MA2210
THE UNIVERSITY OF WARWICK
SECOND YEAR EXAMINATION: JUNE 1995
GEOMETRY AND TOPOLOGY
Time allowed: 2 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on it.

ANSWER 4 QUESTIONS
If you have answered more than the required 4 questions, you will only be given credit for your 4 best answers.

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1. (a) Define a motion of the Euclidean plane $\mathbf{E}^2$. Describe in words and figures the different types of motion, and explain without proof the theorem classifying motions into types.

   (b) Let L be a line, $\mathbf{a}$ a vector along L, and P a point not on L, as in the diagram below. Consider the glide reflection $g = \mathrm{Glide}(L, \mathbf{a})$ and the rotation $r = \mathrm{Rot}(P, 2\theta)$. By considering its effect on points and lines in the diagram below, or otherwise, determine the composite motion $t = r \circ g$.

CONTINUED ...

2. (a) Let X be a topological space and $Y \subset X$ a subset with the subspace topology. Under the assumption that X is compact, prove that Y closed implies Y compact. Under what condition on X is the converse true? (The proof is not required).

(b) Let X and Y be compact metric spaces and $\varphi\colon X \to Y$ a continuous bijective map. Sketch the proof that φ is a homeomorphism.

(c) Let $S^1 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle, and $\varphi\colon S^1 \to \mathbf{R}^n$ a continuous map with the property that for all $(x,y) \in S^1$

$$\varphi(x,y) = \varphi(x',y') \iff (x,y) = \pm(x',y').$$

Prove that the image $\varphi(S^1)$ is homeomorphic to S^1.

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3. (a) Say what it means for a real $n \times n$ matrix A to be orthogonal. If A is a 3×3 orthogonal matrix, prove from first principles that there exists a real vector $x \in \mathbf{R}^3$ such that $Ax = \pm x$.

(b) State without proof the theorem on the normal form of an orthogonal matrix; explain what it means in the case of a 3×3 orthogonal matrix with $\det A = 1$.

(c) Define the quaternion conjugate q^* of a quaternion q, and explain how to find the inverse q^{-1} if $q \neq 0$. Let p, q be quaternions. Prove that

$$p \text{ is pure imaginary} \implies qpq^{-1} \text{ is pure imaginary.}$$

Identify the space of pure imaginary quaternions $p = xi + yj + zk$ with Euclidean space $\mathbf{E}^3$ (with coordinates x, y, z). Assuming that $|q|^2 = 1$ for simplicity, prove that the map $r_q\colon \mathbf{E}^3 \to \mathbf{E}^3$ given by $r_q(p) = qpq^{-1}$ is an element of $\mathrm{SO}(3)$.

(d) Find a unit quaternion q for which r_q fixes the z-axis and rotates the x, y plane through an angle θ.

CONTINUED ...

4. (a) Define the notions of affine linear subspace and linear span in affine space $\mathbf{A}^n_{\mathbf{R}}$. State and prove the theorem on dimension of intersection of affine linear subspaces $E, F \subset \mathbf{A}^n_{\mathbf{R}}$ (results on vector spaces may be used freely). Explain briefly why the lecturer seems to think that this result is more convenient to state in projective space $\mathbf{P}^n_{\mathbf{R}}$.

(b) Consider $\triangle PQR$ and $\triangle P'Q'R'$ in projective 3-space $\mathbf{P}^3$, where

$$P = (0,0,0,1), \quad Q = (1,0,0,1), \quad R = (0,1,0,1),$$

$$P' = (0,0,1,1), \quad Q' = (1,0,1,1), \quad R' = (0,1,1,1).$$

By writing out the lines PP', QQ', RR', show that $\triangle PQR$ and $\triangle P'Q'R'$ are in perspectivity. [Hint: you may want to take coordinates x, y, z, t and to visualise $PQRP'Q'R'$ as a triangular prism in the affine piece $t = 1$.] Now determine the points of intersection of the corresponding sides of the triangles, for example:

$$QR : (x + y = t, z = 0), \quad Q'R' : (x + y = t, z = t),$$

$$\implies QR \cap Q'R' = (1 : -1 : 0 : 0).$$

(c) State and prove Desargues' theorem for two triangles $\triangle PQR$ and $\triangle P'Q'R'$ spanning distinct planes in $\mathbf{P}^3_{\mathbf{R}}$ and in perspectivity from a point O. Verify it in the example of (c).

