Quadrics through a canonical surface

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No profit grows where is no pleasure ta'en.
The Taming of the Shrew, I.1

Introduction

(0.0) Motivation I want to consider the geography of surfaces X of general type, especially the region $K^2 < 4\chi$ (usually with $p_g \gg 0$); my aim is to extend results known if $K^2 < 3\chi$.

Conjecture *For $g = 2, 3, \dots$ there exist rational numbers a_g and b_g with*

$$a_2 < a_3 < \cdots \quad \text{and} \quad \lim a_g = 4 \ \text{as}\ g \to \infty,$$

such that for every surface X of general type,

$$K^2 \le a_g\chi - b_g \implies X \text{ has a pencil of curves of genus} \le g.$$

As in [Reid3, Xiao], this statement would imply that the algebraic fundamental group $\pi_1 X$ of a surface X with $K^2 < 4\chi$ is either finite or a finite extension of π_1 of a curve; and that the Albanese morphism of X maps to a curve.

(0.1) The classical results in this direction are due to Max Noether, Castelnuovo, Horikawa and Xiao Gang: every surface of general type satisfies $K^2 \ge 2p_g - 4$; if $p_g \ge 4$ and $K^2 < 3p_g - 7$ then the 1-canonical map $\varphi_K \colon X \dashrightarrow \varphi(X) \subset \mathbb{P}^{p_g - 1}$ is 2-to-1 onto a surface ruled by lines or conics. The conjecture for $g = 2, 3$ with $a_2 = 2\frac{2}{3}$ and $a_3 = 3$ is closely related to this, and was proved by Horikawa. If a surface X of general type is hyperelliptic, in the sense that it is birational to a 2-to-1 cover of a ruled surface, then Xiao has proved that

$$K^2 < \frac{4s}{s+1}\chi - \frac{9s^2+8s}{2(s+1)} \implies$$
$$X \text{ has a pencil of hyperelliptic curves of genus } \le s;$$

see for example [Horikawa1–2, Xiao, Reid4].

(0.2) There is thus no loss of generality in assuming that X is a surface for which the 1-canonical map $\varphi_K\colon X \dashrightarrow \varphi(X) \subset \mathbb{P}^{p_g-1}$ is birational. The following conjecture of mine is more than 10 years old [Reid2–3].

Conjecture *Suppose that X is a surface for which the 1-canonical map $\varphi_K\colon X \dashrightarrow Y \subset \mathbb{P}^{p_g-1}$ is birational. Suppose that*

$$K^2 < 4p_g - 12.$$

Then $Y \subset W \subset \mathbb{P}^{p_g-1}$, where W is a variety of dimension ≥ 3 contained in the intersection of all quadrics through Y.

Remark It is easy to see that if W is in the conclusion of Conjecture 0.2 exists then necessarily $\dim W = 3$ and $\deg W \le K^2 - 2p_g + 4$; the relation with Conjecture 0.0 is that the general hyperplane section $S = W \cap \mathbb{P}^{p_g-2}$ of the 3-fold W is a surface for which K_S is not quasieffective, so that the methods of [Xiao, Reid4] are applicable to give a pencil of rational curves of bounded degree on S. In fact it follows that $K_W + H$ is not quasieffective on W itself, so that it may be possible to generalise the method of [Reid4] to prove that suitable inequalities imply that W has a pencil of surfaces of small degree cutting out the required pencil on X. If W is smooth then results of Sommese [Sommese] are already available: W is a $\mathbb{P}^1$-bundle over a smooth surface, a quadric bundle over a curve, or a (possibly blown up) Fano 3-fold of index ≥ 2.

(0.3) Contents of the paper Unfortunately, I am still not able to make any substantial advance towards proving Conjecture 0.2, although the paper does contain several numerological considerations each of which points to the inequality $K^2 < 4p_g - 12$ as relevant (see especially (1.5, iii), (2.4), (3.2), (3.5, (7)) and (3.11)). Section 1 tries to explain the background, and Section 2 contains an ad hoc proof of the first case of the key Conjecture 1.5.

This material is about 100 years old (and is for the most part covered in [Castelnuovo, Fano, Babbage, Harris, Ciliberto]), but I enjoyed rediscovering it and writing it out, and I believe that it may serve as a useful introduction to Conjecture 0.2.

Section 3 discusses a hopeful approach to the problem that uses vector bundles, but that doesn't work yet, and raises several open questions (see (3.12)). I should say that Conjecture 1.5 has occupied me on and off for more than 10 years, and the more I fail to prove it, the more I am convinced of its truth; my main aim in publishing this tentative material is to recruit the interest of people with more competence or time than myself; a secondary aim is to get the numerology of (1.5) and (3.2) written down in one place so that I don't have to do it all over again from scratch every time I come back to the problem.

(0.4) Acknowledgements Much of Sections 1–2 of this paper is a reworking of a letter to Tyurin (circulated in [Reid2]); this is closely related to ideas of Tyurin in [Tyurin, Chap. II, Section 5], and I have benefited from many discussions with him since 1972. I am very grateful to Ciro Ciliberto for pointing out the references [Fano, Ciliberto]. The idea of a set being uniform w.r.t. forms of degree k occuring in (1.2) and (2.6) is due to Harris and Eisenbud, and fills a painful gap in my argument; their detailed workout of my letter [Eisenbud] has been useful throughout Section 2, although I dispute that the proof of Theorem 2.2 in (2.7–9) is 'exceedingly ugly'. The material of Section 3 was worked out during a research visit to Pisa in December 1987, and I would like to thank the Italian CNR for financial support, and Fabrizio Catanese for stimulating discussions, and for wonderful hospitality and punctuality.

1 Counting quadrics

(1.1) Terminology and its abuse Let $X \subset \mathbb{P}^N = \mathbb{P}$ be a subscheme with ideal sheaf I_X; I assume throughout that X spans $\mathbb{P}$. Then $H^0(\mathbb{P}, I_X \cdot \mathcal{O}_{\mathbb{P}}(2))$ is the space of quadrics through X, and its dimension is the *number of quadrics through* X; I usually write

$$h^0(\mathbb{P}, I_X \cdot \mathcal{O}_{\mathbb{P}}(2)) = \binom{N+2}{2} - f_X,$$

and say that X *imposes* $f = f_X$ *conditions on quadrics.* Equivalently $H^0(I_X \cdot \mathcal{O}_{\mathbb{P}}(2))$ is the kernel of the restriction map $\rho\colon H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}}(2)) \to H^0(X, \mathcal{O}_X(2))$, and $f = \operatorname{rank} \rho$.

I will say that an irreducible variety $X \subset \mathbb{P}^N$ is *generically an intersection of quadrics* to mean that X is one component of the intersection of all quadrics through X; the opposite possibility is that the intersection of all quadrics through X contains a component of dimension at least $\dim X + 1$ containing X.

(1.2) Reduction to the zero dimensional case Question: how to give lower bounds for f_X, or equivalently, how to find an upper bound for the number of quadrics through X? The difficulty here is that ρ is bilinear in $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}}(1))$, whereas the obvious methods are usually linear.

