MA2430
THE UNIVERSITY OF WARWICK
SECOND YEAR EXAMINATION: JUNE 1998
GEOMETRY
Time allowed: 2 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on it.

ANSWER 4 QUESTIONS
If you have answered more than the required 4 questions, you will only be given credit for your 4 best answers.

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1. (a) Define a motion of the Euclidean plane. Describe in words and figures the different types of motion, and explain without proof the theorem classifying motions into types.

   (b) In the figure below, $PAQB$ and $DPCQ$ are two rectangles having a common diagonal, and the median lines L, M and vectors $\mathbf{a}$, $\mathbf{b}$ are as indicated.

   Set $g_1 = \mathrm{Glide}(L, \mathbf{a})$ and $g_2 = \mathrm{Glide}(M, \mathbf{b})$. By considering its effect on the points P and A in the diagram, determine the composite $g_2 \circ g_1$ of the two glide reflections.

CONTINUED ...

2. (a) The model of the non-Euclidean hyperbolic plane used in the course is the upper sheet $\mathcal{H}^2$ of the hyperbola $x^2 + y^2 - t^2 = -1$ in Lorentz space $\mathbf{R}^{2,1}$. Straight lines of $\mathcal{H}^2$ are the "great hyperbolas" defined as the intersection of $\mathcal{H}^2$ with planes of $\mathbf{R}^{2,1}$. Define the hyperbolic distance $d(P,Q)$ between two points of $\mathcal{H}^2$. Explain how the Lorentz group acts as a group of motions (isometries) on $\mathcal{H}^2$; how transitive is this action on points and lines of $\mathcal{H}^2$? (No proofs are required.)

   (b) Let $\triangle PQR$ be a hyperbolic triangle with right angle at P and $d(P,Q) = s$, $d(P,R) = t$. State and prove from first principles a formula expressing the length of the hypotenuse $d(Q,R)$ in terms of the lengths s and t of the other two sides. [Hint: First use the transitivity described in (a) to put the triangle in a simple form.]

   (c) Use your formula and the approximation $\cosh s \approx 1 + s^2/2 + s^4/4!$ to verify that the hypotenuse is greater than the Euclidean value $\sqrt{s^2 + t^2}$, and approximates it when the distances s and t are small.

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3. (a) What is an affine transformation $t\colon \mathbf{A}^2 \to \mathbf{A}^2$ of the affine plane $\mathbf{A}^2$? Explain without detailed proof how t can be represented in matrix terms.

   (b) Write down if possible an affine transformation t which takes

$$(0,0) \mapsto (2,0), \quad (1,0) \mapsto (0,1) \quad \text{and} \quad (1,1) \mapsto (a,b),$$

   where a, b are two real numbers with $a + 2b \neq 2$. What is the reason for forbidding $a + 2b = 2$?

   (c) Explain briefly why angles and distances are not in general defined in the affine geometry of $\mathbf{A}^2$. What properties of a triple of points can be defined?

CONTINUED ...

4. (a) Define the notions of affine linear subspace and linear span in affine space $\mathbf{A}^n_{\mathbf{R}}$. State and prove the theorem on dimension of intersection of affine linear subspaces $E, F \subset \mathbf{A}^n_{\mathbf{R}}$ (results on vector spaces may be used freely). Explain briefly why this result is more convenient to state in projective space $\mathbf{P}^n_{\mathbf{R}}$.

(b) Consider $\triangle PQR$ and $\triangle P'Q'R'$ in projective 3-space $\mathbf{P}^3$, where

$$P = (0,0,0,1), \quad Q = (1,0,0,1), \quad R = (0,1,0,1),$$

$$P' = (0,0,1,1), \quad Q' = (1,0,1,1), \quad R' = (0,1,1,1).$$

By writing out the lines PP', QQ', RR', show that $\triangle PQR$ and $\triangle P'Q'R'$ are in perspectivity. [Hint: you may want to take coordinates x, y, z, t and to visualise $PQRP'Q'R'$ as a triangular prism in the affine piece $t = 1$.] Now determine the points of intersection of the corresponding sides of the triangles, for example:

$$QR : (x + y = t, z = 0), \quad Q'R' : (x + y = t, z = t),$$

$$\Longrightarrow QR \cap Q'R' = (1 : -1 : 0 : 0).$$

(c) State and prove Desargues' theorem for two triangles $\triangle PQR$ and $\triangle P'Q'R'$ spanning distinct planes in $\mathbf{P}^3_{\mathbf{R}}$ and in perspectivity from a point O. Verify it in the example of (b).

