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§0. Introduction.

This paper is an extract from a forthcoming more complete article [1]; I refer to [1] for some technical results. Varieties are defined over an algebraically closed field of characteristic 0.  $\pi_1 X$  is the algebraic (profinite) fundamental group. (Analogous results to Theorem 1 have been obtained by E. Horikawa: Surfaces with small  $c_1^2$ , v.)

Theorem 1. Let  $X$  be a minimal surface of general type, and suppose that  $K_X^2 < \frac{1}{3}c_2(X)$  (equivalently,  $K_X^2 < 3\chi(O_X)$ ); then either

(i)  $\pi_1 X$  is finite

or (ii) there exists an étale Galois cover  $Y_0 \rightarrow X$ ,  $Y_0$  having a morphism  $Y_0 \xrightarrow{f} C_0$  to a curve of genus  $p > 0$  inducing an isomorphism  $f_*: \pi_1 Y_0 \xrightarrow{\cong} \pi_1 C_0$ .

Furthermore in (ii) the fibres of  $f: Y_0 \rightarrow C_0$  are hyperelliptic curves of genus  $g \leq 5$ .

Corollary 2. (i)  $\pi_1 X$  is an extension of  $\pi_1 C_0$  by a finite group  $G_0$ ; (ii) for every étale cover  $Y \rightarrow X$  with  $q(Y) \neq 0$  the Albanese map  $\alpha: Y \rightarrow \text{Alb } Y$  maps onto a curve; (iii) if  $X \xrightarrow{f} C$  is a non-constant morphism of  $X$  with connected fibres to a curve  $C$  of genus  $p > 0$  then  $q(X) = p$ .

Dividing  $Y_0 \rightarrow C_0$  by the equivariant action of  $G_0$  gives a re-statement of (ii):

(ii)'  $\pi_1 X$  is infinite, and contains a normal subgroup  $A$  of order  $\leq 4$ ; there exists a morphism  $X \rightarrow B$  to a curve  $B$ ; and every étale cover  $Y \rightarrow X$  which corresponds to a finite quotient of  $\pi_1 X/A$  is obtained by making a normalized pull-back diagram

$$\begin{array}{ccc} Y = Y \overset{\sim}{\times}_B C & \longrightarrow & X \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \end{array}$$

over a cover  $C \rightarrow B$  ramified only at points of  $B$  corresponding to multiple fibres of  $X \rightarrow B$ .

Corollary 3. Let  $X$  be a minimal surface with  $K_X^2 < 3p_g(X) - 7$ ,  $K_X^2 < 3\chi(O_X)$ , and  $p_g \geq 8$ ; then one of the following 4 cases hold:

- (i)  $|K_X|$  is composed of a rational pencil  $X \rightarrow P^1$ , and  $q(X) = 0$ ;
- (ii)  $|K_X|$  is composed of an irrational pencil  $X \rightarrow C$  with  $C$  a curve of genus  $p > 0$ , and  $q(X) = p$ ;
- (iii)  $\phi_{K_X} : X \rightarrow F \subset P^p_g(X)^{-1}$  is generically 2-to-1 onto a rational surface  $F$ , and  $q(X) = 0$ ;
- or (iv)  $\phi_{K_X} : X \rightarrow F \subset P^p_g(X)^{-1}$  is generically 2-to-1 onto a ruled surface of genus  $p > 0$ , and  $q(X) = p$ .

It is quite likely that (i) and (ii) cannot actually occur (see Problem R5); I also do not know if fibres of genus 4 or 5 can occur in Theorem 1 - if this happens then  $|K_Y|$  must have a fixed part having large intersection number with the fibres.

There is an element of ineffectivity in Theorem 1, there being no bound for the finite  $\pi_1 X$  which can occur; the problem is to bound the 2-torsion in  $\text{Pic } X$  (see Problem R2).

