Classification of indecomposable Gorenstein Fano 3-folds

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*Lecture notes taken by TAKAGI Hiromichi, English draft by Miles Reid. The current draft is provisional and not intended for general distribution.

1 Introduction

Most of the material of Sections 1–2 is contained in Iskovskikh's two papers on Fano 3-folds [I1], [I2].

1.1 Fano 3-folds

A Fano variety is a projective variety X with ample $-K_X$. Here $-K_X$ is the anticanonical divisor, that is, $\det T_X = \mathcal{O}_X(-K_X)$. Nonsingular Fano 3-folds can be classified. (Note however that nonsingular Fanos with $\dim X \geq 4$ and $\mathbb{Q}$-Fano 3-folds are harder.)

We set

$$\begin{aligned} \rho &:= \operatorname{rank} H^2(X, \mathbb{Z}) = \operatorname{rank} \operatorname{Pic} X, \quad \text{the } \textit{Picard rank} \text{ of } X, \\ r &:= \max\bigl\{r \bigm| -K_X = rH\bigr\}, \quad \text{the } \textit{Fano index} \text{ of } X. \end{aligned}$$

Over an algebraically closed field, the case $\rho = r = 1$ occurs for the first time in dimension 3. (Over nonclosed fields, there are also surfaces with $\rho = r = 1$). Nonsingular Fano 3-folds with $\rho = r = 1$ were classified by Fano and Iskovskikh [I2].

1.2 Reminder

Curves: Clearly $\deg -K_X = 2 - 2g > 0$ happens if and only if $g = 0$, and $g = 0$ implies $X \cong \mathbb{P}^1$. Then $-K_X = 2P$, so that $\rho = 1$ and $r = 2$.

Surfaces: By Noether's formula $c_1^2 + c_2 = 12\chi(\mathcal{O}_X)$, a Fano surface has Hodge diamond

$$\begin{matrix} 0 & 0 & 1 \\ 0 & \rho & 0 \\ 1 & 0 & 0 \end{matrix} \quad \text{with} \quad (-K_X)^2 + \rho = 10.$$

Therefore $\rho = 1$ implies $(-K_X)^2 = 9$. Now Poincaré duality implies that the intersection pairing on $H^2(X, \mathbb{Z})$ is unimodular. Therefore $-K_X = 3H$, so that $r = 3$. In fact one sees that $X \cong \mathbb{P}^2$.

1.3 Iskovskikh's table of Fano 3-folds

In dimension 3, Riemann–Roch gives only $c_1c_2 = 24\chi(\mathcal{O}_X) = 24$, which of course does not exclude the possibility $\rho = r = 1$, and says nothing at all about $(-K_X)^3$. One of the main conclusions of the theory is that

$$(-K_X)^3 = 2g - 2,$$

with g taking only 10 possible values, namely $g = 2, 3, \ldots, 10, 12$.

Table 1 is taken from Iskovskikh and Shokurov [IS]. The varieties in cases 6–10 and 12 were classically constructed by double projection from a line $L \subset V_{2g-2}$. In the final column, $p = \frac{1}{2}B_3$ is the dimension of the intermediate Jacobian of V.

g	anticanonical model	p
2	$V_2 \xrightarrow{\text{2-to-1}} \mathbb{P}^3$ double cover branched in a sextic	52
3	$V_4 \subset \mathbb{P}^4$, or $V_4' \xrightarrow{\text{2-to-1}} Q \subset \mathbb{P}^4$	30
4	complete intersection $(2) \cap (3) \subset \mathbb{P}^5$	21
5	complete intersection $(2) \cap (2) \cap (2) \subset \mathbb{P}^6$	14
6	$V_{10} \subset \mathbb{P}^7$	10
7	etc.	7
8		5
9		3
10		2
12		0

Table 1: Cases 6–10 and 12 are varieties $V_{2g-2}^3 \subset \mathbb{P}^{g+1}$.

1.4 Boundedness

Is there a convincing reason for g to be bounded? Two of the central topics of these lecture notes are the classification of the cases $g \geq 6$ and the proof that g is bounded.

1.5 A preview of the method

We study Fano varieties in cases $g = 6, \dots, 10$ and 12 in terms of vector bundles on surfaces. No specially clever methods of constructing vector bundles are involved (or indeed, available): all the bundles we need come as extensions from linear series on curve sections. The following fairly transparent example illustrated our methods very neatly.

Example 1.1 We prove that a del Pezzo surface of degree 5 is a linear section of the Plücker embedding of the Grassmann variety $\mathrm{Grass}(2,5)$; that is, every S_5 fits into a Cartesian square

$$\begin{array}{ccc} S_5 & \subset & \mathbb{P}^5 \\ \cap & & \cap \\ \mathrm{Grass}(2,5) & \subset & \mathbb{P}^9 \end{array}$$

The main idea is to construct a rank 2 vector bundle $\mathcal{E}$ on S_5 having

$$\det \mathcal{E} = \mathcal{O}_S(1) = \mathcal{O}_S(-K_S) \quad \text{and} \quad h^0(\mathcal{E}) = 5.$$

For this, let $E \in |-K_S|$ be a general element, which is a nonsingular elliptic curve, and let $\xi = P + Q$ be a divisor of degree 2 on $\mathcal{E}$. Then $\mathcal{O}_E(\xi)$ is generated by 2 global sections, which provides a surjective homomorphism $2\mathcal{O}_E \twoheadrightarrow \mathcal{O}_E(\xi)$, and composing with restriction from S gives an exact sequence

$$0 \to \mathcal{E}^\vee \to 2\mathcal{O}_S \to i_*\mathcal{O}_E(\xi) \to 0. \tag{1.1}$$

The kernel $\mathcal{E}^\vee$ is a rank 2 vector bundle on S with $\det \mathcal{E}^\vee = \mathcal{O}_S(-1)$. Rather than $\mathcal{E}^\vee$, we consider the dual $\mathcal{E}$, which has $\det \mathcal{E} = \mathcal{O}_S(1) = \mathcal{O}_S(-K_S)$, and fits into the dual exact sequence

$$\begin{array}{ll} \leftarrow \ \mathcal{E} \ \leftarrow \ 2\mathcal{O}_S \ \leftarrow \ 0 & \\ \qquad\qquad\quad 0 & \leftarrow \mathcal{E}xt^1(i_*\mathcal{O}_E(\xi), \mathcal{O}_S) \leftarrow \end{array} \tag{1.2}$$

Calculating the $\mathcal{E}xt^1$ using the adjunction formula gives $\mathcal{E}xt^1(i_*\mathcal{O}_E(\xi), \mathcal{O}_S) = i_*\mathcal{O}_E(h - \xi)$, where $h = \mathcal{O}_E(1)$ is the hyperplane section divisor. Since $h^0(E, h - \xi) = 3$, we get $h^0(\mathcal{E}) = 5$. From this, it is not hard to complete the proof that $\mathcal{E}$ is generated by its H^0, and that these section define a morphism to $\mathrm{Grass}(2,5)$ that embeds S as a linear section.

2 Basic properties of Gorenstein Fano 3-folds

2.1 Free linear system

Let X be a Fano 3-fold with at worst Gorenstein canonical singularities (see Reid [R1], [YPG]). We set

$$g := \frac{1}{2}(-K_X)^3 + 1, \quad \text{so that} \quad \dim|-K_X| = \frac{1}{2}(-K_X)^3 + 2 = g + 1.$$

Our starting point is the following result:

Theorem 2.1 (Shokurov–Reid) *The anticanonical linear system $|-K_X|$ contains a surface S with at worst rational double points (RDPs).*

By adjunction and vanishing, S is a K3 surface (that is, $K_S \sim 0$ and $q(S) = 0$). Now $(-K_X)_{|S}$ is ample on S, so that $(S, (-K_X)_{|S})$ is a polarized K3 surface. Moreover, $|-K_X|$ cuts out the complete linear system $|(-K_X)_{|S}|$; this follows by $H^1(\mathcal{O}_X) = 0$ using the short exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(-K_X) \to \mathcal{O}_S\left((-K_X)_{|S}\right) \to 0.$$

Principle A *Practically all properties of $|-K_X|$ can be understood in terms of linear systems on the K3 surface S.*

We recall the theory of linear systems on K3 surfaces, as studied for example in Saint-Donat [SD]. Let (S, h) be a polarized K3 surface; that is, S is nonsingular and h is nef and big.[1] Then $|h|$ is a free linear system, with the following exceptional case

$$\text{there exists an elliptic pencil } |E| \text{ on } S \text{ with } h \cdot E = 1. \tag{2.1}$$

In this case, (S, H) is called *monogonal*: since h has degree 1 on each fibre of the elliptic pencil $|E|$, it has a base point on it. It follows that $|h|$ has a base curve Γ that is a section of $|E|$, and then $h \sim (g + 1)E + \Gamma$.

Corollary 2.2 *If $|h|$ has no base curve, it is free.*

[1] At this point we should put a remark that the theory for nonsingular S goes through with only minor changes for S with RDPs. Compare Section 2.2.

Corollary 2.3 *If X is a nonsingular Fano 3-fold with $\rho = r = 1$ then $|-K_X|$ is free.*

This follows by Moishezon's theorem, see [M] and 5.3, page 28.

Corollary 2.4 *If X is an indecomposable Fano 3-fold then $|-K_X|$ is free.*

The indecomposable condition[2] is the natural extension to the singular case of the assumption $\rho = r = 1$:

Definition 2.5 A Fano 3-fold X is *indecomposable* if and only if $|-K_X|$ has no decomposition into mobile linear systems. In other words, there is no decomposition $-K_X \sim A + B$ with A, B Weil divisors such that $\dim |A| > 0$ and $\dim |B| > 0$.

Suppose that (S, h) is not monogonal. Then the general element $C \in |h|$ is nonsingular and $K_C = h_{|C}$ by the adjunction formula. Moreover, by the vanishing $h^1(\mathcal{O}_S) = 0$, the restriction

$$H^0(\mathcal{O}_S(h)) \twoheadrightarrow H^0(\mathcal{O}_C(K_C))$$

is surjective.

Principle B *Practically all properties of $|h|$ on a K3 surface follow from those of the canonical linear system $|K_C|$ on a curve $C \in |h|$.*

Principles A–B amount to a ladder structure:

$$(X, -K_X) \rightsquigarrow (S, h) \rightsquigarrow (C, K_C).$$

For example, at each step of the ladder the degree is given by

$$(-K_X)^3 = h^2 = \deg K_C = 2g - 2.$$

2.2 The projective model of a K3 surface

It is a well known consequence of RR that

$$K_C \text{ is very ample} \iff C \text{ is nonhyperelliptic.}$$

[2]There is some repetition between the material here and that of Definition 5.2 and Section 7; this needs tidying up.

Here by definition a curve of genus $g \geq 2$ is *hyperelliptic* (or *bigonal*) if its canonical linear system defines a double cover onto the rational normal curve R_{g-1}:

$$\Phi_{|K_C|}\colon C \xrightarrow{\text{2-to-1}} R_{g-1} \subset \mathbb{P}^{g-1}.$$

A *canonical curve* $C_{2g-2} \subset \mathbb{P}^{g-1}$ is the isomorphic image $C_{2g-2} = \Phi_{|K_C|}(C)$ of a nonhyperelliptic curve C of genus g.

