MA2430
THE UNIVERSITY OF WARWICK
SECOND YEAR EXAMINATION: JUNE 1996
GEOMETRY
Time allowed: 2 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on it.

ANSWER 4 QUESTIONS
If you have answered more than the required 4 questions, you will only be given credit for your 4 best answers.

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1. (i) What is meant by a motion of Euclidean n-space $\mathbf{E}^n$? State and prove a necessary and sufficient condition on an $n \times n$ matrix A in order that the map defined by

$$\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b} \quad \text{for } \mathbf{x} \in \mathbf{E}^n \tag{$*$}$$

should be a motion of $\mathbf{E}^n$.

(ii) Let L_1 and L_2 be the lines of $\mathbf{E}^2$ defined by the equations

$$L_1 : (y = b) \quad \text{and} \quad L_2 : (\cos\theta)y = (\sin\theta)x,$$

where θ is a fixed angle between 0 and π, and $\beta \in \mathbf{R}$. Write down the reflections R_1, R_2 in L_1 and L_2 in the form $(*)$; proofs are not required.

(iii) Calculate the composite motion $R_2 \circ R_1$ in this form.

(iv) Using the fact that both R_1 and R_2 fix the point

$$P = L_1 \cap L_2 = ((\cot\theta)\beta, \beta),$$

describe the effect of $R_2 \circ R_1$ on $\mathbf{E}^2$ in geometric terms.

CONTINUED ...

2. (a) Define the notions of affine linear subspace and linear span in affine space $\mathbf{A}^n_{\mathbf{R}}$. State and prove the theorem on dimension of intersection of affine linear subspaces $E, F \subset \mathbf{A}^n_{\mathbf{R}}$ (results on vector spaces may be used freely). Explain briefly why the this result is more convenient to state in projective space $\mathbf{P}^n_{\mathbf{R}}$.

(b) Consider $\triangle PQR$ and $\triangle P'Q'R'$ in projective 3-space $\mathbf{P}^3$, where

$$P = (0,0,0,1), \quad Q = (1,0,0,1), \quad R = (0,1,0,1),$$

$$P' = (0,0,1,1), \quad Q' = (1,0,1,1), \quad R' = (0,1,1,1).$$

By writing out the lines PP', QQ', RR', show that $\triangle PQR$ and $\triangle P'Q'R'$ are in perspectivity. [Hint: you may want to take coordinates x, y, z, t and to visualise $PQRP'Q'R'$ as a triangular prism in the affine piece $t = 1$.] Now determine the points of intersection of the corresponding sides of the triangles, for example:

$$QR : (x + y = t, z = 0), \quad Q'R' : (x + y = t, z = t),$$

$$\implies QR \cap Q'R' = (1 : -1 : 0 : 0).$$

(c) State and prove Desargues' theorem for two triangles $\triangle PQR$ and $\triangle P'Q'R'$ spanning distinct planes in $\mathbf{P}^3_{\mathbf{R}}$ and in perspectivity from a point O. Verify it in the example of (b).

CONTINUED ...

3. (i) Define a reflection of Euclidean n-space $\mathbf{E}^n$, and state without proof the theorem of the course expressing any motion of $\mathbf{E}^n$ as a composite of a number of reflections. Explain what it means in particular for a direct motion of $\mathbf{R}^3$ fixing the origin. Deduce that a 3×3 orthogonal matrix A with $\det A = 1$ has $+1$ as eigenvalue. (That is, any element $A \in \mathrm{SO}(3)$ fixes an axis.)

(ii) Define the quaternion conjugate q^* of a quaternion q, and explain how to find the inverse q^{-1} if $q \neq 0$. Let p, q be quaternions. Prove that

$$p \text{ is pure imaginary} \implies qpq^{-1} \text{ is pure imaginary.}$$

Identify the space of pure imaginary quaternions $p = xi + yj + zk$ with Euclidean space $\mathbf{E}^3$ (with coordinates x, y, z). Assuming that $|q|^2 = 1$ for simplicity, prove that the map $r_q\colon \mathbf{E}^3 \to \mathbf{E}^3$ given by $r_q(p) = qpq^{-1}$ is an element of $\mathrm{SO}(3)$.

