Prelimiting flips *†

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Abstract

The paper discusses an inductive approach to constructing log flips. In addition to special termination and thresholds, we introduce two new ingredients: saturation of linear systems, and families of divisors with bounded singularities. We state conjectures concerning these notions in any dimension, and prove these in general in dimension ≤ 2. This allows us to construct prelimiting flips (pl flips) and all log flips in dimension 4, and to prove the stabilization of an asymptotically saturated family of birationally free (b-free) divisors under certain conditions in dimension 3. In dimension 3, this stabilization generalizes special flips and also gives for the first time a proof of the existence of log flips that is algebraic in character, that is, via f.g. algebras, as opposed to geometric flips. It also accounts for all the currently known flips and flops.

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†Current draft, edited by Miles Reid. I haven't understood everything. I've tried to make the table of contents useful, but it could probably still be improved. I've started Appendix 1.12 discussing b-divisors, but it should be expanded. I have started on a list of index keywords, and it will be quite easy to make an index from this. Maybe a glossary to explain the new terms would also be useful. Most of the footnotes are questions to myself or to Slava.

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1 Introduction

1.1 Prelimiting contractions

By $f\colon X \to X_\vee$ we denote a *prelimiting contraction* or *pl contraction* with respect to a divisor S. By this we mean that f is a birational contraction such that

1. $S = \sum S_i$ is a sum of $s \geq 1$ prime Weil divisors $S_1, \dots, S_s$ on X that are $\mathbb{Q}$-Cartier and *proportional*, that is, $S_i \sim_{\mathbb{Q}} r_{i,j} S_j$ for some rational $r_{i,j} > 0$;

2. there is a boundary B such that $\lfloor B \rfloor = 0$ and $K + S + B$ is divisorially log terminal, and purely log terminal if $s = 1$; and

3. $K + S + B$ is numerically[1.1] negative/$X_\vee$.

By Conditions 1, 2 and [9, Corollary 3.8] (see also [4, Chap. 16]) the divisors S_i are normal varieties, with normal varieties in their intersections and normal crossing at the generic points of these intersections. In particular, $Y = \bigcap S_i$ is a normal variety.

Conditions 2 and 3 together mean that $(X/X_\vee, S+B)$ is a weak log Fano contraction[1.2] with only *log terminal* singularities. Thus by Connectedness of LCS (Kollár and others [4, Theorem 17.7]), Y is irreducible of dimension $d = n - s'$ near each fibre of f, where s' is the number of S_i that intersect that fibre. Moreover, f induces a contraction $Y \to f(Y) \subset X_\vee$, and (Y, B_Y) is Klt (Kawamata log terminal) for the adjoint log divisor $K_Y + B_Y = (K+S+B)_{|Y}$. These properties guarantee an induction on Y in our construction of pl flips below (see Section 3 for inductive families).

We always consider the *local* situation, where f is a germ over a neighborhood of a given point $P \in X_\vee$. In particular, all the S_i intersect the *central fibre* $f^{-1}P$ of X/P. Thus $\dim Y = d = n - s \leq n - 1$ by Condition 1. The *core dimension* d measures the difficulty of constructing pl flips, in a sense indicated in Induction Theorem 1.4.

It is sometimes also reasonable to consider a contraction f that is not birational. Then S is not numerically negative/$X_\vee$, in contrast to the case

[1.1] numerically negative means simply that minus it is an ample $\mathbb{Q}$-Cartier divisor. No clever distinction between numerically ample and ample is involved.

[1.2] *contraction* is my addition, for grammar.

of elementary pl flips. This situation occurs naturally in Section 3 in certain applications of Main Lemma 3.39 (see Example 3.34).

However, in Reduction Theorem 1.2 below, we can assume that the pl contractions f of most concern to us are *elementary*, that is, satisfy in addition:

4. S is numerically negative/$X_\vee$;
5. $K + S + B$ is strictly log terminal/$X_\vee$, that is, X is $\mathbb{Q}$-factorial and projective/$X_\vee$;
6. (EL.6) $\rho(X/X_\vee) = 1$, where ρ denotes the relative Picard number; and
7. (EL.7) f is small, or equivalently under the current assumptions, S is not f-exceptional.

We can omit the final assumption (EL.7), but then we meet an additional well-known situation, when the contraction is divisorial, and is its own S-flip (cf. Example 3.41). Note also that by (EL.6) f is nontrivial, that is, not an isomorphism.

A *pl flip* is the S-flip of f ([12, Section 5], see also Example 3.12 below). By a *log flip*, we mean the D-flip of a log contraction $(X/T, B)$ over a[1.3] base T with $D = (K + B)$, assuming (cf. [9, Theorem p. 96]) that

- $K + B$ is Klt, and
- $-(K + B)$ is nef/T.

Thus each elementary pl flip is also the log flip of $(X/T, B) = (X/X_\vee, aS+B)$ for some real number $0 \ll a < 1$, since $-(K+aS)$ is Klt, and $\sim_{\mathbb{R}} rS$ for some real $r > 0$. Note also that a D-flip is uniquely determined up to isomorphism by the class of D up to $\sim_{\mathbb{R}}$ and multiplication by positive real numbers (see Corollary 3.4).

Interest in pl flips rests on the following result:

Reduction Theorem 1.2 *Log flips exist in dimension n provided that:*

[1.3] At present T is not defined. There is a small problem of consistency; you use T, Z interchangeably in Chap 1. On the other hand, you also introduce the slightly crazy notation $X_\vee$, which appears only in a few sections. Is there some reason for using three different notations $X/X_\vee$, $/Z$, $/T$?

$(\mathrm{PLF})_n^{\mathrm{el}}$ elementary *pl flips exist in dimension n; and*

$(\mathrm{ST})_n$ *special termination holds in dimension n.*

We discuss Termination (ST) presently in Section 2, where we sketch a proof of Reduction Theorem 1.2. In Special Termination 2.3, we also prove that $(\mathrm{ST})_n$ follows from LMMP in dimension $n-1$. By our reduction, this is sufficient for the existence of 3-fold and 4-folds log flips.

Corollary 1.3 *In dimension $n \leq 4$, $(\mathrm{PLF})_n^{\mathrm{el}}$ implies the existence of all log flips.*

See the proof at the end of Section 2, p. 14. The following inductive statement is similar in nature, but its proof is more sophisticated. However, we are obliged to drop[1.4] the elementary property in our approach to pl flips, because it is not preserved on passing to covers (cf. Lemma 3.47).

Induction Theorem 1.4 *The statement*

$(\mathrm{PLF})_n^{\mathrm{small}}$ small *pl flips exist in dimension n*

follows from

$(\mathrm{FGA})_m$ *the finite generation (f.g.) of certain sheaves*[1.5] *of algebras over algebraic varieties of dimension $m \leq n-1$.*

More precisely, for any pl flip of core dimension $d = n - s$, we need the local version $(\mathrm{FGA})_d(\mathrm{bir})$ *in dimension d. Moreover, for the existence of these pl flips, it is enough to assume the local restricted version* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *in dimension d (we refer forwards to Definition3.43 and Conjecture 3.44 for* (RFA)*).*

Sections 3–4 explain all the statements (FGA), (RFA) together with variations on them, and relations between them.

[1.4] Slava: every time you say "drop", you introduce an ambiguity: it could mean that you omit the condition in the assumptions, or in the conclusions, or both. In many cases, I have not been able to sort out which you mean to clear up the ambiguity. "drop" $\mapsto$ "omit" is better English.

[1.5] Is singular or plural intended? It might be better to say f.g. of algebras of type $(\mathrm{FGA})_{n-1}$.

Corollary 1.5 *LMMP and* $(\mathrm{FGA})_m$ *in dimension* $m \le n-1$ *imply the existence of log flips in dimension* n*. More precisely, for the existence of these flips, it is enough to assume* $(\mathrm{FGA})_m(\mathrm{bir})$ *or even* $(\mathrm{RFA})_{n,m}(\mathrm{bir})$ *in dimension* $m \le n-1$*.*

See the proof at the end of Section 2. Since LMMP is known in dimension ≤ 3 [12, Theorem 5.2], for 3-fold and 4-folds log flips, we can focus on (FGA) and (RFA).

Corollary 1.6 $(\mathrm{FGA})_m$ *in dimension* $m \le 3$ *implies the existence of log flips in dimension* $n \le 4$*. More precisely, for the existence of these flips, it is enough to assume* $(\mathrm{FGA})_m(\mathrm{bir})$ *or even* $(\mathrm{RFA})_{n,m}(\mathrm{bir})$ *in dimension* $m \le n-1$*.*

Proof Immediate by Corollary 1.3 and Induction Theorem 1.4. $\square$

Main Theorem 1.7 (FGA) *and* (RFA) *hold in low dimension:*

$$(\mathrm{FGA})_m \qquad \textit{holds for } m \le 2\textit{; and}$$

$$(\mathrm{RFA})_{n,m}(\mathrm{bir}) \qquad \textit{holds for all } n \le 4 \textit{ and } m \le n-1.$$

These are among the main results of the paper, and they imply

Corollary 1.8 *Log flips exist in dimension* ≤ 4*.*

Proof Immediate by Corollary 1.6 and the Main Theorem. $\square$

The (RFA) statement of Induction Theorem 1.4 is proved in Section 3, and the (FGA) statement in Section 4. These two sections explain the role of pl flips in LMMP as a link between the geometry and algebra of flips. Section 3 introduces graded divisorial sheaves of algebras, and the restricted algebras appearing in (RFA). Section 4 introduces more general functional algebras, and we state a conjecture that we need about finitely generated algebras (FGA) of a certain type.

The Main Theorem needs more preparations; we carry these out for (FGA) in Sections 5–6, and Section 6 contains the proof of (FGA). For (RFA), Sections 8–10 contain preparations, and Section 11 the proof.

1.9 History

Kulikov, Reid, Mori, Kawamata, Tsunoda, Kollár, Kachi, Takagi.

Remark 1.10 Minor modifications of our arguments prove the existence of 3-fold and 4-fold log flips in the analytic category over a neighborhood of any compact subset of Z.

1.11 Conventions

Since the construction of flips is local, we fix $P \in S_\vee \subset X_\vee$ and view $f\colon X \to X_\vee$ throughout as a germ$/P$. All other varieties and their objects are considered locally$/P$. For example, if D is a divisor on a variety $Y/X_\vee$ then $|D|$ denotes its linear system in a neighborhood of a fibre of Y/P.

$\operatorname{Mov} D$ and $\operatorname{Fix} D$ denote the *mobile* and *fixed* components of an $\mathbb{R}$-Weil divisor D, that is, divisors $M = \operatorname{Mov} D$ and $F = \operatorname{Fix} D$ such that $|D| = |M| + F$ and $|M|$ does not have fixed components. These are well defined if $|D| \neq \emptyset$. Then $D = M + F$ with $F \geq 0$ and M integral (Cartier in codimension 1). If $|D| = \emptyset$ we set $F = +\infty$; then $M = -\infty$ and $\mathcal{O}_X(M) = 0$.

To say that an $\mathbb{R}$-divisor D is nef always means in particular that D is $\mathbb{R}$-Cartier (compare b-nef in Lemma 4.21). This last condition holds for every $\mathbb{R}$-divisor if Y is $\mathbb{Q}$-factorial.

The base field k is of characteristic 0. For example, $k = \mathbb{C}$.

1.12 Temporary Appendix: b-divisors

Much of the paper uses the language of b-divisors; see [12] and Appendix 1.12 for a discussion of these ideas. $\mathcal{A} = \mathcal{A}^X = \mathcal{A}(X, B)$ denotes the *discrepancy* or *coboundary b-divisor* of the pair (X, B) for some $\mathbb{R}$-divisor B on X. (This is R in [12, Definition 1.1.4], with $C = 0$ as a b-divisor, and $C_X = B$ in the formula.) $\mathcal{B} = \mathcal{B}^X = \mathcal{B}(X, B) = -\mathcal{A}$ denotes the *codiscrepancy* or *pseudo-boundary* b-divisor of (X, B).

The "b-" of b-divisor stands for *birational*, that is, a *b-divisor* is a system of divisors on all the different models of X compatible with birational equivalence. If $Y \dashrightarrow X$ is an isomorphism in codimension 1, we can walk a divisor from X to Y by birational transform. Otherwise we can do various things, sensible or otherwise, with the exceptional prime divisors. The notion of b-divisor is introducted in [11, Section 1] to do this systematically.[1.6]

The notion of b-divisor hides an exceptional divisor on each concrete model, which may be extremely infinite.

The definition in [11, Section 1] has at least one defect: "the group of b-divisors is the Abelian group generated by prime Weil b-divisors". Here "generated" is in the same sense as the formal power series ring is generated by monomials. The group of b-divisors is in fact between the sum $\sum ZD_i$ and the product $\times ZD_i$ where D runs through prime b-divisors (geometric or divisorial valuations). So, each b-divisor $D = \sum d_i D_i$ and this sum can be infinite, but on each model Y of X its visible (divisorial) component $D_Y = \sum d_i D_i$ is a usual divisor on Y, where now D_i only runs through prime Weil divisors on Y. Thus, the latter sum D_Y is finite in the algebraic category (locally finite in the analytic one). Then this condition is satisfied on each model Y.

Thus, in a formal sense, a b-divisor is a projective limit of ordinary divisors.

Frequently asked questions

Q Give some examples of b-divisor.

A The total transform of a Cartier divisor, the birational transform of a mobile linear system, the canonical class of Y, or the discrepancy divisor $K_Y - K_X X$ over X are divisors defined on every model $Y \to X$ that are compatible with pushdown.

Q Is the canonical class a bidivisor?

A The canonical b-divisor $\mathcal{K}$ is a typical example. Just as the usual canonical divisor, $\mathcal{K}$ is associated by definition with a rational canonical differential form ω as follows. If D_i is a usual prime divisor on a model Y then d_i is the multiplicity of $K_Y = (\omega)$ in D_i. Thus, on any model Y, $\mathcal{K}_Y = K_Y = (\omega)$ (note that ω is well defined on any model). If in addition K_X is $\mathbb{Q}$-Cartier, then Reid's discrepancy $R = \mathcal{K}_X - \overline{K_X}$ is a b-divisors that is exceptional and measures the deviation of $\mathcal{K}_Y$ from being a Cartier b-divisor $\overline{K_X}$ over X.

[1.6] This appendix will be rewritten formally. What follows is based on an e-mail correspondence with FAQ by Iskovskikh, Kachi, Reid, etc.

Q Can a b-boundary have infinitely many components?

A Yes, it can. Normally a log boundary B is suppose to have multiplicity 1 in all exceptional prime Weil divisors, but multiplicity 0 at almost all nonexceptional divisors. The discrepancy b-divisor has bigger and bigger multiplicities at exceptional divisors on higher and higher models Y/X.

Q Is the following interpretation right? Consider the category (bir)$/X$ of birational morphisms $Y \to X$. Then a Weil b-divisor is any map $\mathcal{D}\colon Y \mapsto D_Y$ that assigns a Weil divisor D_Y to each Y, in a way that is compatible with push down (that is, $f_*(D_{Y'}) = D_Y$ for every morphism $f\colon Y' \to Y$ of (bir)$/X$. For example, we could choose some model Y and a Cartier divisor D_Y on Y that pushes down to D on X, and just take $D_{Y'} = f^*D_Y$ for every $f\colon Y' \to Y$, and the obvious pushdown from a hut $Y' \swarrow W \searrow Y''$ for models not dominating Y. (Or maybe (bir)$/X$ is only considered as a limiting system? Are b-divisors equal (or identified) if they coincide from some Y/X upwards?)

A ...? b-divisors turn up at many points of the paper, notably Definition 3.1, the discussion after Theorem 3.14, Definition 4.9, Section 7, etc., and the correct interpretation of these matters can be determined from the treatment given there. There are accompanying notions of b-nef, b-semiample, b-free, b-Cartier, b-divisorial sheaf, b-divisorial algebra, b-restriction, etc. In short, it's a whole bundle of fun.

2 Special termination

Let $(X/Z, B)$ be a log pair such that

- $f\colon X \to Z$ is a proper morphism; and
- (X, B) is log canonical.

Let $g\colon X \to Y/Z$ be a *birational contraction* of $(X/Z, B)$ that is *log canonical*, that is, g satisfies:

log canonical $\qquad K + B$ is numerically negative$/Y$.

Equivalently, this is a contraction $\mathrm{cont}_F = g$ of an extremal face F of $\overline{\mathrm{NE}}(X/Z$ with $(K+B)\cdot F<0$. We say that the $(K+B)$-flip of g is a *log canonical* flip (see [12, Section 5].)

Caution 2.1 These flips are log flips in the sense of Section 1 only if $K+B$ is Klt.[2.1]

Special termination in dimension n is the statement

$(\mathrm{ST})_n$ for arbitrary $(X/Z,B)$ with $\dim X=n$, any chain of successive log canonical flips in extremal faces F with $|F|\cap\lfloor B\rfloor\neq\emptyset$ terminates.

Here the *support* $|F|$ of an extremal face F is the exceptional subvariety of cont_F, that is, the union of curves C/Z with $\mathrm{cont}_F\, C=\mathrm{pt}$.

Example 2.2 Suppose that we have a chain of successive log canonical flips of $(X/Z,B)$ in extremal rays R with $S\cdot R<0$, where S is reduced and $0<S\leq\lfloor B\rfloor$. Then $|R|\subset\lfloor B\rfloor$ and $|R|\cap\lfloor B\rfloor=|R|\neq\emptyset$. If $(\mathrm{ST})_n$ holds then any chain of this form terminates. We meet this situation below in the proof of Reduction Theorem 1.2.

Special Termination 2.3 *LMMP in dimension* $\leq n-1$ *implies* $(\mathrm{ST})_n$.

Remark 2.4 For the proof of Reduction Theorem 1.2, we need the situation where

- $f\colon X\to Z$ is projective;
- (X,B) is log terminal;
- X is $\mathbb{Q}$-factorial; and
- the contractions of $(X/Z,B)$ and their flips are *extremal*, that is, they are contractions g satisfying

extremal $\quad \rho(X/Y)=1.$

Flips preserve this situation. Thus, in Special Termination 2.3, it is enough to assume log termination in LMMP (cf. Remark 2.5 below).

[2.1] Note that [12] develops LMMP in the log canonical category, more general than that of [4]; in particular, its contractions may involve contracting a locus of log Kodaira dimension ≥ 0 along its Iitaka fibration.

Proof First, as in [9, proof of Theorem 4.1], after a finite number of flips, we can assume that

- for flips, the support $|F|$ does not contain log canonical centers,

because the set of log canonical centers is finite and each flip in F eliminates any log canonical center in $|F|$.

We prove termination by induction on the dimension $\dim S = d \le n-1$ of log canonical centers $S \subset \lfloor B \rfloor$ of (X, B) that intersect $|F|$. We state that such flips terminate, that is, after a finite number of steps, they do not intersect the log canonical centers $S \subset \lfloor B \rfloor$ of dimension d. In particular, this gives $(\mathrm{ST})_n$ for $d = n-1$.

Thus by the above we know this for $d = 0$. Suppose now that

- for flips, the support $|F|$ does not intersect any log canonical center $S \subset \lfloor B \rfloor$ of dimension $< d$.

For our applications, the only difficult case is $d = 1$ and $n = 4$: that is, a curve $S \subset \lfloor B \rfloor$ in a 4-fold X.

Note that, by our assumptions, S is a minimal center, and is normal as a subvariety near $|F|$. By local adjunction, $(K+B)_{|S} = K_S + B_S$ is Klt near each point $P \in |F| \cap S$, and all the possible multiplicities of D_S belong to a finite subset of the real interval $[0, 1)$. This subset only depends on the local structure of (X, B) near S, and includes the multiplicities of B near P. Moreover, under the restrictions of Remark 2.4, these multiplicities are all of the form

$$\frac{m-1}{m} + \sum \frac{l_i}{m} b_i,$$

where b_i are the multiplicities of B and m and l_i are natural numbers. This follows from [9, Corollary 3.10 and Lemma 4.2].

In particular, there is only[2.2] a finite set of such numbers that are ≤ 1 minus the least log discrepancy of (S, B_S) for the starting model (X, B), in the Klt part of (S, B_S). Another approach is to let $(W, B_W) \to S$ be the Iitaka fibration of S, with $\dim W = n-2$, replace S by W and use divisorial adjunction on W for a log terminal resolution of $S \subset \lfloor B \rfloor$. Then

[2.2] Maybe something like: we introduce the notation $\mu = \mathrm{lld}(S, B_S)$ (least log discrepancy); then in the interval $[0, 1-\mu]$, the starting model (X, B) has only a finite set of ...?

finiteness means the finiteness of such models with the above multiplicities $\leq 1 - \mathrm{lld}(S, B_S)$. The only interesting case is $d = 1$ and $n = 4$: then W is a log elliptic Iitaka fibrations over a curve, with a positive lower bound for the least log discrepancy and given boundary multiplicities.

Each flip restricts to a birational transformation of $(S/Z, B_S)$ or of the Iitaka fibration $(W/Z, B_W)$. Unfortunately, restricted to these, it may not be a flip; this happens exactly if the flipped contraction on $(S^+/Z, B_S^+)$ or $(W^+/Z, B_W^+)$ is not small. In this case, it blows up prime divisors that again have multiplicities of B_S^+ or B_W^+ in our finite set, and $< 1 -$ the log discrepancy in them for the previous model $(S/Z, B_S)$ or $(W/Z, B_W)$. However there only exist a finite number of the latter transformations (cf. [9, proof of Theorem 4.1]). The final monotonicity follows from adjunction and the opposite monotonic property of log discrepancies for flips, strict[2.3] in $|F|$ [8, 2.13.3]. Only such transformations occur in our main case $d = 1$ and $n = 4$.

Then we use termination of log flips on $(S/Z, B_S)$ or $(W/Z, B_W)$ by LMMP. $\square$

Remark 2.5 The termination at the end of the proof of special termination is that of any chain of successive log flips. They are *not necessarily extremal*, that is, they may contract extremal faces of dimension ≥ 2. Using LMMP, we can reduce this to a chain of elementary flips. Thus we need LMMP in dimension $n - 1$ in an essential way.

Proof of Reduction Theorem 1.2 We follow the part of the argument of [9, proof of Reductions 6.4–5] that concerns limiting flips, with improvements by Kollár and others [4, (18.12.1.3–4)]. In this reduction, each small contraction cont_R is elementary pl, and then we discard all the reduced irreducible components S_i in B with $S_i \cdot R \geq 0$. Indeed, by construction, B includes a reduced component S_i with $S_i \cdot R < 0$. Thus log flips exists by $(\mathrm{PLF})_n$ and are the same as before discarding some of the S_i. They terminate by Example 2.2 and $(\mathrm{ST})_n$ (cf. [9, Corollary 4.6]). $\square$

Proof of Corollary 1.3 Immediate by Reduction Theorem 1.2, Special Termination 2.3 and LMMP in dimension ≤ 3 [12, Theorem 5.2]. $\square$

[2.3] What? Presumably strictly monotonic under some assumption is intended.

Proof of Corollary 1.5 Immediate by Reduction Theorem 1.2 and Induction Theorem 1.4, and by Special Termination 2.3. $\square$

3 Divisorial $\mathcal{O}_X$-algebras and flips

In this section we explain the relation between the geometry and algebra of D-flips, and prove Induction Theorem 1.4 in the case of (RFA) algebras. We start with geometry. The following clumsy definition aims to unify some standard geometric constructions, in particular, flips and contractions.

Definition 3.1 We say that a Weil $\mathbb{R}$-divisor D on X is *b-sup-semiample/*Z or *bss ample* if one of the following two equivalent conditions hold

(BSS) there is a *rational* 1*-contraction* $g\colon X \dashrightarrow Y/Z$ and a numerically ample $\mathbb{R}$-divisor H on Y/T such[3.1] that $D \sim_{\mathbb{R}} g^*H+E/Z$, where E is effective and *essentially exceptional* on Y; or

(ZDC) there is a hut

$$\begin{array}{ccc} & W & \\ {}^{g}\swarrow & & \searrow^{h} \\ X & & Y \ /Z \end{array}$$

with contractions g, h and[3.2] a numerically ample $\mathbb{R}$-divisor H on Y/T such[3.3] that g is birational, each g-exceptional divisor is *contractible* for h, and $g^{-1}D + F \sim_{\mathbb{R}} h^*H + E/Z$, where F, E are Weil $\mathbb{R}$-divisors on W that are exceptional on X and Y respectively, E is effective, and the complement to $\operatorname{Supp} E$ in the fibre over each divisor in Y is not exceptional on X, so that in particular, $g(E)$ is essentially exceptional on Y.

Proof–Explanation By a *rational contraction* $X \dashrightarrow Y/Z$ we mean a composite $g := h \circ g^{-1}$ of two contractions as in (ZDC). This is a dominant rational map with connected fibres $g(h^{-1}P)$. For any $\mathbb{R}$-Cartier divisor H,

[3.1] must be Y/Z. T and Z seem to be interchangeable

[3.2] Here the letter g is used for two different maps $g\colon X \dashrightarrow Y$ and $g\colon W \to X$. Change one of them to i or γ or something.

[3.3] must be Y/Z.

we can define a Weil $\mathbb{R}$-divisor $g^*H := g(h^*H)$ on X that is independent of the Hironaka hut. In terms of b-divisors, $g^*H = \mathcal{H}_X$, where $\mathcal{H} = \overline{h^*H}$ [12, Example 1.1.1].

By a *rational* 1*-contraction* we mean a rational contraction $g\colon X \dashrightarrow Y$ such that

- g does *not blow up* divisors *downstairs*: every exceptional prime b-divisor E of X is *contractible* on Y, that is,

$$\dim \operatorname{center}_Y E < \dim E = \dim X - 1.$$

Thus each hut in (ZDC) defines a rational 1-contraction and vice versa.

With respect to $g\colon X \dashrightarrow Y$, we say that a divisor E *on* X is *exceptional on* Y if it is contractible and *nonhorizontal*, that is, has $g(E) \neq Y$; we only need this when the rational contraction g is not birational or equivalently, h is not birational. For a rational contraction of fibre type, it contracts to a divisor again. In this case, it is *essentially exceptional* if its support does not contain entire fibres over divisors in Y. The divisor is *really exceptional* if its image is supported in codimension ≥ 2. For birational contractions all flavors of contractibility are equivalent.

The divisor E in (ZDC) gives $E := g(E)$ in the statement (BSS), since F is exceptional on X and $g(E)$ is still essentially exceptional on Y. Conversely, the equivalence $D \sim_{\mathbb{R}} g^*H + E/Z$ gives an equivalence $g^{-1}D + F \sim_{\mathbb{R}} h^*H + E/Z$ in which $E := g^{-1}E \geq 0$ and F are still essentially exceptional on Y and X respectively. $\square$

Each bss ample divisor is effective modulo $\sim_{\mathbb{R}}$:

$$D \sim_{\mathbb{R}} h^*H + E \sim_{\mathbb{R}} h^*H' + E \quad \text{and} \quad \geq 0,$$

where the $\mathbb{R}$-divisor $H' \sim_{\mathbb{R}} H/Z$ is effective. Thus not every divisor D is bss ample. However, the following uniqueness result holds.

Proposition 3.2 *Suppose that D is bss ample. Then the rational contraction g (up to isomorphism) and the divisor E are uniquely determined by $(X/Z, D)$. The divisor H is defined up to $\sim_{\mathbb{R}} /Z$.*

Proof Since $E \sim_{\mathbb{R}} D - \mathcal{H}_X/Z$ is effective and essentially exceptional on Y, by [9, Negativity 1.1], it is unique if $\mathcal{H}$ is unique up to $\sim_{\mathbb{R}}/Z$. Thus it is enough to establish the uniqueness of $\mathcal{H}$. If $\mathcal{H}'$ corresponds to another g', Y' and E', then $\mathcal{H} + \mathcal{E} \sim_{\mathbb{R}} \mathcal{H}' + \mathcal{E}'/Z$, where $\mathcal{E}, \mathcal{E}'$ are effective and essentially exceptional on Y, Y' respectively. Note that $\mathcal{H}'$ is free modulo $\sim_{\mathbb{R}}/Z$, $\mathcal{H}$ is numerically trivial$/Y$, $\mathcal{H} - \mathcal{H}' \equiv \mathcal{E}' - \mathcal{E}/Z$ is numerically seminegative over Y, and the negative components of $\mathcal{E}' - \mathcal{E}$ are exceptional on Y. Thus if h is not birational, that is, h is of fibre type, the whole of $\mathcal{E}' - \mathcal{E}$ is exceptional, and $\mathcal{H}' \equiv 0$ over the generic point of Y. Moreover, the latter holds over generic points of every divisor in Y, because we can assume that $\mathcal{H}'$ is effective without fixed components and nonhorizontal. Hence $\mathcal{E}' - \mathcal{E} \equiv 0$ over generic points of divisors. By (ZDC) $\mathcal{E}$ does not support entire fibres over generic points of divisors. Hence $\mathcal{E}' - \mathcal{E} \geq 0$ over generic points of divisors, and the negative part of $\mathcal{E}' - \mathcal{E}$ is really exceptional. This also holds if h is birational.

Hence, again by [9, Negativity 1.1], $\mathcal{E}' - \mathcal{E} \geq 0$ or $\mathcal{E}' \geq \mathcal{E}$ since, for every model W/Y and $/Y'$ of X, we have that $-(\mathcal{H}_W - \mathcal{H}'_W)/Y$ is nef and $\mathcal{E}' - \mathcal{E}$ is effective on prime divisors that are really nonexceptional on Y, in particular on $\mathcal{E}'$. Similarly, $\mathcal{E} \geq \mathcal{E}'$. Hence $\mathcal{E}' = \mathcal{E}$ and $\mathcal{H} \sim_{\mathbb{R}} \mathcal{H}'$. $\square$

Corollary 3.3 *If $D' \sim_{\mathbb{R}} rD$ for some positive $r \in \mathbb{R}$, then D' is also bss ample with the same rational 1-contraction g, $E' = rE$, and $H' \sim_{\mathbb{R}} rH$. Thus bss ample is a property of the ray $\mathbb{R}^{>0} \cdot D/{\sim_{\mathbb{R}}}$ in the space of divisors.*

Proof Immediate by Proposition 3.2, because $rE > 0$ and $H' = rH$ is ample. $\square$

The most important applications have $E = 0$; then we say that D is *b-semiample$/Z$*, that is, semiample[3.4] on some model W of X after adding certain divisors that are exceptional on X. For example, if $Y =$ pt., then $H = 0$, $E = 0$, $D \sim_{\mathbb{R}} \mathcal{H} = 0$.

