2x_2y_1, x_1x_2^2y_0, x_2^3y_2$			
	$x_1y_0y_1, x_1y_2^2, x_2y_0^2, x_2y_1y_2$		$x_1x_2z_2, x_2^2z_1,$		$x_1R_1, x_2R_0,$
	y_0z_2, y_1z_1				S_2
$6K$	$x_1^6, x_1^3x_2^3, x_2^6$	$x_1^4y_2, x_1^3x_2y_1, x_1^2x_2^2y_0, x_1x_2^3y_2, x_2^4y_1$			
	$x_1^2y_0y_1, x_1^2y_2^2, x_1x_2y_0^2, x_1x_2y_1y_2, x_2^2y_0y_2, x_2^2y_1^2$				
	$y_0^3, y_1^3, y_2^3, y_0y_1y_2$				(f,g)
	$x_1^2x_2z_2, x_1x_2^2z_1, x_1y_1z_1, x_2y_0z_1, x_1y_0z_2, x_2y_2z_2, z_1z_2.$				
	etc.				

Proposition 3.1 *$|2K_Y|$ is without fixed points and defines a birational morphism $\varphi_{2K_Y}\colon Y \to \overline{F} \subset \mathbb{P}^5$. φ_{2K_Y} takes the 3 points $\psi^{-1}(P)$ into three distinct points of the vertex $\mathbb{P}^2$ of $\overline{F}$ not lying on any of the coordinate axes $y_i = 0$.*

Since these 3 points are permuted by the $\mathbb{Z}/3$ action, in suitable coordinates they become the 3 points $P_\omega = (1, \omega, \omega^2)$ with a root of $\omega^3 = 1$.

Proof The 3 points of $\psi^{-1}(P)$ are precisely the fixed points of $|K_Y|$, while none of the y_i vanish at them; the fact that $|2K_Y|$ is without fixed points is an immediate consequence of this.

If φ_{2K_Y} is not birational then it must be either 2–1 onto a rational surface (in which case Y cannot have a fixed point free action by a group of order 3), or it must be 3–1 onto the Veronese surface W, a rational scroll $\mathbb{F}_0$ or $\mathbb{F}_2$ or a cone $\overline{\mathbb{F}}_4$ over a rational normal curve of degree 4; the case of the Veronese surface is impossible, since it leads to Y having a nontrivial 2-torsion element. In the $\mathbb{F}_2$ case, the pencil $|A|$ of $\mathbb{F}_2$ lifts to give a base point free pencil E, and $2K_Y \sim 3E + F$, with $F > 0$. Then $6 = 2K_Y^2 = 3K_Y E + K_Y F$ implies that $K_Y E = 2$, so that $|E|$ is a pencil of genus 2; in this case $|2K_Y|$ must be composed with a 2–1 map, which is a contradiction. In the $\mathbb{F}_0$ and $\overline{\mathbb{F}}_4$ cases a similar purely numerical argument gives an immediate contradiction.

Proposition 3.2 *The image $\varphi_{2K_Y}(Y) = \overline{Y} \subset \overline{F} \subset \mathbb{P}^5$ is contained in two cubics linearly independent from the cubics containing $\overline{F}$; and one of those cubics can be chose to contain the vertex $\mathbb{P}^2$ of $\overline{F}$.*

Proof As indicated in Table 1, $H^0(X, 6K_X)$ contains 18 monomials in the x_i and y_i, whereas $h^0(6K_X) = 16$; the two relations between these monomials can be written as cubics in the elements of (4).

To get more precise information, consider the 3 spaces $H^0(4K+i)$, each of which contains 8 monomials; since $h^0(4K) = 7$, there is one relation between the elements in each, say R_0, R_1 and R_2. The monomials involving the z_i are as follows

$$\begin{aligned} &R_0\colon x_1 z_2,\ x_2 z_1,\\ &R_1\colon x_2 z_2,\\ &R_2\colon \qquad\quad x_1 z_1. \end{aligned}$$

A suitable linear combination of $x_1x_2R_0$, $x_1^2R_1$ and $x_2^2R_2$ does not involve the z_i. If this linear combination is identically zero then it must involve at least one of $x_1^2R_1$ or $x_2^2R_2$, and it follows that (say) R_1 is identically divisible by x_2, leading to a relation Q_2 between the elements of $H^0(3K+2)$, which contradicts the choice of z_2.