CONTINUED ...

5. Give a brief mathematical argument for or against each of the following five statements, which may be true, false, or ambiguously formulated.

   (a) Two distinct reflections of Euclidean space $\mathbf{R}^n$ commute if and only if they generate a group of order 4.

   (b) There are motions of the sphere S^2 having exactly 0, 2 or ∞ fixed points.

   (c) Let S_n be the symmetric group on $\{1, 2, \ldots, n\}$ and $G \subset S_n$ a subgroup; if G contains the transposition (12) and an element of order n, then $G = S_n$.

   (d) The group SO(2) of rotations of the Euclidean plane is a normal subgroup of the Euclidean group Eucl(2).

   (e) The theorems of affine geometry are also valid in Euclidean space.

   [*This is not a multiple choice question*, and no credit will be given for "true" or "false" without justification. Credit will be given for the quality of your argument, irrespective of whether you conclude the statement is "true" or "false". Feel free to add your own interpretation of any statement that seems ambiguous.]

END

Answer to Q1.

(a) $\mathbf{E}^2 = \mathbf{R}^2$ with the usual Euclidean metric. A Euclidean motion is an isometry $t\colon \mathbf{E}^2 \to \mathbf{E}^2$, that is, a bijection that preserves distances.

[Definition, 4 marks]

Theorem. *Classification of motions into 4 types:*

1. *Translation* $\mathrm{Transl}(b)$ *in vector* b
2. *Rotation* $\mathrm{Rot}(P, \theta)$ *through* θ *about centre* P
3. *Reflection* $\mathrm{Refl}(L)$ *in line* L
4. *Glide (or glide-reflection)* $\mathrm{Glide}(L, a)$. *This is the reflection in a line* L *composed with a translation in a vector* a *along* L.

[Bookwork, 9 marks]

Following the indication, in the figure, $g_1(P) = Q$ and $g_2(Q) = P$. Therefore $g_2 \circ g_1$ fixes P. Both g_1 and g_2 are opposite motions, so we expect $g_2 \circ g_1$ to be direct. Thus it should be a rotation about centre P. (We also guess that the angle of rotation should be 2θ.)

Now g_1 takes A to a point $A' = g_1(A)$ along the extension of BQ, with $\angle CQA' = \theta$. In turn g_2 takes C to a point C' on the extension of DP, so that $\angle APC' = \theta$, and takes A' to A'' so that $\angle C'PA'' = \theta$, therefore $\angle APA'' = 2\theta$. Therefore $g_2 \circ g_1 = \mathrm{Rot}(P, 2\theta)$.

[Problem 6+6 marks]

[Standard type of question. $4 + 9 + 6 + 6 = 25$]

Answer to Q2.

(a) Write $(\mathbf{x}\cdot_L\mathbf{y})$ for the Lorentz inner product. Then for $P=\mathbf{x}$, $Q=\mathbf{y}$, the hyperbolic distance $d(P,Q)$ is defined by $\cosh d(P,Q)=(\mathbf{x}\cdot_L\mathbf{y})$. (Not required: r.h.s. >0, so arccosh is defined.)

The Lorentz group $\mathrm{SO}^+(2,1)$ is the orthogonal group of the Lorentz inner product preserving the components of the light cone, that is, the group of 3×3 matrixes preserving $\cdot_L$. It acts on $\mathbf{R}^{2,1}$ preserving everything, therefore acts on $\mathcal{H}^2$ preserving distances, that is, as motions.

$\mathrm{SO}^+(2,1)$ acts transitively on points of $\mathcal{H}^2$, on tangent directions at points, and at orthogonal frames at a point.

[Definitions 8 marks]

(b) By the transitivity just discussed, given a right-angled triangle $\triangle PQR$, we can choose coordinates so that $P=(0,0,1)$ and the sides PQ and PR are the positive x and y axes. Then $Q=(\sinh s,0,\cosh s)$ and $R=(0,\sinh t,\cosh t)$, with $s,t>0$. Hence

$$\cosh d(Q,R)=\cosh s\cosh t.$$

This is the required formula.