Corollary 3.4 *Suppose that $f\colon X \to Z = T$ is a birational contraction. Then any D-flip is uniquely determined by D given up to $\sim_{\mathbb{R}}$ and multiplication by positive real numbers.*

Lemma 3.5 *The D-flip of a contraction $f\colon X \to T$ exists if and only if*

(BSA) *the divisor $f(D)$ on T/T is bss ample$/T$,*

[3.4] Slava: You might consider changing "semiample" $\mapsto$ "eventually free", which is more logical terminology, and sounds better in English to me; at two points later you say "very semiample" for "free", which seems quite bizarre.

and then the flipped contraction $g^{-1}\colon Y \to Z = T$ is small, and the flipping modification is $g \circ f$; we have $E = 0$ in (BSS).

The same holds for D if f is small.

In (BSA) $E = 0$, so $f(D)$ is b-semiample/T.

Proof If the D-flip exists, then it defines a small contraction $Y = X^+/T$ with $H = D^+ = g(f(D))$ numerically ample/T and a flipping modification $X \dashrightarrow X^+ = Y/T$. Then $f(D)$ is bss ample with the rational 1-contraction $g\colon T \dashrightarrow X^+$, given H and $E = 0$. Indeed, $f(D) = g^*H = g^*D^+$.

If $f(D)$ is bss ample, then $Y/T = Z$ is small, $E = 0$ and

$$D^+ = g(f(D)) \sim_{\mathbb{R}} g(g^*H) = H.$$

The same arguments apply to D itself for any small f. □

Proof of Corollary 3.4 Immediate by Lemma 3.5, Proposition 3.2 and Corollary 3.3. □

Corollary 3.6 *Suppose that D is nef/Z. Then D is bss ample if and only if it is semiample* [12, Definition 2.5]. *In addition $E = 0$. In particular, every semiample divisor D is bss ample.*

Proof We can replace $(X/Z, D)$ by $(W/Z, g^*D)$ in (ZDC). Then $E = 0$ by (BSS) and [9, Negativity 1.1]. Thus $D \sim_{\mathbb{R}} h^*H$, $\overline{D} \sim_{\mathbb{R}} \mathcal{H}$ and h contracts all curves C on W/Z and X/Z with $D \cdot C = 0$. □

Thus if D is bss ample, Proposition 3.2 define a unique decomposition

$$D = D^m + D^e.$$

as a sum of two Weil $\mathbb{R}$-divisors $D^e := E \geq 0$ and $D^m := D - E$. If $X/\mathrm{pt.}$ is a surface, this is a *Zariski decomposition*; then g in (BSS) is a regular contraction. It is also useful to consider the unique b-divisors $\mathcal{D}^m \sim_{\mathbb{R}} \mathcal{H}$ with $\mathcal{D}^m_X = D^m$, and $\mathcal{D}^e = E$ (considered as a b-divisor) with $\mathcal{D}^e_X = D^e$. The divisor

$$D^{\mathrm{small}} = \sum (\mathrm{mult}_{D_i} D) D_i, \quad \text{for } D_i \text{ not in } \operatorname{Supp} D^e = \operatorname{Supp} E \tag{3.6.1}$$

is also important. We define generalizations of these for any D presently in Section 4.

Again if Y is a point, then $D^e = \mathcal{D}^e = 0$, $D^{\mathrm{small}} = D^m \sim_{\mathbb{R}} 0$ and $\mathcal{D}^m \sim_{\mathbb{R}} 0$.

3.7 The algebra of flips

Here we interpret a flip in terms of a certain graded functional sheaf of algebras. Its *homogeneous* elements (that is, elements of *pure degree*) are sections of *divisorial* $\mathcal{O}_X$-sheaves $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$ with

$$\Gamma(U, \mathcal{O}_X(D)) = \big\{a \in k(X) \bigm| (a) + D \geq 0\big\},$$

where D is a Weil $\mathbb{R}$-divisor, and $k(X)$ denotes the field of rational functions of X; we write (a) for the divisor of a nonzero element $a \in k(X)$, with the convention that 0 has divisor $(0) = +\infty$. An axiomatic treatment of divisorial sheaves is given by Reid [6, Appendix to §1]. The sheaves $\mathcal{O}_X(D)$ are *functional sheaves*, that is, subsheaves of the constant sheaf $k(X)$, and are coherent. They are well defined if D is *nonsingular*, that is, every generic point of $\operatorname{Supp} D$ is nonsingular in X. (At a prime divisor D_i where X has a nonnormal singularity, the above sheaf condition means that f is regular[3.5] at D_i.) However, since X is assumed to be normal, we can disregard these subtleties. Any Weil $\mathbb{R}$-divisor on a normal variety X is nonsingular, because S is nonsingular in codimension 1.

Definition 3.8 Let $g\colon X \to Z$ be a proper morphism. We define the $\mathbb{N}$-*graded* $\mathcal{O}_Z$*-algebra* of a Weil $\mathbb{R}$-divisor D on X as

$$\mathcal{R}_g\, D = \mathcal{R}_{X/Z}\, D \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} g_* \mathcal{O}_X(iD).$$

This is an $\mathbb{N}$-graded *functional* $\mathcal{O}_Z$-subalgebra of the constant $\mathcal{O}_Z$-algebra

$$k(X)_{\bullet} = \bigoplus_{i=0}^{\infty} k(X).$$

Thus the natural multiplication in $\mathcal{R}_{X/Z}\, D$ is induced by multiplication in $k(X)$: for homogeneous elements $a \in \mathcal{O}_X(iD) \subset k(X)$ and $b \in \mathcal{O}_X(jD) \subset k(X)$, we have $ab \in \mathcal{O}_X((i+j)D) \subset k(X)$. Indeed, $(ab) + (i+j)D = (a) + iD + (b) + jD \geq 0$.

An $\mathbb{N}$-graded $\mathcal{O}_Z$-algebra is *(relatively) divisorial* for X/Z if it is isomorphic to $\mathcal{R}_{X/Z}\, D$ for some D. An $\mathbb{N}$-graded $\mathcal{O}_Z$-algebra is *(relatively) b-divisorial* for X/Z if it is isomorphic to $\mathcal{R}_{Y/Z}\, D$ for some model Y/Z of X/Z and for some divisor D on Y. Section 4 generalizes these algebras and the notation (see Definition 4.9 and Example 4.11 below).

[3.5] This is of course wrong. It's not very subtle either.

Remark 3.9 If $X = Z$, so that $X/Z = X/X$ is the identity, each homogeneous component $g_*\mathcal{O}_X(iD) = \mathcal{O}_X(iD)$ is a divisorial sheaf. The algebras $\mathcal{R}_{X/X}\,D$ have a similar axiomatic description. If X/Z is a birational contraction, then $g_*\mathcal{O}_X(iD)$ is a *b-divisorial* sheaf. They also have an axiomatic description, as do the b-divisorial algebras.

Example 3.10 If D is a Cartier divisor on X, its algebra $\mathcal{R}_{X/X}(D)$ is canonically isomorphic to the *tensor algebra* of D

$$\mathcal{T}(\mathcal{O}_X(D)) = \bigoplus_{i=0}^{\infty} \mathcal{O}_X(D)^{\otimes i}.$$

This graded algebra is defined for any $\mathbb{R}$-divisor D. However the above isomorphism does not hold for general $\mathbb{R}$- or $\mathbb{Q}$-Cartier divisors D, even in the local case X/X, because the tensor algebra is f.g., and, moreover, generated by homogeneous elements of degree 1. For any $\mathbb{R}$-divisor D, the divisorial algebra $\mathcal{R}_{X/X}(D)$ is canonically isomorphic to the *reflexive tensor algebra* of D

$$\mathcal{A}(\mathcal{O}_X(D)) = \mathcal{T}(\mathcal{O}_X(D))^{\vee\vee}.$$

But this algebra may be infinitely generated.

If X/Z is not the identity then divisorial and b-divisorial algebras are more complicated. But they are sometimes f.g., as we see in Section 4.

Definition 3.11 An $\mathcal{O}_Z$-algebra has *finite type* if it is *finitely generated* or *f.g.* The latter means that it is locally$/Z$ generated as an $\mathcal{O}_Z$-algebra by a finite set of sections (*generators*).

Example 3.12 Let $f\colon X \to T$ be a birational contraction and D an $\mathbb{R}$-Weil divisor. Then the *flipping* algebra of $(X/T, D)$ is the divisorial algebra

$$\mathcal{F}\mathcal{R}_{X/T}\,D = \mathcal{R}_{T/T}\,f(D).$$

For any $\mathbb{Q}$-divisor D, by Corollary 3.28, the D-flip exists if and only if $\mathcal{F}\mathcal{R}_{X/T}\,D$ is f.g.

Note that if f is small then

$$\mathcal{F}\mathcal{R}_{X/T}\,D \cong \mathcal{R}_{T/T}\,D$$

by Example 3.17, (b). But this is not true in general. For example, the isomorphism does not hold if f is divisorial and $D = H$ is a hyperplane section of X/T (use Theorem 3.14 below).

Example 3.13 (Stupid Example) Let P be a point on a complete curve C. Take $D = dP$ with nonnegative $d \in \mathbb{R}$. Then $\mathcal{R}_{C/\mathrm{pt.}}\, D$ has finite type if and only if d is rational (cf. Theorem 3.14 below). Nonetheless D is semiample for any real $d \geq 0$ and bss ample by Corollary 3.6.

Note that $\mathcal{R}_{C/\mathrm{pt.}}\, D$ is also f.g. for any real number $d < 0$ because then it is *trivial*, that is, it has no sections of *positive degree*.

Thus f.g. is somewhat different from ample, or from bss ample, although the notions are intimately related. Here the important phenomena are rationality of multiplicities and nonvanishing in positive degrees. However, to explain f.g. in geometric terms of bss ampleness, we need some preparations.

Theorem 3.14 *The divisorial algebra $\mathcal{R} = \mathcal{R}_{X/Z}\, D$ is f.g. and* globally almost generated *if and only if D is bss ample/Z and D^{small} is a $\mathbb{Q}$-divisor (see (3.6.1)); this last* rationality condition *holds in particular if D is a $\mathbb{Q}$-divisor.*

Moreover, if $\mathcal{R}$ is f.g., then, in the notation of Definition 3.1, $Y/Z \cong$ $\mathrm{Proj}_Z\, \mathcal{R}$, *$E$ is the* stable base divisor *of $\mathcal{R}$ and* $\mathrm{Supp}\, E$ *is the* stable divisorial base locus *of $\mathcal{R}$.*

The proof is given on p. 29.

A divisorial algebra $\mathcal{R} = \mathcal{R}_{X/Z}\, D$ is *globally almost generated* or *g.a.g.* if there is a natural number N and sections of $f_*\mathcal{O}(ND)$ that generate the algebra $\mathcal{R}_{X/X}\, ND$ in codimension 1 except for a divisorial subset that is exceptional on $Y = \mathrm{Proj}_Z\, \mathcal{R}$. (In this definition, we need to assume that $\mathcal{R}$ is finitely generated. But one can generalize it to get rid of this assumption.) In particular, this implies that $\mathcal{R}$ is *nontrivial*, that is, it has a nonzero element of some positive degree $d > 0$. For a functional algebra, this implies that the algebra has a nonzero element in each degree *i divisible by d*. By the theorem, we can define divisors E, D^m and b-divisors $\mathcal{D}^m, \mathcal{D}^e$, etc., for $\mathcal{R}$. These are well known: E is the *stable base divisor* of D, and it gives the *stable divisorial base locus* $\mathrm{Supp}\, E$ as the above exceptional divisorial subset. Either of these can be referred to D or to the *divisorial algebra* $\mathcal{R}$, but as an algebra associated with D, not an abstract algebra (cf. Truncation Principle 4.5).

Note that if we replace bss ample by semiample in the theorem, we should replace g.a.g. by the usual global generation of the algebra $\mathcal{R}_{X/X}\, ND$. The next result shows under what circumstances we can add an effective divisor to the mobile part of a multiple linear system.

Lemma 3.15 *Let D be a semiample $\mathbb{Q}$-divisor, and suppose that E is an effective Weil $\mathbb{Q}$-divisor that is not contractible, or is exceptional but not essentially exceptional*[3.6] *for the contraction given by D. Then for some natural numbers $N \gg 0$ and M, the base locus $\operatorname{Bs}(ND + ME)$ is a strict subset of $\operatorname{Supp} E$, that is,*

$$\operatorname{Supp}\operatorname{Fix}(ND + ME) \subsetneq \operatorname{Supp} E.$$

Addendum 3.15.1 *Moreover, if E is integral, we can take $M = 1$ except for the following two cases:*

- *E has no contractible or nonhorizontal components, $\nu(D) = \dim X - 1$ (the numerical dimension) and the generic fibre of the contraction is a curve of genus $g \geq 1$; then we can take $M = g + 1$; or*
- *E is contained in multiple fibres of the contraction; then we can take M to be the least common multiple of the multiplicities.*

Proof We can assume that all components of $\operatorname{Supp} E$ have a single irreducible image for the contraction $h\colon X \to Y/Z$ given by D. Then we have three cases:

Case: E is not contractible and horizontal Then $\nu(D) = \dim \operatorname{Supp} E = \dim X - 1$. The generic fibre of h is a curve of genus g, and $ND + ME$ cuts out a divisor of positive degree $\geq M$. Thus we have the required result in the generic curve by RR for curves. This implies the lemma in this case for $N \gg 0$.

Case: E is not contractible and not horizontal Then $\nu(D) = \dim X$ and h is birational. In this case the lemma follows from the general RR (compare Reid [6, proof of Lemma (1.6)]).

Case: E is contractible but not essentially exceptional This means that $\nu(D) < \dim X$, that is, h is fibred, and $\operatorname{Supp} E$ forms the fibre over the prime divisor $E' = g(\operatorname{Supp} E)$. Let M be the lcm of the multiplicities of g^*E'. Then there exists a natural number m such that $ME - mg^*E'$ is effective

[3.6] See the explanation after 3.1.

and essentially exceptional. Then we obtain this case from the second one with $E = mE'$. $\square$

On the other hand, in certain cases we can't add an effective divisor to the mobile part of a multiple of the linear systems.

Proposition 3.16 *Let $g\colon X \dashrightarrow Y/Z$ be a rational 1-contraction, D an $\mathbb{R}$-Cartier divisor on Y, and E an effective divisor on X that is essentially exceptional on Y. Then g induces an isomorphism*

$$g^*\colon i_*\mathcal{O}_Y(D) \to f_*\mathcal{O}_X(g^*D + E),$$

where $i\colon Y \to Z$.

This statement has the following applications, that can also be proved directly and generalized:

Example 3.17 (a) If $g\colon X \dashrightarrow X^+/T$ is a D-flip$/Z$ and X/T is small then g^* induces an isomorphism

$$g^*\colon f^+_*\mathcal{O}_Y(D^+) \to f_*\mathcal{O}_X(D)$$

where $D^+ = g(D)$, $f\colon X \to Z$ and $f^+\colon X^+ \to Z$. Indeed, $E = 0$ in this case. The proof is a direct verification; it is enough to do it for $Z = T$.

(b) Moreover, we can replace the D-flip by any small birational transformation, that is, assume that it is a 1-contraction in each direction. Here we can omit the assumption that D is $\mathbb{R}$-Cartier.

Thus in either case g^* induces an isomorphism of $\mathcal{O}_Z$-algebras

$$g^*\colon\ \mathcal{R}_{X^+/Z}\, D^+ \to \mathcal{R}_{X/Z}\, D$$

because g^* is compatible with multiplication.

Lemma 3.18 *Let*

$$\begin{array}{ccc} & W & \\ {}^{g}\swarrow & & \searrow^{h} \\ X & & Y\ /Z \end{array}$$

be a hut with contractions g, h, where g is birational, and D an $\mathbb{R}$-Cartier divisor on W such that *(i) each prime divisor E_i with negative multiplicity in D is exceptional on X and is contractible/Y; and*

(ii) $-D$ is nef at general curves of $E_i/h(E_i)$ for all E_i in (i).

Then D is effective.

Proof This follows from the negativity of a proper modification [13, Negativity 2.15]. Indeed, each prime E_i with negative multiplicity has a sufficiently general curve/Z on which D is numerically nonpositive, namely, a general curve in the generic fibre of $E_i/h(E_i)$. By (i–ii), this exists because E_i is contractible/Y. □

Proof of Proposition 3.16 First, g^* induces an inclusion because, for any nonzero rational function $s \in i_*\mathcal{O}_Y(D) \subset k(Y)$, the function $g^*s \in k(X)$ belongs to $f_*\mathcal{O}_X(g^*D + E)$. Indeed, $(s) + D \geq 0$. Hence

$$(g^*s) + g^*D + E \geq g^*(s) + g^*D = g^*((s) + D) \geq 0.$$

Surjectivity uses Lemma 3.18. For this, we decompose g into $h \circ g^{-1}$ as in the hut of Lemma 3.18. Then we need to show that

$$(g^{-1})^* \circ h^* \colon i_*\mathcal{O}_Y(D) \to f_*\mathcal{O}_X(g_*(h^*(D)) + E).$$

is surjective. Thus take a nonzero section $s \in f_*\mathcal{O}_X(g_*(h^*(D)) + E)$. Then $(s) + g_*(h^*(D)) + E \geq 0$. Hence $(g^*s) + h^*(D) + g^{-1}E \geq 0$ in any prime divisor E_i nonexceptional on X. We claim that the horizontal part H of the latter divisor is 0. We can of course suppose that W is nonsingular, so that any $\mathbb{R}$-divisor is $\mathbb{R}$-Cartier. In addition, H satisfies Lemma 3.18, (i), because h contracts each E_i exceptional on X. Finally, since $g^{-1}E$ is exceptional on Y, then $H \equiv (g^*s) \equiv 0$ over the generic point of $Y = h(H)$. Hence $H \geq 0$ by Lemma 3.18, and $H = 0$ because it is $\equiv 0$ over the generic point of Y.

Therefore g^*s does not have horizontal zeros or poles. Thus $g^*s = h^*s'$ for some rational function $s' \in k(Y)$. Moreover, $(g^{-1})^* \circ h^*s' = (g^{-1})^*g^*s = s$, and

$$\begin{aligned} h^*((s') + D) + g^{-1}E &= (h^*s') + h^*(D) + g^{-1}E \\ &= (g^*s) + h^*(D) + g^{-1}E \geq 0 \end{aligned}$$

in any prime divisor E_i nonexceptional on X. In particular in any prime E_i over a prime divisor $h(E_i)$ but not in $g^{-1}E$ and nonexceptional on X. Since $g^{-1}E$ is essentially exceptional, such E_i exists over each prime divisor in Y. Hence $(s') + D \geq 0$ and $s' \in i_*\mathcal{O}_Y(D)$. $\square$

Example 3.19 Let $g\colon X \dashrightarrow X^+/T$ be a D-flip$/Z$, where X/T is a D-contraction$/Z$, that is, D is numerically negative$/T$. Then g^* induces a canonical isomorphism

$$g^*\colon f^+_*\mathcal{O}_Y(D^+) \to f_*\mathcal{O}_X(D)$$

where $D^+ = g(D)$, $f\colon X \to Z$ and $f^+\colon X^+ \to Z$ (see Example 3.17, (a)). Indeed, $D = g^*D^+ + E$, where E is effective by Lemma 3.18. We can decompose g into birational contractions g and h as in Lemma 3.18. Then $g^*D = h^*D^+ + E_W$, where E_W is exceptional on X^+ and $-E_W = h^*D^+ - g^*D$ is nef$/X^+$. Thus the lemma applied to $X^+ \leftarrow W \to X^+$ give $E_W \geq 0$. Hence $E = g(E_W) \geq 0$, and the required isomorphism follows by Proposition 3.16.

Thus g^* induces a canonical isomorphism of $\mathcal{O}_Z$-algebras

$$g^*\colon\ \mathcal{R}_{X^+/Z}\,D^+ \to \mathcal{R}_{X/Z}\,D.$$

In fact we can replace the D-flip X^+/T by any small contraction (cf. Example 3.17, (b)).

By the following result, Proposition 3.16 can also be applied in cases where $g^*D + E$ is replaced by a linearly equivalent divisor.

Lemma 3.20 *Let $D \sim D'$ be two divisors that are linearly equivalent; that is, $D = D' + (s)$, where $s \in k(X)$ is a nonzero rational function. Then the multiplication map $t \mapsto st$ gives an isomorphism*

$$\sim\colon f_*\mathcal{O}_X(D) \to f_*\mathcal{O}_X(D')$$

and an isomorphism of $\mathcal{O}_Z$-algebras

$$\sim\colon\ \mathcal{R}_{X/Z}\,D \to \mathcal{R}_{X/Z}\,D'.$$

These isomorphisms are not unique, and depend on the choice of s. But they are compatible with multiplication provided we take s^i for $iD' \sim iD$.

Proof Immediate from the definitions. □

Corollary 3.21 *Under the assumptions of Proposition 3.16, suppose that* $D' \sim g^*D + E$. *Then* g *and* $\sim$ *induce isomorphisms*

$$\sim g^*\colon i_*\mathcal{O}_Y(D) \to f_*\mathcal{O}_X(D') \quad \textit{and} \quad \sim g^*\colon \mathcal{R}_{Y/Z}\, D \to \mathcal{R}_{X/Z}\, D'.$$

Proof Immediate by Proposition 3.16, since g^* is compatible with multiplication. □

However, we can't replace $\sim$ by $\sim_{\mathbb{R}}$ in Lemma 3.20 and Corollary 3.21, or even by $\sim_{\mathbb{Q}}$. This last equivalence only gives a quasi-isomorphism of algebras (see Example 4.11). Together with an application of the Truncation Principle 4.5 (see Corollary 3.25), this is enough to prove Theorem 3.14. For this, we first clarify one definition.

Definition 3.22 (compare [12, Definition 2.5]) We say that D and D' are $\mathbb{Q}$-*linearly equivalent*, and write $D \sim_{\mathbb{Q}} D'$, if $D - D'$ is a $\mathbb{Q}$-*principal* divisor, that is, a rational linear combination of principal divisors. Equivalently, $iD \sim iD'$ for some nonzero integer i.

Note that $\sim_*$ does not need $/Z$ (even if we have one!).

Since every principal divisor is integral, the $\mathbb{R}$-vector space of $\mathbb{R}$-principal divisors is defined over $\mathbb{Q}$, and all its $\mathbb{Q}$-divisors are $\mathbb{Q}$-principal. Thus any two $\mathbb{Q}$-divisors D and D' are $\sim_{\mathbb{R}}$ if and only if they are $\sim_{\mathbb{Q}}$. Note also that if $D \sim_{\mathbb{Q}} D'$ and D is a $\mathbb{Q}$-divisor, then D' is also a $\mathbb{Q}$-divisor.

Proposition 3.23 *Suppose that* D *is bss ample*$/Z$, *and* D^{small} *is a* $\mathbb{Q}$-*divisor (see (3.6.1)). Then there is a* $\mathbb{Q}$-*divisor* H *on* Y *and a natural number* I *such that*

(i) H *is numerically ample*$/Z$;

(ii) $D \sim_{\mathbb{Q}} g^*H + E$; *and moreover,*

(iii) $iD \sim ig^*H + iE$ *if and only if* $I \mid i$; *equivalently,*

(iv) for all $i \gg 0$, *the* i*th component* $\mathcal{R}_i$ *of* $\mathcal{R} = \mathcal{R}_{X/Z}\, D$ *is nonzero.*

The natural number I *is unique, and* H *is unique up to* $\sim$.

Addendum 3.23.1 *The divisorial algebra $\mathcal{R} = \mathcal{R}_{X/Z}\, D$ is equal to its truncation*

$$\mathcal{R}^{[I]} = \sum_{I|i} \mathcal{R}_i,$$

and isomorphic to the truncation

$$(\mathcal{R}_{Y/Z}\, H)^{[I]},$$

where $[I]$ means that we preserve degrees.

Note that $(\mathcal{R}_{Y/Z}\, H)^{[I]}$ is usually not equal to $\mathcal{R}_{Y/Z}\, H$ for $I \geq 2$.

Lemma 3.24 *Let $f\colon X \to T$ be a contraction and let D, E be $\mathbb{R}$-divisors on T, X respectively such that*

- *D is $\mathbb{R}$-Cartier;*
- *E is vertical, and*
- *$E \sim_{\mathbb{R}} f^*D$.*

Then there is a unique $\mathbb{R}$-divisor F on T such that

- *$F \sim_{\mathbb{R}} D$,*
- *in particular, F is also $\mathbb{R}$-Cartier; and*
- *$E = f^*F$.*

Proof We can assume that $D = 0$. Indeed, if we replace D by $E - f^*D$ and D by 0, then the required F is $F - D \sim_{\mathbb{R}} 0$.

Thus E is $\mathbb{R}$-principal and vertical. In other words, $E = \sum d_i(f_i)$, where all $d_i \in \mathbb{R}$ and all f_i are nonzero rational functions on X. We need to find a presentation in this form with functions $f_i = f^*g_i$, where the g_i are nonzero rational functions on T. Indeed, we can then take $F = \sum d_i(g_i)$.

Since E is vertical, a presentation of the required form is a presentation with $\mathbb{Q}$-linear independent and nonzero (f_i) over the generic point of T, that is, in the horizontal components.

A presentation of this form exists by the following inductive contraction. Suppose that $\sum r_i(f_i) = 0$ over the generic point of T, where all $r_i \in \mathbb{Q}$

and one of them is nonzero, say, $r_0 \neq 0$. Then the same holds for some integral coefficients r_i. Hence $\left(\prod f_i^{r_i}\right) = 0$ over the generic point of T and the rational function $g = \prod f_i^{r_i}$ does not have horizontal zeros or poles over the generic point of T, that is, 0 over the generic point of T. Thus in our presentation, we can replace (f_0) by a rational linear combination of other divisors (f_i) and (g). But $(g) = 0$ over the generic point of T.

Uniqueness follows because f^* is injective, since f is surjective. $\square$

Proof of Proposition 3.23 We first check that the extension $D_W^{\mathrm{small}} = \mathcal{D}_W^{\mathrm{small}}$ is also a $\mathbb{Q}$-divisor. Indeed, by definition $D_W^m \sim_{\mathbb{R}} h^*H$. Thus $D_W^m \sim_{\mathbb{R}} 0$ over the generic point of T and $\equiv 0/T$; again by definition, $D_W^m = D_W^{\mathrm{small}}$ outside $\operatorname{Supp} E$. On the other hand, over the generic point on Y, E is 0 and the prime divisors E_i with nonrational multiplicities in D_W^m are exceptional on X and contractible on Y. They are thus uniquely determined by the linear equations with rational coefficients $D_W^m \cdot C_i = 0$ for general curves C_i of $E_i/h(E_i)$. They are unique and thus rational by Lemma 3.18. Therefore D_W^m is rational and $D_W^m \sim_{\mathbb{Q}} 0$ over the generic point of T. In particular, D_W^m is vertical if D^m is.

Thus over the generic point of any prime divisor in Y, the prime divisors E_i with nonrational multiplicities in D_W^m and not in $\operatorname{Supp} E$ are exceptional on X and exceptional contractible on Y. Again they are uniquely determined by the linear equations with rational coefficients $D_W^m \cdot C_i = 0$ for general curves C_i of $E_i/h(E_i)$. Hence they are rational and D_W^m is rational over such points because E is exceptionally singular.

Since $D_W^m \sim_{\mathbb{Q}} 0$ over the generic point of T and E is vertical, we can suppose that the divisors D_W^m and D are themselves vertical$/Y$ up to $\mathbb{Q}$-linear equivalence. By Lemma 3.24, we can replace H by an $\mathbb{R}$-linearly equivalent divisor such that $D_W^m = h^*H$ and it is also ample$/Z$. Then H is also rational because h^* is $\mathbb{Q}$-linear. This is our choice of H.

Take I as the minimal natural number such that ID_W^{small} is vertical$/Y$ up to $\sim$. The latter vertical divisor is $ID_W^m = Ih^*H = h^*(IH)$. Then H and I satisfy the required conditions. Indeed, for any natural number i divisible by I, $iD \sim iD_W^{\mathrm{small}} + iE = Ih^*H + iE$ on W. This induces

$$iD \sim iD^{\mathrm{small}} + iE = ig^*H + iE \quad \text{on } X.$$

On the other hand if this holds for any integer i, it holds for its nonnegative remainder r by I, and $r = 0$ by our choice of I since then $rD_W^{\mathrm{small}} = rh^*H$ is vertical.

This implies that $\mathcal{R}_i = 0$ whenever $I \nmid i$. Thus Addendum 3.23.1 follows by Corollary 3.21. In addition, $\mathcal{R}^{[I]}$ has a nonzero element in each $i \gg 0$ divisible by I, because this holds for $\mathcal{R}_{Y/Z}\, H$. This gives Proposition 3.23, (iv).