A curve on S with $h \cdot E = 0$ is necessarily a -2-curve; that is, $E \cong \mathbb{P}^1$ and $E^2 = -2$. There are at most finitely many such curves, and there is a birational morphism $(S, h) \to (\overline{S}, \overline{h})$ that contracts all of these to RDPs, and is an isomorphism outside.

According to Saint-Donat [SD], $\overline{h}$ is very ample if and only if the following conditions hold:

(1) $h^2 = 2g - 2$ with $g \geq 3$;

(2) $h \cdot E \geq 3$ for each elliptic curve on S;

(3) if $h^2 = 8$ (that is, $g = 5$) then $h \neq 2h_0$ with $h_0^2 = 2$.

In this case, $\Phi_{|h|}(S) = \Phi_{|\overline{h}|}(\overline{S}) = S_{2g-2} \subset \mathbb{P}^g$ is called the *projective model* of (S, h).

Otherwise, we have the hyperelliptic case:

$$\Phi_{|\overline{h}|}\colon \overline{S} \xrightarrow{\text{2-to-1}} F_{g-1} \subset \mathbb{P}^g,$$

where F_{g-1} is a surface of minimum degree $g - 1$ (see Appendix A). The possible cases for F_{g-1} are as follows:

(a) $g = 2$ and $F_{g-1} = \mathbb{P}^2$. Then S is a double cover of $\mathbb{P}^2$ branched in a sextic.

(b) $g \geq 3$ and F_{g-1} is a $\mathbb{P}^1$-bundle; then the composite $S \to F_{g-1} \to \mathbb{P}^1$ is an elliptic fibration.

(b′) $3 \leq g \leq 5$ and F_{g-1} is the cone over a rational normal curve R_{g-1}. Then the composite rational map $S \to F_{g-1} \dashrightarrow \mathbb{P}^1$ is again an elliptic fibration.

(c) $g = 5$ and $F_{g-1} \subset \mathbb{P}^5$ is the Veronese surface. In this case $h = 2h_0$, and the map Φ_h factors as the composite

$$\Phi_h = v_2 \circ \Phi_{h_0}\colon S \to \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$$

where v_2 is the Veronese embedding.

2.3 The anticanonical model of a Fano 3-fold

In what follows, we assume that X has Gorenstein canonical singularities and $-K_X$ is ample; we suppose also that $|-K_X|$ is free. Then one of the following two conditions holds:

(1) $|-K_X|$ is very ample; or

(2) $\Phi_{|-K_X|}\colon X \xrightarrow{\text{2-to-1}} T_{g-1} \subset \mathbb{P}^{g+1}$ is generically a double cover, and its image is a 3-fold $T_{g-1} \subset \mathbb{P}^{g+1}$ of minimal degree.

In most cases, T_{g-1} is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ (see Appendix A), and then X indecomposable excludes case (2).

Corollary 2.6 *If $\rho = r = 1$ and $g \geq 4$ then $|-K_X|$ is very ample.*

Corollary 2.7 *If $g \geq 4$ and X is indecomposable then $|-K_X|$ is very ample.*

If $|-K_X|$ is very ample, the image

$$\Phi_{|-K_X|}(X) = X_{2g-2} \subset \mathbb{P}^{g+1}$$

is called the *anticanonical model* of X. Note that in this case the ladder structure of Principles A–B takes the form

$$\begin{array}{ccll} X_{2g-2} & \subset & \mathbb{P}^{g+1} & \text{anticanonical Fano 3-fold} \\ \bigcup & & \bigcup & \\ S_{2g-2} & \subset & \mathbb{P}^{g} & \text{K3 surface} \\ \bigcup & & \bigcup & \\ C_{2g-2} & \subset & \mathbb{P}^{g-1} & \text{canonical curve} \end{array}$$

where each step is a hyperplane section.

Appendix A: Varieties of minimal degree

Recall that if $V \subset \mathbb{P}^N$ is an irreducible n-fold spanning $\mathbb{P}^N$, its general zero dimensional section is a finite set of points spanning $\mathbb{P}^{N-n}$, so that we must have $\deg V \geq N + 1 - n$. A *variety of minimal degree* is an irreducible polarized variety (V, L) with

$$\Delta(V, L) := \deg_L V + \dim V + 1 - h^0(L) = 0.$$

It is known that L is then very ample, and that (V, L) is one of the following:

(1) $(\mathbb{P}^n, \mathcal{O}(1))$, that is, projective space itself.

(2) $(Q^n, \mathcal{O}(1))$, that is, a nonsingular quadric hypersurface $Q \subset \mathbb{P}^{n+1}$.

(3) $(\mathbb{P}^2, \mathcal{O}(2))$, that is, the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$.

(4) $(\mathbb{P}^1, \mathcal{O}(d))$, that is, a rational normal curve of degree $d \geq 1$.

(5) A rational normal scroll $(\mathbb{F}(a_1, \dots, a_n), \mathcal{O}(1))$ with all $a_i > 0$, where

$$\mathbb{F}(a_1, \dots, a_n) = \mathbb{P}_{\mathbb{P}^1}\Big(\bigoplus_{i=1}^n \mathcal{O}(a_i)\Big) \to \mathbb{P}^1$$

is the $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ obtained as the projectivization of the vector bundle $\bigoplus_{i=1}^n \mathcal{O}(a_i)$, and $\mathcal{O}(1)$ is the tautological sheaf of this projectivization.

(6) A cone over any of the varieties (2–5).

2.4 Defining equations of X_{2g-2}

By a famous theorem of Castelnuovo and Petri, a canonical curve $C_{2g-2} \subset \mathbb{P}^{g-1}$ is a scheme theoretic intersection of (many) quadrics, with the following exceptional cases:

(1) C is trigonal (that is, C has a g^1_3, or a 3-to-1 cover $C \to \mathbb{P}^1$); or

(2) $g \leq 4$; or

(3) $g = 6$ and C is a plane quintic curve.

This theorem is a consequence of Riemann–Roch plus a little work (see for example Arbarello, Cornalba, Griffiths and Harris [ACGH] or Green and Lazarsfeld [GL]). The result for curves implies similar results for K3 surfaces and Fano 3-folds, and for this reason we can in practice almost always assume that $S_{2g-2} \subset \mathbb{P}^g$ or $X_{2g-2} \subset \mathbb{P}^{g+1}$ is an intersection of quadrics.

For a K3 surface $S_{2g-2} \subset \mathbb{P}^g$, the exceptions to being a scheme theoretic intersection of quadrics are the following:

(a) There exists an elliptic pencil E such that $E \cdot h = 3$.

(b) $g \leq 4$;

(c) $g = 6$ and S is a weighted hypersurface $S_5 \subset \mathbb{P}(1,1,1,2)$ in weighted projective space (w.p.s.) with the stated weights, embedded in $\mathbb{P}^6$ as the cone over the Veronese surface

$$\mathbb{P}(1,1,1,2) = v_2(\mathbb{P}^2) * (0 : \cdots : 0 : 1) \subset \mathbb{P}^6.$$

We define the *quadric hull* of a subscheme $\Sigma \subset \mathbb{P}^n$ to be the intersection of all quadrics containing it:

$$\bigcap_{\Sigma \subset Q} Q.$$

Then in each of cases (a–c), the quadric hull of S_{2g-2} is a 3-fold T of minimal degree (see Appendix A), and S is a divisor $S_{2g-2} \subset T^3_{g-2} \subset \mathbb{P}^g$.

In the same way, a Fano 3-fold $X_{2g-2} \subset \mathbb{P}^{g+1}$ is either an intersection of quadrics, or is contained in a 4-fold of minimal degree

$$X_{2g-2} \subset T^4_{g-2} \subset \mathbb{P}^{g+1}.$$

In most cases T^4_{g-2} is a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$, leading to a mobile decomposition.

Corollary 2.8 *For a Fano 3-fold X of genus $g \geq 5$, X indecomposable implies that $X_{2g-2} \subset \mathbb{P}^{g+1}$ is an intersection of quadrics.*

Using the above, we can see as an exercise that K3 surfaces and Fano 3-folds are characterized as being varieties with canonical curve sections:

- a surface $S_{2g-2} \subset \mathbb{P}^g$ is a K3 surface in its projective embedding if and only if it has at worst RDPs and its hyperplane section is a canonical curve;
- a 3-fold $X_{2g-2} \subset \mathbb{P}^{g+1}$ is a Fano 3-fold in its anticanonical embedding if and only if it has at worst Gorenstein canonical singularities and its linear section by two hyperplanes is a canonical curve.

In fact this characterization gives the title to Fano's original 1937 paper "Sulle varietà à tre dimensioni à curve-sezioni canoniche".

3 Varieties with canonical curve sections

By a *linear section* of a variety $X^n \subset \mathbb{P}^N$ we mean an $n-k$ dimensional variety $Y^{n-k} \subset \mathbb{P}^{N-k}$ obtained as the complete intersection[3] of k hyperplane sections of X. That is,

$$Y = X \cap H_1 \cap \cdots \cap H_k \ \subset \ \mathbb{P}^{N-k},$$

viewed as an $(n-k)$ dimensional subvariety of $\mathbb{P}^{N-k}$. We sometimes write this as

$$\left[X^n \subset \mathbb{P}^N\right] \cap H_1 \cap \cdots \cap H_k = \left[Y^{n-k} \subset \mathbb{P}^{N-k}\right].$$

During a visit to Romania in August 1982, I learned from Ciprian Borcea that K3 surfaces occur as linear sections of the G_2 manifold. On returning to the University of Warwick, I calculated the dimension, degree and Fano index of the projective homogeneous spaces $G/P \subset \mathbb{P}^n$ of Picard rank 1 (due to Borel, Bott, ...). I discovered at the same time that 4 of these have canonical curve sections. These calculations are contained in the preprint [Mu1], see also [Mu2].

Conjecture 3.1 *Let X be a Fano n-fold of rank 1 and coindex 3, with $-K_X = (n-2)H$. Then $|H|$ contains a smooth element.*

This conjecture was solved by Wilson [Wi] and Prokhorov for $\dim X = 4$, and by Mella [Me] for any n.

3.1 Results of the 1982 Warwick calculations

A projective homogeneous space of Picard rank 1 is of the form

$$G/P \subset \mathbb{P}^N$$

where G is a semisimple algebraic group, P a maximal parabolic subgroup, and the embedding into $\mathbb{P}^N$ is given by the projectivization of an irreducible representation of G. The cases of interest to us are when

$$\operatorname{Pic}\Sigma = \mathbb{Z}\cdot\mathcal{O}(1) \quad \text{and} \quad \mathcal{O}(-K_\Sigma) = \mathcal{O}(\dim\Sigma - 2).$$

[3] We assume that the local equations of the H_i in $\mathcal{O}_{X,P}$ form a regular sequence at every point $P \in Y$; since all the varieties X we are interested in are Cohen–Macaulay, this is equivalent to saying that every componet of Y has dimension $n-k$.

genus	dim, deg	description	Dynkin diagram, group
7	$\Sigma^{10}_{12} \subset \mathbb{P}^{15}$	spinor Grassmann in Grass$(5, 10)$	$\widetilde{E}_8^{[4]}$ can't read
8	$\Sigma^{8}_{14} \subset \mathbb{P}^{14}$	8 dimensional Grassmann Grass$(2, 6)$	$\widetilde{E}_7^{[3]}$ can't read
9	$\Sigma^{6}_{16} \subset \mathbb{P}^{13}$	symplectic Grassmann in Grass$(3, 6)$	$F_4^{[2]}$ can't read
10	$\Sigma^{5}_{18} \subset \mathbb{P}^{13}$	G_2 variety	$G_2^{[1]}$

Table 2: Projective homogenous spaces of coindex 3

Figure 1: The Dynkin diagrams E_8, E_7, F_4 and G_2 and their completions

The 4 cases are listed in Table 2; the corresponding Dynkin diagrams and algebraic groups are in Figure 1. [4]

3.2 Main aim

The aim of these lectures is to prove that any indecomposable Fano 3-fold X of genus $g = 7, 8, 9, 10$ is isomorphic to a linear section of the corresponding homogeneous space Σ_{2g-2} from this list. We do this in Sections 4, 5 and 6.1. In 6.2 we also use the same circle of ideas to prove the bound on the genus. One of our main concerns here is to check that we can allow X to have Gorenstein singularities.