First, passing to any hyperplane section $X' \subset \mathbb{P}^{N-1} = \mathbb{P}'$, the number of quadrics can only increase:

$$H^0(I_X \cdot \mathcal{O}_{\mathbb{P}}(2)) \hookrightarrow H^0(I_{X'} \cdot \mathcal{O}_{\mathbb{P}'}(2));$$

because the kernel $H^0(I_X \cdot \mathcal{O}_{\mathbb{P}}(1))$ consists of hyperplanes through X. Now assume that X is irreducible, and the hyperplane sections are chosen generically; then the process can be continued down to a reduced finite set of points $\Sigma \subset \mathbb{P}^n$ spanning $\mathbb{P}^n$, where $n = \operatorname{codim}(X \subset \mathbb{P})$. A traditional argument shows that the finite set Σ consists of points that are *linearly general*, or *in general position with respect to hyperplanes* (see for example [4 authors, pp. 107–113] in characteristic 0; or the argument of [Andreotti, Section 6, p. 815] gives the same result for $\operatorname{char} p$ large compared to the degree). Moreover, Σ is *uniform* (or *in uniform position*) with respect to hypersurfaces of any degree $k \geq 1$: that is, all subsets of Σ with the same number of elements impose the same number of conditions on forms of degree k; this follows from the Lefschetz–Harris principle that if Σ is a generic 0-dimensional section of the irreducible variety X, it is irreducible and its Galois group over its field of definition is the full symmetric group on Σ.

(1.3) Proposition *Let $\Sigma \subset \mathbb{P}^n$ be a finite set of d points spanning $\mathbb{P}^n$; suppose that Σ is linearly in general position.*

(i) If $d \leq 2n+1$ then Σ imposes d conditions on quadrics.

(ii) If $d \geq 2n+3$ and Σ imposes $\leq 2n+1$ conditions on quadrics then Σ is contained in a rational normal curve $C \subset \mathbb{P}^n$.

Suppose in addition that Σ is uniform with respect to quadrics.

(iii) If $d \geq 2n+5$ and Σ imposes $\leq 2n+2$ conditions on quadrics then Σ is contained in a curve $C \subset \mathbb{P}^n$ of degree $\leq n+1$ (necessarily with $p_a C \leq 1$).

Proof of (i) This is rather trivial: I add $2n+1-d$ general points of $\mathbb{P}^n$ to Σ to give a set Σ^+ of $2n+1$ points not contained in any pair of hyperplanes. Then by construction, the (projectivised) space of quadrics through Σ^+ is disjoint from the $2n$-dimensional space of quadrics of rank 2; this implies that $H^0(I_{\Sigma^+} \cdot \mathcal{O}(2))$ has codimension $\geq 2n+1$ in all quadrics, so that Σ^+ imposes independent conditions.

(ii) is a 100-year old result of Castelnuovo (see [Castelnuovo, Babbage, Section 6]) and (iii) (due to Fano [Fano, Ciliberto]) is obtained by working harder with the same ideas. I discuss the proofs later in Section 2.

(1.4) Harmless though it may appear, Proposition 1.3, (i) together with the argument of (1.2) has a large number of consequences. It is essentially equivalent to the free pencil trick of Castelnuovo and Mumford (see [Mumford, Section 2, Segre] and (1.7)).

Corollaries

(i) Let $W \subset \mathbb{P}^N$ be an irreducible variety spanning $\mathbb{P}^N$ of dimension w, and set $n = N - w$; then W imposes

$$\geq \binom{N+2}{2} - \binom{n+2}{2} + \min(\deg W, 2n+1)$$

conditions on quadrics.

(ii) ("Clifford plus") Let C be a smooth curve and $D = g^r_d$ a divisor (of degree d with $h^0(D) = r+1$), such that the rational map φ_D is a birational embedding; then

$$h^0(2D) \geq \min(r + \deg \varphi_D(C), 3r);$$

if D is a special divisor then $\deg \varphi_D(C) \geq 2r$ by Clifford's theorem[1] *so that $h^0(2D) \geq 3r$; if moreover $2D$ is special then $d \geq h^0(2D) - 1 \geq 3r - 1$.*

(iii) Let X be a surface of general type for which the 1-canonical map φ_K is a birational embedding; then $K^2 \geq 3p_g + q - 7$.

[1] This should be rewritten to an include a proof of Clifford's theorem from first principles.

Proof (i) follows directly from (1.3, i) and (1.2).

(ii) Throwing out the fixed part of the linear system, I assume that $|D|$ is free, so $\varphi_D(C) = \Gamma \subset \mathbb{P}^r$ is an irreducible curve of degree d. Now $H^0(C, 2D) \supset \operatorname{im} S^2H^0(C, D)$, which can be identified with the image of $\rho_\Gamma \colon H^0(\mathbb{P}^r, \mathcal{O}(2)) \to H^0(\Gamma, \mathcal{O}_\Gamma(2))$; by (1.3, i), $\operatorname{rank} \rho_\Gamma = f_\Gamma$ is at least

$$r + 1 + \min(d, 2r - 1);$$

Clifford's theorem for D gives $d \geq 2r$, which proves that $h^0(2D) \geq 3r$. Now $2D$ is again effective and special, so Clifford's theorem for $2D$ gives

$$2d = \deg 2D \geq 2(h^0(2D) - 1),$$

as required.

(iii) Suppose that $\varphi_K \colon X \dashrightarrow \varphi(X) \subset \mathbb{P}^{p_g-1}$ is birational. Then quadrics of $\mathbb{P}^{p_g-1}$ cut out on $\varphi(X)$ a space that can be identified with a subspace of $H^0(X, \mathcal{O}(2K))$, so that using (1.2), it follows from (ii) that

$$1 - q + p_g + K^2 = P_2 \geq p_g + 3(p_g - 2). \qquad \text{Q.E.D.}$$

(1.5) The main problem Proposition 1.3 raises the following hope:

Conjecture *Let $\Sigma \subset \mathbb{P}^n$ be a finite set spanning $\mathbb{P}^n$, linearly general and uniform with respect to quadrics; suppose that Σ consists of d points and imposes $f = f_\Sigma$ conditions on quadrics. Then for $p = 0, \dots, n-2$,*

$$d \geq 2n + 2p + 1 \quad \textit{and} \quad f \leq 2n + p \implies$$

the intersection of all quadrics through Σ contains a curve C containing Σ.

Remarks

(i) I don't know that C is irreducible, although that is certainly the case if Σ is a general hyperplane section of a curve. The statement is equivalent to saying that Σ is contained in a curve C of degree $\leq n+p$. Also, (1.4, i) guarantees that no variety of dimension ≥ 2 spanning $\mathbb{P}^n$ can be contained in so many quadrics.

(ii) The bound $d \geq 2n + 2p + 1$ is best possible: let $C \subset \mathbb{P}^n$ be a curve of genus p embedded by a complete linear system of degree $n + p$. In view of $p \leq n - 2$, it follows that $\mathcal{O}_C(1)$ has degree $\geq 2p + 2$ and is necessarily nonspecial, and C is an intersection of quadrics imposing exactly $2n + p + 1$ conditions on quadrics. Therefore, if $\Sigma = C \cap Q$ is the intersection of C with a quadric then Σ consists of $2n + 2p$ points, and imposes $2n + p$ conditions on quadrics.

