Answers to Worksheet 5

A. Choose, e.g., $P := (0,0)$ and $Q := (0,1)$ with images $P' = (3,-4)$ and $Q' = (3,-5)$ under f. Hence $\overrightarrow{PQ} = -\overrightarrow{P'Q'}$, and L must be perpendicular to $PQ||P'Q'$ of equal distance to both P and P'. Therefore, $L := \{(x,y) \in \mathbb{R}^2 \mid y = -2\}$, and $\mathbf{b} := (3,0)$ yields $f = \mathrm{Glide}(L,\mathbf{b})$.

B. Since $\mathrm{Rot}_{P_2}(\theta)$, $\mathrm{Rot}_{P_1}(-\theta)$ and $\mathrm{Trans}_{\overrightarrow{QP_2}}$ are all direct motions, both $f := \mathrm{Rot}_{P_2}(\theta) \circ \mathrm{Rot}_{P_1}(-\theta)$ and $g := \mathrm{Trans}_{\overrightarrow{QP_2}}$ are direct. Hence by Lemma 3.1 it suffices to show that f and g agree on two distinct points. First, $f(Q) = \mathrm{Rot}_{P_2}(\theta) \circ \mathrm{Rot}_{P_1}(-\theta)Q = \mathrm{Rot}_{P_2}(\theta)P_2 = P_2$ and $g(Q) = \mathrm{Trans}_{\overrightarrow{QP_2}}Q = P_2$. Second, $R\colon = f(P_1) = \mathrm{Rot}_{P_2}(\theta)(P_1)$, and it is easy to see that $\square P_1QP_2R$ is a parallelogramme. In particular, $\overrightarrow{QP_2} = \overrightarrow{P_1R}$. Hence $R = \mathrm{Trans}_{\overrightarrow{QP_2}}P_1 = g(P_1)$, proving the claim.

C. The composite of reflecting in two opposite vertical sides gives the horizontal translation by $(2,0)$, and similarly for $(0,2)$. Let T be the translation group by $(2\mathbb{Z})^2$. Then T covers the plane with images of $2 \times \square$, the square of side length 2. Now reflecting in the top horizontal side and the right-hand vertical obviously generates 4 copies of $\square$ that exactly cover $2 \times \square$.

If $\mathrm{Trans}_\lambda \in T$ and $g \in G$, then $g\mathrm{Trans}_\lambda g^{-1} = \mathrm{Trans}_{g(\lambda)} \in T$. This proves that T is normal.

D. The first observation in the hint follows from $[d(\widehat{x},\widehat{y})]^2 = (x_1-y_1)^2 + (x_2-y_2)^2 + 0 = [d(x,y)]^2 = [d(h(x),h(y))]^2$ since $h \in I(\mathbb{E}^3)$.
Hence, since $\widehat{h}(\widehat{x}) = (h_1(x),\, h_2(x))^T$ and analogously for $\widehat{h}(\widehat{y})$,

$$\begin{aligned} [d(\widehat{x},\widehat{y})]^2 - [d(\widehat{h}(\widehat{x}),\widehat{h}(\widehat{y}))]^2 &= [d(h(x),h(y))]^2 - [d(\widehat{h}(\widehat{x}),\widehat{h}(\widehat{y}))]^2 \\ &= (h_3(x) - h_3(y))^2. \end{aligned}$$

Therefore, $\widehat{h} \in I(\mathbb{E}^2)$ iff $h_3(x) - h_3(y) = 0$ for all $x,\, y \in \Pi$, i.e. iff $h(\Pi)||\Pi$.