CONTINUED ...

5. Give a brief mathematical discussion of the truth or otherwise of each of the following statements (a–e). In each section, credit will be given for the quality of your argument rather than for guessing or remembering whether the statement is true. The five sections carry equal marks.

(a) The group of direct motions of Euclidean space contains a non-Abelian finite subgroup.

(b) Let (X, d) be a metric space and $\sim$ an equivalence relation on X; then setting

$$d([x],[y]) = \inf\{d(x',y')\} \quad \text{taken over all } x' \in [x], y' \in [y].$$

defines a metric on the quotient space $Y = X/\sim$.

(c) Let $\varphi\colon [0,1] \to \mathbf{R}^2$ be a loop with $\varphi(0) = \varphi(1)$. Then there exists a point $Q \in \mathbf{R}^2$ such that φ has winding number zero around Q

(d) There exists an affine linear map $\mathbf{A}^1 \to \mathbf{A}^1$ of the affine line taking any set of 3 distinct points into any other.

(e) Given two lines L and M in $\mathbf{E}^3$ with $L \cap M = \emptyset$, there exists a unique line N perpendicular to both L and M.

END

Answer to Q1.

(a) $\mathbf{E}^2 = \mathbf{R}^2$ with the usual Euclidean metric. A Euclidean motion is an isometry $t\colon \mathbf{E}^2 \to \mathbf{E}^2$, that is, a bijection that preserves distances. It can be written $t(x) = Ax + b$ (w.r.t. any basis), where $x \in \mathbf{E}^2$ is the column vector $x = (x_1, x_2)$, A is a 2×2 orthogonal matrix, and b a translation vector.

[Definition, 4 marks]

Theorem. *Classification of motions into 4 types:*

1 . Translation $\mathrm{Transl}(b)$ *in vector* b

2 . Rotation $\mathrm{Rot}(P, \theta)$ *through* θ *about centre* P

3 . Reflection $\mathrm{Refl}(L)$ *in line* L

4 . Glide (or glide-reflection) $\mathrm{Glide}(L, a)$. *This is the reflection in a line* L *composed with a translation in a vector* a *along* L.

[Memorable bookwork, 8 marks]

(b) Then answer is $\mathrm{Glide}(M, 2b)$. First, M is taken to itself: g reflects M into M' and translates it to M'', and the rotation r rotates M'' back to M. Next, points on M are translated by the vector $2b$, for example, $g\colon Q \mapsto$ itself $\mapsto R$ and $r\colon R \to R'$. Finally, vectors orthogonal to M are taken into vectors orthogonal to M (because the motion $r \circ g$ preserves angles), and are reflected.

[Unseen example, similar to ex. 13 marks. $4 + 8 + 13 = 25$] [A bit too easy?]

Answer to Q2.

(a) X is compact and $Y \subset X$ closed. Let U_λ be an open cover of Y indexed by $\lambda \in \Lambda$. By definition of the subspace topology, there exist open sets $V_\lambda \subset X$ with $U_\lambda = V_\lambda \cap Y$. Then $Y \subset \bigcup V_\lambda$, so that $X = CY \cup \bigcup V_\lambda$ is an open cover of X (where CY is the complement of Y). Since X is compact, a finite number of these cover X, say $X = CY \bigcup V_i$, so $Y \subset \bigcup V_i$, that is, $Y = \bigcup (V_i \cap Y) = \bigcup U_i$. This proves Y compact.

[Bookwork. 7 marks]

The converse is true if X is Hausdorff: this means for all $x, y \in X$, there exist disjoint open sets U_x, U_y with $x \in U_x, y \in U_y$.

[Bookwork, I'll buy metric instead. 3 marks]

(b) φ is bijective, so that there is an inverse map φ^{-1}. If we prove that φ is a closed map (that is, takes closed sets of X to closed sets of Y) then φ^{-1} is continuous, so that φ is a homeomorphism.