(iii) The bound $p \leq n - 2$ is also best possible, since the intersection $\Sigma = C \cap Q$ of a quadric with a canonical curve $C \subset \mathbb{P}^n$ of genus $n + 1$ has degree $4n$ and imposes $3n - 1$ conditions on quadrics. This example suggests that $d \geq 4n$ and $f \leq 3n - 1$ imply that *either* Σ is not generically an intersection of quadrics, *or* equalities throughout and Σ is ideal theoretically an intersection of quadrics.

(iv) The conjecture is an easy exercise if Σ is contained in a linearly normal curve $C \subset \mathbb{P}^n$ of degree $\leq 2n - 1$; in Section 2 I will prove that it also holds if Σ is contained in a surface $F \subset \mathbb{P}^n$ of degree $n - 1$, a result presumably known to Fano and Castelnuovo.

(v) It might be interesting to formulate an analogous conjecture in the style of Mark Green for the higher syzygies:

Σ of small degree and an intersection of quadrics $\implies$
higher syzygies aren't generated in lowest degree?

(1.6) Conditional results *Conjecture 1.5 implies the following.*

(i) Let C be a curve and $D = g^r_d$ a divisor with $2D$ special such that the rational map $\varphi_D \colon C \to \Gamma \subset \mathbb{P}^r$ is a birational embedding; suppose that the image $\varphi_D(C) = \Gamma$ is generically an intersection of quadrics in the sense of (1.1); then

$$d \geq h^0(2D) - 1 \quad \textit{and} \quad h^0(2D) \geq 4r - 5.$$

(ii) Let X be a surface of general type for which the 1-canonical image $\varphi_K(X) \subset \mathbb{P}^{p_g - 1}$ is generically an intersection of quadrics; then X satisfies

$$K^2 \geq 4p_g + q - 12.$$

Proof (i) This is exactly the same as (1.4, ii). Without loss of generality I can assume $|D|$ is free, and set

$$d = \deg D \quad \text{and} \quad p = \left[\frac{d-1}{2}\right] - n,$$

where $n = r - 1$ as in (1.2). Then since $\deg \varphi_D(C) \geq 2n + 2p + 1$, Conjecture 1.5 would imply that

$$h^0(\mathcal{O}_C(2D)) \geq 3n + 1 + \min(p, n-2).$$

However, Clifford's theorem for the special divisor $2D$ gives

$$d \geq h^0(\mathcal{O}_C(2D)) - 1,$$

and it follows from these inequalities that $p \geq n - 2$, and thus $h^0(2D) \geq 4n - 1 = 4r - 5$.

For (ii), note that (1.2) and (i) give[2]

$$1 - q + p_g + K^2 = P_2 \geq p_g + 4(p_g - 2) - 5. \quad \text{Q.E.D.}$$

Remark The cases $p = 1$ and $p = 2$ of (1.5) are contained in (1.3), proved in Section 2, so that it follows for example that for a surface X with $K^2 = 3p_g - 6$ and birational φ_K, the image $\varphi_K(X) \subset \mathbb{P}^{p_g - 1}$ is contained in a 3-fold W of degree $p_g - 3$ or $p_g - 2$. When $p_g \geq 12$ the only possibilities for W are a rational normal 3-fold scroll or a double cone over an elliptic curve, or linear projections of these.

(1.7) Relation with the free pencil trick Let $C \subset \mathbb{P}^{n+1}$ be a curve. The free pencil trick is a classical method of giving a lower bound on the rank of the restriction map $\rho_C \colon H^0(\mathcal{O}_{\mathbb{P}}(2)) \to H^0(\mathcal{O}_C(2))$: fix $n+2$ general points $P_0, \dots, P_{n+1} \in C$, and choose the coordinates $x_0, \dots, x_{n+1}$ so that $x_i(P_j) = \delta_{ij}$. Then x_0, x_1 span the pencil of hyperplanes through $\Pi = \langle P_2, \dots, P_{n+1} \rangle = \mathbb{P}^{n-1}$, which cuts out residually on C the linear system $|H - A|$, where H is the hyperplane section and $A = P_2 + \dots + P_{n+1}$; then the exact sequence of sheaves

$$0 \to \mathcal{O}_C(A) \xrightarrow{x_0, x_1} \mathcal{O}_C(1) \oplus \mathcal{O}_C(1) \xrightarrow[x_0]{-x_1} \mathcal{O}_C(2H - A) \to 0$$

[2] There is an error of some sort here – it doesn't add up.

shows that the $2n+4$ quadratic monomials

$$\Big\{x_0x_i, x_1x_i\Big\}_{i=0}^{n+1} \in H^0(\mathcal{O}_C(2))$$

are linearly independent except for relations of the form $s(x_0x_1 - x_1x_0)$ with $s \in H^0(\mathcal{O}_C(A))$. For example, if $g(C) \geq n$ then $h^0(\mathcal{O}_C(A)) = 1$, and this implies that the $3n+3$ monomials

$$\Big\{x_0^2, x_0x_1, x_1^2, x_0x_i, x_1x_i, x_i^2\Big\}_{i=2,\dots,n+1}$$

are linearly independent, so $h^0(\mathcal{O}_C(2H)) \geq 3(n+1)$.

This is the same result as (1.4, ii), and the proof is just a mutation of the proof in (1.2) and (1.3, i): it boils down to saying that after taking a general hyperplane section Σ: $(x_0 = 0) \subset C$, the only quadrics of rank 2 containing Σ with vertex in the linear subspace $(x_1 = 0)$ are of the form $x_1 \cdot \lambda$ where λ is a linear form vanishing at $\Sigma - A$.

(1.8) It's important to understand the weakness of this argument: the bilinear problem of estimating the rank of ρ_C is reduced to a linear one, but at the expense of considering only the $(2n+3)$-dimensional subspace of forms involving x_0 and x_1.

2 Rational normal scrolls and the proof of (1.3)

The remaining assertions (ii) and (iii) of (1.3) will follow from the two following theorems.

(2.1) Theorem *Let $\Sigma \subset \mathbb{P}^n$ be a set of $d \geq 2n+2p+1$ points, linearly general and uniform with respect to quadrics. Suppose that $p \leq n-2$ and that Σ imposes $f \leq 2n+p$ conditions on quadrics. Then Σ is contained in a p-dimensional rational normal scroll F (possibly singular; if $p=1$, F is a rational normal curve).*

(2.2) Theorem *Suppose that $\Sigma \subset F \subset \mathbb{P}^n$ is contained in a rational normal surface scroll F of degree $n-1$. Then Conjecture 1.5 holds for Σ.*

(2.3) Plan of proof of (2.1) Write

$$\Pi = \langle P_1, \dots, P_{n-1} \rangle = \mathbb{P}^{n-2}$$

for the codimension 2 space spanned by $n-1$ elements of Σ. A dimension-count shows that $\Pi \cup \Sigma$ is contained in at least $n-p$ linearly independent quadrics, and the intersection of these will consist of Π together with the required scroll F. This is a classic construction for rational normal scrolls, and the only possible way it can degenerate would contradict the fact that Σ is linearly general. The tricky part is to show that the initial points $P_1, \dots, P_{n-1}$ are also contained in the residual intersection F, and this is where the uniform assumption on Σ is used.

