MA243 Geometry

Worksheet 2

Please hand your solutions in on Thursday, Oct 16, 2003, 2:00pm.

A. **Definition of angles**

Let V denote a finite dimensional real vector space and $\cdot$ a scalar product on V, such that for $v \in V$: $\|v\| = \sqrt{v \cdot v}$. To define the angle ϑ between two non-zero vectors $u, v \in V$ by

$$\cos\vartheta = \frac{u \cdot v}{\|u\|\,\|v\|}, \qquad \text{as in Def. 1.2.,}$$

we need to ensure that the right hand side always gives values in the image of $\cos\vartheta$, $\vartheta \in (-\pi, \pi]$. Prove that this is the case, using the results presented in the lectures. Moreover, show that u can be decomposed according to $u = \lambda v + v'$ with $v \cdot v' = 0$ and $\lambda\,\|v\| = \|u\|\cos\vartheta$. Illustrate the latter observation by a drawing.

B. **Orthonormal basis**

Let $P_0 = (3/\sqrt{5},\, 3)$ and $P_1 = (\sqrt{5},\, 3+1/\sqrt{5}) \in \mathbb{R}^2$; find all points $P_2 \in \mathbb{R}^2$ for which $\overrightarrow{P_0P_1}$, $\overrightarrow{P_0P_2}$ is an orthonormal basis of $\mathbb{R}^2$.

C. **General scalar products**

Show that on $\mathbb{R}^3$ (with Cartesian coordinates x_1, x_2, x_3) the following defines a scalar product:

$$\begin{aligned} \forall\, x = (x_1, x_2, x_3),\ y &= (y_1, y_2, y_3) \in \mathbb{R}^3 : \\ x \cdot_A y &:= x_1y_1 + x_2y_2 + 6x_3y_3 - (x_2y_3 + x_3y_2). \end{aligned}$$

Now let $f_1 := (1, 0, 0)$ and $f_2 := (0, 1, 0) \in \mathbb{R}^3$; find all vectors $f_3 \in \mathbb{R}^3$ for which f_1, f_2, f_3 is an orthonormal basis of $\mathbb{R}^3$ with respect to "$\cdot_A$".

D. **Euclidean two-space**

Suppose that $a, b, c \in \mathbb{R}$ satisfy

$$0 < a < b + c, \quad 0 < b < a + c, \quad 0 < c < a + b.$$

Give a geometric construction (with a pair of compasses and a ruler) to prove that there exists a triangle in the Euclidean plane $\mathbb{E}^2$ (that is, $\mathbb{R}^2$ with the usual metric) with sides of length equal to a, b, c.

What special thing happens in your construction if one of the inequalities becomes equality, say, $a = b + c$?

Observe that this shows: A metric space with exactly 3 points is isometric to a subset of $\mathbb{E}^2$.