A positive answer to Problems R1 and R8 would be one step in the direction of the following:

Conjecture 4. The hypothesis in Theorem 1 can be weakened to

$$K_X^2 < \frac{1}{2}c_2(X) \quad (\text{equivalently } K_X^2 < 4\chi(O_X)).$$

The conjecture can be weakened to only ask that for surfaces with  $K_X^2 < \frac{1}{2}c_2(X)$  (ii) and (iii) of Corollary 2 hold; perhaps the natural approach to this conjecture would be through differential-geometrical methods, which could also shed light on the problem of the topological fundamental group for surfaces in this range.

F. Sakai points out that the hypothesis of Theorem 1 cannot be weakened to  $K_X^2 < c_2(X)$ , since for every surface of general type  $S$  there exists a cyclic branched cover  $T \rightarrow S$  with  $K_T^2 < c_2(T)$ . See however Problem R3.

### §1. Proof of Theorem 1 and its corollaries.

Suppose that  $X$  is as in Theorem 1, and that  $\pi_1 X$  is infinite. I will prove (ii).

Let  $Y \rightarrow X$  be an étale Galois cover such that  $G = \text{Gal } Y/X$  has  $|G| = n$ ;  $Y$  has the invariants

$$K_Y^2 = nK_X^2 ; \quad \chi(O_Y) = n\chi(O_X).$$

Since  $K_X^2 \leq 3\chi(O_X) - 1$  I have

$$K_Y^2 \leq 3p_g(Y) + 3 - 3q(Y) - n.$$

Exactly as in [1], Step 5 of §1, if  $n \geq 11$  then  $K_Y^2 < 3p_g - 7$ , so that by [1], Theorem 8.1 (ii) (see also [2, Lemma (1.1) p. 122])  $\phi_{K_Y}$

cannot be birational; suppose that  $|K_Y|$  is not composed of a pencil. Then  $\phi_{K_Y}$  is generically  $m$ -to-1 onto a surface  $\bar{Y}$ , and the standard argument as in [1] §1, Step 2 gives

$$m \cdot \deg \bar{Y} \leq K_Y^2.$$

Since  $\bar{Y}$  is a surface spanning  $P^{p_g(Y)-1}$ , for  $n \geq 10$  I must have  $n = 2$ , and  $\deg \bar{Y} < \frac{3}{2}(p_g(Y)-1)$ . Thus using [1] Theorem 6.2 and Corollary 6.5 there are the following 3 possibilities for  $\phi_{K_Y}$ :

- (i)  $|K_Y|$  is composed of a pencil;
- (ii)  $\phi_{K_Y} : Y \rightarrow \bar{Y}$  is generically 2-to-1 onto a rational surface  $\bar{Y}$ ;
- (iii)  $\phi_{K_Y} : Y \rightarrow \bar{Y}$  is generically 2-to-1 onto a surface  $\bar{Y}$  having an irrational ruling by lines or conics.

In (ii) there is a biregular involution  $i$  of  $Y$  such that the quotient  $F = Y/i$  is rational. As in [1] §1, Step 3 it follows from Lemma 4.2 (ii) of [1] that  $G = (Z/2)^a$ . It follows that  $(Z/2)^a \subset \text{Pic } X$ .

**Proposition 1.** There is a bound  $T$  (depending on  $X$ ) such that  $(Z/2)^a \subset \text{Pic } X$  implies that  $a \leq T$ .

**Proof.**  $T$  can be taken as  $2q(X)$  plus the number of generators for the (finitely generated) Neron-Severi group.

Thus under the hypothesis that  $\pi_1 X$  is infinite there exist étale Galois covers  $Y \rightarrow X$  for which  $Y$  falls under (i) or (iii) above.

**Proposition 3.** Suppose that for some  $Y \rightarrow X$  with  $p_g(Y) \geq 8$   $|K_Y|$  is composed of a pencil; then  $\phi_{K_Y} : Y \rightarrow C \subset P^{p_g(Y)-1}$  is a morphism with irreducible generic fibres of genus 2.