(2.4) Dimension count Since quadrics of $\Pi = \mathbb{P}^{n-2}$ vanishing at the $n-1$ given points $\{P_1, \dots, P_{n-1}\}$ form a vector space of dimension $\binom{n-1}{2}$, it follows that Π imposes at most this number of conditions on quadrics of $\mathbb{P}^n$ through Σ. Therefore the vector space of quadrics through Π and Σ has dimension

$$\geq \binom{n+2}{2} - (2n+p) - \binom{n-1}{2} = n-p,$$

as required.

(2.5) Classic construction for scrolls Consider the blowup $\sigma\colon F_0 \to \mathbb{P}^n$ of $\mathbb{P}^n$ in Π; with its projection $\pi\colon F_0 \to \mathbb{P}^1$, this is the n-dimensional scroll $F_0 = \mathbb{P}(\mathcal{E}_0)$, where $\mathcal{E}_0$ is the rank n vector bundle over $\mathbb{P}^1$

$$\mathcal{E}_0 = \mathcal{O}(1) \oplus (n-1)\mathcal{O}.$$

F_0 contains a negative divisor $B_0 = \sigma^{-1}\Pi \simeq \mathbb{P}^1 \times \mathbb{P}^{n-2}$, and I write A_0 for the fibre of π. The morphism $\sigma\colon F_0 \to \mathbb{P}^n$ is given by the complete linear system $|A_0 + B_0|$. Let $Q_1, \dots, Q_{n-p}$ be the linearly independent quadrics of $\mathbb{P}^n$ through Π provided by (2.4). Then since $Q_i \supset \Pi$, it follows that each $\sigma^* Q_i = B_0 + Q'_i$, where $Q'_i \sim 2A_0 + B_0$. Now for $k = 1, \dots, n-p$, set

$$F_k = \bigcap_{j=1}^{k} Q'_j \subset F_0.$$

By induction on k, suppose F_k is irreducible and is a $\mathbb{P}^{n-k-1}$-bundle over $\mathbb{P}^1$, having degree $k+1$ under the morphism σ to $\mathbb{P}^n$ defined by the divisor $A_k + B_k$ (where I write A_k and B_k for the restrictions of A_0 and

B_0 to F_k). Then $F_{k+1} \subset F_k$ is in the divisor class $2A_k + B_k$, and so has degree $k+2$ under σ, and has a unique component that is a $\mathbb{P}^{n-k-1}$-bundle over $\mathbb{P}^1$. Thus if reducible, F_{k+1} could be written as a sum of A_k with a divisor in $|A_k + B_k|$; then $\sigma(F_k+1) \subset \sigma(F_k) \subset \mathbb{P}^n$ would be contained in the union of two hyperplane sections, of which the one containing $\sigma(A_k)$ can be chosen to pass through Π. Since Σ is certainly contained in $\sigma(F_{k+1}) \cup \Pi$, this contradicts the assumption that Σ is linearly general.

Therefore $F = \sigma(F_{n-p}) \subset \mathbb{P}^n$ is a p-dimensional scroll as required.

Remark If irreducible, $B_k \subset F_k$ is itself a scroll mapping birationally to $\sigma(B_k) \subset \Pi$ if $k \geq 1$. It can certainly happen that B_k is reducible, but in any case $\sigma(B_k) \subset \Pi$ has codimension $k-1$ and degree k.

(2.6) Key technical point: $\Sigma \subset F$ The points of Σ other than the $\{P_1, \ldots, P_{n-1}\}$ belong to each quadric Q_i and not to Π, and hence are in the residual component F of $\bigcap Q_i = \Pi \cup F$.

I now prove that the points P_i for $i = 1, \ldots, n-1$ also belong to F, following an argument kindly supplied by Eisenbud and Harris. For this, let $P_n, P_{n+1} \in \Sigma$ be two more points, and choose the coordinates x_n, x_{n+1} so that x_n vanishes on $\langle \Pi, P_{n+1} \rangle$ and x_{n+1} vanishes on $\langle \Pi, P_n \rangle$; then of course $\Pi : (x_n = x_{n+1} = 0)$.

Lemma *After possibly reordering $\{P_1, \ldots, P_{n-1}\}$, all the equations of F can be written in the determinantal form*

$$\operatorname{rank} \begin{pmatrix} x_n & \lambda_1 & \lambda_2 & \ldots & \lambda_{n-p} \\ x_{n+1} & \mu_1 & \mu_2 & \ldots & \mu_{n-p} \end{pmatrix} \leq 1,$$

where the λ_i are linear forms such that $\lambda_i(P_j) = \delta_{ij}$ for $i, j = 1, \ldots, n-p$. Here the $n-p$ quadrics through $\Sigma \cup \Pi$ of (2.4) are given by

$$Q_i = \det \begin{vmatrix} x_n & \lambda_i \\ x_{n+1} & \mu_i \end{vmatrix}. \tag{*}$$

First of all, the lemma implies $\Sigma \subset F$, and thus proves Theorem 2.1. In fact, the remaining quadrics of the determinantal are

$$Q_{ij} = \det \begin{vmatrix} \lambda_i & \lambda_j \\ \mu_i & \mu_j \end{vmatrix} \quad \text{for } i, j = 1, \ldots, n-p \text{ with } i \neq j. \tag{**}$$

Now Q_{ij} vanishes on F, hence on all the points of Σ except for the P_i, and the ever-vigilant reader will be able to see from the form of Q_{ij} that it also vanishes at P_k for $k = 1, \dots, n-p$, $k \neq i, j$. Thus in total it vanishes at

$$\geq 2n + 2p + 1 - (n-1) + (n-p-2) = 2n + p$$

points of Σ. Hence by the uniform assumption on Σ, Q_{ij} vanishes at all points of Σ.

Proof of the lemma Every quadric through Π is of the form $x_n\mu - x_{n+1}\lambda$, so the equations of the $n-p$ quadrics Q_i can certainly be put in the form $(**)$ for some linear forms λ_i and μ_i. It follows that F is given by equations $(*)$. Now I claim that the λ_i restrict to $n-p$ linearly independent forms on Π. For otherwise some nonzero linear combination of the Q_i would be of the form

$$Q = \det \begin{vmatrix} x_n & \lambda \\ x_{n+1} & \mu \end{vmatrix}.$$

with λ a linear combination of x_n and x_{n+1}; but $Q(P_n) = 0$ implies that λ would have to be a multiple of x_n, since $x_n(P_n) = 0$ and $x_{n+1}(P_n) \neq 0$. This is absurd, since Σ is not contained in a quadric of rank 2.

Suitable linear combinations of the λ_i give the statement of the lemma. Q.E.D.