Now X and Y are compact and Hausdorff by assumption, so that closed = compact by (a). Now a continuous map φ takes compact sets to compact sets, therefore it takes closed sets to closed sets.

[Bookwork. 6 marks]

(c) Let $\sim$ be the equivalence relation $(x, y) \sim \pm(x, y)$. Then by assumption, φ is constant on the equivalence classes, so defines a continuous injective map $\psi\colon S^1/(\pm 1) \to \mathbf{R}^n$. This map is a homeomorphism of $S^1/(\pm 1)$ to its image $\psi(S^1)$ by the theorem of (b).

The point is to see that $S^1/(\pm 1) \simeq S^1$, for example because it is the upper half-circle with its ends identified.

[Similar to $SO(3) = S^3/\pm 1$ in course. It's a bit long, but making it shorter would make it harder. (c) contains a trap for the unwary. 9 marks. $7 + 3 + 6 + 9 = 25$]

Answer to Q3.

(a) A is orthogonal if ${}^tAA = 1_n$. If A is orthogonal and x any vector, then

$$|Ax|^2 = (Ax) \cdot (Ax) =^t x^t AAx = x \cdot x = |x|^2.$$

A 3×3 real matrix A has at least one real eigenvalue and eigenvector: indeed, the characteristic polynomial $\det(A - x1_3)$ is a real cubic x, so has a real root λ. If $Ax = \lambda x$ with λ real, then

$$\lambda^2|x|^2 = |Ax|^2 = |x|^2, \qquad \text{therefore } \lambda = \pm 1.$$

[Bookwork. 7 marks]

(b) There exists an orthonormal basis w.r.t. which A is the block diagonal matrix, with each block either 1×1 block with entry ± 1, or the 2×2 rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

In the 3×3 case there is at most one rotation block. If $\det A = 1$ (that is, A is direct) then either $A = 1_3$ or A has a block $+1$ and a rotation block.

[Stating result of course. 5 marks]

(c) If $q = a + bi + cj + dk$ then $q^* = a - bi - cj - dk$. Thus q is real resp. pure imaginary if $q^* = q$ or $q^* = -q^*$. If $q \neq 0$ then $qq^* = |q|^2$ is a positive real, so that $q^{-1} = q^*/|q|^2$ is the inverse.

Now for a real, $qaq^* = aqq^* = a|q|^2$ is also real. If p is pure imaginary then $p^* = -p$, so that $(qpq^*)^* = qp^*q^* = -qpq^*$, and p is imaginary.

Setting $r_q(p) = qpq^{-1}$ then r_q is a linear map $\mathbf{R}^3 \to \mathbf{R}^3$, and $|r_q(p)|^2 = |qpq^{-1}|^2 = (qpq^*)(qp^*q)/|q|^4 = pp^* = |p^2|$, so that r_q is a motion or orthogonal map. It is direct because $\det r_q$ is a continuous function from the nonzero quaternions to ± 1.

[Bookwork. 6 marks]

(d) Write $c = \cos\psi$, $s = \sin\psi$ and $q = c + sk$. Then q is a unit quaternion; $r_q(k) = qkq^{-1} = k$ because $\mathbf{R}[k] = \mathbf{C}$ is commutative. And the action on i, j is given by

$$\begin{aligned} r_q(i) &= (c + sk)i(c - sk) = (c^2 - s^2)i + 2csj \\ \text{and} \qquad r_q(j) &= (c + sk)j(c - sk) = -2csi + (c^2 - s^2)j. \end{aligned}$$

This is a rotation of the x, y plane by an angle 2ψ.

[Calculation similar to proof in course. 7 marks. $7 + 5 + 6 + 7 = 25$]

Answer to Q4.