**Proof.** Write

$$|K_Y| = |E^{(r)}| + F,$$

with  $F$  the fixed part and  $E^{(r)} = E_1 + \dots + E_r$ , the  $E_i$  being irreducible fibres of a map  $Y \rightarrow C$ . Let  $E$  be the numerical class of  $E_i$ .

Then as in [1], Step 1 of §1  $K_Y^2 \geq r^2 E^2$ , where furthermore  $r \geq p_g(Y) - 1$ , so that  $E^2 = 0$ ; also

$$K_Y E \leq \frac{1}{r} K_Y^2 \leq \frac{3p_g(Y)+3}{p_g(Y)-1} < 4,$$

so that  $K_Y E = 2$ .

Thus  $\varphi_{K_Y}$  is obtained by composing the morphism  $Y \rightarrow C$  with the rational map  $\varphi_{K_Y}: C \rightarrow \mathbb{P}^{p_g(Y)-1}$  given by a linear system  $|d|$  on  $C$  of degree  $r$ ; if this map is not birational, then  $r \geq 2p_g(Y) - 2$ , again contradicting the numerical conditions. The proposition is proved.

I write  $\varphi_{K_Y}: Y \rightarrow \bar{Y} \subset \mathbb{P}^{p_g(Y)-1}$  for the canonical rational map, regardless of the dimension of  $\bar{Y}$ .

**Proposition 4.** Let  $Y_2 \xrightarrow{\psi_{2,1}} Y_1 \rightarrow X$  be an étale tower, with  $p_g(Y_1) \geq 8$ ; then there exists a rational map  $\bar{Y}_2 \rightarrow \bar{Y}_1$  making the diagram

$$\begin{array}{ccc} Y_2 & \longrightarrow & \bar{Y}_2 \subset \mathbb{P}^{p_g(Y_2)-1} \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & \bar{Y}_1 \subset \mathbb{P}^{p_g(Y)-1} \end{array}$$

commutative. Furthermore (i) if  $\bar{Y}_2$  is a curve then so is  $\bar{Y}_1$ ; (ii) if  $\bar{Y}_2$  is a surface ruled by lines then either  $\bar{Y}_1$  is a curve, or it is a surface ruled by lines.

**Proof.** The composite

$$Y_2 \longrightarrow Y_1 \longrightarrow \bar{Y}_1 \subset \mathbb{P}^{p_g(Y)-1}$$

is defined by the subspace  $\psi_{2,1}^* H^0(K_{Y_1}) \subset H^0(K_{Y_2})$ ; hence the required map is defined by the linear projection from  $\mathbb{P}^{p_g(Y_2)-1}$  onto  $\mathbb{P}^{p_g(Y)-1}$ . The final assertion is obvious.

Thus my surface  $X$  falls under one of the following 3 cases:

**Case 0.** There exists some étale Galois cover  $Y_0 \rightarrow X$  such that for every étale cover  $Y \rightarrow Y_0$   $|K_Y|$  is composed of a pencil of curves of genus 2;

Case i. (for  $i = 1$  or  $2$ ). There exists an étale Galois cover  $Y_0 \rightarrow X$  such that

- (\*) for every étale cover  $Y \rightarrow Y_0$ ,  $\phi_{K_Y} : Y \rightarrow \bar{Y} \subset \mathbb{P}_g^{p_g(Y)-1}$  is a double cover of a surface  $\bar{Y}$  having an irrational pencil of rational curves of degree  $i$ .

It seems quite likely that Case 0 cannot occur (see Problem R5); in any case it is easy to deal with:

Proposition 5. In Case 0 there exists a morphism  $X \xrightarrow{f} B$  inducing an isomorphism  $f_* : \pi_1 X \xrightarrow{\cong} \pi_1 B$ .