(2.7) Plan of proof of (2.2) The proof of (2.2) considers the linear system L with assigned base points cut out on F by quadrics of $\mathbb{P}^n$ through Σ (the case of F a cone causes no problem, just resolve the vertex). Firstly, since $h^0(F, \mathcal{O}(2)) = 3n$ and Σ imposes $\leq 2n + p$ conditions on quadrics, it follows that

$$h^0(F, I_\Sigma \cdot \mathcal{O}(2)) \geq n - p, \quad \text{that is,} \quad \dim L \geq n - p - 1;$$

write $L = M + D$ with M mobile and D the fixed part. Clearly since F is a scroll, $\mathcal{O}(2)$ has degree 2 relative to the projection $\pi \colon F \to \mathbb{P}^1$, and there are 3 possibilities for the decomposition $L = M + D$:

Case 1 M has degree 2 over $\mathbb{P}^1$, and D is a union of $\beta \geq 0$ fibres of π.

Case 2 M and D both have degree 1 over $\mathbb{P}^1$.

Case 3 M is a union of $\geq n-p-1$ fibres of π and the base locus D has degree 2 over $\mathbb{P}^1$.

In Case 3, there is nothing to prove: the base locus $D \supset \Sigma$ and is the intersection of all quadrics through Σ; clearly $\deg D \leq n+p$. It is therefore enough to prove that Case 1 leads to a contradiction; and that in Case 2, $\Sigma \subset D$ and $\deg D \leq n+p$.

(2.8) Idealised proof of (2.2) I first illustrate the proof by deriving a contradiction from the assumptions that L has base locus exactly the reduced set Σ, and that its general element Γ is irreducible and nonsingular (a priori the main case?). Write L_Γ for the free linear system cut out on Γ by L after subtracting the base locus Σ.

Then by (2.7), $\dim L_\Gamma \geq n-p-2$. On the other hand, $(2H)^2 = 4\deg F = 4(n-1)$ (where $H = \mathcal{O}_F(1)$), so that

$$\deg L_\Gamma = 4(n-1) - \deg \Sigma \leq 2n-2p-5.$$

But it's easy to see from properties of scrolls that

$$H^0(\mathbb{P}, \mathcal{O}(2)) \to H^0(\Gamma, \mathcal{O}_\Gamma(2H))$$

is surjective; therefore Σ does not impose independent conditions on the vector space $H^0(\Gamma, \mathcal{O}_\Gamma(2H))$, and so $H^1(\Gamma, \mathcal{O}_\Gamma(2H-\Sigma)) \neq 0$ and the linear system L_Γ is special. Hence the two most recently acquired inequalities contradict Clifford's theorem.

(2.9) Case 1 is impossible Since the fibre A of $\pi\colon F \to \mathbb{P}^1$ satisfies $LA = 2$, $A^2 = 0$, I get $M^2 = 4(n-1) - 4\beta$. Suppose that the points of Σ distribute themselves as

$$2n+2p+1 \leq \deg \Sigma = d = a+b,$$

with a points on the fixed part D and b base points of M outside D. Clearly $a \leq 2\beta$.

Step 1 The general element of M is irreducible.

For if M is composed of a pencil, $M = kE$ with $k \geq 2$ and E has b base points, so

$$M^2 = kE^2 \geq 4E^2 \geq 4b \geq 4(2n+2p+1-2\beta),$$

giving $\beta \geq n+2p+2$, which is absurd.

Step 2 Write Γ for the normalisation of the general element of M; consider the free linear system M_Γ cut out on Γ by M (after subtracting off the base locus). Then

$$\begin{aligned}\deg M_\Gamma \le M^2 - b &\le 4(n-1) - 4\beta - (2n + 2p + 1 - 2\beta)\\ &\le 2n - 2p - 2\beta - 5;\end{aligned}$$

and as before, $\dim M_\Gamma \ge n - p - 2$, which contradicts Clifford's theorem if M_Γ is special.

Step 3 Therefore M_Γ is nonspecial, so I get a bound on the genus of g from RR, which will lead to a contradiction.

I intend to use the classical language, writing $\sum m_i P_i$ for the actual base locus of M, including infinitely near points; the reader who is unduly distressed by this can perform the easy exercise of translating the following argument in terms of successive blowups of the base locus. First,

$$\deg M_\Gamma = M^2 - \sum m_i^2 = 4n - 4 - 4\beta - \sum m_i^2$$

and

$$g(\Gamma) = n - 2 - \beta - \sum \binom{m_i}{2}$$

Here $n-2-\beta$ is the genus of the general element of $|2H-\beta A|$ on F (exercise using the adjunction formula).

Also, since M_Γ is a nonspecial linear system, RR gives

$$\begin{aligned}n - p - 1 \le h^0(\Gamma, M_\Gamma) &= 1 - g(\Gamma) + \deg M_\Gamma\\ &\le 3n - 3 - 3\beta - \sum \binom{m_i + 1}{2}.\end{aligned}$$

However, M has at least $b \ge 2n + 2p + 1 - 2\beta$ assigned base points, and hence

$$2n + 2p + 1 - 2\beta \le b \le \sum \binom{m_i + 1}{2} \le 2n + p - 2 - 3\beta,$$

which is absurd.

(2.10) Proof in Case 2 Write $M \sim H - \beta A$ and $D \sim H + \beta A$ where $H = \mathcal{O}_F(1)$, A is the fibre of π, and $\beta \in \mathbb{Z}$. Let $\Gamma \in M$ be a general element; notice that Γ projects isomorphically to $\mathbb{P}^1$ under π, so that I am spared the cases of M being composed of a pencil or the linear system M_Γ special.

As before, suppose that the points of Σ distribute themselves as

$$2n + 2p + 1 \leq \deg \Sigma = d = a + b,$$

with a points on the fixed part D and b base points of M outside D. Let M_Γ be the free linear system cut out by M on Γ (after subtracting the base points). Now

$$\deg M_\Gamma \leq M^2 - b = n - 1 - 2\beta - b,$$

and

$$h^0(M_\Gamma) \geq h^0(F, I_\Sigma \cdot \mathcal{O}(2)) - 1 \geq n - p - 1.$$

Therefore

$$n - p - 1 \leq n - 2\beta - b, \quad \text{that is,} \quad b + 2\beta \leq p + 1,$$

and hence

$$a = d - b \geq 2n + p + 2\beta.$$

However, since $M \sim H - \beta A$ moves in an irreducible linear system on F it follows that $h^0(F, \mathcal{O}_F(M)) = \chi(\mathcal{O}_F(M)) = n + 1 - 2\beta$, and it is obvious that $D \sim 2H - M$ imposes

$$h^0(F, \mathcal{O}(2)) - h^0(F, \mathcal{O}_F(M)) = 3n - n - 1 + 2\beta = 2n - 1 + 2\beta$$

conditions on quadrics. Therefore the a points of $\Sigma \cap D$ impose dependent conditions on quadrics, and by the uniform assumption, this implies that $\Sigma \subset D$. The remaining assertion, that $\deg D \leq n + p$ follows from the fact that $2\beta \leq p + 1$. Q.E.D.

3 A vector bundle approach

(3.1) This section starts off with some numerology. The following material on quadrics of small rank or containing large linear subspaces is well known and will be used throughout.