As a minimal natural number, I is unique. This implies the uniqueness of H up to linear equivalence $\sim$. □

In Section 4 we derive the following result from Truncation Principle 4.5:

Corollary 3.25 (cf. Stupid example 3.13) *Let H be an $\mathbb{R}$-divisor that is numerically ample/Z. Then H is a $\mathbb{Q}$-divisor if and only if its algebra $\mathcal{R}_{X/Z}\, H$ is f.g. Moreover, it is g.a.g. with the empty stable base locus, and with $E = 0$ whenever it is f.g.*

Proof of Theorem 3.14 Suppose that $\mathcal{R} = \mathcal{R}_{X/Z}\, D$ is f.g. and g.a.g. Then we can take $Y/Z \cong \mathrm{Proj}_Z\, \mathcal{R}$. It is well known that the correspondence g from X to Y/Z is then a rational 1-contraction by Lemma 3.15, and $D = D^m + D^e$, where, for some natural number $N \gg 0$,

$$D^m = \mathrm{Mov}\,(ND)/N, \quad D^e = \mathrm{Fix}\,(ND)/N,$$
$$\text{and} \quad |ND| = |\mathrm{Mov}\,(ND)| + \mathrm{Fix}\,(ND)$$

is a decomposition of the linear system/Z into its *mobile* and *fixed* components. More precisely, this holds for any $N > 0$ such that the Nth component $\mathcal{R}_N = g_*\mathcal{O}_X(ND)$ generates the subalgebra

$$\mathcal{R}^N = \bigoplus_{i=0}^{\infty} \mathcal{R}_{iN}$$

(cf. 4.3). G.a.g. implies that $E = D^e$ is exceptional on Y, and essentially exceptional again by Lemma 3.15. Indeed, in codimension 1, $\mathcal{R}_{X/X}\, ND$ is not generated by global sections exactly in $\mathrm{Supp}\, E$. Then we apply Lemma 3.15 to a divisor $h^*g(D')$ for $D' \in |ND|$ on any model W/Z of X/Z that is regular/Y with a contraction $h\colon W \to Y/Z$. If $E = D^e$ is exceptional but not essentially exceptional on Y, there is an effective but not essentially exceptional $\mathbb{Q}$-divisor $E' \le E$. By Lemma 3.15, this contradicts f.g. Thus D is bss ample and D^{small} is $\mathbb{Q}$-divisor equal to D^m outside $\mathrm{Supp}\, D^e$.

Conversely, suppose that D is bss ample/Z and D^{small} is a $\mathbb{Q}$-divisor. By Proposition 3.23 and the description of its isomorphism, the divisorial algebra $\mathcal{R} = \mathcal{R}_{X/Z}\, D = \mathcal{R}_{X/Z}\,(D - E)$ is isomorphic to its truncation $(\mathcal{R}_{Y/Z}\, H)^{[I]}$. On the other hand, the latter is f.g. by Corollary 3.25, because H is a $\mathbb{Q}$-divisor and is numerically ample/Z. Moreover, H is g.a.g. with the empty stable base locus on Y. Since E is an effective divisor on X and $\mathcal{R}_{X/Z}\, D = \mathcal{R}_{X/Z}\,(D - E)$, g.a.g. holds for D on X with exceptional stable base divisor E on X. $\square$

Remark 3.26 By the proof of the theorem, every nontrivial f.g. algebra $\mathcal{R}_{X/Z}\, D$ gives a decomposition $D = D^m + E$, where D^m is b-semiample/Z, $E = D^e \geq 0$ is a stable base divisor and $\mathcal{R}_{X/Z}\, D = \mathcal{R}_{X/Z}\, D^m$ (cf. Example 4.27). D^m is the *minimal divisor* with these properties; it is of course a $\mathbb{Q}$-divisor.

It is not difficult to show that this is equivalent to a geometric decomposition $D = D^m + E$ for an $\mathbb{R}$-divisor D with $E \geq 0$ and with *maximal* b-semiample D^m, under the assumption that the maximal divisor D^m is a $\mathbb{Q}$-divisor. A decomposition of this form can be defined even if the maximum D^m is not a $\mathbb{Q}$-divisor; it is unique if it exists, and can be considered as a *generalized Zariski decomposition*. In the case D bss ample/Z, E is essentially exceptional. However in general there is no numerical criterion to be maximal, e.g., being exceptional or negative (cf. [13, Examples 1.1–2]). But this can be done in certain situations, namely, if

(1) D is big/Z (cf. Corollary 3.27 below), or

(2) (cf. Theorem 3.29 and Corollary 3.30 below) there is a boundary B on X, such that $(X/Z, B)$ is a 0-*log pair*, that is,

- X/Z is projective,
- (X, B) is Klt, and
- $K + B \equiv 0/Z$.

Moreover, we then expect the existence of the decompositions for all divisors $D \sim_{\mathbb{R}} D' \geq 0$ that are effective up to $\sim_{\mathbb{R}}$.

For example, (1) holds if X/Z is finite or birational; and, by the base point free theorem, (2) holds if X/Z has the structure of a weak log Fano contraction $(X/Z, B)$, with only Klt singularities.

Corollary 3.27 *Suppose that D is big/Z. Then D is bss ample/Z and has a $\mathbb{Q}$-divisor D^{small} if and only if $\mathcal{R}_{X/Z}\,D$ is f.g.*

In particular, we can replace[3.7] *rationality of D^{small} by rationality of D.*

Proof Indeed, for each D big/Z with a f.g. divisorial algebra $\mathcal{R} = \mathcal{R}_{X/Z}\,D$, the algebra $\mathcal{R}$ is big/Z (that is, its field of fractions[3.8] is finite/$k(Z)$) and g.a.g. with empty stable divisorial base locus. This follows by Lemma 3.15 in the noncontractible case. Thus we can omit the g.a.g. assumption in Theorem 3.14. $\square$

We can for example apply the corollary to a birational contraction f.

Corollary 3.28 *Suppose that f is birational and D is a $\mathbb{Q}$-divisor. Then the D-flip exists if and only if $\mathcal{FR}_{X/T}\,D$ is f.g.*

If X/T is small, $\mathcal{FR}_{X/T}\,D = \mathcal{R}_{X/T}\,D$, so that the D-flip exists if and only $\mathcal{R}_{X/T}\,D$ is f.g.

Proof Immediate by Lemma 3.5, Example 3.12 and Corollary 3.27. Indeed, each divisor on X is big/T, since $f\colon X \to T$ is birational. $\square$

Theorem 3.29 *LMMP in dimension n implies the existence of both decompositions of Remark 3.26 under (2) in $\dim X = n$, for any divisor D that is effective up to $\sim_{\mathbb{R}}$.*

In particular, the algebra $\mathcal{R}_{X/Z}\,D$ is f.g. for any $\mathbb{Q}$-divisor D.

This follows essentially because LMMP is compatible with the decompositions.

Proof By Corollary 3.3, we can assume that D is effective and $(X, B + D)$ is still Klt. Moreover, the new D and D^{small} are $\mathbb{Q}$-divisors again whenever the old D and D^{small} are; and the old algebra $\mathcal{R}_{X/Z}\,D$ is a truncation of the new one. Thus it is enough to establish the theorem for effective D.

For this, we apply LMMP to the pair $(X/Z, B + D)$. This is the same as the D-MMP [12, Section 5] since $K + B + D \equiv D/Z$ by Remark 3.26, (2). Thus if D is not nef then we have an (extremal) D-contraction $h\colon X \to Y/Z$, that is birational since D is effective. By LMMP we can make a D-flip

[3.7] Why replace? how replace? replace as an assumption in a theorem?

[3.8] Surely the opposite of what is intended: D is big, so $k(X) = k(\mathcal{R}(D))$?

$g\colon X \dashrightarrow X^+/Z$, where $D = g^*D^+ + E$ with $E \geq 0$ by Lemma 3.18 (cf. Example 3.19). Lemma 3.18 also implies that the maximal D^m, if it exists, is $\leq g^*D^+$; and it exists and is equal to $g^*D^{+\mathrm{small}}$ if and only if it exists for D^+. By Example 3.19, the same holds for f.g. of the algebra $\mathcal{R}_{X/Z}\, D = \mathcal{R}_{X^+/Z}\, D^+$ and the rationality of D^{small}, because the divisorial Supp E is contractible on X^+.

By termination in LMMP, we can suppose that D is nef. Then it is semiample by semiampleness of log canonical divisors, and we have its bss decomposition by Lemma 3.6 with $D^m = D$ and $E = 0$. Thus if $D^{\mathrm{small}} = D$ is a $\mathbb{Q}$-divisor then $\mathcal{R}_{X/Z}\, D$ is f.g. by Theorem 3.14. Semiample follows by Kawamata and [12, Remark 6.23.5], since $D \geq 0$.

Finally, note that if D is not effective up to $\sim_{\mathbb{R}}$ then $\mathcal{R}_{X/Z}\, D = k(X)$ is trivial[3.9] and f.g. but not g.a.g. $\square$

Corollary 3.30 *In dimension $n \leq 3$, we can omit LMMP as an assumption in Theorem 3.29.*

Proof Immediate by Theorem 3.29 and [12, Theorem 5.2]. $\square$

Thus we expect in particular the following:

Conjecture 3.31 *Let $f\colon X \to X_\vee$ be a pl contraction with respect to S. Then*[3.10]

$(\mathrm{PLF})_n$ *the divisorial algebra $\mathcal{R}_{X/X_\vee}\, S$ is f.g.*

On the other hand,

Corollary 3.32 $(\mathrm{PLF})_n$ *implies Induction Theorem 1.4,* $(\mathrm{PLF})_n^{\mathrm{small}}$ *and the existence of elementary pl flips in dimension n.*

Proof Immediate by Corollary 3.28. $\square$

In turn we have the following result.

Theorem 3.33 $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *implies* $(\mathrm{PLF})_n$ *of Conjecture 3.31.*

This is the main result of this section. We explain and prove it below, but we start with the following:

[3.9] Surely not? You must mean k or $k(Z)$ or $\mathcal{O}_Z$

[3.10] Who is S in the display? I think $S = S_1, S_2, \ldots, S_s$ is part of the inductive apparatus, whereas there must be a divisor D and $\mathcal{R}_{X/X_\vee}\, D$ is f.g.

Proof of Induction Theorem 1.4 for (RFA) Immediate by Corollary 3.32 and Theorem 3.33. □

Example 3.34 Let X be a projective variety with an ample, effective and reduced divisor S and H an $\mathbb{R}$-divisor such that $S \sim rH$ for some natural number r and the divisorial $\operatorname{Supp} H \cap \operatorname{Supp} S = \emptyset$. Then it is known that H is a $\mathbb{Q}$-divisor and $\mathcal{R}_X\, H$ is f.g. (cf. Corollary 3.25). However, here the essential ingredient is not just Truncation Principle 4.5 but rather the following induction on $n = \dim X$.

Indeed, restricting rational functions without poles to S gives a canonical homomorphism of graded k-algebras

$$_{|S}\colon\ \mathcal{R}_X\, H \to \mathcal{R}_S\, H_{|S}.$$

Moreover, to prove that $\mathcal{R}_X\, H$ is f.g., it is enough to prove that the image subalgebra $\mathcal{R} = \mathcal{R}_X\, H_{|S} \subset \mathcal{R}_S\, H_{|S}$ is f.g. (use Main Lemma 3.39 below).

On the other hand, by Serre vanishing, the truncation $\mathcal{R}^{[I]}$ of $\mathcal{R}$ is equal to that of $\mathcal{R}_S\, H_{|S}$. Thus by Truncation Principle 4.5 it is enough to verify that $\mathcal{R}_S\, H_{|S}$ is f.g. This gives an induction using the base point free theorem for multiples of $H_{|S}$.

We use the same idea in the diametrically opposite situation, when $-S$ can be ample. But as for pl contractions, this is only possible in the local case if X/Z is generically finite or birational. For a more general situation than the above example, in Main Lemma 3.39 below, the f.g. of $\mathcal{R}_{X/Z}\, D$ reduces to that of its restriction $\mathcal{R} = \mathcal{R}_{X/Z}\, D_{|S}$. It is a subalgebra of the divisorial algebra $\mathcal{R}_{S/Z}\, D_{|S}$. Truncation Principle 4.5 applies again, but now Serre vanishing does not work. Replacing it by Kawamata–Viehweg vanishing is a natural guess; the idea is right, but, rather than divisorial algebras up to truncation, it leads us to the (FGA) algebras of Section 4.

Definition 3.35 A *normal (irreducible) ladder* $\{S^i\}$ in X is a chain $S^0 = X \supset S^1 \supset S^2 \supset \cdots \supset S^s$, where

- each S^i is a normal (irreducible) proper subvariety of pure codimension i in X, and
- the generic point of S^i is a nonsingular point of S^j for all $j \le i$.

Let D be an $\mathbb{R}$-Cartier divisor of X. A sequence $S_1, \dots, S_s$ of reduced Weil divisors is *inductive* with respect to D if

- the subvarieties $S^i = S_1 \cap \dots \cap S_i$ form a normal (irreducible) ladder with normal crossing at the generic point of each S^i;
- $\operatorname{Supp} D$ does not contain any S^i; and
- each $S_i \sim r_i D$ for some natural number $r_i \geq 1$.

The final condition means that for each $i = 1, \dots, s$, there exists a rational function $0 \neq t_i \in k(X)$ such that $(t_i) = S_i - r_i D$. In particular, $r_i D$ must be integral divisors. Thus D itself is a $\mathbb{Q}$-Cartier divisor, possibly not integral, and each S_i is also $\mathbb{Q}$-Cartier.

Note also that $t_i \in \mathcal{O}_X(r_i D)$, and so $t_i \in \mathcal{R}_X\, D$ has degree r_i, because $(t_i) + r_i D = S_i > 0$. Moreover, for all $s \geq i > j \geq 0$, the rational function $0 \neq t_{i|S^j} \in k(S^j)$ (otherwise $S^j \subset S_i$, contradicting the assumption on a normal ladder), and it gives the rational equivalence $S_{i|S^j} \sim r_i D_{|S^j}$. We call t_i a *translation function* for S_i of *degree* r_i.

Example 3.36 Let $(X/T, S + B)$ be a log pair such that

- X/T is a (birational) contraction;
- $S = \sum S_i$ is a sum of $s \geq 1$ prime Weil $\mathbb{Q}$-Cartier divisors $S_1, \dots, S_s$;
- the S_i are *linearly proportional* to an integral divisor D, for $1 \leq t \leq i \leq s$, that is, each $S_i \sim r_i D$ for some natural number $r_i \geq 1$ if $t \leq i \leq s$ (compare 1.1.1 for pl contractions);[3.11]
- $K + S + B$ is divisorially log terminal with $\lfloor B \rfloor = 0$; and
- $K + S + B$ is numerically negative/T.

This is a slight twist on the definition of pl contraction, that we use as follows: the restrictions $S_{i|S^{t-1}}$ for $t \leq i \leq s$ form an inductive family with respect to a divisor $D'_{|S^{t-1}}$, where $D' \sim D$. Moreover, the normal ladder S^j for $t \leq i \leq s$ is *irreducible* and X/T induces a contraction $S^j/f(S^j)$ for each j.

[3.11] You don't need t here. You don't need to repeat $1 \leq t \leq i \leq s$ twice. On the other hand, you intend to restrict everything down to a surface, so $t = n - 2$.

Indeed, the latter follows by induction on s. By [9, Corollary 3.8], $S^1 = S_1$ is normal and irreducible$/T$, even formally so (that is, single branched). The numerical negativity of $K + S + B$ and the other assumptions imply the numerical negativity of $K + S^1 + B'$ for a boundary $S^1 + B' \leq S + B$ with $\lfloor B' \rfloor = 0$. In other words, $(X, S^1 + B')$ is purely log terminal with the reduced divisor S^1 in the boundary. Then any linear system that defines the contraction X/T restricts surjectively to S^1, and thus induces a contraction $S^1/f(S^1)$; and $P \in f(S^1)$ since S_1 intersects the fibre $f^{-1}P$.

Each S_i with $i \geq 2$ gives a prime Weil divisor $S_{i|_{S^1}} = S_i \cap S_1$ in S^1. Indeed, again by [9, Corollary 3.8], this last intersection is normal as a subvariety and transverse at generic points. In other words, it gives a normal ladder $S^1 \supset S_1 \cap S_i$ and $S_{i|_{S^1}} = \sum T_j$ is a sum of prime Weil divisors. In fact, we have a single T_i. First, some T_j exists by Connectedness of LCS (Kollár and others [4, Theorem 17.7]) for $K + S + B - \varepsilon S'$, where $0 \ll \varepsilon < 1$ and $S' = \sum_{j \neq 1,i} S_j$. In other words, S_1 intersects any other S_i in $f^{-1}P$, since each S_i intersects $f^{-1}P$. Second, there is only one T_i for the same reasons: namely, $\bigcup T_i$ is a normal subvariety with disjoint components T_j and by adjunction [9, 3.2.3] $(K + S + B - \varepsilon S')_{|S^1} = K_{S^1} + B'' + \sum T_j$ is divisorially log terminal with $\lfloor B'' \rfloor = 0$, where the LCS is $\bigcup T_j$. Finally, each restrictions $T_i = S_{i|_{S^1}}$ is $\mathbb{Q}$-Cartier.

Thus by adjunction we preserve the situation for $T_i = S_{i|_{S^1}}$ for $s \geq 2$ and for an appropriate choice of D. Since D is integral and the generic point of S^s is nonsingular in X, by the Moving Lemma we can assume up to $\sim$ that $\operatorname{Supp} D$ does not contain S^s or any other S^i with $i \geq 2$.

The new contraction $S^j/f(S^j)$ may not be birational even if X/S is (cf. Example 3.41).

Note also that, by definition, pl contractions do not always give the above situation, since they do not necessarily satisfy linear proportionality. Indeed, the proportionality 1.1.1 only implies that there exists an integral divisor D such that each $S_i \sim_{\mathbb{Q}} r_i D$ for natural number r_i. Fortunately, we can upgrade this to $\sim$ using the covering trick (cf. Lemma 3.47 and the proof of Theorem 3.33).

To generalize Example 3.34, we need to consider restrictions of divisorial algebras.

Definition 3.37 Let $Y \subset X$ be a subvariety and D an $\mathbb{R}$-Cartier divisor such that

- $\operatorname{Supp} D$ does not contain the generic points of Y.

Then restricting functions induces the *restriction* of divisorial algebras

$$_{|Y}\colon \mathcal{R}_{X/Z}\, D \to \mathcal{R}_{Y/Z}\, D_{|Y}$$

(see Remark 3.38 for the restrictions of divisors). This map is an $\mathcal{O}_Z$-homomorphisms of $\mathbb{N}$-graded algebras. In particular, it induces a restriction homomorphism $_{|Y}\colon \mathcal{L} \to \mathcal{R}_{Y/Z}\, D_{|Y}$ of any $\mathbb{N}$-graded $\mathcal{O}_Z$-subalgebra $\mathcal{L} \subset \mathcal{R}_{X/Z}\, D$ (functional in the terminology of Definition 4.1).

Remark 3.38 The above restriction is defined provided that functions in $\mathcal{R}_{X/Z}\, D$ have no poles at generic points of Y, that is, the support of positive components of D does not contain any generic point of Y. The restriction can however be zero; and this may happen even under our assumption on D. The important point in the definition is that if $D_{|Y}$ is defined, then each restricted nonzero function $f_{|Y}$ for $f \in \mathcal{R}_{X/Z}\, D$ belongs to a divisorial algebra in the same degree, so that its zeros and poles are controllable.

Restriction generalizes of course to D that are not $\mathbb{R}$-Cartier, assuming that we restrict $\mathcal{L}$ either

(1) as a subalgebra for a divisorial algebra of $\mathbb{R}$-Cartier $D' \geq D$; or

(2) Y is the final step S^s of a normal ladder (or $_{|Y^\nu}$ on the normalization Y^ν for a nonnormal ladder).

See [9, Section 3] for restricting Weil $\mathbb{R}$-divisors to a normal[3.12] divisor. Note that a $\mathbb{Q}$-divisor restricts to a $\mathbb{Q}$-divisor, but an integral divisor may not restrict to a $\mathbb{Z}$-divisor (cf. D in Definition 3.35). However, an integral divisor restricts to an integral divisor along any nonsingular locus of X (the ambient space). Inequalities between divisors such as $D_1 \geq D_2$ are preserved, and in particular, an effective divisor remains effective. (This follows from [9, Negativity 1.1].) This implies an inclusion $\mathcal{R}_{X/Z}\, D_{2|Y} \subset \mathcal{R}_{X/Z}\, D_{1|Y}$.

Main Lemma 3.39

[3.12] Probably the *normalization of a divisor* is intended.

(SDA) *Let $\mathcal{L}$ be an $\mathcal{O}_Z$-subalgebra of a divisorial algebra $\mathcal{R}_{X/Z}\,D$,*

(IND) *let $\{S_i \mid 1 \le i \le s\}$ be an inductive family with respect to D with translations t_i satisfying:*

(TRL) *$t_i \in \mathcal{L}_{r_i}$ for each i and*

$$t_i^{-1} \colon \mathcal{L}_N(-S_i) \to \mathcal{L}_{N-r_i} \quad \text{for every } N \ge r_i;$$

that is, $at_i^{-1} \in \mathcal{L}_{N-r_i}$ for a function $a \in \mathcal{L}_N(-S_i)$, where

$$\mathcal{L}_j = \mathcal{L} \cap f_*\mathcal{O}_X(ND),$$

is the jth component of $\mathcal{L}$, and

$$\begin{aligned}\mathcal{L}_N(-S_i) &= \mathcal{L}_N \cap f_*\mathcal{O}_X(ND - S_i)\\ &= \{a \in \mathcal{L}_N \mid (a) + ND - S_i \ge 0\}.\end{aligned}$$

Then, for $Y = S^s$, the algebra $\mathcal{L}$ is f.g. near $f(Y)$ if and only if its restriction $\mathcal{L}_{|Y}$ is.

Addendum 3.39.1 *The required translations t_i exist if $\mathcal{L} = \mathcal{R}_{X/Z}\,D$ is divisorial.*

Proof We only need to check that $\mathcal{L}$ is of finite type if $\mathcal{L}_{|Y}$ is, and by induction on s, we only only need to consider the divisorial case: that is, $s = 1$ and $Y = S = S_1 = S^1$ with a single translation $t = t_1$ of degree $r = r_1 \ge 1$. Indeed, for $s \ge 2$, the restrictions

$$\mathcal{L}_{|S^1}, \quad \{S_{i|S^1} \mid 2 \le 1 \le s\}, \quad D_{|S^1} \quad \text{and} \quad t_{i|S^1} \text{ for } 2 \le i \le s$$

again satisfy the assumptions of the lemma (with the same r_i). Thus $\mathcal{L}_{|S^1}$ is f.g. by induction.

The problem is local. Thus we fix a point $P \in f(S) \subset Z$ and check that $\mathcal{L}$ is f.g. near P. Since the algebra $\mathcal{L}_{|S}$ has finite type near P, it has a system of homogeneous generators $s_1, \dots, s_m$. Recall that a homogeneous element of a graded algebra is an element in a homogeneous component; in our case, $(\mathcal{L}_{|S})_j = \mathcal{L}_{j|S}$. The sections can be presented as restrictions $s_i = t_{i|S}$, where

$t_i \in \mathcal{L}_j$. We claim that t, the sections t_i and the generators of the $\mathcal{O}_Z$-modules $\mathcal{L}_j$ for $j < r$ generate $\mathcal{L}$ near P.

Indeed, for any homogeneous $a \in \mathcal{L}$ of degree $N \geq r$, we can find a polynomial $p(x_1, \ldots, x_m) \in \mathcal{O}_{f(S),P}[x_1, \ldots, x_m]$ such that $a_{|S} = p(s_1, \ldots, s_m)$ and is homogeneous, that is, all its monomials are homogeneous of weighted degree N, where $\deg x_i = \deg s_i$. The coefficients of p are in the local ring $\mathcal{O}_{f(S),P}$ of $P \in f(S)$. Thus p can also be obtained as the restriction $p = q_{|S}$ of a similar polynomial $q(x_1, \ldots, x_m) \in \mathcal{O}_{Z,P}[x_1, \ldots, x_m]$ with coefficients in the local ring $\mathcal{O}_{Z,P}$. Then $a - q(t_1, \ldots, t_m)$ vanishes on S, and belongs to $\mathcal{L}_N(-S)$. This last conclusion holds because $S \not\subset \operatorname{Supp} D$. Hence, by (TRL), $b = t^{-1}(a - q(t_1, \ldots, t_m)) \in \mathcal{L}_{N-r}$, and $a = q(t_1, \ldots, t_m) + tb$, where b is homogeneous of degree $(N - r) < N$. Induction on N completes the proof.

For Addendum 3.39.1, we need to verify (TRL) for the divisorial algebra $\mathcal{L} = \mathcal{R}_{X/Z}\, D$ with any translations t_i. By Definition 3.35, $t_i \in \mathcal{R}_{X/Z}\, D$ is homogeneous of degree r_i. Moreover,

$$(a) + ND - S_i \geq 0 \quad \text{for any } a \in \mathcal{L}_N(-S_i) = g_*\mathcal{O}_X(ND - S_i).$$

Thus at_i^{-1} belongs to $\mathcal{L}_{N-r_i} = g_*\mathcal{O}_X((N - r_i)D)$:

$$\begin{aligned}(at_i^{-1}) + (N - r_i)D &= (a) - (t_i) + (N - r_i)D \\ &= (a) - S_i + r_i D + (N - r_i)D \\ &= (a) - S_i + ND \geq 0. \quad \square\end{aligned}$$

Remark 3.40 We have proved more: namely, the kernel of the restriction is f.g. as an algebra (without 1).[3.13] Of course, if $A \twoheadrightarrow B$ is a surjection of algebras and its kernel is f.g. as an algebra then A is f.g. if and only if B is.

Example 3.41 Let $g\colon Y \to g(Y)$ be a contraction of fibre type, and suppose that D is numerically negative on general curves of its generic fibres. Then $\mathcal{L}_{|Y} = \mathcal{O}_{f(Y)}$ is trivial and needs 0 generators. Thus $\mathcal{L}$ is f.g./Z. More precisely, it is generated by the translations t_i and generators of $\mathcal{L}_j$ with $j \leq \max\{r_i\}$.

If we replace Condition (EL.7) for elementary flips by its opposite, that is, by the condition

[3.13] This is nonsense. You probably mean f.g. as an ideal. Even a principal ideal such as $(x) \subset k[x, y]$ is not f.g. as a ring: you need xy^i for every i.

- f is divisorial,

or, more generally, if we consider a divisorial pl contraction with S numerically negative/$X_\vee$, then $S = S_1 = S^1$ and $\mathcal{R}_{X/X_\vee} S = \mathcal{R}_{X/X_\vee} 0$ is f.g., that is, $(\mathrm{PLF})_n$ holds in this situation. This is of course well known. But from our point of view $S \sim D$ with $S \not\subset \operatorname{Supp} D$ and D is numerically negative on $S/X_\vee$. Hence $\mathcal{R}_{X/X_\vee} S$ is isomorphic to $\mathcal{L} = \mathcal{R}_{X/X_\vee} D$, which is generated by t with $(t) = S - D$ and $1 \in \mathcal{L}_0 = \mathcal{O}_{X_\vee}$.

Of course, Main Lemma 3.39 only proves f.g. near $f(S)$, but it holds outside $f(S)$ because $\mathcal{R}_{X/X_\vee} S = \mathcal{R}_{X/X_\vee} 0$ there. In general, *the lemma is entirely sufficient for local purposes*, since $P \in f(S)$, as we assume throughout what follows.

Corollary 3.42 *Under the assumptions of Example 3.36, there exists $D' \sim D$ such that the algebra $\mathcal{L} = (\mathcal{R}_{X/T} D')_{|S^{t-1}}$ is f.g. if and only if $\mathcal{L}_{|S^s}$ is.*

Proof Immediate by Main Lemma 3.39 and Example 3.36, if we take D such that $S^s \not\subset \operatorname{Supp} D'$. The restricted algebra $\mathcal{L}$ is a $\mathcal{O}_{f(S^{t-1})}$-subalgebra of $\mathcal{R}_{S^{t-1}/f(S^{t-1})} D'_{|S^{t-1}}$. The condition (TRL) on S^{t-1} is induced by the same on X by Addendum 3.39.1. $\square$

Definition 3.43 Under the assumptions of Example 3.36, suppose that X/T is birational, $t = 1$ and $S^s \not\subset \operatorname{Supp} D$. Then we say that the *restricted algebra* $\mathcal{L} = (\mathcal{R}_{X/T} D)_{|Y}$ is of type $(\mathrm{RFA})_{n,d}$, where $Y = S^s$, $n = \dim X$ and $d = n - s = \dim Y$. If $Y/f(Y)$ is again birational, we say that $\mathcal{L}$ is of type $(\mathrm{RFA})_{n,d}(\mathrm{bir})$.

Conjecture 3.44

$(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *every algebra of type* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *is f.g.*

Corollary 3.45 $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *of Conjecture 3.44 implies*

$(\mathrm{RFA})_{n,d}$ *every algebra of type* $(\mathrm{RFA})_{n,d}$ *is f.g.*

Proof In the other words, $(\mathcal{R}_{X/T}\,D)_{|Y}$ is still f.g., even if $Y/f(Y)$ is not birational. Indeed, since X/T is birational but $Y = S^s/f(Y) = f(S^s)$ is not, then for some $1 \le i < s$, $S^i/f(S^i)$ is still birational but $E = S^{i+1}/f(S^{i+1})$ is not. Therefore E is a Weil divisor of S^i, and is exceptional on $f(S^{i+1})$. By our assumptions, it is $\mathbb{Q}$-Cartier and $\sim r_{i+1}D$. Thus D is numerically negative on general curves of the generic fibre of $S^{i+1}/f(S^{i+1})$. Therefore the algebras $(\mathcal{R}_{X/T}\,D)_{|S^{i+1}}$ and $(\mathcal{R}_{X/T}\,D)_{|Y}$ are trivial by Example 3.41. $\square$

3.46 The covering trick

Further applications needs the covering trick.

Lemma 3.47 *Let $\pi\colon \widetilde{X} \to X$ be a finite cover that is etale in codimension* 1.