This relation f does not contain any monomials cubic in the y_i, so that it defines an element $Q \in |6A+2B|$ on F, or a cubic of $\mathbb{P}^5$ containing the vertex of $\overline{F}$. The proposition is proved.

Let $\widetilde{Y}$ denote the inverse image of $\overline{Y} \subset \overline{F}$ under $\varphi\colon F \to \overline{F}$. The restriction $\varphi\colon \widetilde{Y} \to \overline{Y}$ consists just of blowing up the intersection of $\overline{Y}$ with the vertex $\mathbb{P}^2$ of $\overline{F}$, that is, the 3 points $P_\omega = (1, \omega, \omega^2)$, with $\omega^3 = 1$, so that the set theoretic intersection of $\widetilde{Y}$ with $B \subset F$ consists of the 3 lines $\mathbb{P}^1 \times P_\omega \subset B$.

The following lemma will ensure that the monomials x_iz_j occur in the relations R_{i+j} with nonzero coefficients, so that after making an obvious normalisation we have

$$\begin{aligned} R_0 &= x_1z_2 + x_2z_1 + \cdots \text{(terms not involving } z_i) \\ R_1 &= x_2z_2 + \qquad\qquad \cdots \\ R_2 &= \qquad\qquad x_1z_1 + \cdots \end{aligned}$$

and hence

$$f = x_1x_2R_0 - x_1^2R_1 - x_2^2R_2.$$

Lemma 3.3 *$\overline{Y}$ is not contained in any divisor $Q' \in |nA+2B|$ on F with $n < 6$.*

Proof $\overline{Y}$ is obviously not contained in any divisor in $|nA+B|$ for any n, since the fibres of $\widetilde{Y} \to \mathbb{P}^1$ are canonical curves of genus 4. And $|nA+mB|$ contains B as a fixed component if (and only if) $n < 2m$, so that I only need to check that Y is not contained in any irreducible element of $|nA+2B|$ with $n = 4$ or 5; I can choose this element to be invariant under the action of $\mathbb{Z}/3$.

The 3 fixed loci of the action of $\mathbb{Z}/3$ on F are as follows:

(i) $\{t = x_1 = y_1 = y_2 = 0\} \cup \{t = x_2 = y_1 = y_2 = 0\}$,

(ii) $\{x_1 = y_0 = y_2 = 0\}$,

(iii) $\{x_2 = y_0 = y_2 = 0\}$.

An invariant element of $|4A+2B|$ or $|5A+2B|$ is given by a linear combination of the monomials in x_i and y_i occurring in one of the $H^0(X, nK+i)$ (for $n = 4$ or 5, $i = 0, 1$ or 2). One checks at once that any such hypersurface must contain one of the loci (ii) or (iii). An irreducible element of $|nA+2B|$ meets each fibre of $F \to \mathbb{P}^1$ in a quadric, and the fixed locus (ii) or (iii) will be a line lying on such a quadric. The fibres of $\widetilde{Y}$ are the canonical images of curves in $|K_Y|$, contained in the fibres of Q; it would follow that $\widetilde{Y}$ must meet the fixed locus, and in turn this implies that the action of $\mathbb{Z}/3$ on Y has a fixed point.

Corollary 3.4 *The two divisors $Q \in |6A+2B|$ and $C \in |6A+3B|$ defined by the cubics f and g of Proposition 3.2 are irreducible, and their intersection consists of $\widetilde{Y}$ together with a number of components of the fibres of $Q \to \mathbb{P}^1$. of total degree* 6.