Proposition *The (projective) space of quadrics of rank $\leq k$ in $\mathbb{P}^n$ is a symmetric determinantal $S_{k,n}$ in the space S_n of quadrics of $\mathbb{P}^n$, and has dimension*

$$k(n - k + 1) + \binom{k+1}{2} - 1;$$

it has a resolution that is a projective bundle over $\mathrm{Gr}(n - k + 1, n + 1)$*, the Grassmanian of $(n - k)$-planes in $\mathbb{P}^n$, with fibre $S^2 E$, where E is the* *tautological rank k quotient bundle on* Gr *(the point of* Gr *is the vertex $\Lambda \subset \mathbb{P}^n$, and the fibre consists of quadrics on a complementary $\mathbb{P}^{k-1}$, the base of the cone).*

If $\Lambda \subset \mathbb{P}^n$ is a fixed $(n-k)$-plane and Q any quadric with $\operatorname{Sing} Q = \Lambda$ *then Q is a nonsingular point of $S_{k,n}$, and the (projective) space of quadrics of $\mathbb{P}^n$ containing Λ is the tangent space $T_{S_{k,n},Q} \subset S_n$ to $S_{k,n}$ at Q; in particular, it has the above dimension.* $\square$

(3.2) Numerology Conjecture 1.5 deals with a set $\Sigma \subset \mathbb{P}^n$ of d points imposing f conditions on quadrics, such that d is 'sufficiently large', and $f = 2n + p$ is in the range

$$2n + 1 \le f \le 3n - 2.$$

This range of values has the following peculiarities:

(a) If $\Pi = \langle P_1, \dots, P_{n-1}\rangle = \mathbb{P}^{n-2}$ is an $(n-2)$-plane spanned by points of Σ, then $h^0(I_\Pi \cup \Sigma \cdot \mathcal{O}(2)) \ge 3n - f \ge 2$; that is, the Castelnuovo argument of Theorem 2.1 can at least start working, and Σ is contained in a scroll of dimension $f - 2n \le n - 2$.

(b) If $\Lambda = \mathbb{P}^{n-3}$ is a general $(n-3)$-plane of $\mathbb{P}^n$ then again

$$h^0(I_{\Pi\cup\Sigma} \cdot \mathcal{O}(2)) \ge 3n - f \ge 2;$$

(c) Quadrics of rank ≤ 3 through Σ form a projective space of dimension ≥ 1.

A curve $C \subset \mathbb{P}^{n+1}$ imposing $f_C \le 4n$ conditions on quadrics is contained in:

(a) a positive dimensional linear system of quadrics through an $(n-2)$-plane $\Pi = \langle P_1, \dots, P_{n-1}\rangle = \mathbb{P}^{n-2}$ spanned by $n-1$ (general) points of C;

(b) a positive dimensional linear system of quadrics through a general $(n-3)$-plane $\Lambda = \mathbb{P}^{n-3} \subset \mathbb{P}^{n+1}$;

(c) a positive dimensional family of quadrics of rank 4.

Likewise, let X be a surface of general type with $K^2 < 4p_g - 12$ and for which the 1-canonical map $\varphi_K \colon X \to \varphi_(X) \subset \mathbb{P}^{p_g-1}$ is birational; set $n = p_g - 3$. Then the canonical image $\varphi(X)$ is contained in:

(a) a positive dimensional linear system of quadrics through an $(n-2)$-plane $\Pi = \langle P_1, \dots, P_{n-1} \rangle = \mathbb{P}^{n-2} \subset \mathbb{P}^{p_g-1}$ spanned by $n-1$ points of X;

(b) a positive dimensional linear system of quadrics through any $(n-3)$-plane $\Lambda = \mathbb{P}^{n-3} \subset \mathbb{P}^{p_g-1}$;

(c) a positive dimensional family of quadrics of rank 5.

(3.3) Since the proof of Proposition 1.3, (i) just used the fact that the set Σ cannot be contained in too many quadrics of rank 2, it seems reasonable to ask if quadrics of small rank couldn't be persuaded to provide some entertainment.

(3.4) Suppose that $X \subset \mathbb{P}^{p_g-1}$ is a nonsingular surface with $K_X = \mathcal{O}_X(1)$ and $K^2 \le 4p_g - 12$; write $n = p_g - 3$, so that $K^2 = \deg X \le 4n$. Then X is contained in a quadric of rank 5,

$$X \subset Q_5 \subset \mathbb{P}^{p_g-1}.$$

I assume in addition that X does not meet the vertex of Q_5.

Remark These assumptions look harmless, since I could presumably blow up and subtract off the base locus, etc. In fact, I would be happy to make progress under even stronger assumptions, such as X projectively normal, and its homogeneous ideal generated by quadrics; unfortunately, these don't seem to be too much help. I am ignoring the case that X is contained in quadrics of rank $k \le 4$ not forced by dimension count. If this happens then I get at once a decomposition of K_X.

(3.5) Motivation for the vector bundle construction of (3.7–11) Let $C = X \cap \mathbb{P}^{n+1}$ be a general hyperplane section of X through $\operatorname{Sing} Q_5$, so that C is a nonsingular curve with $K_C = \mathcal{O}_C(2)$, and is contained in $Q_4 = Q_5 \cap \mathbb{P}^{n+1}$, a quadric of rank 4. The two rulings of Q_4 by generators $\mathbb{P}^{n-1}$ cut out pencils E and E' on C with

$$\mathcal{O}_C(1) = E + E'. \tag{1}$$

The pencils E are free, but not necessarily complete linear systems. I abuse the notation by writing E for a divisor in the pencil $|E|$; a general E is made up of distinct points, and I write $\langle E \rangle$ for their span. Note that $E + E'$ is a hyperplane section of C, so that for any choice of E, E' in their pencils,

$$\left\langle E, E' \right\rangle = \mathbb{P}^n \subset \mathbb{P}^{n+1}. \tag{2}$$

In view of $\mathcal{O}_C(1) = E + E'$, obviously

$$\dim |E| \geq n - \dim \left\langle E' \right\rangle. \tag{3}$$

Now, the two families of generators of Q_4 are interchanged by monodromy as the hyperplane $\mathbb{P}^{n+1}$ moves, so that

$$\deg E = \deg E' = \frac{1}{2} \deg X = \frac{1}{2} K^2, \tag{4}$$

$$\dim \langle E \rangle = \dim \left\langle E' \right\rangle \quad \text{and} \quad h^0(E) = h^0(E'); \tag{5}$$

similarly, points of a general E impose independent conditions on forms of degree k if and only if the same holds for E', and so on.

(3.6) Wake up! The rest of Section 3 aims to prove that E is very special: it moves in a big linear system on C, it spans $\langle E \rangle = \mathbb{P}^\nu$ with ν small, and it imposes few conditions on quadrics. The main result of Section 3 is (3.11); though tentative, this is very striking: roughly speaking, the point set $E \subset \mathbb{P}^\nu$ itself satisfies the assumptions of Conjecture 1.5. Since E has half the degree of Σ and most of its good properties, this looks like an opening for an attack by induction.

