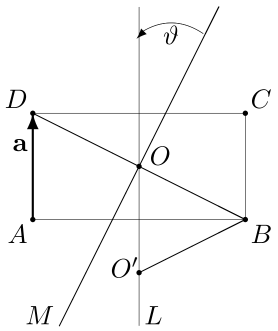
Answers to Worksheet 6

A. $\Pi = \langle L, P\rangle$ and $\Pi' = \langle M, P\rangle$ are two distinct planes in $\mathbb{A}^3$, hence $\langle \Pi, \Pi'\rangle = \mathbb{A}^3$. Π and Π' have a common point P, so by Theorem 2.7, they intersect exactly in a line N. This is the unique line through P coplanar with L and M. Any line through P meeting L (or M) is coplanar with L (or M), therefore is N if it meets both. Usually N meets L and M, but it could be parallel to one or the other.

B.



$g^{-1} = \mathrm{Gli}(L, -a) \circ \mathrm{Refl}_M$, so following the hint, let $g^{-1}(O) = O'$, i.e. $a = \overrightarrow{O'O}$. Moreover, $g^{-1}(B) = \mathrm{Gli}(L, -a)(D) = B$. g is direct and $g(B) = B$, hence $g = \mathrm{Rot}_B(\varphi)$ for some angle φ. AB is the perpendicular bisector of $[O, O']$, hence $\varphi = \angle O'BO = 2\angle ABO$. Since $\vartheta = \angle(M, L)$, using the sum π of angles in a triangle and $AB \perp L$, $BD \perp M$: $\angle OBA = \vartheta$, hence $g = \mathrm{Rot}_B(-2\vartheta)$.

C. If $O = (0,0,0)$ and $A = (1,1,1)$, then $f(O) = A$ and $f(A) = A + \overrightarrow{OA}$, so the line $L = OA$ is taken to itself. Hence the answer is rotation about L, followed by translation through $b := \overrightarrow{OA} = (1,1,1)$. The rotation is given by $(x, y, z) \longmapsto (z, x, y)$, hence it has order three and the angle of rotation is $\vartheta = 2\pi/3$.

D. d is obviously positive definite and symmetric, and the triangle inequality holds.

If you try to put A, B, D isometrically into $\mathbb{E}^n$, then they must be collinear with B the midpoint of AD (since $2 = d(A, D) = d(A, B) + d(B, D) = 1 + 1$). The same for A, C, D. Therefore the isometric condition forces B and C to go to the same point, a contradiction, since $d(B, C) = 1$.

Suppose the sphere S^2_r has radius $r = 2/\pi$, so that the unit of length equals $1/4$ of the circumference; then take the north pole, south pole and two points on the equator. (You can't do it on a sphere of any other radius. It's interesting to see that this finite metric space X already has its own definite "curvature".)