(1) *The pair $(\widetilde{X}, \widetilde{D} = \pi^{-1}D)$ is divisorially log terminal if (X, D) is; and $(\widetilde{X}, \widetilde{D} = \pi^{-1}D)$ is log canonical, respectively Klt or purely log terminal if and only if (X, D) is;*

(2) *$\pi^{-1}D = f^*D$ is $\mathbb{Q}$-Cartier (or $\mathbb{R}$-Cartier) if and only if D is;*

(3) *$\pi^{-1}D \sim_{\mathbb{R}} \pi^{-1}D'$ if and only if the same holds for D, D';*

(4) *$\pi^{-1}D = \pi^*D$ is bss ample/Z (or b-semiample, semiample, ample) if and only if D/Z is; and*

(5) *a pl contraction $f\colon X \to X_\vee$ induces a pl contraction $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}_\vee$ with $\widetilde{S} = \pi^{-1}S$ and finite $\widetilde{X}_\vee/X_\vee$; the same holds for $f\colon X \to T$ of Example 3.36; in addition, f has a pl flip or even a D-flip for any contraction f if and only if the same holds for $\widetilde{f}$.*

Proof (1) is immediate from the pullback formula [9, 1.1] (cf. the proof of [9, Corollary 2.2]). For divisorially log terminal (which means that the log discrepancies are only 0 over normal crossing intersections of $\lfloor B \rfloor$), note that π is etale over the nonsingular points, and preserves the nonsingularity of the log canonical centers and normal crossings of $\lfloor B \rfloor$ in their generic points. The nonsingularity in codimension 1 of irreducible components of $\left\lfloor \widetilde{B} \right\rfloor$ follows from [9, Lemma 3.6 and Corollary 2.2].

For (2–3), see [9, proof of Corollary 2.2]. (4) is immediate by Definition 3.1 and the uniqueness of Proposition 3.2.

(5) follows by the connectedness arguments of Example 3.36, and again by [9, Corollary 2.2]. They imply that $\widetilde{S}_i = \pi^{-1}S_i$ are again prime. The rational numbers $r_{i,j}$ and natural number r_i are the same on $\widetilde{X}$. The final statement on pl and D-flips follows from (4) (cf. [9, Lemma 2.5]). □

Proof of Theorem 3.33 Let $f\colon X \to X_\vee$ be a pl contraction with respect to S. By Corollary 3.27, $\mathcal{R}_{X/X_\vee} S$ is f.g. if and only if S is bss ample. Thus by Lemma 3.47 we can replace X by any finite cover that is etale in codimension 1.

On the other hand, by Condition 1.1.1, the divisors S_i generate an Abelian subgroup of rank 1 in the group of integral Weil divisors modulo $\sim_{\mathbb{Q}}$. This is torsion free. Hence there is a generator of this subgroup that is the class modulo $\sim_{\mathbb{Q}}$ of an integral Weil divisor D. In other words, each $S_i \sim_{\mathbb{Q}} r_i D$ for some natural number r_i; and $S \sim_{\mathbb{Q}} (\sum r_i)D$. By Corollary 3.3, $\mathcal{R}_{X/X_\vee} S$ is f.g. if and only if D is bss ample, and hence if and only if $\mathcal{R}_{X/X_\vee} D$ is f.g.

After a finite cover etale in codimension 1, we can assume that each $S_i \sim r_i D$ for some natural number r_i; and $S \sim (\sum r_i)D$. Indeed, if $S_i \sim_{\mathbb{Q}} r_i D$ then it defines a cyclic cover π that is etale in codimension 1 such that $\pi^{-1}S_i \sim r_i\pi^{-1}D$ [9, Construction 2.3]. Then we use induction on i. In addition, we can choose D up to $\sim$ such that $\operatorname{Supp} D$ does not contain all S^j for $j \geq 1$. We are now in the situation of Definition 3.43 and by Main Lemma 3.39, $\mathcal{R} = \mathcal{R}_{X/X_\vee} D$ is f.g. if $\mathcal{L} = \mathcal{L}_{|S^s}$ is. This last algebra is of type $(\mathrm{RFA})_{n,d}$. Hence if $S^s/f(S^s)$ is birational, then $\mathcal{L}$ is f.g. by our assumption. Otherwise we use Corollary 3.45. □

Corollary 3.48 $(\mathrm{PLF})_n$ *holds for a pl contraction* $X/X_\vee$ *if* $Y = S^i = E$ *or* $Y \subset f^{-1}P$*, where* E *is the exceptional locus for* $X/X_\vee$ *and* $f^{-1}P$ *is the fibre of* $X/X_\vee$ *over* P.

Proof Immediate by the proof of Theorem 3.33, because then $Y = S^s$ is a point whenever it is birational$/f(Y)$. But if Y is a point, any subalgebra of $\mathcal{R}_Y D = k_\bullet$ is f.g.$/k$ (and is quasi-isomorphic to a divisorial algebra whenever it is nontrivial). □

For example, the latter case in the proof applies to Example 3.34 even if f is not a pl contraction and is not birational.

However even if Y is a curve, a subalgebra $\mathcal{L} \subset \mathcal{R}_Y D$ need not be divisorial in general (cf. Examples 4.16–4.17). Thus further investigations are needed to specify restricted subalgebras. We do this in Section 4.

Example 3.49 (cf. [9, flips of type (6.6.2)]) Let $X/X_\vee$ be a pl contraction with respect to S such that:

- S is numerically negative/$X_\vee$.

For example, this holds for an elementary pl contraction. Then $X/P \subset E \subset Y = S^s$, where $X/P = f^{-1}P$ is the fibre/P and E denotes the exceptional locus of f.

Indeed, $C \cdot S_i < 0$ for any curve C/P and for each S_i. Hence $C \subset Y$, and $s \leq n-1$, where $n = \dim X$. Moreover,

(DPT) $s \leq n - \dim E \leq n - \dim X/P$. Moreover, $s = n - \dim E$ holds only if $E = Y$; and $\dim E = \dim X/P$ holds only if $E = X/P$.

Thus if $E \neq \emptyset$ and $s = n - \dim E \leq n-1$, then $\mathcal{R}_{X/X_\vee} S$ is f.g. by Corollary 3.48. In particular, elementary flips exist under this assumption. For example, this holds if $s = n-1$ and f is not an isomorphism.

Another trivial case is when $f\colon X \to T$ is an *isomorphism*. Then any divisorial algebra $\mathcal{R}_{X/T} D$ is f.g. for D a $\mathbb{Q}$-Cartier divisor by Corollary 3.25. Our restriction arguments apply again to this case because the restrictions are surjective.

Example 3.50 The same arguments apply to any toric contraction $(X/T, S)$ provided that

- X is $\mathbb{Q}$-factorial, and
- X/T is birational and extremal, that is, $\rho(X/T) = 1$;

$S = \sum S_i$ denotes the invariant divisor, where the S_i are the invariant prime divisors, and $1 \leq i \leq \dim X + 1 = n + 1$ (cf. [13, Theorem 6.4 and the conjecture after it]). We order the S_i so that those with $1 \leq i \leq s$ are the only divisors that are numerically negative/T. It is known that the exceptional locus E of X/T is then also invariant, and thus of the form

$E = \bigcap_{1\le i\le s} S_i$ with $s \le n-1$, and one of the other divisors, say S_n, is numerically positive/T. Unfortunately, the contraction may not be pl with respect to $S^- = \sum_{1\le i\le s} S_i$ since the pair (X, S^-) may not be divisorially log terminal, having singularities along log centers. However we can eliminate all these after a finite covering (even in the toric category) using the positive S_n and get a pl contraction. Then Example 3.49 implies that S^- is bss ample or equivalently, that $\mathcal{R}_{X/T} S^-$ is f.g. This gives a construction of toric flips, or it would be better to say flops.

This is the best we can do using divisorial algebras.

4 Finitely generated pbd-algebras

In this section[4.1] we focus on $\mathbb{N}$-graded $\mathcal{O}_Z$-algebras over a normal algebraic variety Z. We again consider the local case near $P \in Z$.

Definition 4.1 We say that a $\mathbb{N}$-graded $\mathcal{O}_Z$-algebra $\mathcal{L} = \bigoplus_{i\ge 0} \mathcal{L}_i \subset k(X)_\bullet$ is a (coherent) *functional algebra* if each homogeneous component $\mathcal{L}_i$ (of degree i) is a coherent $\mathcal{O}_Z$-submodule $\mathcal{L}_i \subset k(X)$, where X/Z is a proper morphism. We always assume that $\mathcal{L}_0 = \mathcal{O}_Z$.

In slightly more abstract form, we have:

Proposition 4.2 *Any functional algebra* $\mathcal{L}$

- *is commutative;*
- *is* integral*, that is, has no zerodivisors;*
- *each* $\mathcal{L}_i$ *is a coherent* $\mathcal{O}_Z$*-module, and* $\mathcal{L}_0 = \mathcal{O}_Z$*; and*
- $\mathcal{L}$ *has field of fractions of finite type over* $k(Z)$*.*

Conversely, each $\mathbb{N}$*-graded* $\mathcal{O}_Z$*-algebra with these properties is functional.*

[4.1] I have corrected several errors in Sections 4.1–4.10, but there are possibly several more left. The material is basically very elementary.

Proof–Explanation The homogeneous field of fractions of an integral graded algebra $\mathcal{L}$ is a field L such that $\mathcal{L}$ is a graded subalgebra of $L_\bullet$; each element (function) of L is a fraction l/l', where $l, l' \in \mathcal{L}_i$ have the same degree i and $l' \neq 0$. For a functional algebra $\mathcal{L}$, it is a subfield of $k(X)$ and of finite type over $k(Z)$. Thus the properties of $\mathcal{L}$ are immediate from the definition.

Conversely, every integral graded algebra has a field of fractions L. It has an inclusion $\mathcal{L} \hookrightarrow L_\bullet$ given by fractions:

$$\mathcal{L}_i \hookrightarrow L \quad \text{defined by} \quad l_i \mapsto \frac{l_i}{t^{i/d}} = \frac{l_i l_m^{i/d}}{l_n^{i/n}},$$

where for a nontrivial algebra $\mathcal{L}$, $d \mid i$, $d = \gcd\{i \mid \mathcal{L}_i \neq 0\}$, and $t = l_n/l_m$ with $n - m = d, 0 \neq l_n \in \mathcal{L}_n, 0 \neq l_m \in \mathcal{L}_m$. For a trivial algebra we take $t = 1$.

Since L has *finite type* over $k(Z)$, then $L = k(X)$ for a normal algebraic variety X, and we can assume that X/Z is proper.

Note that any $\mathcal{O}_Z$-subalgebra of finite type of $k(X)$ is coherent, since $\mathcal{O}_Z$ is Noetherian. We give a more explicit form soon (see Proposition 4.13). □

Example 4.3 Let $\mathcal{L} = \bigoplus \mathcal{L}_i$ be a $\mathbb{N}$-graded algebra and I a natural number. The *truncation* of $\mathcal{L}$ is the algebra $\mathcal{L}^{[I]} = \bigoplus_{I|i} \mathcal{L}_i$. Our convention is to give elements the same degree (traditionally, one often gives them degree i/I). Any truncation of a functional algebra is also functional.

Two algebras are *quasi-isomorphic* if they have isomorphic truncations.

Example 4.4 Let $\mathcal{L}$ such graded k-algebra that

$$\mathcal{L}_i = \begin{cases} kx^{i/2} & \text{for } i \text{ even,} \\ k\varepsilon_i & \text{otherwise} \end{cases}$$

with $x^i x^j = x^{i+j}, \varepsilon_i x^j = \varepsilon_i \varepsilon_j = 0$. Then $\mathcal{L}^{[2]}$ is isomorphic to $k[x]$, that is f.g. and generated by $x = x^1$. But $\mathcal{L}$ itself is not f.g. Note that $\mathcal{L}$ is not integral, in particular, not functional.

Theorem 4.5 (Truncation Principle) *Quasi-isomorphism preserves f.g. of a functional algebra.*

Proof It is enough to prove this for a truncation $\mathcal{L}^{[I]} \subset \mathcal{L}$.

If $\mathcal{L}$ is f.g., it is well known that $\mathcal{L}^{[I]}$ is also f.g. If $s_1, \dots, s_m$ generate $\mathcal{L}$, then $\mathcal{L}^{[I]}$ is generated by $s_1^I, \dots, s_m^I$ and the generators of $\mathcal{L}_{Ii}$ with $i < \sum d_i$, where $d_i = \deg s_i$.

Conversely, if $\mathcal{L}^{[I]}$ is f.g., each coherent $\mathcal{O}_Z$-submodule of $\mathcal{L}^{[I]}$ is of finite type because $\mathcal{O}_Z$ is Noetherian. On the other hand, $\mathcal{L}$ f.g. means that $\mathcal{L}$ is of finite type as $\mathcal{L}^{[I]}$-module. This follows from the Noetherian property since it is a direct sum $\mathcal{L} = \bigoplus_{0 \le r \le I-1} \mathcal{L}^r$ of modules $\mathcal{L}^r = \bigoplus_{i \equiv r \mod I} \mathcal{L}_i$ each of which is of finite type. Indeed, we can assume that $\mathcal{L}^r \neq 0$. Then there is some $0 \neq s \in \mathcal{L}^r$ of degree i such that multiplication $x \mapsto s^{I-1}x$ gives an inclusion of $\mathcal{L}^r$ into $\mathcal{L}^{[I]}$. $\square$

Example 4.6 (Blowup of an ideal) Let $\mathcal{I} \subset k(Z)$ be a *fractional ideal*, that is, up to a multiple, a coherent $\mathcal{O}_Z$-ideal $\mathcal{I} \subset \mathcal{O}_Z$. It defines a functional algebra $\mathcal{T} = \mathcal{T}(\mathcal{I})$, the[4.2] tensor algebra of $\mathcal{I}$, with $\mathcal{T}_i = \mathcal{I}^{\otimes i} = \mathcal{I}^i \subset \mathcal{O}_Z^{\otimes i} = \mathcal{O}_Z$. This algebra is always f.g. and its projective spectrum $\sigma\colon \operatorname{Proj}_Z \mathcal{T} \to Z$ is the *blowup* of Z in $\mathcal{I}$ or in its subscheme. It is a projective birational morphism, possibly not small, since any projective birational morphism to Z can be obtained in this form.

If $\mathcal{I} = \mathcal{O}_T(f(D))$ for a birational contraction $f\colon X \to T = Z$ as in Example 3.12, the corresponding tensor algebra $\mathcal{T}(\mathcal{I})$ is a subalgebra of the flipping algebra $\mathcal{FR}_{X/T}\, D = \mathcal{R}_{T/T}\, f(D)$, and is a proper subalgebra in many interesting cases. Indeed, for integral D, $\mathcal{FR}_{X/T}\, D = \mathcal{A}(\mathcal{I}) = \mathcal{T}(\mathcal{I})^{\vee\vee}$ is the reflexive tensor algebra. If this last algebra is f.g., it gives a *small* contraction: namely, the D-flip X^+/T. However if D is not Cartier on X^+, the blowup in $\mathcal{I}$ is bigger, and even divisorial if X^+ is $\mathbb{Q}$-factorial.

Blowups in ideals have other drawbacks: they can lead to nonnormal varieties, which are outside our category. It also relates to other functional algebras and can be improved in terms of functional algebras.

Example 4.7 (Integral closure) Let $\mathcal{L}$ be a functional algebra. Then $\overline{\mathcal{L}} = \bigoplus \overline{\mathcal{L}}_i$ with

$$\overline{\mathcal{L}}_i = \left\{ a \in k(X) \;\middle|\; \begin{array}{l} a^m + l_1 a^{m-1} + \cdots + l_{m-1} a + l_m = 0 \\ \text{for some elements } l_j \in \mathcal{L}_i^j \end{array} \right\},$$

[4.2]This is nonsense. $\mathcal{I}^{\otimes i} \neq \mathcal{I}^i$ in general.

where $\mathcal{L}_i^j$ denotes the product $\mathcal{O}_Z$-submodule $\mathcal{L}_i \cdots \mathcal{L}_i$ (j times) in $k(X)$ (or in any other field F), is also a functional algebra in $k(X)_\bullet$ (or in $F_\bullet$, that depends on F). We call it the *integral closure* of $\mathcal{L}$ in $k(X)$ (resp. in F). We have $\overline{\overline{\mathcal{L}}} = \overline{\mathcal{L}}$. The proof is given in Proposition 4.13 below, together with another description of integral closure. We also prove that $\mathcal{L}$ is f.g. if and only if $\overline{\mathcal{L}}$ is; in this case, the projective spectrum morphism $\operatorname{Proj}_Z \overline{\mathcal{L}} \to \operatorname{Proj}_Z \mathcal{L}$ is a normalization. It is birational (see Corollary 4.15). We say that an algebra is *normal* if it is integrally closed in its field of fractions. By Proposition 4.13, these are the pbd algebras (up to a truncation).

Example 4.8 (Twist) Let $\mathcal{A}$ be a coherent $\mathcal{O}_X$-algebra. It gives a $\mathbb{N}$-graded $\mathcal{O}_X$-algebra $\mathcal{A}_\bullet$ with components $\mathcal{A}_i = \mathcal{A}$ for $i \geq 1$ and $\mathcal{A}_0 = \mathcal{O}_X$. We define its *twist* $\mathcal{A}[D]$ by a Cartier divisor D on X by setting $\mathcal{A}[D]_i = \mathcal{A}_i(iD) = \mathcal{A}_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(iD)$. Since $\mathcal{A}$ is coherent, the twisted algebra is finite as a module over $\mathcal{R}_{X/X}\, D$. Moreover, $f_*\mathcal{A}[D]$ is also finite as a module over the algebra $\mathcal{R}_{X/Z}\, D = f_*\, \mathcal{R}_{X/X}\, D$ if D is ample$/Z$.

The algebra $f_*\mathcal{A}[D]$ is functional if $\mathcal{A}$ does not have torsion: that is, is a submodule of $k(Y)/k(X)$. For example, if $g\colon Y \to X$ is a proper morphism, then $g_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-subalgebra of $k(Y)$ with the natural multiplication. Its twist $g_*\mathcal{O}_Y[D]$ is the divisorial algebra $\mathcal{R}_{Y/X}\, g^*D$ by the Projection Formula.

However, the most important functional algebras for us are geometric, that is, associated to divisors.

Definition 4.9 Let $\mathcal{D}_\bullet = (\mathcal{D}_i \mid i \in \mathbb{N})$ be a system (sequence) of $\mathbb{R}$-b-divisors of X/Z such that:

- $\mathcal{D}_i + \mathcal{D}_j \leq \mathcal{D}_{i+j}$;
- $\mathcal{D}_0 = 0$; and
- each $f_*O_X(\mathcal{D}_n)$ is a coherent $\mathcal{O}_Z$-module,

where the sections of the $\mathcal{O}_X$-sheaf $\mathcal{O}_X(\mathcal{D}_i) = \mathcal{O}_X(\lfloor \mathcal{D}_i \rfloor)$ are

$$\Gamma(U, \mathcal{O}_X(\mathcal{D}_i)) = \left\{ a \in k(X) \,\middle|\, (a) + \mathcal{D}_i \geq 0 \text{ on } U \right\},$$

and the principal divisor (a) is considered as a b-divisor, namely, the closure of the usual (a) [13, Example 1.1.1]. Since X/Y is proper, it is enough[4.3] that the latter sheaf is coherent. Every such sheaf is torsion free of rank 1 (a fractional ideal sheaf), integrally closed and known as *b-divisorial*. In the *non-coherent* case it is reasonable to *set* $\mathcal{D}_i = -\infty$, then again $\mathcal{O}_X(\mathcal{D}_i) = 0$ is coherent (cf. the proof of Proposition 4.13 below). Note also that $f_*\mathcal{O}_X(\mathcal{D}_0) = f_*\mathcal{O}_X = \mathcal{O}_Z$.

Then we can associate to the system a functional $\mathcal{O}_Z$-algebra

$$\mathcal{R}_{X/Z}\,\mathcal{D}_\bullet = \mathcal{R}_f\,\mathcal{D}_\bullet \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} f_*O_X(\mathcal{D}_n)$$

that we call a *pseudo b-divisorial* algebra (*pbd algebra*).

Multiplication is well defined, since it is for X/X: for $a \in \mathcal{O}_X(\mathcal{D}_i)$ and $b \in \mathcal{O}_X(\mathcal{D}_j)$, $ab \in \mathcal{O}_X(\mathcal{D}_{i+j})$, because $(ab) + \mathcal{D}_{i+j} \geq (a) + (b) + \mathcal{D}_i + \mathcal{D}_j$ (cf. Definition 3.8).

This definition gives a functorial homomorphism $\mathcal{R}_{X/Z}(-)$ of systems of b-divisors into functional algebras. It is compatible with truncations, where the *truncation* of $\mathcal{D}_\bullet$ is the system $\mathcal{D}_i^{[I]} = \mathcal{D}_{iI}$. To preserve degrees, we make the convention $\mathcal{D}_i^{[I]} = \mathcal{D}_i$ for $I \mid i$ and $\mathcal{D}_i^{[I]} = -\infty$ otherwise. We say that two systems having identical truncations are *similar*; we can replace identity by linear equivalence of truncations. Two systems $\mathcal{D}_\bullet$ and $\mathcal{D}'_\bullet$ are *linearly equivalent* if $\mathcal{D}_i = \mathcal{D}'_i + i\overline{(a)}$ for some $0 \neq a \in k(X)$. Linear equivalence induces an isomorphism of algebras (cf. Lemma 3.20). Moreover, this is the *identity* on its field of fractions.[4.4] We always assume this condition for isomorphisms of functional algebras (cf. Proposition 4.13, (8)) that is relevant to general isomorphisms if we add isomorphisms of systems.

Corollary 4.10 *Quasi-isomorphism preserves f.g. of a pbd algebra. In particular, a pbd algebra is f.g. if and only if any of its truncation is; or, more generally, similar systems have the same f.g. properties.*

Moreover, if

[4.3] What on earth does this mean? The assumption X/Y proper is enough to ensure that $f_*\mathcal{O}_X(iD)$ is coherent? Or, the assumption that $f_*\mathcal{O}_X(iD)$ is coherent is enough for subsequent purposes?

[4.4] More precisely, it is multiplication by a^i on $\mathcal{R}_i$ and so the identity on the field of homogeneous fractions.

- $D_\bullet$ *is big/Z* (cf. Remark 3.26, (1))*; and*
- $\mathcal{R} = \mathcal{R}_{X/Z}\, D_\bullet$ *is f.g.,*

then the algebra is g.a.g. (cf. Corollary 3.27).

Proof–Explanation F.g. is immediate by Truncation Principle 4.5, and g.a.g. follows by Lemma 3.15 and the following definitions.

A pbd algebra $\mathcal{R} = \mathcal{R}_{X/Z}\, \mathcal{D}_\bullet$ is *globally almost generated* or *g.a.g.* if there is a natural number N and sections of $\mathcal{O}_X(N\mathcal{D})$ that generate the algebra $\mathcal{R}_{X/X}\, \mathcal{D}_\bullet^{[N]}$ in codimension 1 except for a divisorial subset that is exceptional on $Y = \operatorname{Proj}_Z \mathcal{R}$.

A system $D_\bullet$ is *big/Z* if its algebra $\mathcal{R}$ is big, that is, $k(X)/F$ is finite where F is the field of fractions for $\mathcal{R}$; in particular, $F = k(Y)$, where $Y = \operatorname{Proj}_Z \mathcal{R}$ whenever $\mathcal{R}$ is f.g. $\square$

Example 4.11 Let $\mathcal{D}$ be an $\mathbb{R}$-b-divisor such that:

- $O_X(i\mathcal{D})$ is a coherent $\mathcal{O}_X$-module for all $i \in \mathbb{N}$.

Then the system $(\mathcal{D}_i = i\mathcal{D})$ satisfies the assumptions of Definition 4.9, and gives a *b-divisorial algebra* $\mathcal{R}_{X/Z}\, \mathcal{D} = \mathcal{R}_{X/Z}\,(i\mathcal{D})$. The divisorial algebras of Definition 3.8 for some b-divisor $\mathcal{D}$ with $\mathcal{D}_Y = D$ are a particular case. For example, if D is $\mathbb{R}$-Cartier on model Y/Z of X/Z we can take $\mathcal{D} = \overline{D}$ by Proposition 3.16. (For any $\mathcal{D}$, we can replace each $\mathcal{O}_X(i\mathcal{D})$ by its maximal coherent subsheaf $\mathcal{O}_X(\mathcal{D})^{ch}$; cf. Remark 6.15, (4).)

In general, we need infinitely many models, as in the more typical case Example 4.12 below.

Thus the difference between (pbd) b-divisorial and divisorial algebras is the minor one that our divisor is not on X but rather on a different model Y (models X_i). However divisorial is not so helpful from the f.g. point of view (cf. Example 3.12).

Finally, note that as for divisors, $\mathcal{D} \sim_{\mathbb{Q}} \mathcal{D}'$ if and only if the corresponding systems $(i\mathcal{D})$ and $(i\mathcal{D}')$ are similar. The equivalences $\sim_*$ can be defined for b-divisors as for divisors (see Definition 3.22): we replace $*$-principal divisors (a) by their b-divisors $\overline{(a)}$.

Thus an equivalence $D \sim_{\mathbb{Q}} D'$ gives a quasi-isomorphism of the algebras $\mathcal{R}_{X/Z}\, D$ and $\mathcal{R}_{X/Z}\, D'$ provided that both are functional (cf. Corollary 3.3). For the converse, see Proposition 4.13, (8).

Example 4.12 Let $\big(X_i/Z, D_i \bigm| i \in \mathbb{N}\big)$ be a sequence of models X_i/Z of X/Z and their $\mathbb{R}$-Cartier divisors such that:

- $\mathcal{D}_i + \mathcal{D}_j \le \mathcal{D}_{i+j}$, where $\mathcal{D}_i = \overline{D}_i$.

Then the system $(\mathcal{D}_i)$ again satisfies the assumptions of Definition 4.9.

In fact, by Proposition 4.13 below, an integrally closed functional algebra $\mathcal{L}$ is always of this form, but is usually not b-divisorial. Moreover, we can assume that each D_i is Cartier with $\operatorname{Bs}|D_i| = \emptyset$ whenever $\mathcal{L}_i \ne 0$. A system with this property for all $\mathcal{D}_i$ or for some truncation is *birationally free* (b-free). In particular, any b-free system or its truncation is integral, that is, has integral $\mathcal{D}_i$ or D_i.

We now construct an inverse of $\mathcal{R}_{X/Z}(-)$: namely, the functorial homomorphism sending a functional algebra $\mathcal{L}$ to its *mobile system* $\operatorname{Mov}\mathcal{L} = \mathcal{M}_\bullet$. But first we make the convention either to accept $-\infty$ as a divisor and b-divisor, or to define $\mathcal{M}_\bullet$ only as a truncation, and only for nontrivial $\mathcal{L}$.

Proposition 4.13 *For each functional algebra $\mathcal{L}$ in $k(X)_\bullet$ there is a unique system $\mathcal{M}_\bullet$ of b-divisors of X with the following properties:*

(1) each $(\mathcal{M}_i)$ is b-free, and in particular, integral;

(2) $\mathcal{L} \subset \mathcal{R}_{X/Z}\,\mathcal{M}_\bullet$;

(3) for any system $\mathcal{D}_\bullet$, if $\mathcal{L} \subset \mathcal{R}_{X/Z}\,\mathcal{D}_\bullet$, then $\mathcal{M}_\bullet \le \mathcal{D}_\bullet$ (each $\mathcal{M}_i \le \mathcal{D}_i$)*; that is, $\mathcal{D}_\bullet$ is the minimal system that gives the inclusion of $\mathcal{L}$;*

(4) $\overline{\mathcal{L}} = \mathcal{R}_{X/Z}\,\mathcal{M}_\bullet$ and $\operatorname{Mov}\overline{\mathcal{L}} = \operatorname{Mov}\mathcal{L} = \mathcal{M}_\bullet$; thus $\mathcal{L} = \mathcal{R}_{X/Z}\,\mathcal{M}_\bullet$ if $\mathcal{L}$ is integrally closed in $k(X)$;

(5) if $\mathcal{D}_\bullet$ is b-free, then $\operatorname{Mov}\mathcal{R}_{X/Z}\,\mathcal{D}_\bullet = \mathcal{D}_\bullet$;

(6) $\mathcal{L}$ is f.g. if and only if $\mathcal{R}_{X/Z}\,\mathcal{M}_\bullet$ is f.g.;

(7) $\mathcal{R}_{X/Z}\,\mathcal{M}_\bullet$ is g.a.g. with $E = 0$ on each model Y/Z of X/Z; and finally,

(8) quasi-isomorphic algebras give similar systems.

Proof–Construction We first construct $\mathcal{M}_i$ for each $\mathcal{L}_i \neq 0$. (We set $\mathcal{M}_i = -\infty$ if $\mathcal{L}_i = 0$.) We set

$$\mathcal{M}_i = \sup\{-\overline{(s)} \mid 0 \neq s \in \mathcal{L}_i\} = -\inf\{\overline{(s)} \mid 0 \neq s \in \mathcal{L}_i\},$$

where the sup and inf are taken componentwise, as for b-divisors (the same applies to max and min in what follows); in particular, the result is not necessarily a principal divisor. Since $\mathcal{L}_i \subset k(X)$ is a finite $\mathcal{O}_Z$-module, locally$/Z$, there is a finite set of generators $0 \neq s_i \in k(X)$ for $\mathcal{L}_i$, and the sup and inf are in fact the max and min respectively, for b-divisors $\overline{(s_i)}$ considered as functions over prime b-divisors. In particular, it is a b-divisor, and its restriction to each model of X/Z is an ordinary divisor.