The residual intersection has degree 0 in the general fibre, so is contained in fibres of Q. The degree of a surface in a fibre of $F \to \mathbb{P}^1$ is the same as the degree of its image under $\varphi\colon F \to \overline{F} \subset \mathbb{P}^5$ (which is linear on the fibres of F), and the total degree of the residual components is

$$\deg \overline{F} \cdot Q \cdot C - \deg \overline{Y} = 2 \cdot 3 \cdot 3 - (2K_Y)^2 = 18 - 12 = 6.$$

If some fibre of $Q \to \mathbb{P}^1$ splits as a pair of planes, and just one of these is a residual component, then $\widetilde{Y} \subset Q$ will not be a Cartier divisor.

Lemma 3.5 *The fibre of Q over $(x_1 = 0)$ and $(x_2 = 0)$ does not split as a pair of planes.*

Proof $x_i = 0$ defines a divisor on Y which is invariant under the group action; it follows that there is either one component G with $2K_Y G = 6$ (so that the fibre of $\widetilde{Y}$ is irreducible), or 3 components G_i with $2K_Y G_i = 2$ interchanged by the group action. Symmetry considerations show that in this last case the G_i map into 3 conics, no 2 of which are coplanar. This proves the lemma.

Using the fact that Q contains the 3 lines $l_\omega = \mathbb{P}^1 \times P_\omega \subset B$, we see that the R_i and f must have the form

$$\begin{aligned} R_0 &: x_1z_2 + x_2z_1 + a_0(y_0^2 - y_1y_2) + \cdots ; \\ R_1 &: x_2z_2 \qquad\quad + a_1(y_0y_1 - y_2^2) + \cdots ; \\ R_2 &: \qquad\quad + x_1z_1 + a_2(y_0y_2 - y_1^2) + \cdots \end{aligned}$$

and

$$f = a_0x_1x_2(y_0^2 - y_1y_2) + a_1x_1^2(y_2^2 - y_0y_1) + a_2x_2^2(y_1^2 - y_0y_2) + \cdots ,$$

where $\cdots$ denote terms of degree ≤ 1 in the y_i.

Since if (say) $a_1 = 0$ then the restriction of f to $x_2 = 0$ splits as a product of planes, Lemma 3.5 has the following corollary:

Corollary 3.6 $a_i \neq 0$ *for* $i = 1$ *and* 2.

There are certain syzygies relating the R_i and the relations S_i between the monomials in the spaces $H^0(5K + i)$, which will imply also that $a_0 \neq 0$. To obtain these syzygies I have to prove similar statements to Corollary 3.6 for the leading terms of the S_i. These will follow from the following key lemma.

Lemma 3.7 $h^0(F, \mathcal{I}_{\widetilde{Y}} \cdot \mathcal{O}_F(6A + 3B)) = 2.$

In words, the two cubics containing $\overline{Y}$ provided by Proposition 3.2 are the only ones. The proof is longer than it is interesting, and is deferred to the end of this section.

Now consider the relations occurring between the monomials in x_i, y_i, z_i in $H^0(5K + i)$; each of these spaces contains 14 monomials, and is 11-dimensional. There must therefore be one relation S_i in each, in addition to the two relations x_1R_{i-1} and x_2R_{i-2}. By subtracting off suitable multiples of x_1R_{i-1} and x_2R_{i-2} from S_i one can eliminate terms of degree 2 in the x_i and degree 1 in the z_i, and I can therefore assume that the terms involving z_i in the S_i are as follows:

$$\begin{aligned} S_0 &: y_1z_2, y_2z_1, \\ S_1 &: y_0z_1, y_2z_2, \\ S_2 &: y_0z_2, y_1z_1. \end{aligned}$$