If time had allowed it, we would also have liked to treat the case $g = 12$, that is, $X_{22} \subset \mathbb{P}^{13}$. (Not contained in the present draft; compare Mukai and

[4] The right hand column is off the end of the xeroxed page in Takagi's manuscript, and I can't read it. It contains information about the simple algebraic groups. (Also, why are two of the Dynkin diagrams completed, the other two not?

Umemura [MU].)

Appendix B. The indecomposability condition

Suppose that we cut $\mathrm{Grass}(2,6) \subset \mathbb{P}^{14}$ by 5 hyperplane sections:

$$X = \mathrm{Grass}(2,6) \cap H_1 \cap \cdots \cap H_5.$$

If X is nonsingular, it is indecomposable by Lefschetz theory, but this fails for singular X.

Example 3.2 The counterexample is a hyperplane section of $\mathrm{Grass}(2,6)$ of Schubert type. For this, write $\mathbb{C}^6 = \mathbb{C}^2 \oplus \mathbb{C}^4$, with the exact sequence

$$0 \to \mathbb{C}^4 \to \mathbb{C}^6 \to \mathbb{C}^2 \to 0. \tag{3.1}$$

We define the Schubert subvariety Y by

$$Y = \left\{U^2 \subset \mathbb{C}^6 \;\middle|\; U^2 \cap \mathbb{C}^4 \neq 0\right\}.$$

Then Y is expressed by the vanishing of one Plücker coordinate, so is a hyperplane section of $\mathrm{Grass}(2,6)$. However, the linear system $|H_Y|$ of hyperplane sections of Y has a mobile decomposition; to see this, write $\mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4)$ and $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$, and consider the diagram

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}^3 & & \mathbb{P}^1 \end{tikzcd} \tag{3.2}$$

where the oblique arrows are the rational maps induced by the exact sequence (3.1), that is,

$$\varphi\colon U \mapsto [U \cap \mathbb{C}^4] \quad \text{and} \quad \psi\colon U \mapsto [\operatorname{im} U \subset \mathbb{C}^2].$$

Then $H_Y \sim H_1 + H_2$, where H_1 and H_2 are the pullbacks of the linear systems of hyperplane sections of $\mathbb{P}^3$ and $\mathbb{P}^1$. If we cut Y by 4 further hyperplane sections, we obtain a Fano 3-fold having an ordinary double point (node) as its only singularity, but that is not factorial. The resolution of indeterminacies of φ and ψ give rise to 2 small resolutions $X_i \to X$, fitting into the Sarkisov link:

$$\begin{tikzcd}[column sep=small] & X_1 \arrow[dl, "\varphi'"'] \arrow[dr] & & X_2 \arrow[dl] \arrow[dr, "\psi'"] & \\ \mathbb{P}^3 & & X & & \mathbb{P}^1 \end{tikzcd} \tag{3.3}$$

Here φ' is the blowup of a curve in $\mathbb{P}^3$ of genus 4 and degree 7 and ψ' is a fibre space of del Pezzo surfaces of degree 5 (an extremal contraction of type D in Takeuchi's classification).

4 The G_2-manifold $\Sigma^5_{18} \subset \mathbb{P}^{13}$

In this section we give 3 different descriptions of the G_2 manifold $\Sigma = \Sigma^5_{18} \subset \mathbb{P}^{13}$ (for much more on this, see Baez [Ba]):

1. As a homogeneous space under the simple group G_2 (in 4.1–4.2).
2. As the Grassmann variety of 2-planes with null octonion multiplication (in 4.4).
3. In terms of conditions associated to a trivector of $\mathbb{C}^7$ (in 4.5).

The aim is to prepare for the $g = 10$ case of the linear section theorem in Section 5. After this, we study two cases of degenerations of Σ and of its linear sections, both of which lead to decomposable Fano 3-folds.

4.1 Homogenous space

The root system of type G_2 is the star of David illustrated in Figure 2. The

Figure 2: The root system of type G_2

primitive roots are α and β, with $\alpha^2 = 2$, $\beta^2 = 6$ and $\alpha \cdot \beta = 3$. The set of positive roots is

$$\Delta^+ \ = \ \big\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta \text{ and } 3\alpha+2\beta\big\},$$

and as usual, $\Delta = \Delta^+ \sqcup \Delta^-$ with $\Delta^- = -\Delta$ the negative roots.

The Lie algebra of type G_2

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\gamma\in\Delta} \mathfrak{g}_\gamma,$$

is made up of the Cartan subalgebra $\mathfrak{h} = \mathbb{C}\alpha \oplus \mathbb{C}\beta$ and the weight spaces $\mathfrak{g}_\gamma$, which are 1 dimensional eigenspaces. Thus

$$\dim \mathfrak{g} = 2 + \#\Delta = 14.$$

The Borel subalgebra

$$\mathfrak{b} \ := \ \mathfrak{h} \oplus \bigoplus_{\gamma\in\Delta^+} \mathfrak{g}_{-\gamma} \ \subset \ \mathfrak{g},$$

corresponds to a Borel subgroup

$$\begin{array}{ccc} G & \longleftrightarrow & \mathfrak{g} \\ \bigcup & & \bigcup \\ B & \longleftrightarrow & \mathfrak{b} \end{array}$$

Now $\mathfrak{p} := \mathfrak{b} \oplus \mathfrak{g}_\alpha$ is a parabolic Lie subalgebra with $\mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$ that corresponds to a parabolic subgroup $B \subset P \subset G$. Then $\Sigma := G/P$ is a projective manifold, that we call the G_2 manifold.

The tangent space to a point of $\Sigma = G/P$ equals

$$\bigoplus_{\gamma\in\Delta^+\setminus\{\alpha\}} \mathfrak{g}_\alpha,$$

so is 5 dimensional. More precisely,[5] after a little lift, the tangent bundle splits as a direct sum of line bundles corresponding to the right hand side. Now

$$\operatorname{Pic}(G/B) = (\mathbb{Z}\alpha \oplus \mathbb{Z}\beta)^* \quad \text{and} \quad \operatorname{Pic}(G/P) \cong \mathbb{Z}\mathcal{O}(1),$$

where the positive generator $\mathcal{O}(1)$ of $\operatorname{Pic}\Sigma = \mathbb{Z}\mathcal{O}(1)$ corresponds to the weight ω orthogonal to the fundamental weight α and with $\omega \cdot \beta = \frac{1}{2}\beta^2$; the answer is $\omega = 3\alpha + 2\beta$.

[5] This sentence is a colloquial interpretation of something I can't read properly.

4.2 General theory, the Borel–Weil–Bott theorem

In the above notation, $H^0(\Sigma, \mathcal{O}(n))$ is an irreducible representation of G. In the present case, $H^0(\Sigma, \mathcal{O}(1))$ is the adjoint representation of G on its Lie algebra $\mathfrak{g}$. We have

$$\Phi_{|\mathcal{O}(1)|} \colon \Sigma \hookrightarrow \mathbb{P}(\mathfrak{g}) = \mathbb{P}^{13};$$

the image is a 5 dimensional projective variety.

Now $-K_\Sigma$ corresponds to the sum of roots $\gamma \in \Delta^+ \setminus \{\alpha\}$. One calculates this to be

$$\beta + (\alpha+\beta) + (2\alpha+\beta) + (3\alpha+\beta) + (3\alpha+2\beta) = 9\alpha + 6\beta = 3\omega.$$

Therefore $\mathcal{O}(-K_\Sigma) = \mathcal{O}(3) = \mathcal{O}(\dim\Sigma - 2)$, and Σ has index 3, so coindex $\dim\Sigma + 1 - \text{index} = 3$.

How do we compute the degree

$$\bigl(\mathcal{O}(1)\bigr)^5 = \deg\bigl[\Sigma \subset \mathbb{P}^{13}\bigr]?$$

General theory gives

$$5! \times \prod_{\gamma\in\Delta^+\setminus\{\alpha\}} \frac{\gamma\cdot\omega}{\gamma\cdot\rho},$$

where $\rho =$ half the sum of all the positive roots. In the present case $\rho = 5\alpha + 3\beta$, so that

$$\deg\Sigma \;=\; 5! \times \frac{3\cdot 3\cdot 6\cdot 3\cdot 3}{6\cdot 5\cdot 9\cdot 4\cdot 3} = 18.$$

Thus $\Sigma = \Sigma^5_{18} \subset \mathbb{P}^{13}$.

It follows that cutting Σ twice by hyperplanes gives a Fano 3-fold, cutting three times a K3 surface, and cutting 4 times a canonical curve of genus 10.

In Table 3, the first 3 cases are compact Hermitian symmetric spaces, and have corresponding noncompact duals $D \subset D^*$: for example, $\mathrm{Sp}_6 = D^*$ has noncompact dual $D =$ the Siegel upper half D space of genus 3. However, the final case, the G_2 manifold, is not a Hermitian symmetric space.

4.3 Topology of the G_2 manifold

The simple algebraic group G of type G_2 acts on $\Sigma = G/P$. Restricting to the Borel subgroup gives an action of B on $\Sigma = G/P$ that is no longer transitive; instead, it gives the orbit decomposition

$$\Sigma = \mathbb{C}^5 \sqcup \mathbb{C}^4 \sqcup \mathbb{C}^3 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^1 \sqcup \{\text{pt.}\},$$

$g = 7$	$\mathrm{SO}(10)$
$g = 8$	$\mathrm{Grass}(2, 6)$
$g = 9$	Sp_6
$g = 10$	G_2 manifold Σ

Table 3: The 4 projective homogeneous spaces of coindex 3

where the single point is the zero coset P. In particular, as for any projective variety with a flag decomposition, Σ has the same homology or Chow groups as $\mathbb{P}^5$:

$$
\begin{aligned}
&H^{2i+1}(\Sigma) = 0, \\
&\mathrm{CH}^i(\Sigma) \cong H^{2i}(\Sigma) \cong \mathbb{Z} \quad \text{for } 0 \le i \le 5, \quad \text{and} \\
&\mathrm{CH}^1(\Sigma) \cong \operatorname{Pic}\Sigma = \mathbb{Z}h \quad \text{with generator } h = c_1(\mathcal{O}(1)) \text{ having } h^5 = 18.
\end{aligned}
$$

We can draw the multiplicative structure of the Chow ring of Σ in the following graphic form:

$$
\begin{array}{lll}
\circ & \mathrm{CH}^0 & 1 \\
\downarrow & & \\
\circ & \mathrm{CH}^1 & h = \overline{\mathbb{C}^4} \\
\Downarrow\!\!\!\downarrow & & \\
\circ & \mathrm{CH}^2 & \frac{h^2}{3} = \overline{\mathbb{C}^3} \\
\Downarrow & & \\
\circ & \mathrm{CH}^3 & \frac{h^3}{6}, \quad \text{with generator a cubic surface} \\
\Downarrow\!\!\!\downarrow & & \\
\circ & \mathrm{CH}^4 & \frac{h^4}{18}, \quad \text{with generator a line} \\
\downarrow & & \\
\circ & \mathrm{CH}^5 & \frac{h^5}{18} = \text{pt.}
\end{array}
$$

Here we only need the generators $\frac{h^3}{6} \in \mathrm{CH}^3$ and $\frac{h^4}{18} \in \mathrm{CH}^4$ as cycles, but in fact they exist as subvarieties as indicated.

