Answers to Worksheet 7

A. Clearly, for every $t \in (-\pi, \pi]$, $\|c_{PQ}(t)\|^2 = 1/2(\cos^2 t + \sin^2 t + 1) = 1$, and $c_{PQ}(0) = P$, $c_{PQ}(\pi) = Q$, such that c_{PQ} is a curve with properties as claimed.

The length of an arc of c_{PQ} connecting P and Q is

$$\int_0^\pi \left\| \frac{d}{dt} c_{PQ}(t) \right\| dt = \int_0^\pi \frac{1}{\sqrt{2}} \|(-\sin t,\, \cos t,\, 0)^T\| dt = \frac{\pi}{\sqrt{2}}.$$

On the other hand, $\cos d(P,Q) = p \cdot q = 0$ for the spherical distance $d(P,Q)$ between P and Q. Hence $d(P,Q) = \pi/2 < \pi/\sqrt{2}$, and the shortest curve connecting P and Q is the great circle obtained by intersecting $\mathbb{S}^2$ with the plane $\langle O,\, P,\, Q\rangle$, not c_{PQ}. This is hardly a surprise, since c_{PQ} is a circle, but not a "great" circle.

B. The sum of angles in a spherical n-gon (not overlapping itself) is $(n-2)\pi + \text{area}$: Just subdivide and use the fact that both area and sums of angles are simply additive for polygons.

C. Two distinct planes Π_1 and Π_2 through O in $\mathbb{R}^3$ meet along a line L at a dihedral angle ϑ (measured from Π_1 to Π_2). Then $\mathrm{Refl}_{\Pi_2} \circ \mathrm{Refl}_{\Pi_1} = \mathrm{Rot}_L(2\vartheta)$: The points on L are fixed, and all the motion takes place in the plane orthogonal to L, so the composite is a rotation. To check the angle of rotation 2ϑ, consider the image of any point on Π_1.

For the last part, great circles L_1 and L_2 are cut out on $\mathbb{S}^2$ by planes Π_1, Π_2 of $\mathbb{R}^3$, and the reflections are restrictions of the reflections of $\mathbb{R}^3$. Their composite is a rotation of $\mathbb{R}^3$, as just described, so the composite of the two reflections of $\mathbb{S}^2$ is also the rotation through 2ϑ about the axis through the two points of intersection of L_1 and L_2.

D. Set $P' := \sigma^{-1}(P)$ and $P'' := \mathrm{Refl}_L \circ \sigma^{-1}(P)$ and note that O, N, S, P, P', P'', $\iota(P)$ are all coplanar by construction. For $\triangle NPO$ and $\triangle NP''S$ we have $\angle NOP = \angle SP''N = \pi/2$ (Thales circle) and $\angle PNO = \angle NSP''$ (reflection). Hence also $\angle OPN = \angle P''NS = \angle \iota(P)NO =: \vartheta$. Therefore, using $\triangle NPO$,

$$\tan \vartheta = d_{eucl}(O,N)/d_{eucl}(O,P) = 1/d_{eucl}(O,P).$$

Using the triangle $\triangle N\iota(P)O$,

$$\tan \vartheta = d_{eucl}(O,\iota(P))/d_{eucl}(O,N) = d_{eucl}(O,\iota(P))$$

This proves $d_{eucl}(O,P) \cdot d_{eucl}(O,\iota(P)) = 1$.