Proposition 3.8 *(i) After a suitable normalisation,*

$$\begin{aligned} S_0&\colon y_1z_2+y_2z_1+\cdots\\ S_1&\colon y_0z_1+y_2z_2+\cdots\\ S_2&\colon y_0z_2+y_1z_1+\cdots \end{aligned}$$

where $\cdots$ denotes terms not involving z_i;
The R_i and S_i satisfy the following identities:

$$\begin{aligned} x_2S_0+x_1S_1 \qquad\qquad &\equiv y_2R_0+y_1R_1+y_0R_2,\\ x_1S_0 \qquad\qquad +x_2S_2 &\equiv y_1R_0+y_0R_1+y_2R_2. \end{aligned} \tag{0}$$

Furthermore,

$$g=x_1S_2+x_2S_1-y_0R_0-y_2R_1-y_1R_2$$

defines a divisor $C\in|6A+3B|$ containing $\widetilde{Y}$.
The relations

$$\begin{aligned} h_0&=x_1x_2S_0-x_1y_1R_1-x_2y_2R_2,\\ h_1&=x_1x_2S_1-x_1y_2R_1-x_2y_0R_2,\\ h_2&=x_1x_2S_2-x_1y_0R_1-x_2y_1R_2 \end{aligned}$$

satisfy the identities:

$$\begin{pmatrix} y_0f+x_1x_2g\\ y_1f\\ y_2f \end{pmatrix} \equiv \begin{pmatrix} 0 & x_2 & x_1\\ x_1 & 0 & x_2\\ x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} h_0\\ h_1\\ h_2 \end{pmatrix},$$

or equivalently

$$(y_0+\omega^2y_1+\omega y_2)f+x_1x_2g\equiv(\omega^2x_1+\omega x_2)(h_0+\omega^2h_1+\omega h_2) \tag{+}$$

for the 3 roots of $\omega^3=1$.

Proof The syzygies (0) are a formal consequence of (i) and Lemma 3.7, since otherwise these expressions are relations between the monomials in the x_i and y_i in $H^0(6K+i)$ for $i\neq 0$, and Lemma 3.7 assures us that there are two relations f and g in $H^0(6K)$ (the cubics of Poroposition 3.3), and no relations in $H^0(6K+i)$ for $i \neq 0$.

(i) itself is proved by a similar argument: firstly each S_i involves at least one of the monomials $z_1 y_{i-1}$ or $z_2 y_{i-2}$ according to Lemma 3.3. Suppose say that S_0 does not involves $y_1 z_2$, that is $S_0 = y_2 z_1 + \cdots$; then $x_1 S_0 - y_2 R_2$ is either a relation in x_i, y_i, which contradicts Lemma 3.7, or is identically zero, which implies that R_2 is identically divisible by x_1, which is a contradiction. Thus I can assume that $S_0 = y_1 z_2 + y_2 z_1 + \cdots$. An identical argument shows that S_1 must involve $y_2 z_2$.

Suppose that $S_1 = y_2 z_2 + \cdots$; then

$$x_1 S_1 + x_2 S_0 - y_2 R_0 - y_1 R_1$$

cannot be a relation, by Lemma 3.7, and must therefore be identically zero. But this implies that the coefficient of $y_0 y_1$ in R_1 is zero, contradicting Corollary 3.6.

The normalisation to bring the coefficients of $y_i z_j$ to 1 is straightforward.

The relations (+) imply that the divisor $C \in |6A + 3B|$ defined by g meets Q in $\widetilde{Y}$ together with the 3 fibres of Q over $x_1^3 + x_2^3 = 0$, that is $Q \cap C = \widetilde{Y} + Q_{-1} + Q_{-\omega} + Q_{-\omega^2}$; furthermore, inverting the matrix in (+) one sees that

$$(x_1^3 + x_2^3) h_0 = (-x_1 x_2 y_0 + x_1^2 y_1 + x_2^2 y_2) f - x_1^2 x_2^2 g,$$

so that h_0 defines a divisor $C_0 \in |7A + 3B|$ which cuts out $\widetilde{Y} + 2Q_0 + 2Q_\infty$ on Q.