This description alone is already sufficient to achieve our aim of proving that a generic Fano 3-fold of genus 10 is a linear section of Σ.

4.4 Null octonion Grassmann variety

We write $\mathbb{O}$ for the *complexified* octonions or Cayley numbers:

$$\mathbb{O} = \mathbb{C}e_0 \oplus \mathbb{O}_0 \quad \text{with} \quad \mathbb{O}_0 = \bigoplus_{i=1}^{7} \mathbb{C}e_i, \tag{4.1}$$

where $e_0 = 1$ and $\mathbb{O}_0$ is the space of purely imaginary octonions. It is based by 7 elements e_i with the multiplication

$$\begin{aligned} e_i^2 &= -1, \\ e_i \cdot e_j &= -e_j \cdot e_i, \\ e_i \cdot e_j &= \pm e_k, \end{aligned}$$

where k is determined as follows: we draw the seven points and seven lines of the projective plane over $\mathbb{F}_2$ as Figure 3. Then $e_i \cdot e_j = \pm e_k$ if and only

Figure 3: $\mathbb{P}^2(\mathbb{F}_2)$ and multiplication of unit octonions

if i, j, k are collinear; we explain presently how to determine the sign $\pm$ (see 4.5).

The octonions form a noncommutative, nonassociative algebra $\mathbb{O}$. However, it has the following good properties:

(i) Every purely imaginary octonion has real negative square; indeed, by (1) and (2),

$$\alpha = \sum_{i=1}^{7} a_i e_i \in \mathbb{O}_0 \quad \Longrightarrow \quad \alpha^2 = -\Big(\sum_{i=1}^{7} a_i^2\Big) e_0.$$

(ii) $|\alpha \cdot \beta| = |\alpha| \cdot |\beta|$ for all $\alpha, \beta \in \mathbb{O}$; this gives a famous formula of the form

$$\text{(sum of 8 perfect squares)} \cdot \text{(ditto)} = \text{(ditto)}.$$

(iii) The alternative law holds:

$$\alpha(\alpha\cdot\beta)=\alpha^2\beta \quad\text{and}\quad (\alpha\cdot\beta)\beta=\alpha\beta^2 \quad \text{for all } \alpha,\beta\in\mathbb{O}. \tag{4.2}$$

Now the simple algebraic group of type G_2 is the automorphism group

$$\mathrm{Aut}_{\mathrm{alg}}(\mathbb{O}) \;\subset\; \mathrm{GL}(\mathbb{O}_0).$$

Definition 4.1 We write

$$\mathrm{Grass}_{\mathrm{octa}}(2,\mathbb{O}_0)=\left\{U\subset\mathbb{O}_0 \;\middle|\; \alpha\beta=0 \text{ for all } \alpha,\beta\in U\right\}$$

for the set of 2 dimensional subspaces $U\subset\mathbb{O}_0$ such that the multiplication of $\mathbb{O}$ restricted to U is identically 0, and call it the *null octonion Grassmann variety.*

Here $\mathrm{Aut}^0\,\mathbb{O}$ acts on $\mathrm{Grass}_{\mathrm{octa}}(2,\mathbb{O}_0)$, and it can be shown that under this action the G_2 manifold

$$\Sigma^{10}_{18}=\mathrm{Grass}_{\mathrm{octa}}(2,\mathbb{O}_0) \;\subset\; \mathrm{Grass}(2,7).$$

Therefore, for the hyperplane section theorem for a K3 surface S, we should construct an embedding $S\hookrightarrow \mathrm{Grass}(2,7)$, that is, a vector bundle of rank 2 having $h^0=7$. However, octonion multiplication is not quite the best tool for this.

4.5 Trivectors in 7-space and the G_2-manifold

We return to the question of how the sign in octonion multiplication is determined. For this, we view the index on the basis vector e_i as an element $i\in\mathbb{Z}/7$, and write

$$\begin{aligned}\sigma &= \sum_{i\in\mathbb{Z}/7} e_i\wedge e_{i+1}\wedge e_{i+3} \qquad (4.3)\\ &= e_1\wedge e_2\wedge e_4+\cdots+e_7\wedge e_1\wedge e_3.\end{aligned}$$

Then every combination $e_i\wedge e_j$ occurs in exactly one of the 7 terms of (4.3). We set

$$e_ie_j=\frac{\partial^2\sigma}{\partial e_j^*\partial e_i^*} \quad \text{for } i\neq j\in[1,7].$$

Here the operation $\partial/\partial e_i^*$ is the formal partial differential operator, or the contraction in the external algebra. (We can identify $\mathbb{O}_0$ and $\mathbb{O}_0^*$ using the inner product $\sum_{i=1}^7 x_i^2$.) The following criterion for $U \subset \mathbb{O}_0$ to be a null octonion plane follows from the above description of octonion multiplication.

Fact 4.2 *Write a 2-plane $U^2 \subset \mathbb{O}_0$ as $U^2 = \langle \alpha, \beta \rangle$. Then*

$$U \in \mathrm{Grass}_{\mathrm{octa}}(2, \mathbb{O}_0) \iff \frac{\partial^2 \sigma}{\partial\alpha\partial\beta} = 0.$$

Definition 4.3 Consider a 7 dimensional space V and a trivector $\sigma \in \bigwedge^3 V$; for $\alpha, \beta \in V^*$, we have

$$\frac{\partial \sigma}{\partial \alpha} \in \bigwedge^2 V \quad \text{and} \quad \frac{\partial^2 \sigma}{\partial\alpha\partial\beta} \in V.$$

Now set:

(1) $\mathrm{Grass}_\sigma(2, V^*) = \left\{U^2 = \langle \alpha, \beta \rangle \subset V \;\middle|\; \frac{\partial^2\sigma}{\partial\alpha\partial\beta} = 0\right\}$. We call this the *null Grassmann variety of σ*.

(2) Write q_σ for the quadratic form

$$V^* \to \mathbb{C} = \bigwedge^7 V \quad \text{defined by} \quad \alpha \mapsto \frac{\partial\sigma}{\partial\alpha} \wedge \frac{\partial\sigma}{\partial\alpha} \wedge \sigma.$$

(3) We say that the trivector σ is *nondegenerate* if q_σ is nondegenerate, that is, if and only if $\operatorname{disc} q_\sigma \neq 0$.

The 49 dimensional group $\mathrm{GL}(7)$ acts on $\bigwedge^3 \mathbb{C}^7$ (which has dimension $\binom{7}{3} = 35$). This is a special case of a quasihomogeneous variety – the action has an open orbit, consisting of all the nondegenerate trivectors. Now one nondegenerate trivector is obtained by taking $V = \mathbb{O}_0$ and σ given by (4.3). It follows that a nondegenerate (V^7, σ) is isomorphic to $(\mathbb{O}_0, \sum e_i \wedge e_{i+1} \wedge e_{i+3})$. Note that its stabilizer is the group G_2, and has dimension $49 - 35 = 14$. In particular, the null Grassmann $\mathrm{Grass}_\sigma(2, V^*)$ of any nondegenerate σ is the G_2 manifold.

4.6 The G_2-manifold in the 7 dimensional orthogonal Grassmann in $\mathrm{Grass}(2,7)$

Octonions provide the most effective way of interpreting the spinor space of a 7 dimensional inner product space.

We consider the complexified octonions written out as in (4.1), where $\mathbb{O}_0$ is the space of purely imaginary octonions. For a purely imaginary octonion $\alpha \in \mathbb{O}_0$, consider left multiplication

$$L_\alpha\colon \mathbb{O} \to \mathbb{O} \quad \text{given by} \quad \beta \mapsto \alpha\beta.$$

By the alternative law (4.2), this satisfies

$$L_\alpha \circ L_\alpha = \text{scalar multiplication by } \alpha^2 = -\textstyle\sum_{i=1}^7 a_i^2.$$

In particular, if $\alpha^2 = 0$ then $L_\alpha \circ L_\alpha = 0$. In this case, it always happens that $\operatorname{rank} L_\alpha = 4$. Now an null octonion 2-plane $U^2 = \langle \alpha, \beta \rangle_{\mathbb{C}} \subset \mathbb{O}_0$ is defined by the condition that the inner product $\sum x_i^2$ restricted to it is zero. Therefore

$$L_\alpha^2 = L_\beta^2 = 0, \quad \text{and} \quad L_\alpha L_\beta + L_\beta L_\alpha = 0;$$

in other words, L_α and L_β anticommute. It always happens that the intersection $\ker L_\alpha \cap \ker L_\beta$ is 2 dimensional, and we denote this by $S_U \subset \mathbb{O}$. It is obviously independent of the choice of α, β.

Lemma 4.4 $U \subset \mathrm{Grass}_{\mathrm{octa}}(2, \mathbb{O}_0) \iff S_U \subset \mathbb{O}_0$. $\square$

Let (V, q) be a 7 dimensional inner product space. Then we have the inclusion

$$\begin{array}{ccc} \mathrm{Grass}(2, V, q) & \subset & \mathrm{Grass}(2, V) \\ \cap & & \cap \\ \mathbb{P}^{20} = \mathbb{P}(\mathrm{SO}(7)) & & \mathbb{P}^{20} = \mathbb{P}(\bigwedge^2 \mathbb{C}^7) \end{array}$$

where the left hand is the orthogonal Grassmann $\mathrm{Grass}(2, V, q)$, which is a homogeneous space for $\mathrm{SO}(7)$, and the right hand is a homogeneous space for $\mathrm{PGL}(7)$.

Given (V, q), there exists an 8 dimensional inner product space (S, τ) together with a linear map $L\colon V \to \operatorname{End} S$, uniquely determined up to isomorphism by the following properties:

(1) For $v \in V$ the composite $L_v \circ L_v$ is scalar multiplication by $q(v)$;

(2) L_v preserves τ, in the sense that

$$\tau\big(s, L_v(t)\big) + \tau\big(L_v(s), t\big) = 0.$$

Definition 4.5 We say that S is the *spinor space* of (V, q), and τ is its *spinor norm.*

Example 4.6 If we start from the space of pure imaginary octonions $\mathbb{O}_0$ with (minus) the usual norm $-\sum_1^7 x_i^2$, then its spinor space is the space of octonions $\mathbb{O}$, and the spinor norm on it is the usual inner product $x_0^2 - \sum_1^7 x_i^2$.