Proof.  $Y_0$  has a  $G$ -equivariant pencil  $Y_0 \rightarrow C_0$  of curves of genus 2; by [1] Lemma 4.2 (i)  $G$  must act freely on  $C_0$ , so that the following diagram is a pull-back:

$$\begin{array}{ccc} Y_0 & \rightarrow & C_0 \\ \downarrow & & \downarrow \\ X = Y_0/G & \rightarrow & B = C_0/G \end{array}$$

Now for every étale cover  $Y_1 \rightarrow Y_0$   $\phi_{K_{Y_1}}$  defined a pencil  $Y_1 \rightarrow C_1$ , and by Proposition 4 there is a morphism  $C_1 \rightarrow C_0$  compatible with the canonical maps of  $Y_1$  and  $Y_0$ . But these maps then form a pull-back diagram, and  $Y_1$  can also be obtained as the pull-back  $X \times_B C_1$ ; since every étale cover  $Y \rightarrow X$  fits into a Galois tower under some such  $Y_1$  the proposition is proved.

In Cases 1 and 2 the irrational pencil on  $\bar{Y}$  defined a pencil  $Y \rightarrow C$  on  $Y$ .

Lemma 6. In Case 1 the fibres of  $Y \rightarrow C$  have genus  $\leq 3$ ; in Case 2 they have genus  $\leq 5$ .

Proof. Since each fibre of  $Y \rightarrow C$  goes into a rational curve of degree  $i$  each fibre imposes at most  $i + 1$  conditions on a divisor in  $|K_Y|$ , and setting  $r = \left\lfloor \frac{p_g - 2}{i + 1} \right\rfloor$  it follows that a divisor of  $|K_Y|$  can be found containing  $r$  fibres  $Y \rightarrow C$ ; thus

$$3p_g(Y) + 3 - 3q(Y) - n \geq K_Y^2 \geq rK_Y E.$$

In Case 1 it then follows that  $K_Y E \leq 4$ , and in Case 2  $K_Y E \leq 8$ , as required.

I can now make the further requirement on  $Y_0$ :

(\*\*) as in (\*), and for some  $g$ , for every etale  $Y \rightarrow Y_0$  the irrational pencil on  $\bar{Y}$  defined a pencil of curve of genus  $g$  on  $Y$ .

Now consider the multiple fibres of  $Y_0 \rightarrow C_0$ ; by making a pull-back by a cover of  $C_0$  ramified only in the points corresponding to the multiple fibres I arrive at a cover  $Y_0$  which satisfies in addition

(\*\*\*) as in (\*\*), and  $Y_0 \rightarrow C_0$  has no multiple fibres.

Now let  $Y_1 \rightarrow Y_0$  be an etale Galois cover; by Proposition 4 the double covers  $Y_i \rightarrow \bar{Y}_i$  fit into a commutative diagram with a map  $\bar{Y}_1 \rightarrow \bar{Y}_0$ ; on the other hand, both  $\bar{Y}_i$  are ruled by lines or conics, so that the map  $\bar{Y}_1 \rightarrow \bar{Y}_0$ , which is defined by linear projection, induces a map  $C_1 \rightarrow C_0$  between the curves parametrising the lines or conics of  $\bar{Y}_1$  and  $\bar{Y}_0$ . I thus get a diagram:

$$\begin{array}{ccccc} Y_1 & \rightarrow & \bar{Y}_1 & \rightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \rightarrow & \bar{Y}_0 & \rightarrow & C_0. \end{array}$$

Now since both  $Y_i \rightarrow C_i$  have fibres of the same genus it follows that no element of  $\text{Gal } Y_1/Y_0$  acts trivially on  $C_1$ . Thus  $C_1 \rightarrow C_0$  is also Galois, with  $\text{Gal } Y_1/Y_0 = \text{Gal } C_1/C_0$ ; under these circumstances it follows that  $Y_1$  is birational to the pull-back  $Y_0 \times_{C_0} C_1$ , and then since the pencil  $Y_0 \rightarrow C_0$  is without multiple fibres it follows that  $C_1 \rightarrow C_0$  is also etale. This proves Theorem 1, via the assertion

Theorem 7.  $X$  has a cover  $Y_0$  satisfying (\*\*\*) above. For any such  $Y_0$   $f: Y_0 \rightarrow C_0$  induces an isomorphism  $f_*: \pi_1 Y_0 \xrightarrow{\cong} \pi_1 C_0$ .

