Answers to Worksheet 8

A. For all $\tau \in \mathbb{R}$, $c(\tau) \in \Pi$, hence $c(0)$, $\dot{c}(0) \in \Pi$. The plane Π is two dimensional, so it will suffice to show that $c(0)$ and $\dot{c}(0)$ are linearly independent. However, with respect to the Lorentz scalar product, for all $\tau \in \mathbb{R}$ we have $c(\tau) \cdot_L c(\tau) = -1$. Take the derivative of this equation with respect to τ and evaluate at $\tau = 0$ to find that $c(0)$ is Lorentz orthogonal to $\dot{c}(0)$. Since neither of the two vectors vanishes, this proves the claim.

B. For $(t, x_1, x_2)^T \in L$, $\cosh d((t, x_1, x_2)^T, P)) = t > 0$. On L, t is bounded below and increases as $x_1^2 + x_2^2$ increases. Hence by continuity t obtains a minimal value in some $Q \in L$. Since cosh is strictly increasing for $t > 0$, this is a point in L at minimal distance from P.

In the given parametrisation, $\dot{c}(0) = (\dot{t}(0), \dot{x}_1(0), \dot{x}_2(0))^T$, and $\dot{t}(0) = 0$ since t has a local minimum in $\tau = 0$ by construction. Furthermore,

$$\forall\, \tau \in \mathbb{R}: \quad x_1^2(\tau) + x_2^2(\tau) = -1 + t^2(\tau) \geq -1 + t^2(0) = x_1^2(0) + x_2^2(0),$$

such that $x_1^2(\tau) + x_2^2(\tau)$ also obtains a minimum in $\tau = 0$. Hence $0 = \dot{x}_1(0) \cdot x_1(0) + \dot{x}_2(0) \cdot x_2(0) = \dot{x}_1(0) \cdot x_1(0)$. Since $P \neq Q$, we find $x_1(0) \neq 0$, hence $\dot{x}_1(0) = 0$, and the claim follows.

C. Without loss of generality assume that we are in the situation described in exercise 8.B. We also use the notations introduced there. Then $\dot{c}(0)$ is a multiple of $(0, 0, 1)^T$, so that by exercise 8.A, $\Pi = \mathrm{span}_{\mathbb{R}}\{q, (0, 0, 1)^T\}$. Hence without loss of generality $c(t) = \cosh(t)q + \sinh(t)(0, 0, 1)^T$ and in particular $x_2(t) = \sinh(t)$. Since $c(-t) = \cosh(t)q - \sinh(t)(0, 0, 1)^T$ this means that L is mapped into itself under the hyperbolic reflection in PQ which acts by $x_2 \mapsto -x_2$, leaving all other coordinates invariant. This proves that PQ and L intersect at a right angle.

Uniqueness follows since a triangle ΔPQ_1Q_2 as suggested in the hint would have angle sum greater than π, which is impossible in hyperbolic space.

D. Without loss of generality, $L = \Pi \cap \mathcal{H}^2$ with $\Pi = \big\{(t,\, x_1,\, x_2)^T \in \mathbb{R}^{1,2} \mid x_2 = 0\big\}$. Π can be generated by the two null vectors $l_\infty^\pm = (\pm 1,\, 1,\, 0)^T$ which by definition give asymptotic directions of L. Let $\Pi' := \big\langle (1,\, 1,\, 0)^T, (1,\, 0,\, 1)^T \big\rangle$, then $L' := \Pi' \cap \mathcal{H}^2 \neq \emptyset$ also has l_∞^+ as asymptotic direction and therefore is ultraparallel to L by definition.