[Simple case of "main formula", 12 marks]

(c) It is enough to prove: $\cosh s\cosh t>\cosh(\sqrt{s^2+t^2})$, since cosh is increasing. Using the given approximation, the left hand side is

$$(1+\frac{s^2}{2}+\frac{s^4}{24})(1+\frac{t^2}{2}+\frac{t^4}{24})=1+\frac{s^2+t^2}{2}+\frac{s^4}{24}+\frac{t^4}{24}+\frac{s^2t^2}{4}+\cdots$$

and the right hand side

$$1+\frac{s^2+t^2}{2}+\frac{(s^2+t^2)^2}{24}=1+\frac{s^2+t^2}{2}+\frac{s^4}{24}+\frac{t^4}{24}+\frac{2s^2t^2}{24}.$$

We win because the coeff of s^2t^2 is $1/4$ in the first case, and $1/12$ in the second.

[Tricky, 5 marks]

[Easiest possible question on $\mathcal{H}^2$. $8+12+5=20$]

The left hand side is

$$\sum \frac{s^{2i}}{2i!} \times \sum \frac{t^{2j}}{2j!} = \sum_{i,j} \frac{s^{2i}t^{2j}}{(2i!2j!)}$$

The right hand side is

$$\sum \frac{(s^2+t^2)^i}{2i!}$$

Lhs > rhs follows by simple manipulations from $\binom{2d}{2i} > \binom{d}{i}$.

Answer to Q3.

An affine transformation $t\colon \mathbf{A}^2 \to \mathbf{A}^2$ is a bijective map with the affine linear property $t(\lambda x + (1-\lambda)y) = \lambda t(x) + (1-\lambda)t(y)$ for all $x, y \in \mathbf{A}^2$.

t can be written in matrix terms as $t(x) = Ax + b$ where A is a non-singular 2×2 matrix and b a vector.

[Definition 6 marks]

The translation vector is $(2, 0)$. After translating back by $(2, 0)$, the matrix comes from $(1, 0) \mapsto (-2, 1)$ and $(0, 1) \mapsto (a, b-1)$, so that finally

$$t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & a \\ 1 & b-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

If $a + 2b = 2$ then the matrix is singular, so the whole plane is mapped to the line $x + 2y = 1$, so that t is not an affine transformation.

[Problem 8 marks]

Geometric properties are the properties invariant under the allowed transformations. In affine geometry, you can change nonzero distances arbitrarily, e.g., by $(x, y) \mapsto (\lambda x, \mu y)$. Similarly, you can change angles arbitrarily, e.g., by a "shear" $(x, y) \mapsto (x + \lambda y, y)$. Thus distances and angles are not affine properties.

Given 3 points P, Q, R in $\mathbf{A}^2$, whether or not their are collinear is preserved by affine transformations, so is a property defined in affine geometry. If P, Q, R are collinear, the ratio of distances $d(P, Q) : d(P, R)$ is preserved by affine transformations, so likewise, is a property defined in affine geometry.

[Official course ideology, 11 marks]

[Standard type of question 6+8+11=25]

Answer to Q4.

(a) $\mathbf{A}^n = \mathbf{R}^n$. An affine linear subspace is of the form $u_0 + V$ where $u_0 \in \mathbf{R}^n$ and $V \subset \mathbf{R}^n$ is a vector subspace. Given a nonempty set $\Sigma \subset \mathbf{A}^n$, its span $\langle \Sigma \rangle$ is the smallest affine linear subspace containing Σ. In other words, if $u_0 \in \Sigma$ is any point,

$$\langle \Sigma \rangle = u_0 + V, \quad \text{where} \quad V = \langle \Sigma - u_0 \rangle.$$

The theorem says that for E, F affine linear subspaces,

$$\dim E + \dim F = \dim(E \cap F) + \dim \langle E, F \rangle,$$

provided $E \cap F \neq \emptyset$. For the proof, just pick $u_0 \in E \cap F$, so that $E = u_0 + U$, $F = u_0 + V$ for some vector subspaces $U, V \subset \mathbf{R}^n$. Then

$$E \cap F = u_0 + (U \cap V), \quad \langle E, F \rangle = u_0 + (U + V),$$

and the formula therefore follows from that for vector spaces.

