MA243 Geometry

Worksheet 9

Deadline: Thu 2nd Dec 2004, 2:00pm.
Please staple your solutions.

A. **Coordinate description for lines in $\mathbb{P}^2_{\mathbb{R}}$**

Consider the three lines in $\mathbb{P}^2_{\mathbb{R}}$ given by

$$L_1 : 2x_1 - 2x_2 = 3x_0, \quad L_2 : 3x_1 + x_2 + x_0 = 0, \quad L_3 : x_1 - x_2 - 2x_0 = 0.$$

Calculate the points of intersection $L_1 \cap L_2$ and $L_1 \cap L_3$ as ratios $(a : b : c)$.

B. **Intersection of lines in $\mathbb{P}^n_{\mathbb{R}}$** (part of a past exam question)

Argue, with proof, for the truth or falsity of the following statements. No credit will be given for any guess without argument. Results proven in the lectures may be cited without proof.

(a) Any two projective lines in $\mathbb{P}^2_{\mathbb{R}}$ intersect in at least one point.

(b) Any two projective lines in $\mathbb{P}^3_{\mathbb{R}}$ intersect in at least one point.

C. **Projective transformations in an affine chart**

Consider a general projective transformation on $\mathbb{P}^n_{\mathbb{R}}$, i.e., a map $\alpha\colon [x] \mapsto [Ax]$ with $A = (A_{i,j})_{i,j=0,\dots,n} \in \mathrm{Gl}(n+1)$. Where it is well-defined, determine the induced map on the affine chart $U_0 = \{x_0 \neq 0\} \cong \mathbb{A}^n$ with respect to the affine coordinates $\xi_1, \dots, \xi_n$, where

$$U_0 \ni (x_0 : \cdots : x_n) \longmapsto (\xi_1, \dots, \xi_n)^T = (x_1/x_0, \dots, x_n/x_0)^T \in \mathbb{A}^n.$$

Rewrite your result using

$$\begin{aligned} a &:= (A_{i,j})_{i,j=1,\dots,n} \in \mathrm{Mat}(n \times n, \mathbb{R}), \\ b &:= (A_{1,0}, \dots, A_{n,0})^T \in \mathrm{Mat}(n \times 1, \mathbb{R}), \\ c &:= (A_{0,1}, \dots, A_{0,n}) \in \mathrm{Mat}(1 \times n, \mathbb{R}), \qquad d := A_{0,0} \in \mathbb{R}. \end{aligned}$$

D. **Projective transformations in $\mathbb{P}^2_{\mathbb{R}}$**

All projective transformations $\alpha\colon \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ are given by $\alpha([x]) = [Ax]$ with $A \in \mathrm{Gl}(3)$ (where $A \sim \lambda A$ for $\lambda \in \mathbb{R} - \{0\}$). Write down in matrix form all the projective transformations $\alpha\colon \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ (if any) that fix all four of the points

$$(1:0:0), \quad (0:1:0), \quad (0:0:1) \quad \text{and} \quad (1:1:0).$$