Let me now try to explain what's so special about E. Duality in RR says that E imposes $\deg E - \dim |E|$ conditions on $H^0(\mathcal{O}_C(K_C)) = H^0(\mathcal{O}_C(2))$; so a fortiori, E imposes

$$f = f_E \leq \deg E - \dim |E| \tag{6}$$

conditions on quadrics of $\mathbb{P}^{n+1}$. Since

$$d = \deg E = \frac{1}{2} K^2 \leq 2n, \tag{7}$$

there is tension with (1.3, i): E is cut out on C by a pencil of $\mathbb{P}^{n-1} \subset Q_4$, and $E \subset \mathbb{P}^{n-1}$ is a set of $d \leq 2n$ points imposing dependent conditions on quadrics. If $K^2 < 4p_g - 12$ then $d \leq 2n - 1$, and this contradicts Proposition 1.3, (i) if the points of E are linearly general.

On the other hand, E should satisfy some weak uniform position property, coming from the fact that C is irreducible, which should then imply that $\langle E\rangle = \mathbb{P}^\nu$ with $\nu < n-1$. By (3) and (5), this should imply that $|E|$ is more than a pencil, so that in turn, by (6), its points impose even fewer conditions on quadrics,...

Making this argument work in terms of $C \subset \mathbb{P}^{n+1}$ seems to be hard; for example, the first thing to prove is the following:

Conjecture *If $E \subset \mathbb{P}^\nu$ is a set of points defined and irreducible over a field K, and imposing dependent conditions on quadrics, then*

$$\deg E \geq 2\nu + 2. \tag{8}$$

This is a version of (1.3) in which the condition of linear generality is replaced by the weaker uniform condition: if E has a subset $\{P_1, \dots, P_k\}$ of k points spanning a μ-plane $\mathbb{P}^\mu$, then every point $P \in E$ is contained in such a subset. This and other uniform properties of E come from the fact that C is a fairly general hyperplane section of X, and E general in a pencil on C; so it is appropriate to work directly on X.

(3.7) Construction and properties of $\mathcal{E}$ I use the notation of the start of (3.5). First of all, how does one get pleasure or profit out of a quadric of rank 5? I view the nonsingular quadric of rank 5 $Q = Q_5 \subset \mathbb{P}^4$ as a hyperplane section of the Klein quadric $\mathrm{Gr} = \mathrm{Gr}(2,4) = Q_6 \subset \mathbb{P}^5$. Let $\mathcal{E}_0$ be one of the two tautological quotient bundles on Gr, and

$$V = H^0(\mathrm{Gr}, \mathcal{E}_0), \quad \dim V = 4 \tag{9}$$

so that $\mathrm{Gr} = \mathrm{Gr}(2, V)$; then

$$\mathcal{O}_{\mathrm{Gr}}(1) = \bigwedge^2 \mathcal{E}_0, \quad H^0(\mathcal{O}_{\mathrm{Gr}}(1)) = \bigwedge^2 V, \tag{10}$$

and a nonsingular hyperplane section of Gr corresponds to the isotropic spaces of a nondegenerate skew bilinear form $\psi\colon V \times V \to K$.

Now under the assumptions of (3.5), $X \subset Q_5$ and $X \cap \operatorname{Sing} Q_5 = \emptyset$. This means that X has a projection morphism to the nonsingular $Q \subset \mathbb{P}^4$, so that writing E for the restriction to X gives the following.

Proposition *$X \subset Q_5$ and $X \cap \operatorname{Sing} Q_5 = \emptyset$ gives rise to a vector bundle $\mathcal{E}$ of rank 2 on X such that*

$$\bigwedge^2 \mathcal{E} \simeq K_X, \quad \textit{that is,} \quad K_X \otimes \mathcal{E}^* \simeq E, \tag{11}$$

and V a 4-dimensional space of sections spanning $\mathcal{E}$:

$$0 \to \mathcal{F} \to V \otimes \mathcal{O} \to \mathcal{E} \to 0. \tag{12}$$

V has a nondegenerate skew bilinear form $\psi\colon V \times V \to k$ such that at every point, the fibre of $\mathcal{F}$ is isotropic for ψ, inducing an isomorphism

$$\psi\colon \mathcal{F} \to \mathcal{E}^* = \mathcal{E}(-K_X). \tag{13}$$

(3.8) Lemma

(i) Any general section $s \in H^0(\mathcal{E})$ defines a short exact sequence

$$0 \to \mathcal{O} \to \mathcal{E} \to I_E \cdot \mathcal{O}_X(K_X) \to 0, \tag{14}$$

where $E \subset X$ is a reduced set of points, so that

$$E = c_2(\mathcal{E}) \quad \textit{and} \quad c_1(\mathcal{E}) = K_X. \tag{15}$$

(ii)

$$2E = K^2 \quad \textit{as 0-cycles of } X \tag{16}$$

(modulo rational equivalence).

(iii) If $s \in V \subset H^0(\mathcal{E})$ is a general element then E is just a reinterpretation of the construction of (3.5), (1).

Proof The construction on X is the pullback of a tautological construction on Q. The choice of a tautological bundle $\mathcal{E}_0$ on $\mathrm{Gr}(2,4)$ determines one family of generators of $\mathrm{Gr} = \mathrm{Gr}(2,4) = Q_6 \subset \mathbb{P}^5$. A general element $s \in V = H^0(\mathrm{Gr}, \mathcal{E}_0)$ defines an exact sequence over Gr

$$0 \to \mathcal{O}_{\mathrm{Gr}} \to \mathcal{E}_0 \to I_\pi \cdot \mathcal{O}_{\mathrm{Gr}}(1) \to 0, \tag{17}$$

where $\pi = c_2(\mathcal{E}_0)$ is a generator of Gr in the given family. If π_1 and π_2 are generators of Gr in the two families then $\pi_1 + \pi_2$ is a codimension 2 linear section of Gr; on the other hand, each of them restricts down to a generator of Q, so that on Q, the class of a codimension 2 linear section is twice a generator. (ii) follows from this. The generators of Q move around in a free system, so that the nonsingularity in (i) follows from the separability of $X \to Q$. (iii) is an exercise for the reader.

Problem Assume that the projection $X \to Q \subset \mathbb{P}^4$ is a birational embedding (presumably this is the main case?). If K is the field of definition of the generic section $s \in H^0(\mathcal{E})$, then is it true that the Galois group $\mathrm{Gal}(K(E)/K)$ is the full symmetric group on E? This would imply that E is uniform, and is an analog for vector bundles of the Lefschetz–Harris principle for very ample divisors (compare (1.2)).