The above discussion determines the canonical ring of Y and of X for any surface X with invariants $p_g = 0$, $K^2 = 1$ and $\operatorname{Tors} X = \mathbb{Z}/3$. I have not written down all the relations holding between the x_i, y_i and z_i, since there are further relations in $H^0(6K+i)$ which express the quadratic terms z_1^2, $z_1 z_2$, z_2^2 as functions of the x_i and y_i. However, these final relations are determined in an obvious way by the R_i and S_i.

To finish, I have to show that the construction actually works. For this I write down the most general form of the relations $R_0, R_1, R_2, S_0, S_1, S_2$. I choose $z_i \in H^0(3K+i)$ and $y_i \in H^0(2K+i)$ to eliminate several terms in the R_i, and obtain

$$\begin{aligned}
R_0 &= x_2 z_1 + x_1 z_2 + y_0^2 - y_1 y_2 + c x_1^2 y_1 + d x_2^2 y_2 + e x_1^2 x_2^2, \\
R_1 &= \phantom{x_2 z_1 + {}} x_2 z_2 + y_2^2 - y_0 y_1 + a x_1^4, \\
R_2 &= x_1 z_1 + + y_1^2 - y_0 y_2 + b x_2^4;
\end{aligned}$$

and

$$
\begin{aligned}
S_0 &= y_2z_1 + y_1z_2 + cx_1y_1^2 + dx_2y_2^2 + ax_1^3y_0 + bx_2^3y_0 + x_1x_2S,\\
S_1 &= y_0z_1 + y_2z_2 + cx_1y_1y_2 - cx_2y_1^2 + ax_1^3y_1 - ax_1^2x_2y_0 + ex_1x_2^2y_2 - x_2^2S,\\
S_2 &= y_1z_1 + y_0z_2 + dx_2y_1y_2 - dx_1y_2^2 + bx_2^3y_2 - bx_1x_2^2y_0 + ex_1^2x_2y_1 - x_1^2S;
\end{aligned}
$$

here a, b, c, d, e and the 4 coefficients of S (a linear combination of x_1y_2, x_2y_1, x_1^3 and x_2^3) are 9 free parameters. I have used a transformation of the form $x_i \mapsto \lambda x_i$, $y_i \mapsto \mu y_i$, $z_i \mapsto \nu z_i$ to bring the coefficient of $(y_0^2 - y_1y_2)$ in R_0 to 1, and the form of the equations is unique up to a further such transformation with $\mu = 1$, $\nu = \lambda^{-1}$.

One therefore gets

$$
\begin{aligned}
f &= x_1x_2(y_0^2 - y_1y_2) - x_1^2(y_2^2 - y_0y_1) - x_2^2(y_1^2 - y_0y_2)\\
&\quad + cx_1^3x_2y_1 + dx_1x_2^3y_2 - ax_1^6 + ex_1^3x_2^3 - bx_2^6;\\
-g &= y_0^3 + y_1^3 + y_2^3 - 3y_0y_1y_2 + cx_1^2y_0y_1 + dx_2^2y_0y_2\\
&\quad - (c+d)x_1x_2y_1y_2 + dx_1^2y_2^2 + cx_2^2y_1^2 + (a+b+e)x_1^2x_2^2y_0\\
&\quad + (bx_2^3 - (a+e)x_1^3)x_2y_1 + (ax_1^3 - (b+e)x_2^3)x_1y_2 + (x_1^3 + x_2^3)S;\\
-h_0 &= x_1y_1(y_2^2 - y_0y_1) + x_2y_2(y_1^2 - y_0y_2) - cx_1^2x_2y_1^2 - dx_1x_2^2y_2^2\\
&\quad - (ax_1^3 + bx_2^3)x_1x_2y_0 + ax_1^5y_1 + bx_2^5y_2 - x_1^2x_2^2S.
\end{aligned}
$$

One sees easily that for general values of the parameters a, b, c, d and e, the equation f defines a nonsingular divisor $Q \subset F$. In fact, Bertini's theorem shows that it can only have singularities on B for general values of the coefficients, and this will be sufficient in view of the computations to follow. Fixing f, and applying Bertini's theorem to the linear system obtained by varying S, one sees that the singularities of the general $\widetilde{Y}$ are contained in B.