(iii) Determine the rotation r_q for the unit quaternion

$$q = c + si \quad \text{with} \quad c^2 + s^2 = 1,$$

and express the result geometrically.

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4. Let $S^2 \subset \mathbf{R}^3$ be the sphere of radius 1. Explain how to define the spherical distance $d(P, Q)$ between two points $P, Q \in S^2$. State without proof a version of the triangle inequality and explain briefly why it implies that the shortest path from P to Q is an arc of great circle.

Let ΔPQR be a triangle in spherical geometry, with right angle at P. State and prove from first principles a formula expressing the hypotenuse $d(Q, R)$ in terms of the other two sides $d(P, Q)$ and $d(P, R)$. Verify that your formula approximates Pythagoras' theorem when the triangle is small compared with the radius of the sphere. [Hint: You may use the approximation $\cos a \approx 1 - a^2 + \cdots$ (higher order terms) for small a.]

CONTINUED ...

5. Use your mathematical judgement to argue for the truth or otherwise of the following statements.

   (a) Let S_n be the symmetric group on $\{1, 2, \ldots, n\}$ and $G \subset S_n$ a subgroup; if G contains the transposition (12) and an element taking $i \mapsto 1, j \mapsto 2$, for any pair i, j of distinct elements of $\{1, 2, \ldots, n\}$, then $G = S_n$.

   (b) There exists a finite non-Abelian group of direct motions of the Euclidean plane $\mathbf{E}^2$.

   (c) There exists an affine linear map $\mathbf{A}^1 \to \mathbf{A}^1$ of the affine line taking any set of 3 distinct points into any other.

   (d) Given 3 lines $L_1, L_2, L_3 \subset \mathbf{P}^3$, there is a unique line M meeting all 3.

   (e) In hyperbolic geometry $\mathcal{H}^2$, given a line L and a point $P \notin L$, there is a unique line M through P perpendicular to L.

END

Answer to Q1.

(i) A motion of $\mathbf{E}^n$ is a map $t\colon \mathbf{E}^n \to \mathbf{E}^n$ that preserves Euclidean distances, i.e., $d(t(P), t(Q)) = d(P, Q)$ for all P, Q.

[Definition, 3 marks]

If t is given by $x \mapsto Ax + b$ with A and $n \times n$ matrix and b a vector, t is a motion if and only if A is orthogonal. Indeed, the translation by b does not affect distnaces, so that I need to prove that $x \mapsto Ax$ preserves distance from the origin, that is, the quadratic form $|x|^2 = {}^t x \cdot x$. But this holds if and only if it preserves the bilinear form ${}^t xx$ on $\mathbf{R}^n$, that is,

$$ {}^t(Ax) \cdot Ay = {}^t x({}^t A \cdot A)y = {}^t xy \quad \text{for all } x, y \in \mathbf{R}^n, $$

that is, ${}^t AA = 1$.

[Bookwork, 6 marks]

(ii) The reflection in $L_1 : (y = \beta)$ is given by $x \mapsto x$, $y \mapsto 2\beta - y$, that is,

$$ R_1\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2\beta \end{pmatrix}. $$

The reflection in $L_2 : (\cos\theta)y = (\sin\theta)x$ is given by

$$ R_2\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. $$

(Several derivations are possible, for example, by computing the conjugate $\text{Rotation}(\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{Rotation}(-\theta)$, or by finding the fixed axis.)

[4+4 marks]

(iii) So

$$ \begin{aligned} R_2 \circ R_1\colon \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2\beta \end{pmatrix} \right] \\ &= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2\beta \begin{pmatrix} \sin 2\theta \\ -\cos 2\theta \end{pmatrix} \end{aligned} $$

[4 marks]

(iv) $R_2 \circ R_1 = \text{Rotation}(\text{P}, 2\theta)$, where as given, $P = L_1 \cap L_2 = ((\cot\theta)\beta, \beta)$ (see Figure).