Indeed, the analogous results for usual principal divisors imply

- if $s \in \mathcal{O}_Z \subset k(X)$, then $\overline{(s)} \geq 0$; and
- for any $s, s' \in k(X)$, $\overline{(s + s')} \geq \min\{\overline{(s)}, \overline{(s')}\}$.

Therefore, for any $0 \neq s \in \mathcal{L}_i$, since $s = \sum a_i s_i$ with all $a_i \in \mathcal{O}_Z$,

$$\overline{(s)} = \overline{(\sum a_i s_i)} \geq \min\{\overline{(a_i s_i)}\} = \min\{\overline{(a_i)} + \overline{(s_i)}\} \geq \min\{\overline{(s_i)}\}.$$

For each $0 \neq s \in \mathcal{L}_i$ we have $\mathcal{M}_i \geq -(s)$ or $(s) + \mathcal{M}_i \geq 0$, which gives the inclusion $\mathcal{L}_i \subset f_*\mathcal{O}_X(\mathcal{M}_i)$ of (2). However, for $\mathcal{R}_{X/Z}\,\mathcal{M}_\bullet$ to be functional, we need it to be coherent. For this, (1) is enough.

Next, $\mathcal{M}_i$ is the least divisor for which the inclusion $\mathcal{L}_i \subset f_*\mathcal{O}_X(\mathcal{M}_i)$ holds, which proves (3). Indeed, if the same holds for $\mathcal{D}_i$ then $\mathcal{D}_i \geq -(s)$ or $\mathcal{D}_i + (s) \geq 0$ for any $0 \neq s \in \mathcal{L}_i$. Hence $\mathcal{D}_i \geq \mathcal{M}_i$.

On each model X_i/Z of X/Z, the restriction $M_i = (\mathcal{M}_i)_Y$ is *mobile*$/Z$, provided that $\mathcal{M}_i \neq -\infty$ or $\mathcal{L}_i \neq 0$. (In a certain sense this always holds.) This means that $\operatorname{Mov} M_i = M_i$ or, equivalently, the linear system $|M_i|$ does not have fixed components. This proves (7) with $E = 0$ on each model of X/Z. Moreover, $\mathcal{L}_i$ gives its nonempty linear subsystem L_i also without fixed components (in that sense any functional algebra $\mathcal{L}$ is g.a.g. with $E = 0$).

The linear system L_i only depends on $\mathcal{L}_i$: every element of

$$L_i = \{(s) + M_i \mid 0 \neq s \in \mathcal{L}_i\} \subset |M_i|$$

is the restriction to X_i of $\overline{(s)} + \mathcal{M}_i$, and the general element does not have fixed components. Indeed, by the definitions $\operatorname{Bs} L_i = \min L_i = 0$.

If we take a model X_i/Z on which L_i is free (which exists by Hironaka), then M_i is also free on X_i, and $\mathcal{M}_i = \overline{M_i}$. Indeed, for any other model $g\colon X_i' \to X_i$ and general $D_i = (\mathcal{D}_i)_{X_i}$ in L_i, the divisor does not contain the center of any exceptional divisor, and its birational transform $D_i' = (\mathcal{D}_i)_{X_i'}$ does not contain any exceptional divisor. Thus

$$D_i' = g^{-1}D_i = g^*D_i, \quad \mathcal{D}_i = (s) + \mathcal{M}_i = \overline{D_i} \quad \text{and} \quad \mathcal{M}_i = \overline{M_i}.$$

Hence the components $\mathcal{L}_i \neq 0$ of functional algebras are equivalent to linear systems L_i without fixed components.

Now $\mathcal{M}_{i+j} \geq \mathcal{M}_i + \mathcal{M}_j$ for all i, j, which completes the proof of (1). Indeed, since $\mathcal{L}_{i+j} \supset \mathcal{L}_i\mathcal{L}_j$, then

$$\begin{aligned}
\mathcal{M}_{i+j} &= \sup\Big\{-\overline{(s)} \;\Big|\; 0 \neq s \in \mathcal{L}_{i+j}\Big\} \geq \sup\Big\{-\overline{(s)} \;\Big|\; 0 \neq s \in \mathcal{L}_i\mathcal{L}_j\Big\} \\
&\geq \sup\Big\{-\overline{(ss')} = -\overline{(s)} - \overline{(s')} \;\Big|\; 0 \neq s \in \mathcal{L}_i \text{ and } 0 \neq s' \in \mathcal{L}_j\Big\} \\
&= \sup\big\{-\overline{(s)} \mid 0 \neq s \in \mathcal{L}_i\big\} + \sup\big\{-\overline{(s')} \mid 0 \neq s \in \mathcal{L}_j\big\} = \mathcal{M}_i + \mathcal{M}_j.
\end{aligned}$$

Each $\overline{\mathcal{L}_i} = f_*\mathcal{O}_X(\mathcal{M}_i)$, which proves the first statement in (4). Indeed, $\overline{\mathcal{L}_i} \subset \overline{f_*\mathcal{O}_X(\mathcal{M}_i)} = f_*\mathcal{O}_X(\mathcal{M}_i)$ by (2). This last equation corresponds to the completeness of the linear system for $f_*\mathcal{O}_X(\mathcal{M}_i)$: if $0 \neq s \in k(X)$ such that $s^m + l_1s^{m-1} + \cdots + l_{m-1}s + l_m = 0$ with all $l_j \in f_*\mathcal{O}_X(j\mathcal{M}_i)$, or equivalently, $\overline{(l_j)} + j\mathcal{M}_i \geq 0$, then

$$\begin{aligned}
m(\overline{(s)} + \mathcal{M}_i) = \overline{(s^m)} + m\mathcal{M}_i &= \overline{(l_1s^{m-1} + \cdots + l_{m-1}s + l_m)} + m\mathcal{M}_i \\
&\geq \min\Big\{\overline{(l_1s^{m-1})}, \ldots, \overline{(l_{m-1}s)}, \overline{(l_m)}\Big\} + m\mathcal{M}_i \\
&= \min\Big\{\overline{(l_1)} + \mathcal{M}_i + (m-1)(\overline{(s)} + \mathcal{M}_i), \ldots, \\
&\qquad \overline{(l_{m-1})} + (m-1)\mathcal{M}_i + (\overline{(s)} + \mathcal{M}_i), \overline{(l_m)} + m\mathcal{M}_i\Big\} \\
&\geq \min\Big\{(m-1)(\overline{(s)} + \mathcal{M}_i), \ldots, \overline{(s)} + \mathcal{M}_i\Big\},
\end{aligned}$$

and $\overline{(s)} + \mathcal{M}_i \geq 0$. Hence $s \in f_*\mathcal{O}_X(\mathcal{M}_i)$.

On the other hand, each element $s \in f_*\mathcal{O}_X(\mathcal{M}_i)$ is integral for $\mathcal{L}_i$, which gives the opposite inclusion $f_*\mathcal{O}_X(\mathcal{M}_i) \subset \overline{\mathcal{L}_i}$. The linear systems L_i and $|M_i|$ induce a morphism $g\colon \operatorname{Proj}_Z \mathcal{R}_{X/Z}\,\mathcal{M}_i = Y' \to \operatorname{Proj}_Z \bigoplus \mathcal{L}_i^j = Y$ that is finite, because it is quasi-finite and proper. Hence the b-divisorial algebra $\mathcal{R}_{X/Z}\,\mathcal{M}_i$, isomorphic to $\mathcal{R}_{Y/Z}\,\mathcal{A}[D]$, is finite as a module over the product functional algebra $\bigoplus \mathcal{L}_i^j \hookrightarrow \mathcal{R}_{Y/Z}\,D$ (under the above isomorphism) by Example 4.8, where $\mathcal{A} = g_*\mathcal{O}_Y = h_*\mathcal{O}_{X_i}$, $h\colon X_i \to Y$, and $\mathcal{O}_Y(D) = \mathcal{O}_Y(1)$ or $\mathcal{M}_i \sim \overline{h^*D}$ with very ample D on Y/Z. Indeed, $\bigoplus \mathcal{L}_i^j \cong \mathcal{R}_{Y/Z}\,D$ for all degrees $i \gg 1$ because $\mathcal{L}_i$ is isomorphic to $e_*\mathcal{O}_Y(1)$, $e\colon Y \to Z$. In particular, each $s \in f_*\mathcal{O}_X(\mathcal{M}_i)$ is integral, because the sequence

$$(1) \subset (1, s) \subset \cdots \subset (1, s, \ldots, s^{m-1}) = (1, s, \ldots, s^{m-1}, s^m)$$

of $\bigoplus \mathcal{L}_i^j$-submodules terminates by the Noetherian property.

We have $\operatorname{Mov} \overline{\mathcal{L}} \le \mathcal{M}_\bullet$ by (3), since $\mathcal{R}_{X/Z} = \overline{\mathcal{L}}$. Conversely, $\mathcal{M}_\bullet \le \operatorname{Mov} \overline{\mathcal{L}}$ by (3) again, because $\mathcal{L} \subset \overline{\mathcal{L}}$ implies $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L} \le \operatorname{Mov} \overline{\mathcal{L}}$. This completes the proof of (4).

(5) follows by construction: a b-free divisor is the minimal one giving any linear system.

If $\mathcal{L}$ is f.g., by Truncation Principle 4.5, we can assume that it is generated by $\mathcal{L}_1$. In addition, we can assume that $\mathcal{R}_{X/Z}\,\mathcal{M}_1$ gives the normalization $\operatorname{Proj}_Z \mathcal{R}_{X/Z}\,\mathcal{M}_1$ of $Y = \operatorname{Proj}_Z \mathcal{L}$ in $k(X)$. Then $\overline{\mathcal{L}} = \mathcal{R}_{X/Z}\,\mathcal{M}_\bullet = \mathcal{R}_{X/Z}\,\mathcal{M}_1$ up to a finite number of components (by normality for high multiples of a very ample divisor). Hence $\mathcal{R}_{X/Z}\,\mathcal{M}_\bullet$ is a finite module over $\mathcal{R}_{X/Z}\,\mathcal{M}_1$. But in turn, this last algebra is a finite $\mathcal{L}$-module by Example 4.8 and the above arguments. Hence $\mathcal{R}_{X/Z}\,\mathcal{M}_\bullet$ is also a finite $\mathcal{L}$-module and is f.g. by the Hilbert basis theorem.

Conversely, suppose that $\mathcal{R}_{X/Z}\,\mathcal{M}_\bullet$ and $f_*\mathcal{O}_X(M_1)$ generate the algebra; in particular, $\mathcal{R}_{X/Z}\,\mathcal{M}_\bullet = \mathcal{R}_{X_1/Z}\,M_1$. Then each $\mathcal{L}_i$ defines a finite morphism $Y = \operatorname{Proj}_Z \mathcal{R}_{X_1/Z}\,M_1 \to Y_i = \operatorname{Proj}_Z \bigoplus \mathcal{L}_i^j$. Moreover, the inclusions $\mathcal{L}_i^j \subset \mathcal{L}_{ij}$ define finite morphisms $Y_{ij} \to Y_i$. However, this is only possible under a stabilization: $Y_{ij} \cong Y_i$ for some $i \gg 1$ and all $j \ge 1$. This implies that $\mathcal{L}^{[i]}$ is f.g. because $\mathcal{L}_i \cong e_*\mathcal{O}_{Y_i}(1)$ generates $\mathcal{L}_{ij}$ for all $j \gg 1$, and completes the proof of (6).

Finally, suppose that algebras $\mathcal{L}$ and $\mathcal{L}'$ are quasi-isomorphic, that is, isomorphic up to a truncation. Since we assume that the isomorphism induces the identity on their fields of fractions, each nontrivial $\mathcal{L}_i$ is isomorphic to $\mathcal{L}_i'$ under a multiplication $t \mapsto s_i t$ for a unique $0 \ne s_i \in k(X)$. Since this is a homomorphism of algebras, $s_i s_j = s_{ij}$. Then we can find $s_d \in k(X)$ (even if $\mathcal{L}_d = 0$) such that each $s_{id} = s_d^i$, where d is the gcd of j with $\mathcal{L}_j \ne 0$. This implies the linear equivalence $\mathcal{M}_\bullet \sim \mathcal{M}'_\bullet = \operatorname{Mov} \mathcal{L}'$ and completes the proof of (8). $\square$

On the way we have proved the following two corollaries:

Corollary 4.14 *$\mathcal{L}$ is f.g. if and only if its mobile system $\mathcal{M}_\bullet$ is b-divisorial up to a truncation: there exists a natural number I such that $\mathcal{M}_{iI} = i\mathcal{M}_I$ for all $i \geq 1$.*

Corollary 4.15 *If $\mathcal{L}$ or $\overline{\mathcal{L}}$ is f.g., the operation of taking projective spectrum induces the normalization $\operatorname{Proj}_Z \overline{\mathcal{L}} \to \operatorname{Proj}_Z \mathcal{L}$ of $\operatorname{Proj}_Z \mathcal{L}$ in $k(X)$. In particular, if it is birational, it is the normalization of $\operatorname{Proj}_Z \mathcal{L}$.*

By Proposition 4.13, (6), it is enough to consider pbd algebras to establish a criterion for f.g. (cf. Theorem 4.26). By Corollary 4.14 most pbd algebras are not f.g.

Example 4.16 Let C/C be a curve locally. If $D_\bullet = (-P, -P, \ldots, -P, \ldots)$, the pbd algebra $\mathcal{R}_{C/C}\, D_\bullet$ is not f.g.

Example 4.17 Let C/C be a curve locally. If $D_\bullet = (P, 2^2P, \ldots, n^2P, \ldots)$, the pbd algebra $\mathcal{R}_{C/C}\, D_\bullet$ is not f.g.

In this last case, the algebra is not f.g. since its mobile system grows too fast.

Definition 4.18 We say that a functional algebra $\mathcal{L}$ is *bounded* by a b-divisor $\mathcal{D}$ of X if $\mathcal{L}$ is a subalgebra of $\mathcal{R}_{X/Z}\, \mathcal{D}$.

We will see in Proposition 4.20 that it is equivalent for $\mathcal{L}$ to be bounded by a divisor.

Example 4.19 (Restricted algebra) If we compare Definitions 3.37 and 3.43, we see that in the first, the restriction to Y of the divisorial algebra $\mathcal{R}_{X/Z}\, D$ is bounded: $\mathcal{R}_{X/Z}\, D_{|Y} \subset \mathcal{R}_{Y/Z}\, D_{|Y}$. Moreover, in this inclusion, we can replace D by its b-divisor $\mathcal{D} = \overline{D}$, and $D_{|Y}$ by $\mathcal{D}_{|Y} = \overline{D}_{|Y}$ (see 7.2 for fixed restriction of b-divisors).

In particular, each algebra of type $(\mathrm{RFA})_{n,d}$ is bounded. This is a good sign, but not surprising, because it holds for any bounded algebra (cf. Corollary 4.25).

Proposition 4.20 *Let $\mathcal{L}$ be a functional algebra and $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L}$ its mobile system. Then $\mathcal{L}$ is bounded if and only if any one of the following holds:*

(1) *$\mathcal{L}$ is* divisorially bounded*, that is, there exists a Cartier divisor D on X such that $\mathcal{L} \subset \mathcal{R}_{X/Z}\, D = \mathcal{R}_{X/Z}\, \overline{D}$;*

(2) *the* characteristic system $(\mathcal{D}_i) = (\mathcal{M}_i/i)$ *is bounded, that is,* there exists a b-divisor $\mathcal{D}$ such that all $\mathcal{D}_i \le \mathcal{D}$*; and*

(3) *the algebra is* convergent*, that is,* the limit

$$\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i,$$

exists*, where we consider only $\mathcal{D}_i \ne -\infty$.*

Addendum 4.20.1 *In addition, the limit $\mathcal{D}$ in (3) satisfies*

(MXD) *each $\mathcal{D}_i = \mathcal{M}_i/i \le \mathcal{D}$; and*

(BSD) *all the $\mathcal{D}_i$ are supported in a b-divisor, that is,* $\operatorname{Supp} \mathcal{D}_i \le S$ *for a fixed reduced b-divisor S. (Here we set* $\operatorname{Supp}(-\infty) = 0$*.)*

Thus to say that an algebra is bounded restricts its characteristic divisors $\mathcal{D}_i$ in two ways: i.e., in multiplicities and in supports.

Lemma 4.21 *Let $\mathcal{M}$ be a $\mathbb{R}$-b-divisor that is* b-nef$/X$*, and H an $\mathbb{R}$-Cartier divisor on X such that $\mathcal{M} \le H$. Then $\mathcal{M} \le \overline{H}$.*

In particular, this holds if $\mathcal{M}$ is b-semiample.

Proof–Explanation In general, we say that an $\mathbb{R}$-b-divisor M is *b-nef$/Z$* (or *b-semipositive*) if it is nef $\mathbb{R}$-Cartier, that is, there is a model Y/Z of X/Z such that

- $\mathcal{M}_Y$ is a nef$/Z$ $\mathbb{R}$-Cartier divisor on Y, and
- $\mathcal{M} = \overline{M_Y}$.

b-semiample (see Section 3) implies b-nef. *Numerically b-seminegative* is defined in a similar way.

For the lemma, we need to prove that $\overline{H} - \mathcal{M}$ is effective. This is immediate by Negativity Lemma 3.18, because $\overline{H} - \mathcal{M}$ is numerically b-seminegative$/X$, and $(\overline{H} - \mathcal{M})_X = H - \mathcal{M}_X \ge 0$. □

Lemma 4.22 *Let $\mathcal{M}_\bullet$ be a system of $\mathbb{R}$-b-divisors satisfying*

$$\mathcal{M}_i + \mathcal{M}_j \le \mathcal{M}_{i+j}.$$

(compare Example 4.12). Then the system $(\mathcal{D}_i) = (\mathcal{M}_i/i)$ has the following properties:

convexity $\mathcal{D}_{i+j} \ge (i\mathcal{D}_i + j\mathcal{D}_j)/(i+j)$ *for all i, j; thus*

arithmetic monotonicity $\mathcal{D}_{ij} \ge \mathcal{D}_i$*; and*

convergence $\sup\{\mathcal{D}_i\} = \lim_{i\to\infty} \mathcal{D}_i$*; in particular, $\mathcal{D}_\bullet$ is convergent if and only if it is bounded from above. In this case, all the $\mathcal{D}_i$ have common support as in* (BSD) *of Addendum 4.20.1*

Caution 4.23 We should really write lim sup rather than lim; so, to be more precise, we make the convention that we *omit all* $\mathcal{D}_i = -\infty$ in the sequence! The limits are taken componentwise and possibly *not uniformly*; and the limit itself may not be a b-divisor (cf. (BSD) of Addendum 4.20.1).

If $\mathcal{M}_\bullet$ is a b-free system, that is, $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L}$ for a pbd algebra $\mathcal{L} = \mathcal{R}_{X/Z}\, \mathcal{M}_\bullet$, then by arithmetic monotonicity $\mathcal{L}$ has subalgebras

$$\mathcal{L}^i = \mathcal{R}_{X/Z}\, (j\mathcal{M}_i/i)',$$

where $(j\mathcal{M}_i/i)'$ is the system $(j\mathcal{M}_i/i)$ with $j\mathcal{M}_i/i$ replaced by $-\infty$ whenever $i \nmid j$. The algebra is quasi-isomorphic to a divisorial algebra $\mathcal{R}_{X/Z}\, \mathcal{M}_i$, and is bounded by $\mathcal{R}_{X/Z}\, \mathcal{M}_i/i$. The total algebra $\mathcal{L}$ is an inductive limit $\lim_{-|-} \mathcal{L}^i$ for inclusions $\mathcal{L}^i \subset \mathcal{L}^j$ with $i \mid j$. From this point of view, it is *pseudo b-divisorial*, but not b-divisorial, since the limit of b-divisors does not always exist, and, if it exists under the convergence of the lemma, its algebra may be different from $\mathcal{L}$ (even up to quasi-isomorphism, cf. Example 4.16), or worse, may not be defined. The only case the inductive limit is well defined as a b-divisorial algebra up to a quasi-isomorphism is when it *stabilizes*, that is, $\mathcal{L}$ is quasi-isomorphic to $\mathcal{L}^i$ for some $i \gg 1$ (cf. Limiting Criterion below). This means f.g.

Proof Convexity means

$$\mathcal{M}_{i+j} = (i+j)\mathcal{D}_{i+j} \ge i\mathcal{D}_i + j\mathcal{D}_j = \mathcal{M}_i + \mathcal{M}_j.$$

Thus by induction on j

$$\mathcal{D}_{ij} \geq (i(j-1)\mathcal{D}_{i(j-1)} + i\mathcal{D}_i)/ij \geq (i(j-1)\mathcal{D}_i + i\mathcal{D}_i)/ij = \mathcal{D}_i.$$

It is enough to prove convergence componentwise, that is, for a numerical sequence $\mathcal{D}_i$. Moreover, it follows from the existence of $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ and from its upper bound (cf. Proposition 4.20, (MXD)), that is, for all i,

$$\mathcal{D}_i \leq \mathcal{D},$$

where in the limit we consider only for $\mathcal{D}_i \neq -\infty$, but the limit can be $+\infty$. The case that all $\mathcal{D}_i = -\infty$ is trivial.

Otherwise, by monotonicity, for the gcd d of the indices i with $\mathcal{D}_i \neq -\infty$, all $\mathcal{D}_j \neq -\infty$ with $d \mid j \gg 1$, and all $\mathcal{D}_j = -\infty$ with $d \nmid j$. After truncating by d, we can assume that all $\mathcal{D}_i \neq -\infty$ with $i \gg 1$.

Again by monotonicity, we have an increasing subsequence $\mathcal{D}_{i^j}$ for $\mathcal{D}_i \neq -\infty$. It converges to $\mathcal{D}$ that is a real number if the sequence is bounded from above, or to $+\infty$. Thus we have a convergent subsequence $\mathcal{D}_{i^j} \to \mathcal{D}$.

This gives the required limit. For any $\varepsilon > 0$, there exists $j \geq 1$ such that $\mathcal{D}_j > \mathcal{D} - \varepsilon/2$ (or $> \varepsilon/2$ if $\mathcal{D} = +\infty$), and all $\mathcal{D}_i \neq -\infty$ with $i \geq j$. Then, for all $i \gg 1$, $i = jq + r$ with $j \leq r \leq 2j$, $0 < jq$, and

$$\mathcal{D}_i \geq \frac{jq}{i}\mathcal{D}_{jq} + \frac{r}{i}\mathcal{D}_r \geq \frac{jq}{i}\mathcal{D}_j + \frac{r}{i}\mathcal{D}_r > \frac{jq}{i}(\mathcal{D} - \frac{\varepsilon}{2}) + \frac{r}{i}\mathcal{D}_r > \mathcal{D} - \varepsilon$$

(or $> \frac{jq}{i}\frac{\varepsilon}{2} + \frac{r}{i}\mathcal{D}_r > \varepsilon$). Thus $\liminf_{i\to\infty} \mathcal{D}_i \geq \mathcal{D}$.

Since this holds for any convergent subsequence, $\lim_{i\to\infty} \mathcal{D}_i$ exists, is equal to $\mathcal{D}$, and by arithmetic monotonicity we have the inequality $\mathcal{D}_i \leq \mathcal{D}$.

Finally, to find the common support $\mathcal{S}$ of the b-divisors $\mathcal{D}_i$, we can again assume that $\mathcal{D}_i \neq -\infty$ for all $i \gg 1$, or say, for all $i \geq j \geq 1$. Then as above, for all $i \gg 1$

$$\mathcal{D}_i \geq \frac{jq}{i}\mathcal{D}_j + \frac{r}{i}\mathcal{D}_r,$$

where $j \leq r \leq 2j$. On the other hand, if the characteristic system is bounded by $\mathcal{D}$, then all $\mathcal{D}_i \leq \mathcal{D}$. Thus for all $i \gg 1$, $\operatorname{Supp}\mathcal{D}_i$ is in the union of $\operatorname{Supp}\mathcal{D}$ and of all $\operatorname{Supp}\mathcal{D}_r$. (If $\mathcal{D} \geq \mathcal{D}_i \geq \mathcal{D}'$, then $\operatorname{Supp}\mathcal{D}_i$ is in the union of $\operatorname{Supp}\mathcal{D}$ and $\operatorname{Supp}\mathcal{D}'$.) Taking the union of $\operatorname{Supp}\mathcal{D}_i$ not included in the $i \gg 1$ gives $\mathcal{S}$. $\square$

Proof of Proposition 4.20 Let $\mathcal{L}$ be an algebra bounded by a b-divisor $\mathcal{D}$. Then, by Proposition 4.13, (3) each $\mathcal{M}_i \le i\mathcal{D}$, which implies (2). Conversely, (2) together with Proposition 4.13, (2) imply the inclusion $\mathcal{L} \subset \mathcal{R}_{X/Z}\mathcal{D}$, provided that $\mathcal{D}$ satisfies the coherence of Example 4.11. In fact, we can choose the required $\mathcal{D}$ as the closure $\overline{D}$, where D is a Cartier divisor on X such that $D \ge D_X$. Moreover such D exists among hyperplane sections of X/Z.

Thus each $(\mathcal{M}_i)_X \le (i\mathcal{D})_X = i(\mathcal{D}_X) \le iD$. Hence each $\mathcal{M}_i \le \overline{iD} = i\overline{D}$ by Lemma 4.21, which implies (1) and boundedness.

Finally, by Lemma 4.22, (2) is equivalent to (3) and Addendum 4.20.1 with a b-divisor $\mathcal{D}$. □

Corollary 4.24 *Quasi-isomorphism of algebras preserves their boundedness. Moreover, the corresponding* characteristic limit *satisfies* $\mathcal{D}' \sim_{\mathbb{Q}} r\mathcal{D}$ *for some positive* $r \in \mathbb{Q}$ (cf. Corollary 3.3).

Proof Immediate by Proposition 4.13, (8) and Proposition 4.20, (2–3). □

Corollary 4.25 *The restriction of a bounded algebra is also bounded* (provided it is well defined).

Proof–Explanation Immediate by Proposition 4.20, (1), because $\mathcal{L}_{|Y} \subset \mathcal{R}_{Y/Z}\, D_{|Y}$. The existence of a Cartier divisor D such that $D_{|Y}$ is well defined follows from the proof of Proposition 4.13, (1). Indeed, $\mathcal{L}_{|Y}$ is well defined if $Y \subset X$ is not contained in the poles of any $0 \ne s \in \mathcal{L}$. We then say that Y is *in general position* with respect to $\mathcal{L}$, or vice versa, that $\mathcal{L}$ is in general position with respect to Y (cf. Definitions 3.37 and 3.43). Equivalently, the positive component of each b-divisor $\mathcal{M}_i$ in $\operatorname{Mov}\mathcal{L}$ or of its characteristic system does not contain Y. The same should hold for the limit $\mathcal{D}$. Then we can find $D \ge \mathcal{D}_X$ such that its positive component does not contain Y.

Note that the restriction $\mathcal{D}_{|Y}$ is usually not well defined, but $\lim_{i\to\infty} \mathcal{D}_{i\,\vert Y}$ exists if $\mathcal{L}$ is bounded.[4.5] □

[4.5] cross ref to Fixed Restriction 7.2 for the future notation $_{\vert Y}$.

Theorem 4.26 (Limiting Criterion) *A functional algebra $\mathcal{L}$ is f.g. if and only if the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ stabilizes, that is, $\mathcal{D} = \mathcal{D}_i$ for some $i \gg 1$.*

Under either condition, the limit $\mathcal{D}$ is a b-semiample $\mathbb{Q}$-divisor; the algebra and its characteristic system are bounded.

Proof Immediate by Corollary 4.14 and the arithmetic monotonicity of Lemma 4.22. Note that stabilization is preserved under similarity of systems.

In general, stabilization does not mean that $\mathcal{D} = \mathcal{D}_i$ for all $i \gg 1$. However, this holds up to a truncation, essentially by the above monotonicity (cf. Corollary 5.13). $\square$

By Example 4.16, for an algebra to be f.g., it is not enough that its limit $\mathcal{D}$ is a b-semiample $\mathbb{Q}$-divisor.

Example 4.27 We now discuss the case of b-divisorial algebras. Let $\mathcal{D}$ be an $\mathbb{R}$-b-divisor as in Example 4.11. Its *characteristic limit* is the characteristic limit of its algebra $\mathcal{R}_{X/Z}\,\mathcal{D}$, that is, $\mathcal{D}^m \overset{\text{def}}{=} \lim_{i\to\infty} \mathcal{D}_i$. This exists because $\mathcal{R}_{X/Z}\,\mathcal{D}$ is bounded by itself, and $\mathcal{D}^m \le \mathcal{D}$ by Proposition 4.13, (3) (but not always $= \mathcal{D}$). Thus we have a decomposition

$$\mathcal{D} = \mathcal{D}^m + \mathcal{D}^e,$$

where $\mathcal{D}^e = \mathcal{D} - \mathcal{D}^m \ge 0$, and $\mathcal{D}^m$ is the *maximal pseudo b-semiample* part (or b-divisor) of $\mathcal{D}$ (or *maximal pbs ample* part of $\mathcal{D}$ for short), and $\mathcal{D}^e$ is the *fixed part* of $\mathcal{D}$. Indeed, the *b-semiample part* of $\mathcal{D}'$ is a b-semiample $\mathbb{R}$-b-divisor $\mathcal{D}' \le \mathcal{D}$. A *pseudo b-semiample* divisor is a limit of these. In particular, $\mathcal{D} = \mathcal{D}^m$ itself is pbs ample if and only if $\mathcal{D}^e = 0$.

This generalizes the decompositions of Remark 3.26. Indeed, if $D = D^m + E$ is as in the remark, then $D = \mathcal{D}^m_X$ for some b-semiample $\mathbb{R}$-b-divisor $\mathcal{D}^m$. Thus if we take $\mathcal{D} = \mathcal{D}^m + E$, then $D = \mathcal{D}_X$, $D^m = \mathcal{D}^m_X$, and $E = \mathcal{D}^e$.