Now obviously, for any values of the parameters, the intersection of B with the variety defined by $f = g = h_0 = 0$ is set theoretically contained in the 3 lines $l_\omega = \mathbb{P}^1 \times P_\omega \subset B$.

I write down the derivatives of f, g and h_0 with respect to the coordinates (x_1, x_2), (t, y_0, y_1, y_2) of F, and evaluate them at the point (x_1, x_2),

$(0, 1, \omega, \omega^2)$ of l_ω.

	f	$-g/(c+d)$	h_0
t	$\omega c x_1^3 x_2 + \omega^2 d x_1 x_2^3$	$\omega x_1^2 - x_1 x_2 + \omega^2 x_2^2$	$\omega^2 c x_1^2 x_2 + \omega d x_1 x_2^2$
x_1	0	0	0
x_2	0	0	0
y_0	$(\omega^2 x_1 + \omega x_2)^2$	0	$\omega^2 x_1 + \omega x_2$
y_1	$(\omega^2 x_1 + \omega x_2)(\omega x_1 - 2x_2)$	0	$\omega x_1 - 2x_2$
y_2	$(\omega^2 x_1 + \omega x_2)(-2x_1 + \omega^2 x_2)$	0	$-2x_1 + \omega^2 x_2$

Thus one sees immediately that f is nonsingular along l_ω, provided that $c + d \neq 0$; and that the intersection $Q \cap C$ defined by $f = g = 0$ is nonsingular along l_ω except at the points where $x_1^3 + x_2^3 = 0$. At the point $(\omega, -\omega^2), (0, 1, \omega, \omega^2)$ (the point of l_ω where $\omega^2 x_1 + \omega x_2 = 0$), the derivatives $\frac{\partial f}{\partial t}$ and $\frac{\partial h_0}{\partial y_1}$ are both nonzero, whereas if $\omega^2 x_1 + \omega x_2 \neq 0$ and $x_1 x_2 \neq 0$ the derivatives $\frac{\partial(f, h_0)}{\partial(y_0, y_1, y_2)}$ provide a nonzero 2×2 minor. Thus the intersection $\widetilde{Y} = Q \cap C \cap C_0$ is nonsingular for general values of the parameters. The tangent space to $\widetilde{Y}$ is contained in the tangent space to B at any point of l_ω as follows from the above derivatives. Finally, the fixed points of the $\mathbb{Z}/3$ action on F are listed in the proof of Lemma 3.3, and it is easy to see that for general values of the parameters $\widetilde{Y}$ does not meet these loci.

I have proved:

Theorem 3.9 *There exist surfaces X having $p_g = 0$, $K^2 = 1$ and* $\operatorname{Tors} X = \mathbb{Z}/3$ *and these form an irreducible moduli space.*

Furthermore if X is any such surface and $Y \to X$ the $\mathbb{Z}/3$ cover corresponding to $\operatorname{Tors} X$*, then the canonical ring of Y can be generated by elements x_1, x_2, y_0, y_1, y_2, z_1, z_2 as above, and the relations are $R_0, R_1, R_2, S_0, S_1, S_1, S_2$ together with certain relations T_0, T_1, T_2 experssing $z_1 z_2$, z_2^2 and z_1^2 in terms of the x_i and y_i; the R_i and S_i can be written as above.*

