MA243 Geometry

Worksheet 7

Deadline: Thu 18th Nov 2004, 2:00pm.
Please staple your solutions.

Let $\mathbb{S}^2 := \{(x,\, y,\, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere, with the spherical metric d.

A. **Circles in spherical geometry**

Let $P := \frac{1}{\sqrt{2}}(1,\, 0,\, 1)$, $Q := \frac{1}{\sqrt{2}}(-1,\, 0,\, 1)$ and show that the circle

$$c_{PQ}(t) := \tfrac{1}{\sqrt{2}}(\cos t,\, \sin t,\, 1), \quad t \in (-\pi, \pi]$$

is a curve in $\mathbb{S}^2$ connecting P and Q. Calculate the length of an arc of c_{PQ} between P and Q. Is this the shortest curve connecting P and Q? Why or why not?

B. **Sum of angles in spherical polygons**

By subdividing, prove that the sum of angles in a spherical quadrilateral is $2\pi +$ its area, and generalise to a spherical n-gon.

C. **Composite of spherical reflections**

Let Π_1, Π_2 be two planes of $\mathbb{R}^3$ through the origin, intersecting in a line L. Calculate the composite $\mathrm{Refl}_{\Pi_2} \circ \mathrm{Refl}_{\Pi_1}$ as a motion of $\mathbb{E}^3$. Show how to calculate the composite of two reflections in spherical geometry.

D. **Stereographic projection** (part of a past exam question)

Denote the north pole of $\mathbb{S}^2$ by $N = (0, 0, 1)$ and the equator by $\Pi_0 := \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} \cong \mathbb{R}^2$. Let $\sigma : \mathbb{S}^2 - \{N\} \to \Pi_0$ denote the stereographic projection from the north pole; that is, for $P \in \mathbb{S}^2 - \{N\}$, NP the straight line through N and P in $\mathbb{R}^3$, $\sigma(P) := NP \cap \Pi_0$. For the great circle $L := \Pi_0 \cap \mathbb{S}^2$, Refl_L is the spherical reflection in L. Let

$$\iota := \sigma \circ \mathrm{Refl}_L \circ \sigma^{-1} \colon \quad \Pi_0 - \{O\} \longrightarrow \Pi_0 - \{O\}.$$

Observe: $\overrightarrow{O\iota(P)} = \lambda \overrightarrow{OP}$ with $\lambda > 0$ for all $P \in \Pi_0 - \{O\}$ (no proof required). Argue that ι is the inversion in the unit circle L, i.e. show

$$\forall\, P \in \Pi_0 - \{O\} \colon \quad d_{\mathrm{Eucl}}(O, P) \cdot d_{\mathrm{Eucl}}(O, \iota(P)) = 1,$$

where d_{eucl} is the Euclidean distance on $\Pi_0 \subset \mathbb{R}^3$. [Hint: Observe that O, N, S, P, $\iota(P)$ are coplanar ($S := (0, 0, -1) =$ south pole), and show $\angle OPN = \angle \iota(P)NO$. Hence express $\tan(\angle OPN)$ in two different ways.]