In general, $\mathcal{D}^m$ is not always b-semiample.

If $\mathcal{R}_{X/Z}\,\mathcal{D}$ is f.g., then $\mathcal{D}^m$ is b-semiample by Limiting Criterion 4.26 (cf. Remark 3.26); and, moreover, $\mathcal{D}^m$ is then a $\mathbb{Q}$-b-divisor. However g.a.g. can fails, even if $\mathcal{D}$ is big/Z, because Lemma 3.15 does not hold for any prime b-divisor E. The latter is related to saturation, as we discuss below.

We do not expect f.g. for an arbitrary b-divisor $\mathcal{D}$, even in the situation of Remark 3.26, (2) (cf. Theorem 3.29). This can again be attributed to saturation (cf. Conjecture 4.35 and Example 4.31 below).

But before we finish

Proof of Corollary 3.25 For $\mathbb{Q}$-divisors, immediate by Corollary 4.10; and g.a.g. holds with empty base divisorial locus or $E = 0$ (cf. Proposition 4.13, (7)).

For a non $\mathbb{Q}$-divisor D, ample$/Z$, $\mathcal{R}_{X/Z}\, D$ is never f.g. (cf. Stupid Example 3.13). Indeed, since $\mathcal{D} = \overline{D}$ is (b-semi-)ample, and $\mathcal{R}_{X/Z}\,\mathcal{D} = \mathcal{R}_{X/Z}\, D$ is f.g., then $\mathcal{D} = \mathcal{D}^m$ is a $\mathbb{Q}$-divisor by Example 4.27. $\square$

4.28 Saturation of linear systems

Although saturation can sometimes be explained in terms of sheaves and their algebras, essentially in the situations we need, a better explanation (arguably the only possible explanation) uses divisors and linear systems. Let D be an $\mathbb{R}$-Weil divisor on X/Z. We recall that its linear system is the set

$$|D| = |D|_{X/Z} = \{D' \mid D' \geq 0 \text{ and } D' \sim D/Z\}.$$

In its local version over a point $P \in Z$, $|D|$ is defined up to the equivalence relation that identifies divisors D' that are equal over a neighborhood of P. The most important thing for us is its decomposition into mobile and fixed parts.

Similarly, for any $\mathbb{R}$-b-divisor, we can define $\mathbb{R}$-b-divisors $\operatorname{Mov}\mathcal{D}$ and $\operatorname{Fix}\mathcal{D}$, the *mobile* and *fixed parts* of $\mathcal{D}$ (or of its linear system $|\mathcal{D}|$, or sheaf $f_*\mathcal{O}_X(\mathcal{D}) \subset k(X)$); for the definition, see the proof–construction of Proposition 4.13). By definition, $(\operatorname{Mov}\mathcal{D})_Y \leq \operatorname{Mov}(\mathcal{D}_Y)$ on any model Y/Z of X/Z. However, if the sheaf $f_*\mathcal{O}_X(\mathcal{D})$ is coherent, then, by Proposition 4.13, (1) and (4), we have the following stabilization on a *sufficiently high* resolution X_{hr}. There is a model $g\colon X_{\mathrm{hr}} \to Z$ of X/Z such that $f_*\mathcal{O}_X(\mathcal{D}) = g_*\mathcal{O}_{X_{\mathrm{hr}}}(M)$, where $M = (\operatorname{Mov}\mathcal{D})_{X_{\mathrm{hr}}} = \operatorname{Mov}(\mathcal{D}_{X_{\mathrm{hr}}})$, $\operatorname{Mov}\mathcal{D} = \overline{M}$, $\operatorname{Bs}|M| = \emptyset$, and $|M| + \operatorname{Fix}\mathcal{D}_{X_{\mathrm{hr}}} = |\mathcal{D}_{X_{\mathrm{hr}}}| = |\mathcal{D}|_{X_{\mathrm{hr}}}$.

Definition 4.29 Let D and C be two $\mathbb{R}$-divisors on X. We say that D is *saturated* with respect to C or *C-saturated* if $M = \operatorname{Mov}\lceil D + C\rceil \leq D$. By convention, we agree that the condition holds if $|\lceil D + C\rceil| = \emptyset$, i.e., $\operatorname{mult}_{D_i} M = -\infty$; we also don't need to verify the condition if $\lceil D + C\rceil \leq D$.

We can attribute the same saturation to the linear system $|D|$, or even to any (incomplete) linear system, as well as to a functional sheaf $\mathcal{F} \subset k(X)$. In this last case, we take $D = \operatorname{Mov}\mathcal{F}$. Moreover, if $\mathcal{F} = f_*\mathcal{O}_X(D)$ then C-saturation of D implies the same for $\mathcal{F}$ (cf. Remark 4.30, (1) below).

For $\mathbb{R}$-*b-divisors* $\mathcal{D}$ and $\mathcal{C}$, the *saturation* of $\mathcal{D}$ with respect to $\mathcal{C}$ or $\mathcal{C}$-saturation means the $C = \mathcal{C}_{X_{\mathrm{hr}}}$-*saturation* of $D = \mathcal{D}_{X_{\mathrm{hr}}}$ on any *sufficiently high models* X_{hr}/Z of X/Z. This means that there is a model Y/Z of X/Z such that saturation holds for any model X_{hr}/Z that is also $/Y$.

For asymptotic saturation, we consider a system $\mathcal{D}_\bullet$ of $\mathbb{R}$-b-divisors. This system is *asymptotically saturated* with respect to $\mathcal{C}$ or *asymptotically* $\mathcal{C}$-*saturated* if there exists a natural number I, the *saturation index*, such that, for all natural numbers i,j with $I \mid i,j$, and on any sufficiently high model X_{hr}/Z the decomposition into mobile and fixed parts satisfies:[4.6]

$$\operatorname{Mov}\left(\lceil j\mathcal{D}_i + \mathcal{C}\rceil_{X_{\mathrm{hr}}}\right) \le j(\mathcal{D}_j)_{X_{\mathrm{hr}}}$$

(cf. Remark 4.30, (2–3) below).

A functional algebra is *asymptotically* $\mathcal{C}$-*saturated* if this holds for its characteristic system $\mathcal{D}_\bullet$. Moreover, a pbd algebra associated with a system $\mathcal{M}_\bullet$ is asymptotically $\mathcal{C}$-saturated if this holds for its normalized system $(\mathcal{D}_i = \mathcal{M}_i/i)$ (cf. Remark 4.30, (1)).

Finally, we say that an $\mathbb{R}$-b-divisor $\mathcal{D}$ is *asymptotically* $\mathcal{C}$-*saturated* if this holds for its system $(\mathcal{D}_i = \mathcal{D})$ (cf. Example 4.11 above). Even for a Cartier b-divisor $\mathcal{D} = \overline{D}$, we need to run through all sufficiently high models X_{hr}/Z (cf. Example 4.31 below). The point is that we have no universal X_{hr}/Z unless the b-divisorial algebra of $\mathcal{D}$ is f.g. (cf. Remark 4.30, (2) below).

Remark 4.30 (1) It is enough to verify the inequality

$$M = \operatorname{Mov}\lceil D + C\rceil \le D$$

over $\lceil D + C\rceil > D$, that is, to check that $\operatorname{mult}_{D_i} M \le \operatorname{mult}_{D_i} D$ for any prime divisors D_i with $\lceil \operatorname{mult}_{D_i} D + \operatorname{mult}_{D_i} C\rceil > \operatorname{mult}_{D_i} D$. The same holds for the other saturations.

This implies that $\operatorname{Mov}\lceil D + C\rceil \le \operatorname{Mov} D$. In most of our applications, D is mobile, or even free. Then $\operatorname{Mov} D = D$ and saturation is equivalent to $\operatorname{Mov}(D + \lceil C\rceil) = \operatorname{Mov}\lceil D + C\rceil \le \operatorname{Mov} D = D$. Thus if in addition $\lceil C\rceil \ge 0$, then $D \le \operatorname{Mov}(D + \lceil C\rceil) \le D$, and $\operatorname{Mov}(D + \lceil C\rceil) = D$ or, in terms of linear systems,

$$|D + \lceil C\rceil| = |D| + \lceil C\rceil\,,$$

where $\lceil C\rceil = \operatorname{Fix}(D + \lceil C\rceil)$.

[4.6]Should it be $j\mathcal{D}_j \mapsto j\mathcal{D}_i$ on the right-hand side?

(2) In general, $(\operatorname{Mov}\lceil j(\mathcal{D}_i)+\mathcal{C}\rceil)_{X_{\rm hr}} \le \operatorname{Mov}(\lceil j(\mathcal{D}_i)+\mathcal{C}\rceil_{X_{\rm hr}})$, but equality does not necessarily hold. In important cases, equality holds on any sufficiently high model $X_{\rm hr}$; this means that $\operatorname{Mov}(\lceil j(\mathcal{D}_i)+\mathcal{C}\rceil_{X_{\rm hr}})$ stabilizes on *all* sufficiently high models, but only for given i,j (cf. Examples 4.31 and 4.43 below). In applications, however we need *some* sufficiently high model $X_{\rm hr}$ on which, say, some divisors have normal crossing, and some b-divisors are Cartier (cf. the proof of Proposition 6.26). But we prefer to consider all sufficiently high models $X_{\rm hr}$ because in most cases it is easy to verify saturation for all such $X_{\rm hr}$ by stabilization (cf. Proposition 4.42 below).

(3) If $\mathcal{D}_\bullet$ is the characteristic system of a functional algebra with mobile system $\mathcal{M}_\bullet$, and $i \mid j$, then as in (1) above, asymptotic saturation means that

$$M = \operatorname{Mov}(jD_i + \lceil C\rceil) = \operatorname{Mov}\lceil jD_i + C\rceil \le \operatorname{Mov} jD_j = jD_j$$

with $C = \mathcal{C}_Y$, $D_i = (\mathcal{D}_i)_Y$, $D_j = (\mathcal{D}_j)_Y$ on any sufficiently high model $X_{\rm hr} = Y/X$, where $iD_i = \operatorname{Mov}(\mathcal{M}_i)_Y$, $jD_j = \operatorname{Mov}(\mathcal{M}_j)_Y$ and both linear systems $|iD_i|$ and $|jD_j|$ are free (cf. Proposition 4.13, (1)). Thus in terms of linear systems,

$$|\operatorname{Mov}(jD_i + \lceil C\rceil)| = |M| + F,$$

where $M \le M_j = jD_j$ and $F = \operatorname{Fix}(jD_i + \lceil C\rceil)$. This can be interpreted in terms of sheaves: namely

$$\begin{aligned} f_*\mathcal{O}_Y(M_i)^{\otimes \frac{j}{i}} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(\lceil C\rceil) &= f_*\mathcal{O}_Y(jD_i + \lceil C\rceil) \\ &\subset f_*\mathcal{O}_Y(jD_j) = f_*\mathcal{O}_Y(M_j), \end{aligned}$$

where $M_i = iD_i$ and $\lceil C\rceil$ is assumed to be Cartier. However in most applications we consider $i \gg j$, and then asymptotic saturation has no sheaf theoretic interpretation (cf. Example 4.31 below) because the integral parts depend on divisors up to a linear equivalence, not up to $\mathbb{Q}$-linear equivalence.

(4) Again, if $\mathcal{D}_\bullet$ is the characteristic system of a functional algebra, then by the arithmetic monotonicity of Lemma 4.22, we have inequalities

$$\operatorname{Mov}(\lceil j(\mathcal{D}_i)+\mathcal{C}\rceil_{X_{\rm hr}}) \le j(\mathcal{D}_j)_{X_{\rm hr}} \le j(\mathcal{D}_l)_{X_{\rm hr}} \le j(\mathcal{D})_{X_{\rm hr}}$$

with any $j \mid l$ and limiting divisor $\mathcal{D}$. In fact, this last inequality for $\mathcal{D}$ is the most important (cf. Example 4.37).

(5) For $j = i$ the asymptotic saturation inequality implies

$$\mathrm{Mov}\,(\lceil \mathcal{M}_i + \mathcal{C} \rceil_{X_{\mathrm{hr}}}) = \mathrm{Mov}\,(\lceil i(\mathcal{D}_i) + \mathcal{C} \rceil_{X_{\mathrm{hr}}}) \le i(\mathcal{D}_i)_{X_{\mathrm{hr}}} = (\mathcal{M}_i)_{X_{\mathrm{hr}}}$$

that means the $\mathcal{C}$-saturation for $\mathcal{M}_i = i\mathcal{D}_i$. The same holds for each component $\mathcal{L}_i$ of a functional algebra provided that $\mathcal{L}$ is asymptotically saturated.

(6) In general, asymptotic $\mathcal{C}$-saturation does not imply that $\mathcal{D}_\bullet$ is bounded (cf. Example 5.14). But this does holds for functional algebras and their characteristic systems. By Lemma 4.21, it is enough to establish this on any model Y/Z of X/Z, for example, on X itself. Then $\mathcal{D}_\bullet$ is convergent and has other properties (cf. $(\mathrm{FGA})_n$ and Conjecture 4.35 below). However we always consider bounded algebras, as they appear in our applications.

(7) Note also that *similarity* of characteristic systems is induced by similarity of $(\mathcal{M}_i = i\mathcal{D}_i)$: e.g.,

- $\mathcal{D}_\bullet \sim \mathcal{D}'_\bullet$, where $\mathcal{D}_\bullet = \mathcal{D}'_\bullet + \overline{(a)}$; and
- truncation means $\mathcal{D}_\bullet^{[I]} = I(\mathcal{D}_{iI})$, that is, *each* $\mathcal{D}_i^{[I]} = I\mathcal{D}_{iI}$.

Similarity preserves asymptotic $\mathcal{C}$-saturation for any system $\mathcal{D}_\bullet$ (see Proposition 9.17 below).

We discuss other properties of saturations below. In most applications, the saturation is *essentially exceptional*: i.e., $\mathcal{C}$ or $\lceil \mathcal{C} \rceil$ are exceptional on X.

Example 4.31 (Exceptional saturation) We say that the saturation of Definition 4.29 is *exceptional*/X if it holds for any $\mathcal{C}$ that is exceptional on X.

Exceptional saturation holds in particular for any *integral* D on X/X (cf. Example 4.37 and[4.7] the remarks below); this means that it is also exceptionally saturated as a b-divisor. Since $\mathcal{C}$ is quite arbitrary, we only need to worry about the inequality $\mathrm{Mov}\,\lceil D + \mathcal{C}_{X_{\mathrm{hr}}} \rceil = \mathrm{Mov}\,(D + \lceil \mathcal{C}_{X_{\mathrm{hr}}} \rceil) \le D$ for a prime divisor nonexceptional on X. Thus on X_{hr} we add to D any big divisor that is exceptional on X. This usually taken for granted in classical algebraic geometry when we consider Cartier divisors (cf. Lemmas 3.15

[4.7] I don't understand what this has to do with Example 4.37 and I can't find any related "remark below".

and 4.21). Indeed, if D is Cartier, then its closure $\mathcal{D} = \overline{D}$ is saturated and $\mathcal{O}_X(D) = \mathcal{O}_X(\mathcal{D})$ but this last equality does not hold if D is considered as a Weil b-divisor. Exceptional saturation makes it.[4.8]

The sheaf $\mathcal{O}_X(D)$ is also saturated. For divisorial sheaves, this occurs as

$$\mathcal{O}_X(D)^{\vee\vee} = \mathcal{O}_X(D)$$

(or [6, Proposition 2, (iv)]). The divisorial algebra $\mathcal{R}_{X/X}\,D$ is also exceptionally asymptotically saturated with $I = 1$. For an $\mathbb{R}$-divisor D, this last condition holds with some $I \geq 1$ if and only if D is a $\mathbb{Q}$-divisor (cf. Stupid Example 3.13 and Theorem 3.14).

This holds also for X/Z, and we can explain exceptional saturation for D/Z and its algebra $\mathcal{R}_{X/Z}\,D$ in the same style in terms of double dual. However for divisorial functional sheaves, it has three different versions, namely:

$$(f_*\mathcal{O}_X(D))^{\vee\vee} \subset (f_*(\mathcal{O}_X(D)^{\vee}))^{\vee} \subset f_*(\mathcal{O}_X(D)^{\vee\vee}) = f_*\mathcal{O}_X(D).$$

In general, for a functional subsheaf $\mathcal{L} \subset k(X)$, we can take any of these. We use the *minimal saturation* $\mathcal{L}^{\vee\vee} = \mathcal{L}$ (independent of X) or the *maximal saturation* $f_*\mathcal{L}^{\vee\vee} = \mathcal{L}$, where we take twice $^{\vee}$ for $\mathcal{O}_X$. Thus the maximal saturation depends on the choice of X. However the minimal saturation corresponds to *essentially exceptional* saturation on divisors, that is, we can add any divisors that are essentially exceptional on Z. Usually, $\mathcal{R}_{X/Z}\,D$ is not maximally or essentially exceptionally saturated: for example, if X/Z is birational, because there is a difference between divisorial and b-divisorial sheaves and algebras. The maximal saturation agrees with the exceptional asymptotic saturation/X, and it holds for any integral or $\mathbb{Q}$-divisor D (only integral in the case of sheaves).

This does not hold in general for b-divisorial, pbd-divisorial or functional algebras $\mathcal{L} \subset k(X)$. But in the cases we are interested in, maximal saturation and exceptionally asymptotic saturation hold *over another birational model* Y/Z of X/Z: for this, see predictions, triples, Addendum 5.12.2 and (BIR) of Conjecture 6.14.

- the flipping algebra of Example 3.12 with $Y = X$ and the $\mathbb{Q}$-divisor D (here $I = 1$ if D is integral);

[4.8] This does not make sense: makes it ... do what? Possibly: Assuming that D is exceptionally saturated/X makes the inequality $\mathcal{O}_X(D) = \mathcal{O}_X(\mathcal{D})$ true?

- the (pre-restriction) algebra $\mathcal{R}_{X/T}\, D$ with $Y = X$ and $I = 1$ of Example 3.36 (cf. proof of Induction Theorem 1.4 below);
- any b-divisorial algebra as in Example 4.11 with a Cartier $\mathbb{Q}$-b-divisor $\mathcal{D}$, on a model Y over which $\mathcal{D} = \overline{D}$ for a $\mathbb{Q}$-Cartier divisor D on Y; and
- any f.g. functional algebra $\mathcal{L}$ on any model of $Y/\operatorname{Proj}_Z \mathcal{L}$, by Corollary 4.14.

In this sense, asymptotic saturation is a necessary but not sufficient condition for f.g. The latter also needs certain conditions on X/Z (cf. Remark 3.26, (2), Theorem 3.29, Conjectures 3.31, 4.35, Proposition 4.38 and Remark 4.36, (2)). However, it helps to find a model Y where we expect f.g. Indeed, if $\mathcal{L}$ is exceptionally asymptotically saturated/Y and f.g., then Proj defines a rational contraction of X to $\operatorname{Proj}_Z \mathcal{L}$ that is rational 1-contraction on Y; in particular, Y and $\operatorname{Proj}_Z \mathcal{L}$ are isomorphic in codimension 1 when $\mathcal{L}$ is big. In the case of a pbd algebra it gives the g.a.g. of $\mathcal{D}^m$ or even $\mathcal{D}_\bullet$ over Y (cf. bss ampleness in Theorem 3.14, where the $\mathbb{Q}$-divisor condition for D^{small} can be replaced by exceptionally asymptotic saturation/X); this generalizes Zariski decomposition (cf. Example 4.27 above).

Unfortunately, saturation conditions are no good for induction on the dimension. It is well known that the usual saturation is not preserved under restrictions when they are not surjective (cf. reasons (2–3) in our motivation for introducing log canonical saturations below). The same goes for our saturations for an arbitrary $\mathcal{C}$.

Definition 4.32 A $\mathcal{C}$-saturation is *log canonical*/X or, to be more precise, log canonical over (X, B) if $\mathcal{C} = \mathcal{A} = \mathcal{A}^X = \mathcal{A}(X, B)$ is the *discrepancy* b-divisor of the pair (X, B) for some $\mathbb{R}$-divisor B on X.

In the divisorial case we take $C = \mathcal{A}_X = -B$.

We abbreviate log canonical asymptotic saturation to *lca* saturation.

Caution 4.33 For another pair (Y, B_Y) with a model Y/Z of X/Z, in general $\mathcal{A}^Y = \mathcal{A}(Y, B_Y) \neq \mathcal{A}$, and $\mathcal{B}^Y = \mathcal{B}(Y, B_Y) \neq \mathcal{B}$. Thus the log canonical saturation can be different over (Y, B_Y).

As it used[4.9] to be, $\mathcal{A}_Y$ denotes the restriction divisor of the b-divisor $\mathcal{A}$ on a model Y of X. In general, $\mathcal{B}_Y = B^Y \neq B_Y$. The latter is the

[4.9] What? Does this mean “as before” or “as usual”?

codiscrepancy of (Y, B_Y) on Y, and $\mathcal{B}_Y = B^Y = B_Y$ if and only if (Y, B_Y) is a *crepant* birational transform of (X, B) (cf. Definition 6.9).

By Example 4.16, for a nondivisorial algebra, saturation on its own is not enough for f.g. But we hope that asymptotic saturation works better, especially lca saturation (cf. Conjecture 4.35 and Example 4.37 below). There are four reasons for this according to our applications:[4.10]

(1) the restriction of a flip can be a nonflip;

(2) canonical saturation is compatible with restrictions by Kawamata–Viehweg vanishing (cf. Proposition 4.44);

(3) $\mathcal{A}$ is sufficiently effective: e.g., for a Klt pair (X, B), $\lceil \mathcal{A} \rceil \geq 0$ is exceptional, and at the same time grows at least linearly with blowups (cf. Proposition 4.42, (CGR), and (LGD) for predictions in Section 5 below); and

(4) $\mathcal{A}$-saturation is well defined (cf. Example 4.43 below).

Definition 4.34 Suppose that a pair $(X/T, B)$ satisfies:

(i) X/T is a contraction;

(ii) (X, B) is Klt;

(iii) $-(K + B)$ is nef and big$/Z$; and, finally,

(iv) $\dim X = n$.

The two middle conditions mean that $(X/T, B)$ is a weak log Fano contraction (cf. (WLF) in Proposition 4.38).

Then we say that a functional $\mathcal{O}_T$-algebra $\mathcal{L} \subset k(X)_\bullet$ or its system $\mathcal{D}_\bullet$ is *of type* $(\mathrm{FGA})_n$ if

$(\mathrm{FGA})_n$ it is bounded and *lca saturated* over $(X/T, B)$.

It is of type $(\mathrm{FGA})_n(\mathrm{bir})$ if X/T is birational.[4.11]

[4.10] I don't understand "according to": *in view* of our applications, or *depending on* the application we have in mind, or what?

[4.11] I have tried to untangle the definition and the following Conjecture: "We also use these to refer to the conjecture (cf. Induction Theorem 1.4)."

Conjecture 4.35 *An algebra of type* (FGA)$_n$ *is f.g./T.*

By Limiting Criterion 4.26, an equivalent condition is the stabilization of $\lim_{i\to\infty}\mathcal{D}_i$ for the characteristic system $\mathcal{D}_\bullet$ of $\mathcal{L}$. Thus the system stabilizes if

(LIM) there is a (finite) limit $\mathcal{D} = \lim_{i\to\infty}\mathcal{D}_i$; and

(LCA) it is lca saturated.

However, we prefer to work with b-divisors (cf. Remark 4.36, (5) below). Note for this that the system $\mathcal{D}_\bullet$ also satisfies

(MXD) each $\mathcal{D}_i = \mathcal{M}_i/i \le \mathcal{D}$ by Addendum 4.20.1;

(LBF) each $\mathcal{M}_i = i\mathcal{D}_i$ is b-free, by Proposition 4.13, (1); and

(AMN) *arithmetic monotonicity*: for any $j \mid i$ we have $\mathcal{D}_i \ge \mathcal{D}_j$ by Lemma 4.22.

Moreover, one can expect that LMMP in dimension $m \le n$ implies (FGA)$_n$ (cf. Theorem 3.29).

By Corollary 1.5, the conjecture implies the existence of log flips in dimension n modulo LMMP in dimension $m \le n-1$ (cf. Remark 4.36, (6) below). This would be an inductive construction of log flips.

Remark 4.36 Some of the assumptions in the conjecture can be relaxed or modified:

(1) We can replace the contraction X/T by any proper morphism X/Z of normal algebraic varieties or normal algebraic spaces (certainly), and probably also normal analytic spaces (use the Zariski decomposition of $X \to Z$).

(2) We expect also that Conjecture 4.35 holds for the 0-log pairs $(X/Z, B)$ of Remark 3.26, (2) (cf. Theorem 3.29).

(3) Moreover, Conjecture 4.35 may hold for pairs $(X/Z, B)$ with $\mathcal{D}_X = D = K + B + H$, where H is a nef and big/Z $\mathbb{R}$-divisor; or the same on some crepant model $(Y/Z, B_Y)$ of $(X/Z, B)$.

(4) Possibly everything works for more general gradings, e.g., $\mathbb{N}^r$-gradings.

(5) It can also be generalized to worse singularities, that is, when (X, B) is not Klt, B is not a boundary, and X is nonnormal, or not even seminormal. However we still need to assume that B is effective, and that the algebra $\mathcal{L}$ behaves well (f.g.) or the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ stabilizes near the locus of log canonical singularities LCS $(K + B)$ (see [9, Definition 3.14] and Example 4.37 below).

(6) The restriction of a special flip or a pl flip is not necessarily a flip. However, we expect that $(\mathrm{FGA})_{n-1}(\mathrm{bir}) = (\mathrm{RFA})_{n,n-1}(\mathrm{bir})$. For algebras, this means that each normal algebra $\mathcal{L}$ of type $(\mathrm{FGA})_{n-1}(\mathrm{bir})$ up to a quasi-isomorphism is a restricted algebra of type $(\mathrm{RFA})_{n,n-1}(\mathrm{bir})$ as in Definition 3.43 (cf. proof of Induction Theorem 1.4 below). Then the existence of n-fold flips would be equivalent to $(\mathrm{FGA})_{n-1}(\mathrm{bir})$.

(7) Finally, if $\mathcal{D}$ is big, the *stable* crepant model $(X_{st}/T, B_{st})$ of $(X/Z, B)$ in the conjecture with $X_{st}/T = \mathrm{Proj}_T\, \mathcal{L}$ is isomorphic in codimension 1 to a crepant weak log Fano contraction (cf. Addendum 5.12.2). According to conjectures of Alexeev and Borisov,[4.12] X_{st} should then have bounded modulus (cf. canonical boundedness and triples in (BIG) of Conjecture 6.14 below).

Example 4.37 (A Pythagorian dream) Suppose that $X = C$ is a normal algebraic curve and $B = \sum b_m P_m$ a Klt $\mathbb{R}$-boundary; in other words, each $b_i \in [0, 1)$. Then any lca saturated/(C, B) algebra $\mathcal{L}$ is f.g., or the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ stabilizes. Note that in this case $D = \sum d_m P_m = \mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ is a limit of divisors $D_i = \mathcal{D}_i = \sum d_{m,i} P_{m,i}$. Equivalently, each $d_m = \lim_{i\to\infty} d_{m,i}$ with $d_{m,i} \in \mathbb{Q}$. In particular, stabilization means that each d_m is *also* $\in \mathbb{Q}$. This is the point.

All the divisors D and D_i are supported in a finite subset of C by Lemma 4.22. Thus the limit takes place for the vector of coefficients in $\mathbb{R}^n$.

On the other hand, the discrepancy $\sum a_m P_m = \mathcal{A} = A = -B \le 0$. Moreover, by Klt, all $a_m = -b_m \in [0, -1)$. In the one dimensional case, any $X_{\mathrm{hr}} = C_{\mathrm{hr}} = C$. After truncating $\mathcal{L}$, we can assume that the saturation has index $I = 1$. Thus lca saturation gives the inequality $\mathrm{Mov}\,\lceil jD_i + A\rceil \le$

[4.12] Reference? Possibly Valery Alexeev, Boundedness and K^2 for log surfaces, Internat. J. Math. **5** (1994) 779–810?

jD_j for any natural numbers i, j. In our situation, we can assume that $\operatorname{Mov}\lceil jD_i + A\rceil = \lceil jD_i + A\rceil$ for all $i, j \gg 1$.

By RR on curves, this holds unless C is complete, $Z = \mathrm{pt.}$, and $\deg D \leq 0$. Moreover, then $\deg D = 0$, because we consider the limit only for nonempty linear systems $|M_i| = |iD_i|$. Thus $\deg D_i \geq 0$. Hence $D = 0$, and we have the stabilization by (MXD)(see Addendum 4.20.1). The stable model in this case is $C_{st} = \operatorname{Proj}_Z \mathcal{L} = \mathrm{pt.}/Z = \mathrm{pt.}$

Otherwise, we have the inequality $\lceil jD_i + A\rceil \leq jD_j$ or, componentwise, for each P_m and all $i, j \gg 1$,

$$\lceil jd_{m,i} + a_m\rceil \leq jd_{m,j}.$$

Again by (MXD) this gives, for each P_m and all $j \gg 1$,

$$\lceil jd_m + a_m\rceil = \lceil jd_{m,i} + a_m\rceil \leq jd_{m,j} \leq jd_m,$$

when we take $i \gg j$! By Diophantine geometry, for any real number $a = a_m > -1$, any real number $d = d_m$ satisfying

$$\lceil jd + a\rceil \leq jd$$

for all natural numbers $j \gg 1$ is rational. (See Cassels [2, Lemma 1A]; take the best upper approximations, for $\theta = d$; cf. proof of Approximation Lemma 5.15 below.) This explains also that positive $a = a_m$ are also not good (cf. Remark 4.36, (5) above). However A with $\lceil A\rceil = 0$, if even the total $A = 0$, does have an effect!