Proof of Corollary 0.2 (ii). Since  $Y_0 \rightarrow C_0$  induces an isomorphism on  $\pi_1$  it also induces an isomorphism  $f_*: \text{Alb } Y_0 \xrightarrow{\cong} \text{Alb } C_0$ , so that the Albanese map of  $Y_0$  is just the composite of  $Y_0 \rightarrow C_0$  with the embedding of  $C_0$  in its Jacobian  $\text{Alb } C_0$ .

For  $X$  itself the Albanese map of  $X$  fits into a commutative diagram

$$\begin{array}{ccc} Y_0 \rightarrow C_0 \subset \text{Alb } Y_0 & & \\ \downarrow & & \downarrow \psi_* \\ \alpha: X \longrightarrow \text{Alb } X, & & \end{array}$$

so that  $\alpha(X) \neq \psi_*(C_0)$ ; this proves (ii).

Proof of Corollary 0.2 (iii). Given  $f: X \rightarrow C$  consider the diagram

$$\begin{array}{ccc} \alpha: X & \rightarrow & \text{Alb } X \\ \downarrow & & \downarrow f_* \\ C & \rightarrow & \text{JC} \end{array};$$

the image  $\alpha(X)$  is a curve  $D$  which maps onto  $C$  under  $f_*$ ; since  $f$  has irreducible fibres it follows that  $D = C$ , proving (iii).

Proof of Corollary 0.3. In view of Corollary 0.2 (iii) and [2, Lemma (1.1), p. 122] the only non-trivial assertion remaining to prove is that if  $q(X) \neq 0$  then (i) and (iii) are impossible.

Let  $Y_0 \rightarrow X$  be the cover as in Theorem 1; since  $q(X) \neq 0$  the curve  $C_0 \subset \text{Alb } Y_0$  has a non-trivial map to the curve  $\alpha(X) \subset \text{Alb } X$ , and this implies that  $C_0/G_0$  has genus  $> 0$ .

The map  $Y_0 \rightarrow X$  provided by Proposition 4 fits into a diagram

$$\begin{array}{ccc} Y_0 & \dashrightarrow & Y_0 \subset \mathbb{P}^{p_g(Y_0)-1} \\ \downarrow & \nearrow & \downarrow \\ X & \dashrightarrow & X \subset \mathbb{P}^{p_g(X)-1} \end{array},$$

together with the quotient map  $X = Y_0/G_0 \rightarrow Y_0/G_0$ . Now split into cases: if  $Y_0$  is a curve then  $Y_0$  is birational to  $C_0$ ;  $Y_0/G_0$  is a non-rational curve, birational to  $X$  according to Proposition 3.