[4 marks]

[Standard type of question. $3 + 6 + 4 + 4 + 4 + 4 = 25$]

Answer to Q2.

(a) $\mathbf{A}^n = \mathbf{R}^n$. An affine linear subspace is of the form $u_0 + V$ where $u_0 \in \mathbf{R}^n$ and $V \subset \mathbf{R}^n$ is a vector subspace. Given a nonempty set $\Sigma \subset \mathbf{A}^n$, its span $\langle \Sigma \rangle$ is the smallest affine linear subspace containing Σ. In other words, if $u_0 \in \Sigma$ is any point,

$$\langle \Sigma \rangle = u_0 + V, \quad \text{where} \quad V = \langle \Sigma - u_0 \rangle.$$

The theorem says that for E, F affine linear subspaces,

$$\dim E + \dim F = \dim(E \cap F) + \dim\langle E, F \rangle,$$

provided $E \cap F \neq \emptyset$. For the proof, just pick $u_0 \in E \cap F$, so that $E = u_0 + U$, $F = u_0 + V$ for some vector subspaces $U, V \subset \mathbf{R}^n$. Then

$$E \cap F = u_0 + (U \cap V), \quad \langle E, F \rangle = u_0 + (U + V),$$

and the formula therefore follows from that for vector spaces.

The same formula holds for all E, F without exception in $\mathbf{P}^n$, provided we write $\dim \emptyset = -1$. The lecturer seemed to think this was really good.

[Bookwork. 11 marks]

(b) $PP' : (x = y = 0)$, $QQ' : (x = t, y = 0)$ and $RR' : (x = 0, y = t)$ intersect in $O = (0010)$. (By the hint, the sides of the prism are vertical and parallel so meet at (0010).)

Now $PQ : (y = z = 0)$, $P'Q' : (y = 0, z = t)$ intersect in $C = (1000)$,
$PR : (x = z = 0)$, $P'R' : (x = 0, z = t)$ intersect in $B = (0100)$,
and $QR \cap Q'R'$ intersect in $A = (1 : -1 : 0 : 0)$ is given.

[Easier than lectured material. 7 marks]

(c)

Desargues' theorem. *If $\triangle PQR$ and $\triangle P'Q'R'$ are in perspectivity from a point O then the 3 points of intersection*

$$A = QR \cap Q'R', \quad B = PR \cap P'R', \quad C = PQ \cap P'Q'$$

of the corresponding sides are collinear.

Proof. If the two triangles span distinct planes $\Pi = \langle \triangle PQR \rangle \neq \Pi' = \langle \triangle P'Q'R' \rangle$ then this is obvious: $OPQP'Q'$ is a plane so PQ and $P'Q'$ intersect in a point C, etc., and so C is contained in lines in the planes Π, Π', therefore is contained in the line $L = \Pi \cap \Pi'$. Similarly for B, C. Q.E.D.

The above 3 points $(1 : -1 : 0 : 0), (0100), (1000)$ are obviously on $z = t = 0$.

[The easy case of Desargues theorem. 3 marks. $11 + 7 + 7 = 25$]

Answer to Q3.

(i) The reflection of $\mathbf{E}^n$ in a hyperplane Π is the motion that fixes every $P \in \Pi$, and that takes a point Q into $t(Q)$ constructed as the mirror image, that is, drop a perpendicular from Q to Π and set $t(Q)$ to be the point the same distance the other side. (In other words, Π is the perpendicular bisector of PQ.)