Since d_m is rational, then, for any $j \gg 1$ with integral jd_m, we get stabilization

$$\lceil jd_m + a_m\rceil = jd_{m,j} = jd_m.$$

By (AMN) this implies stabilization: each $d_{m,i} = d_m$ for all i divisible by some j.

Finally, since the divisors are bounded in their supports, we get stabilization by a truncation of $D_\bullet$.

In particular, this establishes $(\mathrm{FGA})_1$.

By these arguments, we can generalize f.g. for any effective B assuming that the limit stabilizes near the $\operatorname{LCS}(K + B) = \{P_m \mid b_m \geq 1\}$; e.g., $d_m = \lim_{i\to\infty} d_{m,i}$ stabilizes in each such point P_m.

For nonnormal C, stabilization near bad singularities means that, up to a similarity, all D_i and D are nonsingular, that is, are supported in nonsingular points of C.

To establish even $(\mathrm{FGA})_2$ is not so simple; this is discussed below after some preparations. Now we turn to the proof of Induction Theorem 1.4. For this we need some results on saturations.

Proposition 4.38 *For any effective $\mathbb{R}$-divisor B such that $K+B$ is $\mathbb{R}$-Cartier, any exceptional saturation/X implies lca saturation/(X,B).*

Thus $(\mathrm{FGA})_n$ *implies f.g. of any algebra $\mathcal{R}_{X/T}\,D$, provided that*

(WLF) $(X/T,B)$ *is a* weak log Fano contraction*, that is, (X,B) is Klt, and $-(K+B)$ is nef and big/T*

(cf. Theorem 3.29 and Remark 4.36, (2) above).

Addendum 4.38.1 *Any exceptional saturation/X implies the corresponding saturation for $\mathcal{A}'=\mathcal{A}(X,B)+\mathcal{E}$, where $\mathcal{E}$ is a reduced b-divisor such that:*

- *$\mathcal{E}_X$ is supported over* $\operatorname{Supp}\mathcal{A}_X=\operatorname{Supp}B$*, and*
- *$\mathcal{A}$-saturation is* integral *over $\mathcal{E}$.*

Definition 4.39 The $\mathcal{C}$-saturation is *integral* over $\mathcal{E}$ means that $j\mathcal{D}_i+\mathcal{C}$ is integral over $\mathcal{E}$.

The same applies to other types of saturations for $\mathcal{D}+\mathcal{C}$.

The proposition applies in particular to a flipping algebra $\mathcal{FR}_{X/T}\,D$. Models of algebras, with exceptional asymptotic saturations, as models for bss ample divisors are obtained by a rational 1-contraction of X. The former $(\mathrm{FGA})_n$ algebras are more general and their models can blow up some divisors exceptional on X. This is the main difference between (FGA) (former) and Zariski (latter) approaches to f.g. algebras.

Lemma 4.40 *Let $C_1\ge C_2$ be divisors on X/T, and $\mathcal{C}_1\ge\mathcal{C}_2$ b-divisors on X/T. Then C_1-saturation implies C_2-saturation, $\mathcal{C}_1$-saturation implies $\mathcal{C}_2$-saturation, and asymptotic $\mathcal{C}_1$-saturation implies asymptotic $\mathcal{C}_2$-saturation.*

Addendum 4.40.1 *Asymptotic $\mathcal{C}_1$-saturation implies asymptotic $\mathcal{C}_2'$-saturation with $\mathcal{C}_2'=\mathcal{C}_2+\mathcal{E}$, where $\mathcal{E}$ is a reduced b-divisor such that:*

- *$\mathcal{C}_1>\mathcal{C}_2$ over $\mathcal{E}$; and*
- *the $\mathcal{C}_2$-saturation is integral over $\mathcal{E}$.*

The same holds for the other types of saturation.

Proof We use two facts:

- $C_1 \geq C_2$ implies that $\lceil D + C_1 \rceil \geq \lceil D + C_2 \rceil$; and
- $C_1 \geq C_2$ implies the inclusion $|\lceil D + C_1 \rceil| \supset |\lceil D + C_2 \rceil| + E$ with $E = \lceil D + C_1 \rceil - \lceil D + C_2 \rceil \geq 0$.

This last fact implies that if $|\lceil D + C_2 \rceil| \neq \emptyset$, then $|\lceil D + C_1 \rceil| \neq \emptyset$ and $D \geq \operatorname{Mov} \lceil D + C_1 \rceil \geq \operatorname{Mov} \lceil D + C_2 \rceil$. This gives the first statement.

The b-divisorial case and asymptotic saturation follows from the same estimate. $\mathcal{C}_2'$-saturation follows from two facts:

- we can replace $\mathcal{C}_2$ by $\mathcal{C}_2 + \sum \varepsilon_i E_i$ with $0 < \varepsilon_i \ll 1$ by the lemma; and
- the effect of this is equivalent to replacing $\mathcal{C}_2$ by $\mathcal{C}_2'$.

This last fact follows from equations $\lceil d + \varepsilon_i \rceil = d + 1 = \lceil d + 1 \rceil$ for any integer d. $\square$

Proof–Explanation of Proposition 4.38 By any saturation in the statement, we mean any type of saturation in Definition 4.29: e.g., for a divisor, for a b-divisor, etc.

Since B is effective, the codiscrepancy $\mathcal{B} = \mathcal{B}(X, B)$ is *also* effective up to components exceptional on X. There is thus an exceptional b-divisor $\mathcal{C}$ such that $\mathcal{C} \geq \mathcal{A} = -\mathcal{B}$; in the divisorial case, we take $C = \mathcal{C}_X = 0$. Then the proposition follows immediately from Lemma 4.40, and Addendum 4.38.1 follows from Addendum 4.40.1. $\square$

We apply this presently in the following situation.

Example 4.41 Let $(X/T, B)$ and $(X/T, B^+)$ be two log pairs such that

- (X, B) is Klt;
- (X, B^+) is log canonical;
- $B^+ \geq B$, or equivalently, $A = \mathcal{A}_X \geq A^+_X = \mathcal{A}^+_X$, where $\mathcal{A}^+ = \mathcal{A}(X, B^+)$; and
- $\mathcal{A}^+$-saturation is integral over divisors in $\operatorname{LCS}(X, B^+)$.

Then lca saturation/(X, B), e.g., of Proposition 4.38, implies asymptotic $\mathcal{A}'$-saturation with $\mathcal{A}' = \mathcal{A}^+ + \mathcal{E}$, where $\mathcal{E} = \sum E_i$ is a sum of prime b-divisors with discrepancy -1 for (X, B^+). The integral saturation for $\mathcal{A}^+$ means that each divisor $jD_i = j\mathcal{D}_{i|X}$ is integral over LCS(X, B^+). For example, this holds when each $\mathcal{D}_i$ is in general position with respect to each divisorial component of LCS(X, B^+) (see (GNP) in Proposition 4.44 below).

Indeed, by monotonicity [9, 1.3.3] we have $\mathcal{B}^+ = \mathcal{B}(X, B^+) \geq \mathcal{B}$, or equivalently, $\mathcal{A} \geq \mathcal{A}^+$; and $A > A^+$ everywhere over LCS(X, B^+). Hence lca saturation implies $\mathcal{A}^+$ and $\mathcal{A}'$-saturations by Lemma 4.40 and Addendum 4.40.1 respectively.

In general, the b-divisorial sheaf $\mathcal{O}_X(\mathcal{D})$ may not be a coherent sheaf (see Definition 4.9 and Examples 4.11–4.12). In other words, this means a stabilization, that is, $=$ for inclusions $\mathcal{O}_Y(\mathcal{D}_Y) \supseteq \mathcal{O}_X(\mathcal{D})$ on a sufficiently high model Y/X. For example, it is true for any $\mathbb{R}$-Cartier divisor $\mathcal{D}$ by Proposition 3.16 over any model Y, where $\mathcal{D} = \overline{D}$ for some $\mathbb{R}$-Cartier divisor D on Y.

Proposition 4.42 *Let $\mathcal{D}$ be an $\mathbb{R}$-Cartier b-divisor on X, $\mathcal{E}$ a* finite *reduced b-divisor, and $\mathcal{C}$ an $\mathbb{R}$-b-divisor on X such that:*

(INO) *$\mathcal{D}$ and $\mathcal{C}$ (or just $\mathcal{D} + \mathcal{C}$) are* integral over $\operatorname{Supp} \mathcal{E}$*; and*

(CGR) *$\mathcal{C}$* grows canonically: *$\mathcal{C}_{X_{\mathrm{hr}}} \geq g^*(\mathcal{C}_Y) + \sum a_i E_i$ for any further resolution $g\colon X_{\mathrm{hr}} \to Y$ of some (sufficiently high) resolution Y of X, where $a_i = a(Y, 0, E_i)$ are the standard discrepancies.*

Then the sheaf $\mathcal{O}_X(\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil)$ is coherent.

Proof Take a sufficiently high resolution Y such that:

- $\mathcal{D} = \overline{D}$ for an $\mathbb{R}$-divisor D on Y;
- (CGR) is satisfied for Y;
- $\mathcal{C}_Y$ has normal crossing support on Y (a property of all b-divisors on some resolution);
- moreover, the supports of all divisors under consideration have normal crossings; and

- $\mathcal{E} = \mathcal{E}_Y$ is divisor on Y whose prime components are all pairwise disjoint.

It is enough to verify that, for any other model $g \colon X_{\mathrm{hr}} \to Y$,

$$g^*(\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_Y) \leq \lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_{X_{\mathrm{hr}}},$$

because then

$$\mathcal{O}_Y(\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_Y) \subset g_* \mathcal{O}_{X_{\mathrm{hr}}}(\lceil (\mathcal{D} + \mathcal{C} + \mathcal{E})_{X_{\mathrm{hr}}} \rceil)$$

by Proposition 3.16, which even gives $=$, the stabilization and the coherence of $\mathcal{O}_X(\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil)$.

Set

$$\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_Y = (\mathcal{D} + \mathcal{C} + \mathcal{E})_Y + F,$$

where $F = f_i E_i$ is the *fractional* part, that is, all $f_i \in [0, 1)$. Note that $(Y, \mathcal{E} + F)$ is purely log terminal by (INO), the fact that the $\mathcal{E}$ are disjoint, and the normal crossings.

Thus by (INO), because $\mathcal{D}$ is Cartier, and by (CGR), we get the inequality

$$\begin{aligned} g^*(\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_Y) &= g^*(\mathcal{D}_Y + \mathcal{C}_Y + \mathcal{E}_Y) + g^* F \\ &= g^*(\mathcal{D}_Y) + g^*(\mathcal{C}_Y) + g^*(\mathcal{E} + F) \\ &\leq \mathcal{D}_{X_{\mathrm{hr}}} + \mathcal{C}_{X_{\mathrm{hr}}} + g^*(\mathcal{E} + F) - \sum a_i E_i \\ &= (\mathcal{D} + \mathcal{C} + \mathcal{E})_{X_{\mathrm{hr}}} + g^*(\mathcal{E} + F) - \mathcal{E} - \sum a_i E_i. \end{aligned}$$

Since we need to verify the above inequality for integral parts over the exceptional E_i, it is enough to verify that

$$\mathrm{mult}_{E_i}\left(g^*(\mathcal{E} + F) - \mathcal{E} - \sum a_i E_i\right) = \mathrm{mult}_{E_i}\left(g^*(\mathcal{E} + F) - \sum a_i E_i\right) < 1$$

for each exceptional E_i. (For any integer i and real numbers $f < 1$ and r, the inequality $i \leq r + f$ implies $i \leq \lceil r \rceil$.) But this means that $(Y, \mathcal{E} + F)$ is purely log terminal: $\mathcal{A}(Y, \mathcal{E} + F) = \mathcal{A}(Y, 0) - g^*(\mathcal{E} + F)$ has each exceptional multiplicity > -1. $\square$

Example 4.43 An discrepancy divisor $\mathcal{C} = \mathcal{A} = \mathcal{A}(X,B)$ is the typical example when (CGR) in Proposition 4.42 holds. Indeed,

$$\begin{aligned}\mathcal{A}_{X_{\mathrm{hr}}} &= K_{X_{\mathrm{hr}}} - (\overline{K+B})_{X_{\mathrm{hr}}} = K_{X_{\mathrm{hr}}} - g^*(\overline{K+B_Y}) \\ &= K_{X_{\mathrm{hr}}} - g^*(K_Y - \mathcal{A}_Y) = g^*\mathcal{A}_Y + K_{X_{\mathrm{hr}}} - \overline{K}_Y = g^*\mathcal{A}_Y + \mathcal{A}(Y,0)_{X_{\mathrm{hr}}}.\end{aligned}$$

Thus for $\mathcal{C} = \mathcal{A}$ in Proposition 4.42 we get that $\mathcal{O}_X(\lceil \mathcal{D} + \mathcal{A} + \mathcal{E} \rceil)$ is coherent.

In particular, for $\mathcal{D} = 0$ and $\mathcal{E} = 0$, the fractional ideal $\mathcal{O}_X(\lceil \mathcal{A} \rceil)$ is always coherent. If B is effective, it is the well-known ideal sheaf for LCS (X,B).

Proposition 4.44 *Let $(X/Z,B)$ be a pair with B an $\mathbb{R}$-divisor, Y a prime divisor of X, and $\mathcal{D}_\bullet$ a system of $\mathbb{R}$-b-divisors such that:*

(GLF) *$(X/Z,B)$ is a* general log Fano contraction: *$-(K+B)$ is nef and big/Z;*

(LCC) *Y is a* log canonical center: *$a(X,B,Y) = -1$;*

(ASA$'$) *the system $\mathcal{D}_\bullet$ is asymptotically saturated with respect to $\mathcal{A}' = \mathcal{A}(X,B) + Y$;*

(BSA) *each $\mathcal{D}_i$ is* b-semiample: *$\mathcal{D}_i = \overline{D_i}$ for a semiample $\mathbb{R}$-divisor D_i on a model W/Z of X/Z* (cf. Lemma 3.5); *and*

(GNP) *$\mathcal{D}_\bullet$ is* in general position *with respect to Y: for each $\mathcal{D}_i$, $Y \not\subset \operatorname{Supp} \mathcal{D}_i$.*

Then the restriction $\mathcal{D}_{\bullet}{}_{|Y^\nu}$ is lca saturated over (Y^ν, B_{Y^ν}), where $\nu\colon Y^\nu \to Y$ is the normalization of Y, $\mathcal{D}_{\bullet}{}_{|Y^\nu}$ is the fixed restriction *(see the proof and Fixed Restriction 7.2 below), and $B_{Y^\nu} = (B-Y)_{Y^\nu}$ is the different of $B-Y$* (see [9, §3] and [4, Prop. 16.5]).

Addendum 4.44.1 *$\mathcal{L} \subset k(X)_\bullet$ be a functional algebra such that:*

(ASA$'$) *$\mathcal{L}$ is asymptotically saturated with respect to $\mathcal{A}'$; and*

(GNP) *$\mathcal{L}$ is* in general position *with respect to Y, as is the mobile system $\mathcal{M}_\bullet$.*

Then $\mathcal{L}_{|Y^\nu}$ is lca saturated/(Y^ν, B_{Y^ν}). If, in addition, (X,B) is purely log terminal, then $\mathcal{L}_{|Y^\nu}$ is normal/$f(Y)$ if $\mathcal{L}/Z$ is.

For a log canonical center Y of a higher codimension, we need an adjunction formula.

Proof Take a log resolution W/Z of $(X/Z, B)$ that is sufficiently high over X and over Y^ν, and take any natural number i and j. Let Y_{hr} be the birational transform of Y on W. By construction, Y_{hr} is normal and we have a decomposition $g\colon Y_{\mathrm{hr}} \to Y^\nu \to Y$. Then the restriction

$$|\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil| \xrightarrow{|_{Y_{\mathrm{hr}}}} \left|\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil_{|Y_{\mathrm{hr}}}\right| = \left|\left\lceil j(\mathcal{D}_i)_{W|Y_{\mathrm{hr}}} + (\mathcal{A}^{Y^\nu})_{Y_{\mathrm{hr}}}\right\rceil\right|$$

is surjective, where $_{|X_{\mathrm{hr}}} = g^*$, and $\mathcal{A}^{Y^\nu} = \mathcal{A}(Y^\nu, B_{Y^\nu})$. Indeed, by adjunction, $K_{Y^\nu} + B_{Y^\nu} = (K+B)_{|Y^\nu}$ ([9, 3.1] and [4, Prop. 16.5]). On the other hand, by definition $\mathcal{A} = \mathcal{A}(X, B) = \mathcal{K} - \overline{K+B}$ and $\mathcal{A}^{Y^\nu} = \mathcal{K}^{Y^\nu} - \overline{K_{Y^\nu} + B_{Y^\nu}}$, where $\mathcal{K}$ and $\mathcal{K}^{Y^\nu}$ are canonical b-divisors of X and Y^ν respectively [12, Example 1.1.3], that is, $\mathcal{K}_W = K_W$ and $\mathcal{K}^{Y^\nu}_{Y^\nu} = K_{Y^\nu}$. Hence

$$\begin{aligned}
&\mathcal{A}'_{W|Y_{\mathrm{hr}}} = (\mathcal{A}_W + Y_{\mathrm{hr}})_{|Y_{\mathrm{hr}}} = (K_W + Y_{\mathrm{hr}} - \overline{K+B})_{|Y_{\mathrm{hr}}} = \\
&(K_W + Y_{\mathrm{hr}})_{|Y_{\mathrm{hr}}} - (\overline{(K+B)_{|Y_\nu}})_{Y_{\mathrm{hr}}} = K_{Y_{\mathrm{hr}}} - (\overline{K_{Y_\nu} + B_{Y_\nu}})_{Y_{\mathrm{hr}}} = (\mathcal{A}^{Y_\nu})_{Y_{\mathrm{hr}}}.
\end{aligned}$$

(In terms of fixed restrictions, this amounts to $\mathcal{A}'_{|Y^\nu} = \mathcal{A}^{Y_\nu}$; see 7.2 below.)

By normal crossings of the supports in the following formula, the roundup $\lceil\ \rceil$ commutes with the restriction:

$$\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil_{|Y_{\mathrm{hr}}} = \left\lceil j(\mathcal{D}_i)_{W|Y_{\mathrm{hr}}} + \mathcal{A}'_{W|Y_{\mathrm{hr}}}\right\rceil = \left\lceil j(\mathcal{D}_i)_{W|Y_{\mathrm{hr}}} + (\mathcal{A}^{Y^\nu})_{Y_{\mathrm{hr}}}\right\rceil.$$

For surjectivity, it is enough to establish the vanishing

$$R^1 h_* \mathcal{O}_W(\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil - Y_{\mathrm{hr}}),$$

where $h\colon W \to Z$. It is natural to use the Kawamata–Viehweg vanishing for this:

$$\begin{aligned}
\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil - Y_{\mathrm{hr}} &= \lceil j(\mathcal{D}_i)_W + \mathcal{A}_W + Y_{\mathrm{hr}}\rceil - Y_{\mathrm{hr}} \\
&= \lceil j(\mathcal{D}_i)_W + \mathcal{A}_W\rceil = \lceil j(\mathcal{D}_i)_W + K_W - r^*(K+B)\rceil \\
&= K_W + \lceil j(\mathcal{D}_i)_W - r^*(K+B)\rceil,
\end{aligned}$$

because Y_{hr} and K_W are integral, and $\mathcal{A}_W = K_W - r^*(K+B)$, where $r\colon W \to X$ is the resolution. Now we get the required vanishing, since $(\mathcal{D}_i)_W$ is nef/Z by (BSA), and $-(K+B)$ is nef and big /Z by (GLF). The former holds on W over any model where $\mathcal{D}_i = \overline{D_i}$ for semiample D_i.

Thus by the surjectivity of the above restriction,

$$|\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W \rceil| \neq \emptyset \quad \text{provided that} \quad \left| \left\lceil j(\mathcal{D}_i)_{W|_{Y_{\mathrm{hr}}}} + (\mathcal{A}^{Y^\nu})_{Y_{\mathrm{hr}}} \right\rceil \right| \neq \emptyset.$$

Moreover, in this case, by Hironaka, and because $\mathcal{O}_X(\lceil j(\mathcal{D}_i) + \mathcal{A}' \rceil)$ is coherent by Proposition 4.42 and Example 4.43, we can assume that

$$\mathrm{Bs}\,|\mathrm{Mov}\,\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W \rceil| = \emptyset$$

on a sufficiently high resolution W (cf. the proof of Proposition 4.13, (1)) and both Mov are stabilized. Therefore the surjectivity gives

$$\mathrm{Mov}\left\lceil j(\mathcal{D}_i)_{W|_{Y_{\mathrm{hr}}}} + (\mathcal{A}^{Y^\nu})_{Y_{\mathrm{hr}}} \right\rceil = \Big(\mathrm{Mov}\,\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W \rceil\Big)_{|Y_{\mathrm{hr}}},$$

(in general, only $\leq$ holds), and

$$\Big(\mathrm{Mov}\,\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W \rceil\Big)_{|Y_{\mathrm{hr}}} \leq j(\mathcal{D}_j)_{W|_{Y_{\mathrm{hr}}}} = j\Big(\mathcal{D}_{j\,|Y^\nu}\Big)_{Y_{\mathrm{hr}}}$$

provided that

$$\mathrm{Mov}\,\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W \rceil \leq j(\mathcal{D}_j)_W.$$

The previous equality holds by definition of fixed restrictions of b-divisors (see 7.2 below) if Y_{hr} lies over a model such that $\mathcal{D}_i = \overline{D}_i$ for semiample D_i. This implies the lca saturation for $\mathcal{D}_{\bullet\,|Y^\nu}$ and on any model$/Y_{\mathrm{hr}}$ by (ASA$'$) with the same index I.

Addendum 4.44.1 follows from the proposition applied to the characteristic system $\mathcal{D}_\bullet$ of $\mathcal{L}$. Indeed, the characteristic system of $\mathcal{L}_{|Y^\nu}$ is the restriction $\mathcal{D}_{\bullet\,|Y^\nu}$ (cf. Fixed restriction 7.2), and (BSA) holds by Proposition 4.13, (1). The normality of restricted algebras follows from the surjectivity of mobile parts when $j = i$.

Indeed, (GLF) and the connectedness of LCS $(X,B)/Z$ (Kollár and others [4, Theorem 17.7]) implies that $\mathcal{A}$ has (locally) only one divisor Y with $a(X,B,Y) = -1$ if (X,B) is purely log terminal. Then $\lceil \mathcal{A}' \rceil \geq 0$, but $i\mathcal{D}_i = \mathcal{M}_i$ is integral as a b-free b-divisor in Proposition 4.13, (1). Hence $|\lceil i(\mathcal{D}_i)_W + \mathcal{A}'_W \rceil| = |(\mathcal{M}_i)_W| + \lceil \mathcal{A}'_W \rceil$ by saturation (cf. Remarks 4.30, (1) and (3) above).

On the other hand, by the adjunction of [9, 3.2.3], (Y^ν, B_{Y^ν}) is Klt. Thus $\mathcal{A}^{Y^\nu} \geq 0$ and

$$i(\mathcal{D}_i)_{W|_{Y_{\rm hr}}} = (\mathcal{M}_i)_{W|_{Y_{\rm hr}}} = (\operatorname{Mov} \mathcal{L}_{|Y^\nu})_{Y_{\rm hr}}$$

is integral and mobile. Thus again

$$\left|\left\lceil i(\mathcal{D}_i)_{W|_{Y_{\rm hr}}} + (\mathcal{A}^{Y^\nu})_{Y_{\rm hr}}\right\rceil\right| = \left|(\operatorname{Mov} \mathcal{L}_{|Y^\nu})_{Y_{\rm hr}}\right| + \left|\lceil(\mathcal{A}^{Y^\nu})_{Y_{\rm hr}}\rceil\right|.$$

Thus we get surjectivity of the restriction

$$|(M_i)_W| \xrightarrow{|_{Y_{\rm hr}}} \left|(\operatorname{Mov} \mathcal{L}_{|Y^\nu})_{Y_{\rm hr}}\right|,$$

and by Proposition 4.13, (4) this implies that $\mathcal{L}_{|Y^\nu}$ is normal if $\mathcal{L}$ is. □

Example 4.45 Consider a pl contraction $X/X_\vee$, and assume that

- $X/X_\vee$ satisfies the inductive assumptions of Definition 3.35 with respect to a divisor D on X; and
- let $\mathcal{L}$ be a functional algebra bounded by D and satisfying the other assumptions of Main Lemma 3.39.

We say that such an algebra $\mathcal{L}$ is of type $(\mathrm{FGA})_n^{pl}$ if, in addition,

- $\mathcal{L}$ is lca saturated over $(X, B + S - \varepsilon S)$ for some $\varepsilon > 0$; and
- $\dim X = n$.

As in Conjecture 4.35, we expect such algebras to be f.g. More precisely,

$$(\mathrm{FGA})_d \implies (\mathrm{FGA})_n^{pl},$$

where $d = \dim Y$ is the dimension of the irreducible normal variety $Y = S^s = \bigcap S_i$ (cf. Example 3.36 with $t = 1$ and $s = n - d$). The implication means that the first conjectural f.g. implies the second.

Indeed, by Example 4.41 with $B^+ = B + S - \varepsilon S'$ and $S' = \sum_{i\geq 2} S_i$, lca saturation of $\mathcal{L}$ implies the asymptotic saturation of $\mathcal{L}$ for $\mathcal{A}' = \mathcal{A}^+ + S_1$. Thus by Addendum 4.44.1, this implies lca saturation$/(S_1, B_{S_1})$ with the different $B_{S_1} = (B^+ - S_1)_{S_1} = (B + S' - \varepsilon S')_{S_1}$. Since $K + B + S$ is numerically negative/$X_\vee$, (GLF) holds for $B^+/X_\vee$ if $0 < \varepsilon \ll 1$. Because (X, B) is divisorially log terminal, (X, B^+) is purely log terminal for such ε, and S_1 is the only log canonical center, with $a(X, B^+, S_1) = -1$. The algebra $\mathcal{L}$ is in general position to S_1 by our choice of D (cf. Definition 3.35 and Main Lemma 3.39).

Then, by Example 3.36, we can use induction on d because

$$B_{S_1} = (B + S' - \varepsilon S')_{S_1} = (B + S')_{S_1} - \varepsilon(S'_{|S_1})$$

by semiadditivity [9, 3.2.1]. The pair $(S_1/f(S_1), (B+S')_{S_1})$ satisfies the same assumption as $(X/X_\vee, B)$ by Example 3.36; in particular,

$$(S_1, (B + S')_{S_1} - \varepsilon(S'_{|S_1}))$$

is Klt, and the restriction $\mathcal{L}_{|S_1}$ is of type $(\mathrm{FGA})^{pl}_{n-1}$.

Hence by induction, $\mathcal{L}_{|Y}$ is lca saturated/(Y, B_Y), where B_Y corresponds to the successive adjunction. Moreover (Y, B_Y) is Klt. Thus by $(\mathrm{FGA})_d$, the algebra $\mathcal{L}_{|Y}$ is f.g. Then $\mathcal{L}$ is f.g. by Main Lemma 3.39 (of course, locally near $P \in X_\vee$). In addition, $\mathcal{L}_{|Y}$ is normal if $\mathcal{L}$ is.

This is a generalization of Induction Theorem 1.4 for (FGA) algebras (cf. its proof below).

If, for $d = n$ and for big $\mathcal{L}$, we knew the equivalence

$$(\mathrm{FGA})_n \iff (\mathrm{FGA})^{pl}_n,$$

this would imply the existence of log flips in dimension n by induction on n (cf. the Reduction Theorem and Conjecture 4.35).

In Section 6, as an application of the example, we prove $(\mathrm{FGA})^{pl}_3$ (cf. Corollary 6.43 and Remark 11.8 below).

Now we are ready to prove the main result of this section.

Corollary 4.46 $(\mathrm{FGA})_d \implies (\mathrm{FGA})_{n,d}$.

Proof–Explanation We verify that a restricted algebra $\mathcal{L} = (\mathcal{R}_{X/T}\, D)_{|Y}$ is of type $(\mathrm{RFA})_{n,d}$ and satisfies $(\mathrm{FGA})_d$. Indeed, the bounded algebra $\mathcal{R}_{X/T}\, D$ is exceptionally asymptotically saturated/X by Example 4.31. Thus by Proposition 4.38, $\mathcal{R}_{X/T}\, D$ is lca saturated/$(X, B+S-\varepsilon S)$ for any $\varepsilon \in [0, 1]$. In addition, by Addendum 3.39.1, the algebra $\mathcal{R}_{X/T}\, D$ is of type $(\mathrm{FGA})^{pl}_n$. Thus by Example 4.45, the restriction $\mathcal{L}$ is of type $(\mathrm{FGA})_d$. $\square$

Proof of Induction Theorem 1.4 for (FGA) Immediate by the corollary and by case (RFA) in Section 3 or by Theorem 3.33. Moreover, we only need $(\mathrm{FGA})_d(\mathrm{bir})$ as for (RFA). $\square$

As a first application of Induction Theorem 1.4 and its proof, by Examples 4.37, 4.45 and by Corollary 4.46, we get the f.g. of algebras of type $(\mathrm{FGA})_n^{pl}$ and $(\mathrm{PLF})_n$ with $d = 1$. By the Reduction Theorem this gives all log flips in dimension 2, which is well known since they are divisorial contractions. However, the same methods also allows us to construct log flips for seminormal surfaces and these in fact may not be contractions (cf. modifications of surfaces D of the semistable models for D [10]).

In higher dimensions, pl flips with $d = 1$ are also trivial since they are negative on Y (cf. Example 3.49). Nonetheless $(\mathrm{FGA})_1$ also gives f.g. for $\mathcal{L}$ in $(\mathrm{FGA})_n$ over the generic points $P \in T$ when $\dim f^{-1}P = 1$, and over the generic points of the prime divisors $D \subset T$ when X/T is birational (cf. condition (BED) in Proposition 4.47 and in Theorem 9.9). This follows by induction on n after taking general hyperplane sections.