Proof of Lemma 3.7 Let $Q \subset F$ be the unique irreducible divisor $Q \in |6A + 2B|$ containing $\widetilde{Y}$. An equivalent formulation of Lemma 3.7 is that the image of the restriction map

$$H^0(F, \mathcal{I}_{\widetilde{Y}} \cdot \mathcal{O}_F(6A + 3B)) \to H^0(Q, \mathcal{I}_{\widetilde{Y}} \cdot \mathcal{O}_Q(6A + 3B))$$

is 1-dimensional. Suppose otherwise. Then I can find two elements p and q in $H^0(F, \mathcal{I}_{\widetilde{Y}} \cdot \mathcal{O}_F(6A + 3B))$ that are eigenforms of the $\mathbb{Z}/3$ action, and restrict to give linearly independent elements of $H^0(Q, \mathcal{I}_{\widetilde{Y}} \cdot \mathcal{O}_Q(6A+3B))$. Since the linear system $L \subset |6A+3B|_Q$ defined by p and q is made up of divisors containing $\widetilde{F}$, it is of the form

$$L = M + Z,$$

where $Z \in |(6-r)A+3B|$ is the fixed part (containing $\widetilde{Y}$), and $M \subset |rA|$ is a linear system without fixed part; clearly $1 \leq r \leq 3$. The divisors M_p and M_q corresponding to p and q are invariant elements of $|rA|$ without common components.

Case (i), $r = 1$**.** Consider first of all the simplest case $r = 1$; the only invariant elements of $|A|$ are Q_0 and Q_∞, so that I can assume $M_p = Q_0$ and $M_q = Q_\infty$. But then both of $x_2 p$ and $x_1 q$ define the same divisor $Z + Q_0 + Q_\infty$, so that some linear combinations, say $x_2 p - x_1 q$, is divisible by f:

$$x_2 p - x_1 q \equiv (ax_1 + bx_2)f, \quad \text{so that} \quad x_2(p - af) \equiv x_1(q + bf).$$

This identity implies, say, that $p - af$ is divisible by x_1, which contradicts Lemma 3.3.

Case (ii), $r = 2$**.** In the same way, I can assume $M_p = 2Q_0$ and $M_q = 2Q_\infty$, so that $x_2^2 p$ and $x_1^2 q$ both define $Z + 2Q_0 + 2Q_\infty$ in Q, leading to an identity

$$x_2^2 p + x_1^2 q \equiv lf,$$

where l is a linear combination of $x_1^2, x_1x_2, x_2^2, y_0, y_1, y_2$. If $x_2^2 p$ and $x_1^2 q$ belong to different eigenspaces of the group action then the above identity splits into two, and each of $x_2^2 p$ and $x_1^2 q$ is separately divisible by f, which gives an easy contradiction.

Write $p = p_i$ to indicate that p belongs to the ith eigenspace of the group action. Since the relation g provided by Proposition 3.2 is invariant, I can assume $p = p_0$ and $q = q_2$, and the identity is

$$x_2^2 p_0 + x_1^2 q_2 \equiv (ax_2^2 + by_1)f, \quad \text{or} \quad x_2^2(p_0 - af) + x_1^2 q_2 \equiv by_1 f.$$

Now $b \neq 0$ for otherwise one gets a contradiction to Lemma 3.3 as above. But this relation implies that the coefficient a_0 of $x_1x_2(y_0^2 - y_1y_2)$ in f is zero (compare Corollary 3.6), and that the coefficients of y_0^3 in p_0 vanishes. It follows from this that the subspace of F defined by $f = p_0 = q_2$ contains the fixed points

$$\{t = x_1 = y_1 = y_2 = 0\} \quad \text{and} \quad \{t = x_2 = y_1 = y_2 = 0\}$$

of the $\mathbb{Z}/3$ action. But one sees easily using Lemma 3.5 that in this case the equations $f = p_0 = q_2 = 0$ define $\widetilde{Y}$ exactly in a neighbourhood of *either* $x_1 = 0$ *or* $x_2 = 0$. Thus the group action on $\widetilde{Y}$ has a fixed point, which is a contradiction.

$r = 3$ splits into several cases.

Case (iii). $M_p = 3Q_0$**,** $M_q = 3Q_\infty$**, and again assume that** $p = p_0$**.** We get a relation

$$x_1^3 q + x_2^3 p \equiv (a x_1^3 + b x_2^3 + c x_1 y_2 + d x_2 y_1) f,$$

or

$$x_1^3 (q - af) + x_2^3 (p - bf) \equiv (c x_1 y_2 + d x_2 y_1) f;$$

not both c and d vanish by Lemma 3.3. But now Corollary 3.6 implies that at least one of $x_1^2 x_2 y_1^3$ or $x_1 x_2^2 y_2^3$ appears on the right hand side with nonzero coefficient, which is a contradiction.

