MA243 Geometry

Worksheet 8

Deadline: Thu 25th Nov 2004, 2:00pm.
Please staple your solutions.

Let $\mathcal{H}^2 = \{(t, x_1, x_2)^T \in \mathbb{R}^{1,2} \mid -t^2 + x_1^2 + x_2^2 = -1,\, t > 0\}$.

A. **Parametrisation of hyperbolic lines**

Let $c : \mathbb{R} \to \mathcal{H}^2$ denote the parametrisation of a hyperbolic line $L = \Pi \cap \mathcal{H}^2$ (where $\Pi \subset \mathbb{R}^3$ is a plane through 0), such that $\dot{c}(\tau) := \frac{d}{d\tau} c(\tau)$ never vanishes. Show that Π is generated by $c(0)$ and $\dot{c}(0)$.

[Hint: Study the derivative of $c(\tau) \cdot_L c(\tau)$ with respect to τ.]

B. **Distance between line and point in hyperbolic space**

Let $L = \Pi \cap \mathcal{H}^2$ denote a hyperbolic line, and $P = (1, 0, 0)^T \in \mathcal{H}^2$ with $P \notin L$. Show that there is a point $Q \in L$ at minimal distance from P.

Without loss of generality let $Q = (\cosh\alpha, \sinh\alpha, 0)$ for some $\alpha \in \mathbb{R}$. Now assume that L is parametrised by $c\colon \mathbb{R} \to \mathcal{H}^2$ with $c(\tau) = (t(\tau), x_1(\tau), x_2(\tau))^T$ such that $\dot{c}(\tau)$ never vanishes, and with $Q = c(0)$. Show that $\dot{c}(0) = (0, 0, \xi)$ for some $\xi \in \mathbb{R} - \{0\}$.

[Hint: Show that $t(\tau)$ and therefore also $x_1^2(\tau) + x_2^2(\tau)$ has a local minimum in $\tau = 0$. Then calculate the derivative of the latter with respect to τ.]

C. **Dropping a perpendicular in hyperbolic space**

Let $L = \Pi \cap \mathcal{H}^2$ denote a hyperbolic line ($\Pi \subset \mathbb{R}^3$ a plane through 0), and $P \in \mathcal{H}^2$ with $P \notin L$. Show that there is a unique point $Q \in L$ such that the hyperbolic line PQ and L intersect in a right angle.

[Hint: Without loss of generality, $P = (1, 0, 0)^T$. Then the point Q is the one constructed in Question B. To show this, use the results of Question A to parametrise L conveniently and to show that it is mapped to itself by reflection in PQ. Prove uniqueness by contradiction: If two such points Q_i exist, calculate the angle sum of the hyperbolic triangle ΔPQ_1Q_2.]

D. **Ultraparallel lines** (part of a past exam question)

Give an example of a pair of ultraparallel hyperbolic lines $L, L' \subset \mathcal{H}^2$ (that is, L, L' "intersect in their point at infinity" or "approach each other asymptotically") by specifying planes $0 \in \Pi$, $\Pi' \subset \mathbb{R}^{1,2}$ with $L = \Pi \cap \mathcal{H}^2$, $L' = \Pi' \cap \mathcal{H}^2$, $L \neq L'$.