Recall that, more generally, an inner product space V of dimension $2n+1$ has a spinor space of dimension 2^n given by the even part of the Clifford algebra $S = \mathrm{Cliff}(V, q)^+$. An element $s \in S$ is called a *spinor.*

Definition 4.7 For $U^2 = \langle \alpha, \beta \rangle_{\mathbb{C}}$ such that $q_{|U} = 0$, we set

$$S_U = \ker L_\alpha \cap \ker L_\beta \subset S. \tag{4.4}$$

For a nonzero spinor $s \in S$, we set

$$\mathrm{Grass}(2, V, q, s) := \left\{ U \in \mathrm{Grass}(2, V, q) \;\middle|\; \begin{array}{l} S_U \text{ is orthogonal} \\ \text{to } s \text{ under } \tau. \end{array} \right\}$$

Example 4.8 If $s = e_0 \in \mathbb{O}$ then by Lemma 4.4, $\mathrm{Grass}(2, V, -\sum x_i^2, e_0)$ is the null octonion Grassmann variety.

We can globalize (4.4) to obtain a tautological rank 2 bundle $\mathcal{F}$ over $G = \mathrm{Grass}(2, V, q)$ with fibre at $[U] \in G$ the dual of S_U. In more detail, we start from the trivial rank 8 bundle $S \otimes \mathcal{O}_G$ over G; a point $U \in G$ is a 2 dimensional isotropic subspace $U^2 \subset V$, and defines a 2 dimensional subspace S_U by (4.4); taking $S_U \subset S$ as the fibre over $[U]$ gives a rank 2 subbundle $\mathcal{F}^\vee \subset S \otimes \mathcal{O}_G$. Finally, we write $\mathcal{F}$ for the dual of this subbundle.

Then $\mathcal{F}$ is a homogeneous rank 2 vector bundle with $\det \mathcal{F} = \mathcal{O}_G(1)$, and $S \otimes \mathcal{O}_G = S^\vee \otimes \mathcal{O}_G \twoheadrightarrow \mathcal{F}$ inducing $S \cong H^0(\mathcal{F})$.

$\mathrm{Grass}(2, V, q, s) \subset \mathrm{Grass}(2, V, q)$ is the set of zeros of the global section $s \in H^0(\mathcal{F})$. Now $\mathcal{F}$ has rank 2 and the dimension goes down by exactly 2, so that $\mathrm{Grass}(2, V, q, s)$ is a locally complete intersection in $\mathrm{Grass}(2, V, q)$. Thus we have

$$\begin{array}{ccccc} \mathrm{Grass}(2,V) & \supset & \mathrm{Grass}(2,V,q) & \supset & \mathrm{Grass}(2,V,q,s) \\ & & \| & & \| \\ & & \text{zeros of } q \in S^2\mathcal{E} & & \text{zeros of } s \in \mathcal{F} \end{array}$$

where each step is a c.i. with respect to a section of a vector bundle.

4.7 Zero locus of a degenerate spinor and degeneration of Σ^5_{18}

There are a priori 3 logical cases for a spinor $s \in S$:

$$(1) \quad \tau(s) \neq 0; \qquad (2) \quad s \neq 0 \text{ but } \tau(s) = 0; \qquad (3) \quad s = 0.$$

Note that if $\sigma_1, \sigma_2 \in \bigwedge^3 V$ are linearly independent then the pencil of trivectors $\{a\sigma_1 + b\sigma_2 \mid a, b \in \mathbb{C}\}$ contains degenerate elements.

We considered the zero locus $Z(s)$ in the nondegenerate case (1) and obtained the null octonion Grassmann Σ. The degenerate case (2) is an analogue of a tensor of rank 1, and its zero locus leads to an interesting degeneration Σ' of Σ. The Fano 3-folds arising as its linear sections of Σ' are an analogue of Example 3.2.

For a 3 dimensional subspace $W = \langle \alpha, \beta, \gamma \rangle_{\mathbb{C}} \subset V$ with $q_{|W} = 0$, the intersection $S_W = \ker L_\alpha \cap \ker L_\beta \cap \ker L_\gamma \subset S$ is 1 dimensional. A nonzero element $s_W \in W$ is called a *pure spinor.*

Fact 4.9 *A nonzero element $s \in S$ is a pure spinor if and only if $\tau(s) = 0$. That is, s is a null vector with respect to the spinor norm.*

Then for $s = s_W$, the null Grassmann variety $\mathrm{Grass}(2, V, q, s)$ coincides with the following Schubert subvariety.

$$\mathfrak{S}_W := \big\{U \in \mathrm{Grass}(2, V, q) \bigm| U \cap W \neq 0\big\}.$$

This is a 5 dimensional projective variety of degree 18, a degeneration of Σ^5_{18}:

$$\mathfrak{S}_W =: \Sigma'^5_{18} \subset \mathbb{P}^{13} \quad \text{with} \quad \mathrm{Sing}\, \mathfrak{S}_w = \big\{U \in \mathrm{Grass}(2, V, q) \bigm| U \subset W\big\} \cong \mathbb{P}^2.$$

It has ordinary double points along $\mathrm{Sing}\, \mathfrak{S}_W$.

Cutting $\mathfrak{S}_W$ by generic hyperplanes H_1, H_2 gives

$$X'_{18} = \Sigma'^5_{18} \cap H_1 \cap H_2 \subset \mathbb{P}^{11}$$

which is *decomposable* Fano 3-fold of genus 10 (see Definition 2.5). We have two rational maps analogous to (3.2)

$$\begin{tikzcd}[column sep=small] & \mathfrak{S}_W \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}_*(W) = \mathbb{P}^2 & & \mathbb{P}^3 = \mathbb{P}_*(V/W) \end{tikzcd}$$

defined by

$$\varphi(U) = [U \cap W] \quad \text{and} \quad \psi(U) = \mathrm{im}(U) \subset V/W \quad \text{for } U \in \mathfrak{S}_W.$$

Cutting once more by a hyperplane gives a K3 surface

$$S_{18} = \Sigma'^5_{18} \cap H_1 \cap H_2 \cap H_3 \subset \mathbb{P}^{10} \quad \text{with} \quad \mathcal{O}(1)_{|S} = h \ \sim \ a + b,$$

where $a = (\varphi_{|S})^*\mathcal{O}(1)$ has $h^0(a) \geq 4$ and $b = (\psi_{|S})^*\mathcal{O}(1)$ has $h^0(b) \geq 3$. This contradicts indecomposable and also Brill–Noether general (see Definition 5.2), since

$$h^0(a)h^0(b) \geq 12 > g = 10.$$

However, it remains Clifford general (see Definition 5.2), because

$$h^0(a) + h^0(b) = 3 + 4 = 2 + \frac{g}{2} = 7.$$

As in Example 3.2, $X'_{18} = \Sigma'^5_{18} \cap H_1 \cap H_2$ is a Fano 3-fold having just one ordinary double point, but nonfactorial. It has 2 small algebraic resolutions

$$\begin{tikzcd}[column sep=small] & X'_{18,1} \arrow[dl, "\varphi'"'] \arrow[dr] & \overset{\text{flop}}{\longleftarrow\!\!\rightarrow} & X'_{18,2} \arrow[dl] \arrow[dr, "\psi'"] & \\ \mathbb{P}^2 & & X'_{18} & & \mathbb{P}^3 \end{tikzcd} \tag{4.5}$$

Here ψ' is the blowup of a curve $\Gamma \subset \mathbb{P}^3$ of genus 2 and degree 6; Γ has a single quadrisecant line, which gets flopped. $\varphi' \colon X'_{18,1} \to \mathbb{P}^2$ is a conic bundle with discriminant curve of degree 4.

4.8 Hyperplane section of Schubert type

Here we give another analogue of Example 3.2, which is a different way of degenerating the Fano 3-fold X_{18}. This example arises by taking a hyperplane section of Schubert type of the nondegenerate G_2 manifold Σ_{18}.

The G_2 manifold is the 5 dimensional projective variety

$$\begin{array}{ccc} \Sigma_{18}^5 = \mathrm{Grass}(2,V,q,\sigma) & \subset & \mathbb{P}^{13} \\ \bigcap & & \bigcap \\ \mathrm{Grass}(2,V) & \subset & \mathbb{P}^{20} \end{array}$$

Here V is a 7 dimensional vector space (typically $V = \mathbb{O}_0$), q a nondegenerate form or inner product (typically $\sum_1^7 a_i^2$), σ a spinor (typically $e_0 = 1 \in \mathbb{O}$), and $\mathrm{Grass}(2,V)$ is the 10 dimensional Grassmann in its Plücker embedding.

We describe how to define a hyperplane section of Σ_{18}^5 of Schubert type. For this, let $W^5 \subset V$ be a 5 dimensional vector subspace such that $q_{|W}$ is nondegenerate, and let

$$\mathrm{Grass}(2,W) \subset \mathfrak{S}_W = \left\{ [U] \mid U \cap W \neq 0 \right\} \subset \mathrm{Grass}(2,V).$$

Here $\mathfrak{S}_W$ is 9 dimensional and its singular locus $\mathrm{Grass}(2,W) = \mathrm{Sing}\,\mathfrak{S}_W$ is 6 dimensional.

$$\begin{array}{ccc} & \mathfrak{S}_W & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ \mathbb{P}^4 = \mathbb{P}(W) & & \mathbb{P}(V/W) = \mathbb{P}^1 \end{array}$$

Then $\mathfrak{S}_W \cap \mathrm{Grass}(2,V,q,s)$ contains $\mathrm{Grass}(2,W) \cap \mathrm{Grass}(2,V,q,s) = \mathrm{Grass}(2,W,q_{|W},s_{|W})$

Here $\mathrm{Grass}(2,W,q_{|W},s_{|W}) = \mathbb{P}^3 =$ the spinor space of $\mathbb{P}(W,q_{|V})$, and $\mathrm{Sing}\,\mathfrak{S}_W \cap \mathrm{Grass}(2,V,q,s)$ is a conic curve.[6] $\mathfrak{S}_W \cap \mathrm{Grass}(2,V,q,s)$ is an index 4 Fano 4-fold with transverse singularities of type $\frac{1}{2}(1,1,1)$ along this conic.

[6] I'm guessing. The original seems muddled here.

If we cut by one more hyperplane we get another example of a decomposable Fano 3-fold:

$$\begin{tikzcd}[column sep=small] & X''_{18,1} \arrow[dl, "\varphi'"'] \arrow[dr] \arrow[rr, leftrightarrow, "\text{flop}"] & & X''_{18,2} \arrow[dl] \arrow[dr, "\psi'"] & \\ \mathbb{P}^4 & & X''_{18} & & \mathbb{P}^1 \end{tikzcd} \tag{4.6}$$

where $X''_{18} = H_1 \cap H_2 \cap \Sigma^5_{18}$ is a Fano 3-fold with a nonfactorial ordinary double point, obtained by taking $H_1 := \langle \mathfrak{S}_W \rangle \subset \mathbb{P}^{13}$.

This is decomposable because $\mathcal{O}_{X''}(1) = A + B$ with $h^0(A) = 5$ and $h^0(B) = 2$. In this case $2 \times 5 = 10$ and $2 + 5 = 7$, so that this is BN general as well as Clifford general. The K3 surface section is of type D (in the sense of the case division of 5.3 below).