Theorem. *Any motion of $\mathbf{E}^n$ is a composite of at most $n+1$ reflections. Any motion fixing the first k elements of a Euclidean frame of reference is a composite of at most $n+1-k$ reflections.*

In particular, if $n=3$ and $k=1$, any motion $t\colon \mathbf{R}^3 \to \mathbf{R}^3$ is a composite of ≤ 3 reflections. If t is direct then the number of reflections is even, that is, 0 (the identity) or 2. But the composite of 2 distinct reflections Reflection(Π_1), Reflection(Π_2) fixes the line $L = \Pi_1 \cap \Pi_2$ (vector subspaces), as required.

[Some students will probably do the last part "Otherwise" for fewer marks: There exists an orthonormal basis w.r.t. which A is the block diagonal matrix, with each block either 1×1 block with entry ± 1, or the 2×2 rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

In the 3×3 case there is at most one rotation block. If $\det A = 1$ (that is, A is direct) then either $A = 1_3$ or A has a block $+1$ and a rotation block.]

[Statement of theorem. Deconstructing to $\mathbf{R}^3$ is unseen. 6+4 marks]

(ii) If $q = a + bi + cj + dk$ then $q^* = a - bi - cj - dk$. Thus q is real resp. pure imaginary if $q^* = q$ or $q^* = -q^*$. If $q \neq 0$ then

$$qq^* = |q|^2 = a^2 + b^2 + c^2 + d^2$$

is a positive real, so that $q^{-1} = q^*/|q|^2$ is the inverse.

Now for a real, $qaq^* = aqq^* = a|q|^2$ is also real. If p is pure imaginary then $p^* = -p$, so that $(qpq^*)^* = qp^*q^* = -qpq^*$, and p is imaginary.

Setting $r_q(p) = qpq^{-1}$ then r_q is a linear map $\mathbf{R}^3 \to \mathbf{R}^3$, and $|r_q(p)|^2 = |qpq^{-1}|^2 = (qpq^*)(qp^*q)/|q|^4 = pp^* = |p^2|$, so that r_q is a motion or orthogonal map. It is direct because $\det r_q$ is a continuous function from the nonzero quaternions to ± 1.

[Bookwork. 7 marks]

(d) Write $c = \cos\psi$, $s = \sin\psi$ and $q = c + si$. Then q is a unit quaternion; $r_q(i) = qiq^{-1} = i$ because $\mathbf{R}[i] = \mathbf{C}$ is commutative. And the action on j, k is given by

$$\begin{aligned} & r_q(j) = (c+si)j(c-si) = (c^2-s^2)j + 2csk \\ \text{and} \qquad & r_q(k) = (c+si)k(c-si) = -2csj + (c^2-s^2)k. \end{aligned}$$

This is a rotation of the y, z plane by an angle 2ψ.

[Calculation similar to proof in course. 8 marks. $10 + 7 + 8 = 25$]

Answer to Q4.

$S^2 \subset \mathbf{R}^2$ is the sphere of radius 1 centred at the origin. If $P = Q$ then define $d(P,Q) = 0$; if $P = -Q$ (antipodal points) then define $d(P,Q) = \pi$. Otherwise $P, Q \in S^2$ are linearly independent points of $\mathbf{R}^3$ and span a 2-plane which intersects S^2 in a great circle. Then define $d(P,Q)$ to be the length of the shorter arc from P to Q. Since the arc length is $a = \angle POQ$, we get that

$$d(P,Q) = \arccos(P \cdot Q) \qquad (*)$$

(vector product).

[Definition. 4 marks]

The triangle inequality:

$$d(P,R) \leq d(P,Q) + d(Q,R) \quad \text{for } P, Q, R \in S^2,$$

with equality if and only if $Q \in PR$, that is, Q is contained in a shorter arc of greater circle from P to R.

[Memorable statement. 4 marks]

If C is any curve from P to Q then by definition of the Riemann integral, for any $\varepsilon > 0$, it can be approximated by a path $P = P_0, P_1, \ldots, P_N = Q$ made up of arcs of great circles, so that

$$\text{length of } C \geq d(P_0, P_1) + d(P_1, P_2) + \cdots + d(P_{N-1}, P_N) - \varepsilon.$$

Omitting the intermediate steps decreases the right hand side, so by induction, length of $C \geq d(P,Q)$. Thus the arc of great circle is the shortest path.