On the other hand if  $Y_0$  is a surface then it is ruled over  $C_0$ , so that  $Y_0/C_0$  is ruled over  $C_0/G_0$ ; if  $X \rightarrow X$  is generically 2-to-1 onto a surface then this surface is birational to  $Y_0/G_0$ . If  $X$  is a curve then by Proposition 3  $X \rightarrow X$  has irreducible fibres, so that the same holds for  $Y_0/G \rightarrow X$ . The ruled surface  $Y_0/G$  is birationally a product  $P^1 \times C$ , with  $C = C_0/G_0$  of genus  $p > 0$ . There are thus just two possibilities for the rational map  $Y_0/G \rightarrow X$ , the projections on the two factors. I am home if it is projection onto  $C$ . But the other case is impossible: for  $X \xrightarrow{\varphi} Y_0/G$  is a double cover, and  $\varphi$  composed with the first projection is birationally the canonical map of  $X$ , and thus by Proposition 3 has fibres curves of genus 2. On the other hand  $\varphi$  composed with the second projection is the map  $X \rightarrow C_0/G_0$  deduced from  $Y_0 \rightarrow C_0$  by quotienting by  $C_0$ , and has thus fibres of genus  $\leq 5$ . Thus the branch locus of the birational double cover  $X \xrightarrow{\varphi} P^1 \times C$  has degree  $\leq 12$  on the first factor, degree 2 on the second factor, and  $C$  has genus 1. It follows easily that then  $p_g \leq 7$ .

References.

- [1] M. Reid, Surfaces with  $p_g = 0$ ,  $K^2 = 2$ , to appear.  
 [2] E. Horikawa, Surfaces with small  $c_1^2$ , II, Invent. Math., 37 (1976) 121 - 155.

ProblemsR.1

A subscheme  $X \subset \mathbb{P}^n$  imposes  $r$  conditions on quadrics if  $h^0(I_X \cdot \mathcal{O}_{\mathbb{P}^n}(2)) = \binom{n+2}{2} - r$ .

Problem: Let  $C \subset \mathbb{P}^{n+1}$  be an irreducible curve of degree  $d$ . Determine the function  $f(n,r)$  such that  $C$  imposes  $\leq r$  conditions on quadrics implies either

(i)  $d \leq f(n,r)$

or

(ii)  $C \subset F \subset \mathbb{P}^{n+1}$ , with  $F$  a component of dimension  $\geq 2$  of the intersection of all quadrics through  $C$ .

If  $r \leq 2n$  then  $f = r$  and if  $r = 2n + 1$  or  $r = 2n + 2$  then  $f = 2r - 2n$ .

Conjecture: For  $2n \leq r \leq 3n - 2$  one has  $f(n,r) = 2r - 2n$ ,  
 and for  $r = 3n - 1$  one has  $f(n,r) = 4n$ .

This conjecture implies the following 3<sup>rd</sup> Castelnuovo Inequality

Conjecture: If  $X$  is a surface such that  $\phi_{K_X}$  is birational, then either

(i)  $K_X^2 \geq 4p_g - 12$

or

(ii)  $\phi_{K_X}(X) = \bar{X}$  is contained in a 3-fold component of the intersection of all quadrics through  $\bar{X}$ .

References. D.W. Babbage, A note on quadrics through a canonical curve, J. London Math. Soc. 14: 4 (1939), 310-314.

A. Tjurin, Prym Varieties, Izv. Akad. Nauk 39: 5 (1975), 1003-1043.

M. Reid, Surfaces with  $p_g = 0$ ,  $K^2 = 2$  (to appear).

R.2

Find an effective bound (or better still, the correct bound)

$T(p_g, q, K^2)$  such that for a minimal surface of general type  $X$  with given  $p_g(X)$ ,  $q(X)$ ,  $K_X^2$  one has

$$(Z/2)^a \subset \text{Pic } X \rightarrow a \leq T.$$

Example: For  $p_g = q = 0$ ,  $K^2 = 2$ , Miyaoka gives the bound  $T = 3$ .

Conjecture: For  $p_g = q = 0$ ,  $K^2 = 3$ ,  $T = 4$  or  $3$ .

Reference. Y. Miyaoka, Tricanonical maps of numerical Campedelli surfaces, in Complex Analysis and Algebraic Geometry, ed. Baily and Shioda, Iwanami Shoten, Tokyo, 1977.

### R.3

Conjecture: If  $X$  is a surface of general type such that the Albanese map  $\alpha: X \rightarrow \text{Alb } X$  is birational, then  $c_1^2(X) \geq c_2(X)$ .