The hyperplane section H is of Schubert type implies that

$$2 \geq \dim \operatorname{Sing}(\Sigma^5_{18} \cap H) \geq 1.$$

(Here we can also allow $q_{|W}$ to be degenerate.) The converse also holds:

Proposition 4.10 *If H is not of Schubert type, then $\Sigma_{18} \cap H$ has isolated singularities.*

5 The linear section theorem (case $g = 10$)

Theorem 5.1 *Any indecomposable Fano 3-fold $X_{18} \subset \mathbb{P}^{11}$ is isomorphic to a linear section of the G_2-manifold. That is,*

$$\big[X_{18} \subset \mathbb{P}^{11}\big] = \big[\Sigma^5_{18} \subset \mathbb{P}^{13}\big] \cap H_1 \cap H_2.$$

Our treatment consists of two parts:

(A) In this section, we prove the linear section theorem for K3 surfaces.

(B) We prove Theorem 5.1 in the following Section 6, using (A) together with the Lefschetz principle.

5.1 Generality conditions on a polarized K3 surface

We write (S, h) for a weakly polarized nonsingular K3 surface and $(\overline{S}, \overline{h})$ for its "canonical model", which is a polarized K3 with at worst RDPs. Write $h^2 = 2g - 2$, where g is the genus of S.

We consider the following 3 generality conditions on (S, h):

Definition 5.2 We say that $(\overline{S}, \overline{h})$ is

(1) *Picard general* if $\operatorname{Pic} \overline{S} = \mathbb{Z}\overline{h}$;

(2) *Brill–Noether general* (BN general) if every mobile decomposition $h = a + b$ (with Weil divisors a and b) satisfies $h^0(a)h^0(b) \leq g$.

(3) *Clifford general* if every mobile decomposition satisfies $h^0(a) + h^0(b) \leq 2 + \frac{g}{2}$.

Note that

$$\text{Picard general} \implies \text{BN general} \implies \text{Clifford general}$$

(Because $xy - 2(x + y - 2) = (x - 2)(y - 2) \geq 0$ for integers $x, y \geq 2$, hence $x + y > 2 + g/2$ implies $xy > g$.)

Remark 5.3 For a nonsingular Fano 3-fold with $\rho = r = 1$, a "very general" element $S \in |-K_X|$ satisfies $\mathbb{Z}(-K_X) = \operatorname{Pic} X \cong \operatorname{Pic} S$ (Moishezon's theorem, [M]). Thus the classification in the case $\rho = r = 1$ and X nonsingular is easy (this is contained in my 1989 announcement [Mu2]).

Theorem 5.4 *A polarized K3 surface $S_{18} \subset \mathbb{P}^{10}$ that is BN general is a linear section of the G_2 manifold.*

5.2 Generality and gonality of a K3 surface

Suppose that S contains an elliptic curve E. Then

$$H^0(\mathcal{O}_C(E)) = 2, \quad E^2 = 0 \quad \text{and}$$
$$h^0(\mathcal{O}_C(h - E)) \geq \frac{1}{2}(h - E)^2 + 2 = g + 1 - h \cdot E.$$

Thus if $h = E + (h - E)$ has $h \cdot E \geq g - 1$, we have a mobile decomposition. On the other hand,

$$\text{BN general} \implies \text{Clifford general}$$
$$\implies h^0(\mathcal{O}_S(h - E)) \leq \frac{1}{2}g \implies h \cdot E \geq \frac{g}{2} + 1.$$

Thus if S is Clifford general, in particular if it is BN general, then it is not monogonal. Therefore $|h|$ is free.

Moreover, if $g \geq 3$ then it not bigonal (hyperelliptic), and the case $h = 2h_0$ with $h_0^2 = 2$ cannot occur (see 2.2). Therefore, after the double cover $S \xrightarrow{\text{2-to-1}} Q^2 \subset \mathbb{P}^3$ in genus 4, for $g \geq 5$, $\overline{h}$ is very ample and $\overline{S} \subset \mathbb{P}^g$ is an intersection of quadrics.

5.3 Case division

Since $(\overline{S}, \overline{h})$ is BN general, we have

$$h^0(\mathcal{O}(a))h^0(\mathcal{O}(b)) \leq 10 \quad \text{and} \quad h^0(\mathcal{O}(a)) + h^0(\mathcal{O}(b)) \leq 7$$

for every mobile decomposition $h \sim a + b$. Our analysis treats the following two cases separately (D stands for decomposition):

D type: There exists a decomposition $h \sim a + b$ with $h^0(\mathcal{O}_S(a)) = 2$ and $h^0(\mathcal{O}_S(b)) = 5$;

non D type: no such decomposition exists.

5.4 Construction of vector bundle in Case D

In this case we perform the construction on S rather than on $\overline{S}$. Write $h \sim a + b$ for the decomposition. By throwing away its fixed components, we can assume that $|a|$ is free. Then $h^0(\mathcal{O}(a)) = 2$ implies $a^2 = 0$, that is, $|a|$ is a free pencil. This gives the exact sequence

$$0 \to \mathcal{O}(-a) \to 2\mathcal{O}_S \to \mathcal{O}_S(a) \to 0$$

Tensoring with b and substituting $a + b = h$ gives

$$0 \to \mathcal{O}(b - a) \to 2\mathcal{O}_S(b) \to \mathcal{O}_S(h) \to 0.$$

Now $h^0(2\mathcal{O}(b)) = 2 \times 5 = 10$ and $h^0(\mathcal{O}_S(h) = 11$, so that it follows that $h^1(\mathcal{O}(b-a)) \neq 0$.

By duality, $h^1(\mathcal{O}(b-a)) \neq 0$ if and only if $h^1(\mathcal{O}(a-b)) \neq 0$. Therefore there exists a nonsplit exact sequence

$$0 \to \mathcal{O}(a) \to \mathcal{E} \to \mathcal{O}(b) \to 0,$$

with $\det \mathcal{E} = h$, $h^0(\mathcal{E}) = 2+5 = 7$. In this case, we can use $\mathcal{E}$ to construct an embedding of S into the G_2 manifold inside $\operatorname{Grass}(2,7)$. We omit the details, which are similar to those in the non D case (see Remarks 5.7 and 5.16).

5.5 Proof of linear section theorem in the non D case

From now on, we assume that we are in the non D case; in this case, we prefer to work over $\overline{S}$ rather than S.

Our starting point is Riemann's theorem. Recall that if a curve C of genus g is represented as an n-fold cover $C \to \mathbb{P}^1$, ramified in b points, then $-2n + b = 2g - 2$, and the curves arising as such covers depend on $b - 3 = 2n + 2g - 5$ moduli. When $g = 10$ and $n = 6$, this number is 27, which coincides with the dimension of the moduli space of curves of genus 10.

Theorem 5.5 (Riemann) *A curve C of genus 10 has a g^1_d with $d \leq 6$; in other words, C is a cover of $\mathbb{P}^1$ of degree at most 6.*

For an arbitrary curve of genus g, a g^1_d always exists if $d \leq \left[\frac{g}{2}\right] + 1$. For modern proofs (of slightly more general results) see Kempf [K], and Kleiman and Laksov [KL1]–[KL2].

What we still have to do: construct a morphism $\overline{S} \to \sum^5_{18}$. For this, we first construct $\overline{S} \to \operatorname{Grass}(2,7)$, or equivalently, a rank 2 vector bundle $\mathcal{E}$ having 7 global sections.

Construction of $\mathcal{E}$ Let $(\overline{S}, \overline{h})$ be a genus 10 polarized K3 surface that is not monogonal (this follows from BN general). Then $|h|$ contains a nonsingular element $C \in |h|$. By Riemann's theorem, there exists a line bundle ξ on C with $h^0(\xi) \geq 2$ and $\deg \xi \leq 6$. Throwing away any possible base points, we can assume that $|\xi|$ is free. Write $i\colon C \hookrightarrow \overline{S}$ for the inclusion, and $i_*\xi$ for the torsion sheaf on S corresponding to ξ on C. Then $H^0(i_*\xi) = H^0(\xi)$ has dimension ≥ 2 and $i_*\xi$ is generated by any two general sections s_1, s_2. These define a surjective homomorphism

$$\varphi\colon 2\mathcal{O}_{\overline{S}} \twoheadrightarrow i_*\xi \quad \text{defined by} \quad x, y \mapsto xs_1 + yx_2.$$

Then $\ker\varphi = \mathcal{E}^\vee$ is a vector bundle on $\overline{S}$, and we write $\mathcal{E} = \mathcal{E}(C, \xi)$ for its dual. The short exact sequence

$$0 \to \mathcal{E}^\vee \to 2\mathcal{O}_{\overline{S}} \to i_*\xi \to 0 \tag{5.1}$$

is dual to the short exact sequence

$$0 \to 2\mathcal{O}_{\overline{S}} \to \mathcal{E} \to i_*(\xi^{-1} \otimes K_C) \to 0, \tag{5.2}$$

where, under the functor $\mathcal{E}xt^*(-, \mathcal{O}_{\overline{S}})$, the final sheaf is $\mathcal{E}xt^1(i_*\xi, \mathcal{O}_{\overline{S}})$ (we calculate the $\mathcal{E}xt^1$ using the locally free resolution $0 \to \mathcal{O}_{\overline{S}}(-C) \to \mathcal{O}_{\overline{S}} \to \mathcal{O}_C \to 0$).

Now the sequence (5.2) together with $q(\overline{S}) = 0$ implies that

$$0 \to \mathbb{C}^2 \to H^0(\mathcal{E}) \to H^0(\xi^{-1} \otimes K_C) \to 0$$

is exact. Moreover,

$$\begin{aligned} h^0(\xi^{-1} \otimes K_C) = h^1(\xi) &= h^0(\xi) - \chi(\xi) \\ &\geq 2 - (d + 1 - 10) \geq 2 - (6 - 9) = 5, \end{aligned}$$

where $d = \deg\xi \leq 6$. It follows that $h^0(\mathcal{E}) \geq 7$.

Now notice the following two facts:

(1) $\mathcal{E}$ is generated by its H^0 outside a finite set of points, namely, the base point of $|K_C - \xi|$.

(2) In (5.1), $H^0(2\mathcal{O}_{\overline{S}}) \to H^0(i_*\xi)$ is injective.

It follows that $H^0(\mathcal{E}^\vee) = 0$, that is, $\operatorname{Hom}(\mathcal{E}, \mathcal{O}_{\overline{S}}) = 0$.

Proposition 5.6 *$h^0(\mathcal{E}) = 7$, and the second exterior power map*

$$\lambda_2\colon \bigwedge^2 H^0(\mathcal{E}) \to H^0(\bigwedge^2 \mathcal{E}) = H^0(\mathcal{O}(\overline{h}))$$

is surjective.