[Bookwork. Vague argument OK. 4 marks]

Let $P = (1,0,0)$ be the North Pole. It is given that $\angle QPR = 90°$, so that I can assume that Q is in the plane $y = 0$, and R in $z = 0$. Suppose $d(P,Q) = a$ and $d(P,R) = b$. Then $Q = (\cos a, 0, \sin a)$, $R = (\cos b, \sin b, 0)$. Therefore $Q \cdot R = \cos a \cos b$, and formula $(*)$ gives

$$d(P,Q) = \arccos(\cos a \cos b).$$

[Easiest form of trig calculation used in notes. 8 marks]

In other words, if c is the hypothenuse then $\cos c = \cos a \cos b$. Now since $\cos a = 1 - a^2 + \cdots$ (higher order terms), when a, b, c are small, we get $(1 - c^2) = (1 - a^2)(1 - b^2)$, that is, $c^2 = a^2 + b^2$ to good approximation.

[Unseen. 5 marks]

$[4 + 4 + 4 + 8 + 5 = 25$

Answer to Q5.

(a) True. By the given information, G contains (12), and $g = g_{ij} : i \mapsto 1, j \mapsto 2$, so by the conjugacy principle, it contains $g(12)g^{-1} = (ij)$. Obviously S_n is generated by all the transpositions.

[Tests conjugacy, 4.2–3 of book. 5 Marks]

(b) False. A finite group $G \subset \mathrm{Eucl}(2)$ fixes a point P_0 (the centroid of any orbit $G \cdot P$); therefore G has only reflections and rotations. If it's direct it only contains rotations around P_0, therefore is commutative. Thus the statement is false.

[Tests various aspects of Euclidean groups. 5 Marks]

(c) False. The ratio of distances is defined. The affine linear map $\varphi\colon \mathbf{A}^1 \to \mathbf{A}^1$ given by $x \mapsto ax + b$ is determined by its effect on $P = 0$ and $Q = 1$: because $\varphi(0) = b$, and $\varphi(1) = a + b$. Therefore, given another point $R = c$, the image $\varphi(R) = ac + b$ is determined by $\varphi(P)$ and $\varphi(Q)$, so can't be changed.

[Easy but unseen; like cross-ratio. 5 Marks]

(d) False. There are infinitely many. For a counterexample, if L_1 and L_2 are disjoint, they span $\mathbf{P}^3$ (by dimension of intersection), so every point of L_3 is on a line joining L_1, L_2, and this make infinitely many lines

[Simple trap. 5 Marks]

(e) True. Several arguments are possible, one by choosing $Q \in L$ to minimise distance $d(P, Q)$ (a continuous function on a compact set takes its inf), etc. Or consider a point $Q \in L$ moving from far over to the left (when the angle is very small) to far over to the right (when it is close to π. The angle is a continuous function so must take the value $\pi/2$ at some point. Or taking the reflection $t = \mathrm{Reflection}(L)$ and the line $Pt(P)$. Or use coordinate geometry in Lorentz space $\mathbf{R}^{-1,+2}$, with the Lorentz inner product $v \cdot_L w$: suppose L is given by $x_2 = 0$ and $P = (a, b_1, b_2)$ with $a^2 = 1 + b_1^2 + b_2^2$; then the point

$$Q = (c, d, 0) = \frac{1}{a^2 - b_1^2}(a, b_1, 0) \in L.$$

Then $\{(c, d, 0), (d, c, 0), (0, 0, 1)\}$ is a new Lorentz basis in which $L : (x_2 = 0)$ and $M : (x_1 = 0)$.

[The coordinate calc is hard unseen. 5 Marks]

[≤ 1 mark for true/false. $5 \times 5 = 25$]