(3.9) Lower bound for $H^0(\mathcal{E})$ If follows immediately from (13), or from RR that

$$\chi(\mathcal{E}) = 2\chi(\mathcal{O}_X) - \deg E. \tag{18}$$

Assuming $K^2 < 4p_g - 12$ and $q = 0$, this gives

$$\chi(\mathcal{E}) = 2p_g + 2 - \frac{1}{2}K^2 > 8. \tag{19}$$

Now Serre duality together with (10) gives $h^2(\mathcal{E}) = h^0(\mathcal{E}^*(K_X)) = h^0(\mathcal{E})$, so

$$h^0(\mathcal{E}) = h^2(\mathcal{E}) = p_g + 1 - \frac{1}{4}K^2 + \frac{1}{2}h^1(\mathcal{E}) \geq 5. \tag{20}$$

On the other hand, the number of hyperplanes of $\mathbb{P}^{p_g-1}$ containing E is $h^0(I_E \cdot \mathcal{O}_X(K))$, and from (13), this is

$$h^0(I_E \cdot \mathcal{O}_X(K)) = h^0(\mathcal{E}) - 2 \tag{21}$$

so that (19) implies

$$\dim \langle E \rangle = p_g - h^0(\mathcal{E}) \leq \frac{1}{4}K^2 - 1 \leq n - 2. \tag{22}$$

This shows that E is not linearly general.

(3.10) E is not quadratically general (11) and (12) give an exact sequence

$$0 \to V \to H^0(\mathcal{E}) \to H^1(\mathcal{E}^*) \to 0. \tag{23}$$

Hence

$$h^1(\mathcal{E}^*) = h^1(\mathcal{E}(K)) = h^0(\mathcal{E}) - 4 > 0. \tag{24}$$

Now tensoring (13) with K_X gives

$$0 \to \mathcal{O}(K_X) \to \mathcal{E}(K_X) \to I_E \cdot \mathcal{O}_X(2K_X) \to 0, \tag{25}$$

leading to the cohomology exact sequence

$$\begin{aligned} H^1(K_X) = 0 \to H^1(\mathcal{E}(K_X)) \to H^1(I_E \cdot \mathcal{O}_X(2K)) \to \\ \to H^2(\mathcal{O}_X(K_X)) = k \to 0 \end{aligned} \tag{26}$$

Therefore

$$h^1(I_E \cdot \mathcal{O}_X(2K)) = h^1(\mathcal{E}(K_X)) + 1 = h^0(\mathcal{E}) - 3. \tag{27}$$

This proves that E imposes

$$f_E = \deg E - h^0(\mathcal{E}) + 3 \tag{28}$$

conditions on quadrics.

(3.11) Curious conclusion *E satisfies*

$$\langle E \rangle = \mathbb{P}^\nu, \qquad \deg E \geq 2\nu + 2\pi + 1, \qquad f_E = 2\nu + \pi, \tag{29}$$

where $1 \leq \pi \leq \nu$. That is, the numerical assumptions of Conjecture 1.5 hold for E (except possibly for the cases $\pi = \nu - 1$, ν).

Proof Set $\langle E \rangle = \mathbb{P}^\nu$; then by (21), (15) and (27),

$$\nu = n + 3 - h^0(\mathcal{E}); \tag{30}$$

$$\deg E = \frac{1}{2}K^2; \tag{31}$$

$$f_E = \deg E - h^0(\mathcal{E}) + 3; \tag{32}$$

Now set

$$\pi = \deg E - n - \nu; \tag{33}$$

$$= f_E - 2\nu; \tag{34}$$

adding $2n - \frac{1}{2}K^2$ to both sides of (31) gives

$$n - \nu = h^0(\mathcal{E}) - 3 \tag{35}$$

$$= \pi + 2n - \frac{1}{2}K^2. \tag{36}$$

Therefore

$$\deg E = \frac{1}{2}K^2 = n + \nu + \pi = 2\nu + \pi + h^0(\mathcal{E}) - 3 \tag{37}$$

$$= 2\nu + 2\pi + (2n - \frac{1}{2}K^2) \geq 2\nu + 2\pi + 1 \tag{38}$$

and

$$f_E = 2\nu + \pi. \tag{39}$$

The final inequality $\pi \le \nu$ comes easily from $\pi = \deg E - n - \nu$ and $\deg E \le 2n - 1$, since (1.2) imply that the two copies of $\langle E \rangle = \mathbb{P}^\nu$ span $\mathbb{P}^n$, so that $2\nu + 1 \ge n$.

Remark The exceptional cases $\pi = \nu - 1$, ν of (3.11) only occur if $\deg E$ is close to $2n - 1$ and ν close to $(n-1)/2$; by (19–26), this corresponds to $h^0(\mathcal{E})$, $h^1(\mathcal{E})$, $h^1(I_E \cdot \mathcal{O}_X(K))$ and $h^1(I_E \cdot \mathcal{O}_X(2K))$ close to their maximum. It's quite likely that there are clever bounds on these groups than the trivial one using (1), (2) I have used.

(3.12) Final remarks My feeling is that there is a vague analogy between Conjectures 0.2 and 1.5 and the subject of special linear systems on curves on K3 surfaces, another area of research that has lain dormant for around 10 years, and has recently been opened up again by Lazarsfeld's ideas using vector bundles [Reid1, Lazarsfeld, Green and Lazarsfeld].

In fact, an important part of the original motivation for [Reid1] was the idea (which goes back to Petri [11]) of trying to capture a special linear system on a curve C in terms of a variety of small degree $C \subset V \subset \mathbb{P}^{g-1}$ through the canonical curve; then if C is a hyperplane section of a K3 surface $X \subset \mathbb{P}^g$, the problem is to extend V to a variety $W \subset \mathbb{P}^g$ containing X, and intersections of quadrics are very relevant to this.

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[3]Believe it or not, that is what UK e-mail addresses looked like in 1989.

Scrap

and $\psi g(\Gamma) = n - 2 - \beta - \sum$

The hyperplane problem An unpleasant problem that occurs frequently enough to be a standard error is the following: suppose $\Sigma = C \cap \mathbb{P}^n$ is a general section of an irreducible curve $C \subset \mathbb{P}^{n+1}$, and that I have some construction to produce a variety W_Σ with $\Sigma \subset W_\Sigma \subset \mathbb{P}^n$; then can the W_Σ be glued together into a variety W with $C \subset W \subset \mathbb{P}^{n+1}$ such that $W \cap \mathbb{P}^n = W_\Sigma$. For example, take a pencil of hyperplanes through an axis $\Pi = \mathbb{P}^{n+1}$ that meets C in n points, and just take the union of W_Σ over hyperplane sections in the pencil; unfortunately, this just doesn't work: the union will in general contain an extra component contained in $\mathbb{P}$i.

On the other hand, E should satisfy some weak uniform position property, coming from the fact that C is irreducible, which should then imply that $\langle E \rangle = \mathbb{P}^\nu$ with $\nu < n - 1$. By (3) and

(5), this should imply that —E— is more than a pencil, so that in turn, by (6), its points impose even fewer conditions on quadrics,..

the span $\langle E \rangle = \mathbb{P}^\nu$ has dimension only half that of $\mathbb{P}^n$,

and E k and

Question: can this argument ever be used? e.g. for special linear systems on curves on a K3, some form of it is true a posteori.

In particular if $\Sigma \subset C \subset X \subset \mathbb{P}^{n+2}$ as in (3.2), with Σ contained in (say) a cubic codim 2 scroll F,

and the 3 quadrics through F coming from $\mathbb{P}^{n+1}$ or $\mathbb{P}^{n+2}$, how can it happen that C, X are not also contained in a cubic scroll?