Proof We use the Castelnuovo linear-bilinear trick, playing off the inclusion

$$\mathrm{Grass}(2, H^0(\mathcal{E})) \subset \mathbb{P}_*(\bigwedge^2 H^0(\mathcal{E}))$$

against the linear map

$$\lambda_{2*} \colon \mathbb{P}_*(\bigwedge^2 H^0(\mathcal{E})) \dashrightarrow \mathbb{P}_*(H^0(\mathcal{O}_{\overline{S}}(1))) = \mathbb{P}^{10}.$$

Now if $h^0(\mathcal{E}) \geq 8$ then $\mathrm{Grass}(2, H^0(\mathcal{E}))$ has dimension ≥ 12, so that it must intersect the kernel of λ_2. In other words, there must be a 2-plane $U \in \mathrm{Grass}(2, H^0(\mathcal{E}))$ that is a common zero of all the linear forms in $H^0(\mathcal{O}_{\overline{S}}(1)))$. If $U = \langle s, t\rangle_{\mathbb{C}}$, this means that s, t are linearly dependent in the fibre $\mathcal{E}_P$ at every point $P \in \overline{S}$, or equivalently, that they only generate a subsheaf of $\mathcal{E}$ of rank 1. Its saturation is a reflexive subsheaf $\mathcal{L} \subset \mathcal{E}$ of rank 1 fitting in an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0. \tag{5.3}$$

Now set $\mathcal{L} = \mathcal{O}(a)$ and $\mathcal{M}^{\vee\vee} = \mathcal{O}(b)$. Then $a + b = h$, and $|a|$ is mobile because $s, t \in H^0(\mathcal{O}_{\overline{S}}(a))$; note also that $|a|$ does not have fixed parts, because $\mathcal{L}$ is the saturation of $\mathcal{O}\cdot s + \mathcal{O}\cdot t \subset \mathcal{E}$. On the other hand, $\mathcal{E}$ is generated by its H^0 outside finitely many points, so that the same hold for $\mathcal{M}$ and $\mathcal{O}_{\overline{S}}(b)$. Now $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_{\overline{S}}) = 0$ ensures that $\mathcal{M}^{\vee\vee} \neq \mathcal{O}_{\overline{S}}$. Therefore $|b|$ is also mobile.

However, now Clifford general implies that $h^0(\mathcal{E}) \leq h^0(a) + h^0(b) \leq 7$. This contradiction proves that $h^0(\mathcal{E}) = 7$.

If λ_{2*} has image of dimension ≤ 10, the same argument gives a 2-plane $U = \langle s, t\rangle_{\mathbb{C}} \in \mathrm{Grass}(2, H^0(\mathcal{E}))$ with s, t generating an exact sequence (5.3). Now $h^1(\mathcal{L}) = 0$ gives $h^0(a) + h^0(b) = 7$, and BN general gives $h^0(a)h^0(b) \leq 10$. Thus $h^0(a) = 2$ and $h^0(b) = 5$, and this contradicts the non D case assumption. □

Remark 5.7 In case D, we can show that the vector bundle $\mathcal{E}$ constructed in 5.3 satisfies $\dim\operatorname{coker}\lambda_2 \leq 1$: note that in this case $h^0(\mathcal{E}) = 7$. If λ_2 has dimension ≤ 10 then as above, we obtain an exact sequence (5.3). Now as above, $|a|$ without fixed part gives $h^1(\mathcal{L}) = 0$ and $h^0()\dots$[7]

Corollary 5.8 *$h^0(\xi^{-1} \otimes K_C) = 5$ and $d = 6$. Thus C does not have any $g^1_{d'}$ with $d' \leq 5$.*

[7]The argument seems to break off at this point.

Lemma 5.9 *$\mathcal{E}$ is generated by its H^0.*

In fact it is enough to see that $|K_C - \xi|$ is free on C. [8]

Lemma 5.9 gives the following commutative diagram:

$$\begin{array}{ccccc} \overline{S} & \xrightarrow{\Phi_{\mathcal{E}}} & \mathrm{Grass}(H^0(\mathcal{E}),2) & = & \mathrm{Grass}(2,H^0(\mathcal{E})^\vee) \\ \bigcap & & \bigcap \text{Plücker} & & \\ \mathbb{P}^{10} & \xrightarrow{\mathbb{P}^*\lambda_2} & \mathbb{P}^*(\bigwedge^2 H^0(\mathcal{E}) & & \end{array}$$

Corollary 5.10 *$\Phi_{\mathcal{E}}$ is an embedding.*

Lemma 5.11 *$h^0(S^2\mathcal{E}) = 27$.* $\square$

Consider the map $S^2\colon S^2(H^0(\mathcal{E})) \to H^0(S^2\mathcal{E})$; because the domain has dimension $\binom{8}{2} = 28$, it follows that there exists a nonzero $q \in S^2(H^0(\mathcal{E}))$ with $S^2(q) = 0$. This q is a quadratic form on $H^0(\mathcal{E})^\vee$ with the property that it restricts to zero on each fibre $\mathcal{E}_P^\vee \subset H^0(\mathcal{E})^\vee$. It follows that the image of $\Phi_{\mathcal{E}}$ is contained in the isotropic Grassmann $\mathrm{Grass}(2, H^0(\mathcal{E})^\vee, q)$.

Lemma 5.12 *q is nondegenerate.*

For otherwise we obtain a contradiction to BN general, compare 4.8. $\square$

Now consider the isotropic Grassmann $\mathrm{Grass}(2, H^0(\mathcal{E})^\vee, q)$ and the spinor bundle $\mathcal{F}$ over it. The restriction map

$$H^0(\mathcal{F}) \to H^0(\mathcal{F}_{|S})$$

goes from the 8 dimensional space $H^0(\mathcal{F})$ of spinors to $H^0(\mathcal{F}_{|S})$, and one calculates that the latter has dimension ≤ 7.

Therefore there is a nonzero spinor $s \in H^0(\mathcal{F})$ that vanishes at every point $P \in S$. Hence $\Phi_{\mathcal{E}}(\overline{S})$ is contained in $\mathrm{Grass}(2, V, q, s)$.

Lemma 5.13 *s has nonzero spinor norm, that is, $\tau(s) \neq 0$.*

---

[8] The missing argument seems to be very easy: if $P \in C$ is a base point of $|K_C - \xi|$ then $\xi + P$ is a g^2_7, which cannot be very ample, so we get g^1_5, contradicting the preceding corollary.

Proof If $\tau(s) = 0$ then $s = s_W$ for some 3-dimensional isotropic subspace $W^3 \subset (V, q)$. Then

$$\mathrm{Grass}(V, q, \tau, s_w) = \mathfrak{S}_W$$

is of Schubert type, and as in 4.7, this gives a mobile decomposition $h \sim a+b$ with $h^0(a) \geq 3$, $h^0(b) \geq 4$, which contradicts BN general. $\square$

It follows from this that the image of $\Phi_{\mathcal{E}}$ is contained in the G_2 variety. This gives a commutative diagram

$$\begin{array}{ccccc} S_{18} & \xrightarrow{\Phi_{\mathcal{E}}} & \mathrm{Grass}(2, V, q, s) & = & \Sigma^5_{18} \\ \bigcap & & \bigcap & & \\ \mathbb{P}^{10} & \xrightarrow{\mathbb{P}^*\lambda_2} & \mathbb{P}^{13}. & & \end{array}$$

It remains to prove that this diagram is *Cartesian*, that is, S_{18} is the scheme theoretic intersection of Σ^5_{18} and $\mathbb{P}^{10}$.

Write H_1, H_2, H_3 for a basis of the hyperplanes of $\mathbb{P}^{13}$ containing S_{18}. Now set

$$N = \langle H_1, H_2, H_3\rangle \cong \mathbb{P}^2 \quad \text{and} \quad T := \Sigma^5_{18} \cap H_1 \cap H_2 \cap H_3.$$

Claim 5.14 *In the non D case, no $H \in N$ can be of Schubert type (see 4.8).*

Therefore $H \cap \Sigma^5_{18}$ has at most isolated singularities (see Proposition 4.10).

It is clear that for $x \in T$, the three planes H_i intersect transversally at x if and only if x is a nonsingular point of $H \cap \Sigma^5_{18}$ for every $H \in N$.

Claim 5.15 $\dim T = 2$.

Proof For suppose that a component T' of T has $\dim T' \geq 3$. Then for every $x \in T'$ there is an $H = H_x \in N$ such that $H_x \cap \Sigma$ is singular. But by the preceding claim, $H_x \cap \Sigma$ has only isolated singularities. This is a contradiction, since the set of $x \in T'$ has dimension ≥ 3, whereas the set of singular $H_x \in N$ has dimension ≤ 2. $\square$

This completes the proof of the linear section theorem (Theorem 5.1).

In establishing the linear section theorem in other cases, in order to prove that the intersection is a Cartesian diagram, we can use the refined Bezout theorem of Fulton [Fu], Chapter 12. Alternatively, we can use the known fact that the coordinate ring of $\Sigma \subset \mathbb{P}$ is Cohen–Macaulay.

Remark 5.16 In the type D case, we can show from the construction of $\mathcal{E}$ that there are at most finitely many hyperplane sections of Σ of Schubert type containing S. After this, from the fact that $\dim \operatorname{Sing} \Sigma \cap H \le 2$, we prove that the intersection is a Cartesian diagram as in the above claim.

Remark 5.17 For each $g \le 9$, a *general* curve C of genus g occurs as the linear section of a K3 surface S. This fails already for $g = 10$.

6 The Lefschetz principle

6.1 Bott vanishing

Let G be a semisimple algebraic group, $B \subset G$ a Borel subgroup, and P a parabolic subgroup with $B \subset P \subset G$. Then the flag variety G/B has a projection $\pi\colon G/B \to G/P$ to the partial flag variety G/P.

The best way of seeing all this is in terms of representation theory.[9]

Write

$$1 \to (\text{unipotent}) \to B \to (\mathbb{C}^*)^\ell \to 0$$

For example, in the case $\mathrm{SL}(n)$, B consists of upper triangular matrices with invertible diagonal terms, whereas the unipotent subgroup consists of upper triangular matrices with 1s down the diagonal.

Then a line bundle over G/B is determined by a weight as follows:

$$\operatorname{Pic}(G/B) = \operatorname{Hom}(B, \mathbb{C}^*) = \operatorname{Hom}((\mathbb{C}^*)^\ell, \mathbb{C}^*) = \mathbb{Z}^\ell$$

given by $L_w \mapsto w \in \mathbb{Z}^\ell$ (the weight lattice).

Theorem 6.1 (Bott [Bo]) *$H^i(G/B, L_w)$ vanishes for all but at most one value of i. This value $i = i_0$ can be computed explicitly.*

This results extends in an obvious way to projective homogeneous spaces G/P with a parabolic subgroup P. Write

$$1 \to (\text{unipotent}) \to P \to P^{\mathrm{ss}} \to 1$$

The semisimple part P^{ss} is a torus $(\mathbb{C}^*)^m$ times a semisimple group.

[9] I've permuted the material around a bit, and I don't know where this sentence comes from.

Corollary 6.2 *Let E_ρ be the homogeneous vector bundle over G/P induced by a rank r irreducible representation $\rho\colon P^{\mathrm{ss}} \to \mathrm{GL}(r)$ of the semisimple part P^{ss} of P. Then the cohomology of E_ρ satisfies the same property.*

This is clear because $\pi_* L_w = E_\rho$. $\square$

6.2 Twistings of the tangent bundle

We apply Bott vanishing to calculate the cohomology of $T_\Sigma(n)$.

$$\Sigma^5_{18} := \mathrm{Grass}_{\mathrm{octa}}(2, \mathbb{O}_0) = G/P$$

where G is of type G_2 and $P^{\mathrm{ss}} \cong \mathrm{GL}(2)$. The tangent bundle T_Σ is a rank 5 homogeneous vector bundle.