### R.4

For a surface  $X$  let  $s \in H^0(S^m \Omega_X^1 \otimes \mathcal{O}_X(D))$  be a non-zero section; let  $X_s$  denote the divisor in the 3-fold  $\mathbb{P}(\Omega_X^1)$ :

$$\begin{array}{ccccc} Y_s & \rightarrow & X_s & \subset & \mathbb{P}(\Omega_X^1) \\ & \searrow & \downarrow \pi & \swarrow & \\ & & X & & \end{array}$$

$\psi$

and let  $Y_s$  be a non-singular model of one component of  $X_s$ . Then  $\psi^*s$  has an expression as a product of  $m$  meromorphic sections of  $\Omega_{Y_s}^1$ ,  $s_i$ .

Problem: Estimate the poles of  $s_i$  in terms of the poles of  $s$  and the ramification of  $\psi$ .

### R.5

Conjecture: Let  $X$  be a minimal surface of general type for which  $|K_X|$  is composed of a pencil of curves of genus 2.

Then  $K_X^2 \geq 4p_g - 6$ .

(This may have been proved by E. Horikawa in case  $q(X) = 0$ .)

Reference: E. Horikawa, Surfaces with a pencil of curves of genus 2, in Complex Analysis and Algebraic Geometry, ed. Baily and Shioda, Iwanami Shoten, Tokyo, 1977.

R.6

The problem is to define a birationally invariant value " $K_X^3$ " for  $X$  a 3-fold of general type.

If the canonical ring  $R(X, K_X)$  is finitely generated (which is totally unknown) then I define

$$"K_X^3" = \frac{1}{2} \cdot \lim_{n \rightarrow \infty} \frac{P_n}{n^3}.$$

This limit exists by the algebra of finitely generated graded rings.

Otherwise the correct " $K_X^3$ " could lie between  $\frac{1}{2} \overline{\lim} \frac{P_n}{n^3}$  and  $\frac{1}{2} \underline{\lim} \frac{P_n}{n^3}$ .

R.7

**Conjecture:** For  $X$  a 3-fold of general type one has  $P_2 \geq 4p_g - 6$  and  $P_3 \geq 10p_g - 20$ .

This is a very easy to check if  $\varphi_{K_X}(X)$  is a 3-fold; and the limit cases correspond to double coverings of normal rational scrolls.

R.8

**Conjecture:** Let  $F \subset \mathbb{P}^n$  be a 3-fold spanning  $\mathbb{P}^n$ , of degree  $d \leq 2n - 4$ . Then  $F$  is birational to  $\mathbb{P}^3$  or to  $\mathbb{P}^2 \times C$ , with  $C$  a curve of genus  $p > 0$ .

R.9

Let  $X \subset \mathbb{A}^4$  be a 3-fold with an isolated singularity at  $0 \in X$ ; let  $f: X' \rightarrow X$  be a resolution, so that duality provides an injection  $f_* \omega_{X'} \subset \omega_X$ . Define the  $n^{\text{th}}$  adjunction ideal

$$I_n = \text{Hom}_{O_X}(\omega_X^{\otimes n}, f_* \omega_{X'}^{\otimes n}) \subset O_X,$$

for  $n \geq 1$ .

**Problem:** Is the sequence  $\{I_n\}_{n \geq 1}$  finitely generated?

This problem seems to be difficult, but is a rather weak form of the general problem of knowing if the canonical ring of 3-folds of general type is finitely generated.

For curves,  $I_1 = G$  is the conductor of  $f_* O_{X'}$  in  $O_X$  and  $I_n = I_1^n$ ; for surfaces one can also prove (using a relatively minimal non-singular model for  $X'$ ) that for  $p, q \geq 3$ :  $I_p \cdot I_q = I_{p+q}$ .

Conjecture. For  $X$  quasi-homogeneous  $\{I_n\}_{n \geq 1}$  is finitely generated.

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