Case (iv). $M_p = Q_{-1} + Q_{-\omega} + Q_{-\omega^2}$**,** $M_q = 2Q_0 + Q_\infty$**, and** $p = p_0$**.** In this case one gets the identity

$$x_1^2 x_2 (p - af) + (x_1^3 + x_2^3) q \equiv (b x_1 y_0 + c x_2 y_2) f;$$

now if $b \neq 0$ Corollary 3.6 implies that the right hand side contains $x_1 x_2^2 y_0 y_1^2$ with nonzero coefficient, which is a contradiction. Hence $b = 0$, $c \neq 0$; but this implies that the coefficient of $x_1 x_2 (y_0^2 - y_1 y_2)$ in f vanishes, as well as the coefficient of y_0^3 in p. This implies that both f and p vanish at the fixed points

$$\{t = x_1 = y_1 = y_2 = 0\} \quad \text{and} \quad \{t = x_2 = y_1 = y_2 = 0\}$$

whereas $f = p = 0$ defines $\widetilde{Y}$ exactly in a neighbourhood of both $x_1 = 0$ and $x_2 = 0$.

Case (v). $M_p = Q_{-1} + Q_{-\omega} + Q_{-\omega^2}$**,** $M_q = 2Q_0 + Q_\infty$**, and** $q = q_0$**.** In this case one gets the identity

$$x_1^2 x_2 p + x_1^3(q - af) + x_2^3(q - bf) \equiv (cx_1y_2 + dx_2y_1)f;$$

if $c = d = 0$ then $q - bf$ is divisible by x_1, contradicting Lemma 3.3. But if $c \neq 0$ then the right hand side contains $x_1^3 y_2^3$ with nonzero coefficient by Corollary 3.6. Thus $c = 0$, $d \neq 0$. But then the coefficient of $x_1x_2(y_0^2 - y_1y_2)$ in f is zero, and again, f and p both vanish on fixed points of the group acion, and define $\widetilde{Y}$ is a neighbourhood of them.

This completes the proof of Lemma 3.7.

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Final remarks (2015).

Correction to introductory remarks

My reference on p. 1 to [7] is not correct. The papers of Rebecca Barlow [8], [9] contain the first constructions of simply connected Godeaux surfaces and Godeaux surfaces with $\text{Tors} = \pi_1 = \mathbb{Z}/2$.

Remark on Theorem 2.1

The general surface $Y(8,8) \subset \mathbb{P}(1,1,4,4,4)$ is a surface with $K_Y = \mathcal{O}(A)$ with the same invariants $p_g = 3$, $K^2 = 4$ and deforms to a surface with $|K_Y|$ free. It can be given a free $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ group action. However, it has 4 cyclic quotient singularities of type $\frac{1}{4}(1,1)$, and these do not deform away with the group action.

Remark on construction of Section 3

One constructs the main family of surfaces of Section 3 much more easily by unprojection methods, which give them as section of a large key variety. This trick does not avoid the need for the proof of irreducibility of the moduli space. For details, see [10]. The main idea is to change coordinates from the eigencoordinates x_i, y_i, z_i of the 1977 Tokyo paper to permutation coordinates by a cyclotomic coordinate change

$$y_i \mapsto y_0 + \omega^i y_1 + \omega^{2i} y_2 \quad \text{(say)},$$

after which the z_i become parallel unprojection coordinates. The key variety is a standard parallel unprojection from the hypersurface

$$y_0 y_1 y_2 = s x_0 x_1 x_2 + r_0 x_1 x_2 y_0 + r_1 x_0 x_2 y_1 + r_2 x_0 x_1 y_2.$$

contained in the product of 3 codimension 2 ideals (x_i, y_i).