(a) $\mathbf{A}^n = \mathbf{R}^n$. An affine linear subspace is of the form $u_0 + V$ where $u_0 \in \mathbf{R}^n$ and $V \subset \mathbf{R}^n$ is a vector subspace. Given a nonempty set $\Sigma \subset \mathbf{A}^n$, its span $\langle \Sigma \rangle$ is the smallest affine linear subspace containing Σ. In other words, if $u_0 \in \Sigma$ is any point,

$$\langle \Sigma \rangle = u_0 + V, \quad \text{where} \quad V = \langle \Sigma - u_0 \rangle.$$

The theorem says that for E, F affine linear subspaces,

$$\dim E + \dim F = \dim(E \cap F) + \dim\langle E, F \rangle,$$

provided $E \cap F \neq \emptyset$. For the proof, just pick $u_0 \in E \cap F$, so that $E = u_0 + U$, $F = u_0 + V$ for some vector subspaces $U, V \subset \mathbf{R}^n$. Then

$$E \cap F = u_0 + (U \cap V), \quad \langle E, F \rangle = u_0 + (U + V),$$

and the formula therefore follows from that for vector spaces.

The same formula holds for all E, F without exception in $\mathbf{P}^n$, provided we write $\dim \emptyset = -1$. The lecturer seemed to think this was really good.

[Bookwork. 11 marks]

(b) $PP' : (x = y = 0)$, $QQ' : (x = t, y = 0)$ and $RR' : (x = 0, y = t)$ intersect in $O = (0010)$. (By the hint, the sides of the prism are vertical and parallel so meet at (0010).)

Now $PQ : (y = z = 0)$, $P'Q' : (y = 0, z = t)$ intersect in $C = (1000)$,
$PR : (x = z = 0)$, $P'R' : (x = 0, z = t)$ intersect in $B = (0100)$,
and $QR \cap Q'R'$ intersect in $A = (1 : -1 : 0 : 0)$ is given.

[Unseen, easier than lectured material. 7 marks]

(c)

Desargues' theorem. *If $\triangle PQR$ and $\triangle P'Q'R'$ are in perspectivity from a point O then the 3 points of intersection*

$$A = QR \cap Q'R', \quad B = PR \cap P'R', \quad C = PQ \cap P'Q'$$

of the corresponding sides are collinear.

Proof. If the two triangles span distinct planes $\Pi = \langle \triangle PQR \rangle \neq \Pi' = \langle \triangle P'Q'R' \rangle$ then this is obvious: $OPQP'Q'$ is a plane so PQ and $P'Q'$ intersect in a point C, etc., and so C is contained in lines in the planes Π, Π', therefore is contained in the line $L = \Pi \cap \Pi'$. Similarly for B, C. Q.E.D.

The above 3 points $(1 : -1 : 0 : 0), (0100), (1000)$ are obviously on $z = t = 0$.

[The easy case of Desargues theorem. 3 marks. $11 + 7 + 7 = 25$]

Answer to Q5.

(a) A finite group $G \subset Eucl(2)$ fixes a point P_0 (the centroid of any orbit $G \cdot P$); therefore G has only reflections and rotations. If it's direct it only contains rotations around P_0, therefore is commutative. Thus the statement is false.

[Mainly on the ex sheet]

(b) False, the inf can be zero. For example, in $\mathbf{R}$, set $x \sim y$ iff either both nonzero or both zero. Then $x \not\sim y$ means $d(x, y) > 0$ but not bounded below.

[Copied from past exam made available to students]

(c) True. The image of φ is contained in some disc $D : (|x| < R)$. Thus if Q is chosen outside D, the path φ is contractible in D, therefore in $\mathbf{R}^2 \setminus Q$, and so the winding number of φ around Q is 0 (by homotopy invariance).

[Easy but unseen]

(d) False. The ratio of distances is defined. The affine linear map $\varphi \mathbf{A}^1 \to \mathbf{A}^1$ given by $x \mapsto ax + b$ is determined by its effect on $P = 0$ and $Q = 1$: because $\varphi(0) = b$, and $\varphi(1) = a + b$. Therefore, given another point $R = c$, the image $\varphi(R) = ac + b$ is determined by $\varphi(P)$ and $\varphi(Q)$, so can't be changed.

[Easy but unseen; like cross-ratio]

(e) False, because L and M could be parallel lines in the plane, and then they have infinitely many transversals. Any reasonably correct proof of the statement under the implicit assumption that L and M not parallel scores most of the points.

[By the warning, ≤ 1 mark for true/false. $5 \times 5 = 25$]