Remark 6.3 In contrast to the cases $g \leq 9$, T_Σ does *not* come from a representation of P^{ss}. This is related to the discussion of symmetric spaces in 4.1 (see Table 3), since one of the characterisations of a symmetric space is that the tangent bundle comes from an irreducible representation.

However, we have the following exact sequence:

$$0 \to S^3\mathcal{E}(1) \to T_\Sigma \to \mathcal{O}_\Sigma(1) \to 0,$$

displays T_Σ as a nontrivial extension of the rank 4 bundle $S^3\mathcal{E}(1)$ and the line bundle $\mathcal{O}_\Sigma(1)$.

Here $\mathcal{O}_\Sigma(1) = \det \mathcal{E}$ is the tautological line bundle and $\mathcal{E}$ is the universal quotient bundles induced by $\rho\colon P^{\mathrm{ss}} = \mathrm{GL}(2)$.

Bott vanishing applied to this gives that $H^i(T_\Sigma(n))$ is zero except for the following:

- $H^0(T_\Sigma(n))$ for $n \geq 0$;
- $H^5(T_\Sigma(n))$ for $n < -5$;
- $H^4(T_\Sigma(-4))$.

6.3 The Lefschetz principle

Let (S, h) be a polarized K3 surface of genus 10 with at worst RDPs, assumed to be BN general. Then as we know from Section 5,

$$S = \Sigma^5_{18} \cap H_1 \cap H_2 \cap H_3.$$

Notation We write

$$\big[\mathbb{P}^{g+1} \supset X^3_{2g-2}\big] \cap H = [\mathbb{P}^g \supset S_{2g-2}]\,.$$

We want to say that the Fano 3-fold X is also a linear section. That is, given $S \subset X$ and knowing that $S \hookrightarrow \Sigma$, we want to show that also $X \hookrightarrow \Sigma$. For this, we first extend S to the formal neighbourhood

$$\widehat{S}_X := \lim_{n\to\infty} S_n, \quad \text{where} \quad S_n := \operatorname{Spec} \mathcal{O}_X/\mathcal{O}_X(-nS).$$

For $n \geq 1$, the obstruction to extending an embedding of S_{n-1} to S_n is an element of $H^1(S, T_\Sigma(-nS)_{|S})$. All of these groups vanish by Bott's theorem, so that the inclusion $S \hookrightarrow \Sigma$ extends to $\widehat{S}_X \hookrightarrow \Sigma$, giving the map $\widehat{\Phi}$ in the diagram below:

$$\begin{array}{ccccccc}
\mathbb{P}^{10} & \supset & S & \subset & \widehat{S}_X & \xrightarrow{\widehat{\Phi}} & \Sigma^5_{18} \subset \mathbb{P}^{13} \\
\bigcap & & & & \bigcap & \nearrow & \\
\mathbb{P}^{11} & & \supset & & X_{18} & &
\end{array}$$

The map we need to construct is the diagonal arrow $X_{18} \to \Sigma^5_{18}$.

The restriction map

$$H^0(X, \mathcal{O}_X(-K_X)) \xrightarrow{N} H^0(\widehat{S}_X, \mathcal{O}_{\widehat{S}_X}(-K_X))$$

is an isomorphism. Indeed, at the first step

$$0 \to \mathbb{C} \to H^0(\mathcal{O}_{S_2}(-K_X)) \to H^0(\mathcal{O}_S(-K_X)) \to 0$$

where the first copy of $\mathbb{C}$ is based by the equation x_0 of the hyperplane $S \subset X$. And for all $n \geq 2$, the kernel

$$\begin{aligned}
\ker\big[\mathcal{O}_{S_{n+1}}(-K_X) \twoheadrightarrow \mathcal{O}_{S_n}(-K_X)\big] &= \mathcal{I}_S^n/\mathcal{I}_S^{n+1} \otimes \mathcal{O}(-K_X) \\
&\cong \mathcal{O}_S(-(n-1));
\end{aligned}$$

has $H^0 = H^1 = 0$, so that $H^0(\mathcal{O}_{S_{n+1}}(-K_X)) \to H^0(\mathcal{O}_{S_n}(-K_X))$ is an isomorphism.

The inclusion $\widehat{\Phi}\colon \widehat{S}_X \hookrightarrow \Sigma^5_{18}$ induces the arrow α in the diagram

$$\begin{array}{ccccc}
 & & \mathbb{C}^{14} = H^0(\Sigma, \mathcal{O}(1)) & & \\
 & & {}^{\alpha}\big\downarrow & \searrow^{\beta} & \\
H^0(X, \mathcal{O}_X(-K_X)) & \cong & H^0(\widehat{S}_X, \mathcal{O}(1)) & \to & H^0(S, \mathcal{O}(1))
\end{array}$$

Since we know that β is surjective, there are just two cases for α:

Case: α is surjective This is the case we want; α defines vertical inclusions

$$\begin{array}{ccccc} \mathbb{P}^{11} & \supset & X & \supset & \widehat{S}_X \\ \cap & & & & \cap \\ \mathbb{P}^{13} & & \supset & & \Sigma \end{array}$$

Indeed, β gives the inclusion $S \subset \mathbb{P}^{10}$, and α provides the remaining coordinate x_0 at every point of $\widehat{S}_X$. Write $\mathbb{P}^{11} = H_1 \cap H_2$ and $\mathbb{P}^{10} = H_0 \cap H_1 \cap H_2$ for the projective subspace containing S. Then $H_1 \cap H_2 \cap \Sigma$ contains $\widehat{S}_X$, and thus contains every infinitesimal neighbourhood of S in X; therefore $X \subset H_1 \cap H_2 \cap \Sigma$. Moreover, since $S = H_0 \cap H_1 \cap H_2 \cap \Sigma = H_0 \cap X$, it follows that $X = H_1 \cap H_2 \cap \Sigma$, as required.

Case: α is not surjective In this case, the rational map $\mathbb{P}^{11} \dashrightarrow \mathbb{P}^{10} \subset \mathbb{P}^{13}$ induced by α is the projection from $P_{x_0} = (1, 0, \dots, 0)$. Thus $\widehat{S}_X$ maps into the cone over S with vertex P_{x_0}, and so, as before X is contained in this cone, and must coincide with it. This contradicts the assumption that X has canonical singularities.

7 Indecomposable Fano 3-fold

7.1 The generality of a K3 surface $S \in |-K_X|$ on an indecomposable X

Definition 7.1 Let X be a Fano 3-fold. A *mobile decomposition* of $-K_X$ is an expression $-K_X \sim A + B$ with A, B Weil divisors such that $\dim |A|$ and $\dim |B| \geq 1$. We say that X is *indecomposable* if $-K_X$ does not admit any mobile decomposition.

Let $S \in |-K_X|$ be a K3 surface with at worst RDPs. We write $h = (-K_X)_{|S}$ for the polarization of S.

Lemma 7.2 *Any mobile decomposition $h \sim a + b$ of h satisfies*

$$h^0(\mathcal{O}_S(a)) \cdot h^0(\mathcal{O}_S(b)) \leq g + 1.$$

Proof For otherwise there is a mobile decomposition $h \sim a + b$ with $r := h^0(a)$, $s := h^0(b)$ and $rs \geq g + 2$.

Write $i\colon S \hookrightarrow X$. We can assume without loss of generality that $|a|$ has no fixed components. We use r sections of $\mathcal{O}_S(a)$ to define a homomorphism

$$r\mathcal{O}_X \to i_*\mathcal{O}_S(a) \tag{7.1}$$

that is surjective outside a finite set of points. We consider the kernel of (7.1), which is a vector bundle outside a finite set, and the dual sequence

$$0 \to r\mathcal{O}_x \to \mathcal{E} \to i_*\mathcal{O}_S(h - a) \to 0,$$

which is exact outside finite set. Then

$$h^0(X, \mathcal{E}) \geq r + h^0(\mathcal{O}_S(b)) = r + s.$$

Using Castelnuovo's linear-bilinear trick, we consider the intersection of

$$\operatorname{Grass}(r, H^0(\mathcal{E})) \subset \mathbb{P}_*(\bigwedge^r H^0(\mathcal{E}))$$

with $\mathbb{P}_*\big(\ker(\bigwedge^r H^0(\mathcal{E}) \to H^0(\bigwedge^r \mathcal{E})\big)$. Since $\bigwedge^r \mathcal{E} = \mathcal{O}(-K_X)$, its H^0 equals $\mathbb{C}^{g+2}$. Also, $\dim(\operatorname{Grass}(r, H^0(\mathcal{E})) \geq rs \geq g + 2$. It follows that there is a nonzero intersection $U^r \in \operatorname{Grass}(r, H^0(\mathcal{E}))$, that is, an r dimensional subspace $U^r \subset H^0(\mathcal{E})$ that generates a subsheaf $\mathcal{F} \subset \mathcal{E}$ of rank $\leq r - 1$.

Write

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0.$$

Then $\mathcal{O}(-K_X) = \det \mathcal{E} = \det \mathcal{F} + \det \mathcal{G}$ is a mobile decomposition (see Appendix B). This contradicts the assumption and completes the proof.

7.2 The bound on the genus

Proposition 7.3 *There does not exist any indecomposable Fano 3-fold of genus $g = 10 + 4n$ with $n \geq 1$.*

Remark 7.4 To prove the bound on the genus in the other cases, we prove, in a basically similar way, that there do not exist Fanos of genus $9 + 4n$, $7 + 4n$ and $12 + 4n$ with $n \geq 1$.

Proof To exclude $X_{2g-2} \subset \mathbb{P}^{g+1}$, we argue on its one-point blowup $\widetilde{X} = \mathrm{Bl}_P X \to X_{2g-2}$. We also write $P = \mathbb{P}^2$ for the exceptional locus of the blowup. Then

$$-K_{\tilde{X}} = -K_X - 2P$$

We choose the point $P \in X_{2g-2}$ not to lie on any line of X, and on at most finitely many conics.

Write T^3 for the projective tangent plane to $P \in X_{2g-2}$. Then for $g \gg 0$, we have $T \cap X = \{P\}$, and the linear system $|-K_X - 2P|$ of hyperplanes with multiplicity ≥ 2 at P is free. It defines the linear projection from T^3, or in other words, the double projection from P:

$$\begin{array}{ccccc} & P \in & X_{2g-2} & \subset & \mathbb{P}^{g+1} \\ & \nearrow & \downarrow & & \downarrow \\ \widetilde{X} & & & & \\ \Phi_{|-K_{\tilde{X}}|} & \searrow & \widetilde{X}'_{2g'-2} & \subset & \mathbb{P}^{g-3} \end{array}$$

Then $g' = g - 4$, and $(-K_{\tilde{X}})^3 = (-K_X)^3 - 8$.

Now X indecomposable implies that X' is also. We see that $g = 14$ does not exist. Indeed

$$\begin{array}{ccc} P \in X_{26} & \subset & \mathbb{P}^{15} \\ \downarrow & & \downarrow \\ \widetilde{X}'_{18} & \subset & \mathbb{P}^{11} \end{array}$$

Here X_{18} must contains the image of P, which is a Veronese surface of degree 4, and at the same time, it is contained in the G_2 manifold Σ^5_{18}. However, it follows from the flag decomposition of the G_2 manifold (4.3, pp. 17), that it does not contain any surface of degree 4. This contradiction proves that $g = 14$ does not occur.

The case $g = 10 + 4n$ with $n \geq 1$ is similar.

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