The same formula holds for all E, F without exception in $\mathbf{P}^n$, provided we write $\dim \emptyset = -1$. The lecturer seemed to think this was really good.

[Bookwork. 11 marks]

(b) $PP' : (x = y = 0)$, $QQ' : (x = t, y = 0)$ and $RR' : (x = 0, y = t)$ intersect in $O = (0010)$. (By the hint, the sides of the prism are vertical and parallel so meet at (0010).)

Now $PQ : (y = z = 0)$, $P'Q' : (y = 0, z = t)$ intersect in $C = (1000)$,
$PR : (x = z = 0)$, $P'R' : (x = 0, z = t)$ intersect in $B = (0100)$,
and $QR \cap Q'R'$ intersect in $A = (1 : -1 : 0 : 0)$ is given.

[Easier than lectured material. 7 marks]

(c)

Desargues' theorem. *If $\triangle PQR$ and $\triangle P'Q'R'$ are in perspectivity from a point O then the 3 points of intersection*

$$A = QR \cap Q'R', \quad B = PR \cap P'R', \quad C = PQ \cap P'Q'$$

of the corresponding sides are collinear.

Proof. If the two triangles span distinct planes $\Pi = \langle \triangle PQR \rangle \neq \Pi' = \langle \triangle P'Q'R' \rangle$ then this is obvious: $OPQP'Q'$ is a plane so PQ and $P'Q'$ intersect in a point C, etc., and so C is contained in lines in the planes Π, Π', therefore is contained in the line $L = \Pi \cap \Pi'$. Similarly for B, C. Q.E.D.

The above 3 points $(1 : -1 : 0 : 0), (0100), (1000)$ are obviously on $z = t = 0$.

[The easy case of Desargues theorem. 3 marks. $11 + 7 + 7 = 25$]

Answer to Q5.

(a) True. Two possible arguments: by abstract groups $r_1^2 = r_2^2$ and $r_1 r_2 = r_2 r_1$ gives the multiplication table of a 4-group. Or two distinct reflection commute if and only if their defining hyperplanes are perpendicular, and then in suitable coordinates the group is $x, y \mapsto \pm x, \pm y$.

[Easy start, but groups is the last chapter of notes. 5 Marks]

(b) True. A rotation about an axis has 2 fixed points. A reflection in a plane has a whole equator of fixed points. The antipodal inversion has no fixed points.

[Tests $\mathrm{O}(3)$ as group of motions of S^2. 5 Marks]

(c) False, or True depending on interpretation. False: if $n = 2k$ with k odd then an element $g =$ (2-cycle)(k-cycle) has order n. Obviously $G = \langle g, (12) \rangle$ is not transitive on $\{1, 2, \dots, n\}$.

True: if you interpret element g of order n to mean n-cycle, then $G = \langle g, (12) \rangle = S_n$.

Partial proof: Messing around g and (12), you eventually get $g_{ij} : i \mapsto 1, j \mapsto 2$ for each i, j so by the conjugacy principle, it contains $g_{ij}(12)g_{ij}^{-1} = (ij)$. Obviously S_n is generated by all the transpositions.

[Tests conjugacy, 5.4–5 of book. 5 Marks]

(d) False. SO(2) means rotation about the origin 0. If a Euclidean motion g takes $0 \mapsto P = g(0)$ then $g\,\mathrm{SO}(2)g^{-1}$ is the group of rotations around P, so $g\,\mathrm{SO}(2)g^{-1} \neq \mathrm{SO}(2)$, and it's not normal.

[Conjugacy was stressed in 5.4 of book. 5 Marks]

(e) To argue for true: The Erlangen program says that theorems of affine geometry are statements invariant under the affine group. Therefore these statements are invariant under the subgroup Eucl(n), so Euclidean theorems.

To argue for false: in affine geometry, it is a theorem that the affine group is 2-transitive, but that is false for the Euclidean group.

[Test Erlangen program and common sense. 5 Marks]

[No credit for true/false. $5 \times 5 = 25$]