Proposition 4.47 *Over codimension d points* $(\mathrm{FGA})_n$ *follows from* $(\mathrm{FGA})_d$. *Thus by induction on n, it is enough to prove* $(\mathrm{FGA})_n$ *over closed points in T assuming after a truncation that $\mathcal{L}$ locally has stabilized characteristic system:*

(BED) *for all i, $\mathcal{D} = \mathcal{D}_i$ outside $f^{-1}P$ over T.*

The same holds for (RFA) *in place of* (FGA).

For example, $\mathcal{L} = \mathcal{R}_{X/T}\, D$ has (BED) in codimension 1 when D is integral and X/T is birational. After a truncation this holds even in codimension 2, since normal X is $\mathbb{Q}$ factorial in codimension 2. (Any flipping algebras satisfies this over $X = T$.) By $(\mathrm{FGA})_1$ and the proposition this holds, but only in codimension 1, for any $(\mathrm{FGA})_n$ algebra when X/T is birational. In Corollary 6.42 this is established in codimension 2 on a similar basis.

Proof Take a general hyperplane section H of T and a point $P \in T$ and $Y = f^{-1}H$. Then $\mathcal{L}_{|Y}$ is of type $(\mathrm{FGA})_{n-1}$. The lca saturation$/(Y/H, B_{|Y})$ follows from Addendum 4.44.1 with $X/Z = X/T$ and $B := B + H$, where (ASA') holds by Example 4.41. Then local f.g. of $\mathcal{L}_{|Y}$ near $H \cap P$ implies local f.g. of $\mathcal{L}$ near P by general properties of restrictions and the definition of stabilization. Then we use induction on n.

For (RFA), note that the restriction $_{|Y}$ also preserves (RFA). $\square$

We remark finally that in the proof of $(\mathrm{FGA})_1$ it was very important that every prime divisor, and every effective divisor of rather high degree up to linear equivalence, was nonsingular, that is, each is a reduced, nonsingular subvariety (cf. Example 6.24 below). In Section 5 we explain to what extent we need this property ("predictions"), and in Section 6 we state when we may expect this property to hold ("triples" and Conjecture 6.14 below).

5 Saturation and descent

5.1 Descent of divisors

In this section we consider the following, which is a key problem in our construction of pl flips. Let $f\colon Y \to X$ be a morphism of algebraic varieties and D an $\mathbb{R}$-Cartier divisor on Y. When can we find an $\mathbb{R}$-Cartier divisor D_X on X such that $D = f^*D_X$? If we can do this, we say that D_X is the *descent* of D; this is thus a descent problem. Note that we solve this problem in a very special circumstances when our divisor is a limit or a rather high element in a sequence of divisors on birational models of X. Thus in fact we consider this problem for b-divisors [12]. Therefore we replace D by its completion $\mathcal{D} = \overline{D}$ or any other $\mathbb{R}$-b-divisor, and we can replace Y by any birational model. The problem is to know whether $\mathcal{D} = \overline{\mathcal{D}_X}$ then holds.

Remark 5.2 If we replace equality of divisors by linear equivalence, we get a different problem, which is better considered in terms of sheaves: $\mathcal{O}_Y(\mathcal{D}) = f^*\mathcal{O}_X(\overline{\mathcal{D}_X})$. If Y/X is birational, this is equivalent to the above problem. Otherwise the latter is quite different and more flexible. However we can relax the problem with $=$ up to the problem with $\sim$ by considering $\mathcal{D}$ up to linear equivalence: $\mathcal{D}' \sim \mathcal{D}$ and $\mathcal{D}'$ has a descent (cf. Lemma 3.24). It is even reasonable to consider $\sim_{\mathbb{Q}}$ or $\sim_{\mathbb{Q}}$ (cf. the bss and b-semiampleness above, as well as the triples of Lemma 8.12 below).

The obstruction to the problem is easy to find.

5.3 Descent data

The descent of $\mathcal{D}$ on X exists if only if $\mathcal{E} = \mathcal{E}(\mathcal{D}) = \overline{\mathcal{D}_X} - \mathcal{D}$ is 0 as a b-divisor. The b-divisor $\mathcal{E}$ is the *descent data* of $\mathcal{D}$ *over* X.

Proposition 5.4 (Properties of descent data) *Let $\mathcal{D}$ be an $\mathbb{R}$-b-divisor of X. Then*

(EXI) *the descent data $\mathcal{E}$ of $\mathcal{D}$ over X* exists *if $\mathcal{D}_X$ is $\mathbb{R}$-Cartier, in particular, if X is $\mathbb{Q}$-factorial;*

(EXC) *$\mathcal{E}/X$ is* exceptional *on X;*

(ADD) additivity: *$\mathcal{E}(\mathcal{D}+\mathcal{D}') = \mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{D}')/X$;*

(HOM) homogeneity: *for any real number r, $\mathcal{E}(r\mathcal{D}) = r\mathcal{E}(\mathcal{D})/X$;*

(DEP) *the data $\mathcal{E}$ depends on $\mathcal{D}$ only* up to $\mathbb{R}$-linear equivalence, *and even only up to numerical equivalence/X; and*

(EFF) *$\mathcal{E}$ is* effective *if $\mathcal{D}$ is b-semiample, or even if $\mathcal{D}$ is b-nef/X.*

Thus for $\mathcal{D}$ as in (EFF), if we know that $\mathcal{E} \leq 0$ then $\mathcal{E} = 0$ and the descent exists.

Proof (EXI), (EXC), (ADD) and (HOM) follow directly from the definitions; (EXC) means that $\mathcal{E}_X = 0$.

The assertion (DEP) for $\sim_{\mathbb{R}}$ follows because the descent data for a principal $\mathbb{R}$-b-divisor is trivial. In turn by (ADD) and (HOM), this follows from the same fact for principal b-divisors:

$$\mathcal{E}(\overline{(f)}) = \overline{\overline{(f)}_X} - \overline{(f)} = \overline{(f)} - \overline{(f)} = 0.$$

For numerical equivalence, by (ADD) we can apply (EFF) to the difference.

For (EFF), it is enough to check that $E_Y = f^*\mathcal{D}_X - \mathcal{D}_Y$ is effective for any resolution $f\colon Y \to X$. For b-semiample $\mathcal{D}$, this follows from (DEP) and the effectiveness of f^*D for any effective D. For a b-divisor $\mathcal{D}$ that is b-free/X, we get this by Lemma 4.21 or by the negativity [13, 2.15]. $\square$

For a general divisor D, as well as for a general b-divisor $\mathcal{D}$, descent may not exist even on a sufficiently high model. However, we are interested in two related cases: $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$, where each $\mathcal{D}_i$ is the completion of some $\mathbb{R}$-Cartier divisor D_i on a model Y_i/X, that is, $\mathcal{D}_i = \overline{D}_i$ (cf. Example 4.12). Thus each $\mathcal{D}_i$ has a descent D_i on Y_i. We expect also that under certain conditions the b-divisor $\mathcal{D}$ may also have a descent; or even that all $\mathcal{D}_i$ together with $\mathcal{D}$ have descents on one model.

5.5 Asymptotic descent problem

Suppose that $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ is a limit of b-divisors on X/Z. Find a model Y/Z of X/Z such that

- $\mathcal{D} = \overline{(\mathcal{D})_Y}$;
- the limit *stabilizes*, that is, $\mathcal{D}_i = \mathcal{D}$ for some $i \gg 0$; in particular,
- for every such i, also $\mathcal{D}_i = \overline{(\mathcal{D}_i)_Y} = \mathcal{D}$.

Thus in this case, infinitely many $\mathcal{D}_i$ have descents on a single model Y (cf. Remark 5.6, (4) below). Our approach gives a *strict* asymptotic descent: e.g., stabilization holds for some $i \gg 0$ after any truncation of $\mathcal{D}_\bullet$. But, in fact, in applications we get a *complete* asymptotic descent: e.g., stabilization holds for all i after a truncation of $\mathcal{D}_\bullet$ (cf. Theorem 5.12 vs Corollary 5.13).

A sheaf version of this conditions is related to f.g. and is given in our standard Example 5.21 below.

The *minimal* assumptions on the system $\mathcal{D}_\bullet$ are the following:

(FDS) *finite divisorial support*: the divisors $(\mathcal{D}_i)_X$ are supported in one reduced divisor F; and

(MXD) of Addendum 4.20.1: each $\mathcal{D}_i \leq \mathcal{D}$.

However these assumptions with two standard ones (saturation and boundedness) are only enough for the *rationality* of $\mathcal{D}$ (cf. Examples 4.37, 5.14, and Theorem 5.12).

Descent needs more. The *additional* assumption on the system $\mathcal{D}_\bullet$ is:

(BNF) each $\mathcal{D}_i$ is *b-nef* in the sense of Lemma 4.21; in particular, this includes

(CAR) each $\mathcal{D}_i$ is $\mathbb{R}$-*Cartier*: $\mathcal{D}_i = \overline{D_i}$ for the $\mathbb{R}$-Cartier divisor $D_i = (\mathcal{D}_i)_{X_i}$ over some model X_i/Z of X/Z (cf. Example 4.12).

In the above assumptions *each* means *up to a truncation.*

Remark 5.6 (1) As in Remark 4.30, (7), similarity of characteristic type preserves the minimal and additional assumptions.

(2) In our applications b-nef follows from b-semiample; and usually each $\mathcal{D}_i$ is $\mathbb{Q}$-Cartier (cf. Example 5.21 below).

(3) Since every modification blows up at most a finite number of divisors, the assertion (FDS) is birationally invariant: if it holds on one model of X then it also holds on any other. It is equivalent to (BSD) in Addendum 4.20.1. But it is weaker than

(FSP) *finite support:* there exists a proper subvariety $S \subset X$ such that all $\mathcal{D}_i$ are supported over S.

However under (BNF), which holds in most of our applications, both conditions are equivalent by Lemma 10.8 below when each $\mathcal{D}_i \geq 0$.

(4) A limit of $\mathbb{R}$-Cartier b-divisors is not necessarily $\mathbb{R}$-Cartier in general. For example, this used to be[5.1] for the limit of the characteristic system of an infinitely generated functional algebra. However under (FDS), if (CAR) holds on a single model $Y = X_i$ for infinitely many i, the limit $\mathcal{D}$ is also $\mathbb{R}$-Cartier/Y. By definition it is enough to establish this (locally) for principal divisors. These are $\mathbb{R}$-Cartier/Y because the $\mathbb{R}$-principal divisors, with support in a fixed reduced divisor, form an $\mathbb{R}$-subspace (defined/$\mathbb{Q}$) in the space of all $\mathbb{R}$-divisors within the given support.

We can resolve the asymptotic descent problem for $\mathcal{D}_\bullet$ under asymptotic saturation and the following boundedness.

Definition 5.7 (Boundedness for descent data) Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathbb{R}$-b-divisors of X. We say that the descent data $\mathcal{E}$ of $\mathcal{D}$ *over* X is *bounded* by $\mathcal{C}$ if

- the data $\mathcal{E} = \overline{\mathcal{D}_X} - \mathcal{D}$ is *defined*, in particular, $\mathcal{D}_X$ is $\mathbb{R}$-Cartier, and
- $\mathcal{E} \leq \mathcal{C}$ *over* Y, that is, $\operatorname{mult}_{E_i} \mathcal{E} \leq \operatorname{mult}_{E_i} \mathcal{C}$ in each prime b-divisor E_i that is *exceptional* on X.

Now let $\mathcal{D}_\bullet$ be a system of $\mathbb{R}$-b-divisors of X: e.g., the sequence of a limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$. The system $\mathcal{D}_\bullet$ is *asymptotically bounded* by $\mathcal{C}$ if there exists a sequence of positive real numbers r_i for some $i \gg 0$ such that:

- $\lim_{i\to\infty} r_i = +\infty$, and
- for each r_i, the descent data $\mathcal{E}_i = \overline{(\mathcal{D}_i)_X} - \mathcal{D}_i$ over X is bounded by $\mathcal{C}/r_i$, or equivalently, $r_i\mathcal{E}_i \leq \mathcal{C}$ over Y.

[5.1] probably "This is usually the case for ... "

We say that an asymptotic bound is *strict* if we can choose a subsequence of real numbers r_i with the required property for any truncation of $\mathcal{D}_\bullet$.

Remark 5.8 (1) Under (FDS), $\lim_{i\to\infty}\mathcal{E}_i = \mathcal{E} = \overline{\mathcal{D}_X} - \mathcal{D}$ is the descent data for $\mathcal{D}$ on X. Indeed, $\mathcal{D}_X$ is also $\mathbb{R}$-Cartier by Remark 5.6, (4), and passing to the limit commutes with restriction $(\ )_X$ and Cartier closure $\overline{(\ )}$.

(2) If, in addition, the sequence of the limit $\mathcal{D} = \lim_{i\to\infty}\mathcal{D}_i$ satisfies (BNF), then by (EFF) in Proposition 5.4 each $\mathcal{E}_i \geq 0$, and so $\mathcal{E} \geq 0$. On the other hand, for chosen i, $\mathcal{E}_i \leq \mathcal{C}/r_i$ over X, and $\mathcal{E} = \lim_{i\to\infty}\mathcal{E}_i \leq 0$. (This means that the limits $\mathcal{E} = \lim_{i\to\infty}\mathcal{E}_i$ and $\mathcal{D} = \lim_{i\to\infty}\mathcal{D}_i$ are uniform with respect to $\mathcal{C}$.) Hence $\mathcal{E} = 0$ and $\mathcal{D} = \overline{\mathcal{D}_X}$ has the descent $\mathcal{D}_X$ on X. Thus the limit $\mathcal{D} = \lim_{i\to\infty}\mathcal{D}_i$ has asymptotic descent if and only if it stabilizes; and the latter descent is also on X.

(3) Asymptotic boundedness is also necessary for the existence of asymptotic descent on X, namely, for any $\mathcal{C} \geq 0$, and any sequence of r_i with i such that $\mathcal{E}_i = 0$.

(4) Similarity of the characteristic type as in Remarks 4.30, (7) preserves strict asymptotic bounds. For the truncation $I\mathcal{D}_{iI}$, we can take the same $\mathcal{C}$ and new real numbers $r_{i/I} := r_i/I$, where we can take the same (truncated) real numbers $r_{i/I} := r_i$ for the usual truncation $\mathcal{D}_{iI}$.

Remark 5.8, (2) is very close to the stabilization that we require for f.g. The only difference is in a *universal* descent: i.e.,

- infinitely many $\mathcal{D}_i$ have descent on the single model X.

To establish this we need asymptotic $\mathcal{C}$-saturation (cf. Theorem 5.12). In addition, asymptotic $\mathcal{C}$-saturation gives

- *rationality:* $\mathcal{D}$ is a $\mathbb{Q}$-divisor.

Both of the properties are essential for f.g. of functional algebras (cf. Limiting Criterion 4.26 above and Example 5.21 below).

For b-divisors, the choice of an appropriate model for descent is crucial (cf. Example 5.14, triples and Conjecture 6.14). A criterion for an appropriate model is asymptotic boundedness over it (cf. Remarks 5.8, (2) and (3)).

5.9 Choice of Y in the asymptotic descent

In our approach to the descent problem (cf. Theorems 5.12 and 8.23 below), it is easier to present a b-divisor by a semiample $\mathbb{R}$-Cartier divisor than by any other. Because by definition, semiample gives some model and an $\mathbb{R}$-Cartier divisor on it! Thus we need a model Y/Z of X/Z such that:

(SAM) $\mathcal{D}_Y$ is *semiample*$/Z$, in particular, nef$/Z$.

Semiampleness$/Z$ means that of $\mathbb{R}$-divisors [12, Definition 2.5]. If $D = \mathcal{D}_Y$ is a $\mathbb{Q}$-divisor, it is equivalent to *stably free*: $\operatorname{Bs}|ND| = \emptyset$ for some natural number N.

We also need an $\mathbb{R}$-divisor $\mathcal{C}$ on X, a reduced $\mathbb{Q}$-Cartier divisor F on Y, and a positive real number γ such that:

(EEF) *exceptional effectiveness:* $\mathcal{C} \geq 0/Y$, that is, $\operatorname{mult}_{E_i} \mathcal{C} \geq 0$ in each exceptional$/Y$ prime b-divisor E_i; and

(LGD) *linear growth for divisor:* in each prime b-divisor P_i

$$\operatorname{mult}_{P_i} \mathcal{C} > -1 + \gamma \operatorname{mult}_{P_i} F,$$

or, equivalently,

(AEF) $\mathcal{C} - \gamma\overline{F}$ is *almost effective*: $\lceil \mathcal{C} - \gamma\overline{F} \rceil \geq 0$.

The latter implies that $\mathcal{C}$ is also almost effective: $\lceil \mathcal{C} \rceil \geq 0$, whereas (EEF) states more: exceptional effectiveness$/Y$.

Definition 5.10 We say that the asymptotic descent problem has a *prediction* $(Y/Z, \mathcal{C}, F, \gamma)$ if $Y, \mathcal{C}, F$ and γ exist, and if

(SAB) the descent data for $\mathcal{D}_\bullet$ is *strictly asymptotically bounded*$/Y$ by $\mathcal{C}$; and

(UAD) F includes the supports of the irrational part of $\mathcal{D}_Y$, and the *divisorial* locus where the *strict asymptotic descent* is *unknown (not given)* (cf. Remark 5.11, (3) below); that is, for any prime divisor P_i in Y, $\operatorname{mult}_{P_i} \mathcal{D}$ is irrational, or after any truncation $\operatorname{mult}_{P_i} \mathcal{D}$ is an accumulation point of $\operatorname{mult}_{P_i} \mathcal{D}_i$.

Under (FDS), by Definition 5.7 and Remark 5.6, (4), (SAB) implies (SAM).

Remark 5.11 (1) In applications under (BNF), boundedness implies that $\beta\mathcal{C} \geq \mathcal{E}_i \geq 0/Y$ for some natural number i and positive real β, (cf. Proposition 5.4, (EFF) and Example 5.21 below). Thus $\mathcal{C} \geq 0/Y$ as in (EEF).

(2) If $\mathcal{C} \geq \varepsilon\overline{F}/Y$ for some $\varepsilon > 0$, and $\lceil \mathcal{C}_Y \rceil \geq 0$ then (LGD) holds for some $0 < \gamma \ll \varepsilon$.

(3) For simplicity, we can assume in (LGD) that for *any* F, there exists a positive γ for which it holds. In fact, we need a bounded family of F (cf. triples), or just a single F in most applications. The minimal assumption on F is (UAD), but it is enough that F contains the support of the fractional part for every divisor $(\mathcal{D}_i)_Y$ (cf. (FDS) in the minimal assumptions above). We need this assumption even though a posteriori $F = 0$, as in Theorem 5.12 below.

(4) As in Remarks 4.30, (7), similarity of characteristic types preserves boundedness of descent data for predictions with new $\mathcal{C} := \mathcal{C} - \overline{(a)}$. For (UAD) we can use the invariance of irrational part of $\mathcal{D}$ under similarity (cf. Lemma 6.30).

Now we are ready to state the main result of the section.

Theorem 5.12 *Let $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ be an asymptotic descent problem such that*

- *the problem has a prediction $(Y/Z, \mathcal{C}, F, \gamma)$;*
- *$\mathcal{D}_\bullet$ satisfies the minimal assumptions of 5.5:* (FDS) on Y (maybe, for some reduced divisor $\neq F$) *and* (MXD)*; and*
- *$\mathcal{D}_\bullet$ is asymptotically $\mathcal{C}$-saturated.*

Then $\mathcal{D}_Y$ is a $\mathbb{Q}$-divisor, and strict stabilization *holds on Y: after any truncation, $(\mathcal{D}_i)_Y = \mathcal{D}_Y$ for some $i \gg 0$.*

Addendum 5.12.1 *If* (BNF) *also holds, then the strict asymptotic descent problem has a solution on Y.*

Addendum 5.12.2 *Moreover, if* (CGR) *of Proposition 4.42 also holds, then* $\mathcal{D}_Y$ *(or* $\mathcal{D}$*) contracts all prime divisors* P_i *on* Y *with* $\operatorname{mult}_{P_i}\mathcal{C} > 0$. *Thus on the* stable *model*

$$Xst/Z = \operatorname{Proj}_Z \mathcal{R}_Z \overline{\mathcal{D}_Y} = \operatorname{Proj}_Z \mathcal{R}_Z \mathcal{D},$$

$\mathcal{C}_{Xst} \leq 0$ *(cf. Conjecture 6.14).*

The main part of the theorem is a rationality theorem, while 5.12.1–2 treat the descent and the structure of the stable model.

Corollary 5.13 *Under arithmetic monotonicity (assertion* (AMN) *of Conjecture 4.35),* complete stabilization *on* Y *holds: i.e., for all* i *after a truncation of* $\mathcal{D}_\bullet$. *More precisely, there exist a stabilization index* I *such that* $(\mathcal{D}_i)_Y = \mathcal{D}_Y$ *for all* $I \mid i \gg 0$.

Addendum 5.13.1 *Again under arithmetic monotonicity* (AMN), complete stabilization *holds for* $\mathcal{D}_\bullet$. *More precisely, there exist a stabilization index* I *such that* $\mathcal{D}_i = \mathcal{D} = \overline{\mathcal{D}_Y}$ *for all* $I \mid i \gg 0$.

Proof Immediate by the theorem, (MXD) and (AMN). □

Example 5.14 Let $Y \to X$ be a divisorial contraction with an exceptional Cartier divisor E. For example, it might be a blowup of a smooth locus in X. Take a system $\mathcal{D}_i = d_i\overline{E}$, where d_i is a sequence of real numbers with $\lim_{i\to\infty} d_i = d$. If all $d_i \leq d$ and $d > 0$ then $\mathcal{D}_\bullet$ satisfies all the conditions of Theorem 5.12 on X, except for (BNF). The descent problem has a solution on Y but it does not stabilize and may not be rational. Nonetheless, for a prediction X, $\mathcal{D}_X = 0$ is always rational as in the main part of the theorem.

Note that the same holds even for $d = +\infty$.

Lemma 5.15 (Approximation) *Let* $\mathfrak{Q} \subset \mathbb{R}^N = \mathbb{Z}^N \otimes_{\mathbb{R}} \mathbb{R}$ *be a rational polyhedron,*[5.2] *and* $D = (d_1, \ldots, d_N) \in \mathfrak{Q}$ *a point in it. Then there exist positive real numbers* r *and* ε, *an infinite set of natural numbers* m *and a sequence of rational points* $Q_m \in \mathfrak{Q}$ *such that:*

- *each* mQ_m *is* integral: $mQ_m \in \mathbb{Z}^N$,

[5.2] Convex is probably is intended.

- $\|D - Q_m\| < \varepsilon/m^{1+r}$.

Moreover, if D is irrational then there exists $D' \in \mathfrak{Q}$, and a sequence $Q_m = (q_{m,1}, \ldots, q_{m,N})$ as above for D' such that:

- *$D' \leq D$; but*
- *$q_{m,i} > d_i$ at least in one coordinate (the ith); and*
- *$q_{m,i} = d_i$ for the rational coordinates d_i.*

Notation 5.16 We write $\|(x_1, \ldots, x_N)\| = \max\{|x_i|\}$ for the *maximal absolute value* norm in $\mathbb{R}^N$; it induces the usual real topology of $\mathbb{R}^N$. The inequality

$$D' = (d'_1, \ldots, d'_N) \leq D = (d_1, \ldots, d_N)$$

means that each $d'_i \leq d_i$.

Proof First of all, we can assume that D is irrational; that is, it has an irrational coordinate d_i. Otherwise we take $Q_m = D$ for m such that mQ_m are integral, and take any r and $\varepsilon > 0$. In particular, $N \geq 1$ if D is irrational.

Secondly, we can assume that each coordinate d_i of D is irrational, and, moreover, that D does not belong to any proper rational affine subspace H. (If d_i is rational then $D \in H = \{x_i = d_i\}$.) Indeed, if $D \in H$ then there is a rational affine embedding $A\colon \mathbb{R}^m \hookrightarrow \mathbb{R}^N$ with

- $A(\mathbb{R}^m) = H$, and
- $M \leq N - 1$.

Note that $\mathfrak{P} = A^{-1}\mathfrak{Q} \subset \mathbb{R}^m$ is also a rational polyhedron with $C = A^{-1}D \in \mathfrak{P}$. Thus if $P_m \in \mathfrak{P}$ is a sequence with the required property for C, then $Q_{Im} = A(P_m)$ gives the required sequence for $D = A(C) \in \mathfrak{Q}$, where I is an *index* for A, that is, IA is integral. The set of natural numbers m is replaced by Im (truncation). Hence $ImQ_m = (IA)(mP_m) \in \mathbb{Z}^N$; and

$$\|D - Q_{Im}\| = \|A(C - P_m)\| < \|A\|\varepsilon/m^{1+r} = \|A\|I^{1+r}\varepsilon/(Im)^{1+r};$$

and we can take $\varepsilon := \|A\|I^{1+r}\varepsilon$ for Q_{Im}, where $\|A\| = \max\|Ax\|/\|x\|$, for $x \neq 0$, is the usual norm of the linear part of A. However we need to replace the conditions on D' by geometric ones: namely, for a polyhedral cone $\mathfrak{D}$ with the single vertex D

- $D' \in \mathfrak{D}$; but
- each $Q_m \notin \mathfrak{D}$; and
- D' is still in each rational face of $\mathfrak{D}$.

For example, the original cone $\mathfrak{D}$ is given by the inequalities $x_i \leq d_i$. But the cone $\mathfrak{C} = A^{-1}\mathfrak{D}$ with the only vertex C is more general. Nonetheless, if $C' \in \mathfrak{C}$ has a required sequence P_m, then Q_{Im} satisfies the lemma with $D' = A(C')$. Note that each rational face of $\mathfrak{D}$ gives a (nonempty) rational face of $\mathfrak{C}$ (because C belongs to it).

Thirdly, we can assume that $\dim \mathfrak{Q} = N$ near D, and D is in interior of $\mathfrak{Q}$. Thus we can assume that $\mathfrak{Q} = \mathbb{R}^N$ locally near D. Otherwise there is a rational face $H \subset \mathbb{R}^N$ of $\mathfrak{Q}$ such that $D \in H$. Then we can reduce the lemma to H as in Step Two. Therefore by induction on N we get the interior.

The required approximations then exist by Cassels [2, Chap. I, Theorem VII] with any $0 < r < 1/N$ and $\varepsilon \geq N/(N+1)$, where $N \geq 1$ is the dimension of the approximation space $\mathbb{R}^N$.

Fourth, by the arguments of Step Three, we can assume that the cone $\mathfrak{D}$ has no rational faces.

Finally, for the last statement of the lemma we need a more accurate choice of m and Q_m, namely, $Q_m \notin \mathfrak{D}$. Otherwise there is an infinite subsequence $Q_m \in \mathfrak{D}$. Their affine span H is rational and contains D since $D = \lim_{m\to\infty} Q_m$. Thus by Step Two, $H = \mathbb{R}^N$, and the cone $\mathfrak{D}$ is *big*: e.g., $\dim \mathfrak{D} = N$. Since $N \geq 1$, there is a rational line L in $\mathbb{R}^N$ that the polyhedron $\mathfrak{D} \cap L = [D', \infty)$ is a ray, and its vertex D' is close to D and irrational on L. By Step Two we can reduce the last statement of the lemma to L with D'. In the latter case we use continued fractions for the best approximations in L from outside $[D, \infty)$ [2, Chap. I, Theorem II]. $\square$

We apply the lemma to a polyhedron in an $\mathbb{R}$-vector space

$$\mathfrak{D}_F = \bigoplus_{i=1}^{N} \mathbb{R}P_i,$$

of $\mathbb{R}$-divisors on X supported in F, where $F = \sum_{i=1}^{N} P_i$ is a reduced divisor on X. Multiplicities of divisors define a canonical isomorphism/$\mathbb{Q}$ of the space with $\mathbb{R}^N$. It induces the well-known topology in $\mathfrak{D}_F$ and norm, the *maximal absolute value* of the multiplicities in prime divisors (or in prime b-divisors in the case of $\mathbb{R}$-b-divisors). In particular, the limits of *divisors* (but not b-divisors) are limits in the norm (cf. Caution 4.23). The isomorphism transforms the ordering of divisors into $\le$ in $\mathbb{R}^N$.

In the space we take a polyhedron $\mathfrak{D}$ of semiample/Z divisors under the following extra assumption.

Definition 5.17 For a given set of divisors D, e.g., for a cone of divisors in Example 5.18 below,

(BND) the semiampleness is *bounded (effective)* if there exists a natural number M such that $N \le MI$ for the stable base freedom of D, where I is the index D

(cf. triples and Lemma 8.12, (d)). Of course, this concerns *only* $\mathbb{Q}$-divisors in the set.

Example 5.18 Any rational polyhedral cone $\mathfrak{D}$ generated by a finite set of semiample/Z divisors D_i satisfies (BND) of Definition 5.17. Indeed, we can assume that

- each D_i is a $\mathbb{Q}$-divisor;
- F is the common support of all divisors D_i; and
- the cone is simplicial.

Thus each $D \in \mathfrak{D}$ has a unique presentation $D = \sum r_i D_i$ with *coordinates* $r_i \in \mathbb{R}$, and each $r_i \ge 0$ and D is rational if and only if each r_i is rational. There is a natural number J such that each JID has integer coordinates r_i, where I is the index of D. Then we can take $M := JM$ in (BND) where M is the minimal natural number for which every $|MD_i|$ is free.

Corollary 5.19 (Approximation) *Let D be a semiample/Z $\mathbb{R}$-divisor on X. Then there exist positive real numbers r and ε, an infinite set of natural numbers m and a sequence of $\mathbb{Q}$-divisors Q_m on X such that:*

- *each $|mQ_m|$ is free, and* in particular, each Q_m is semiample;
- *each* $\operatorname{Supp}(D - Q_m) \subset \operatorname{Supp} D_{\mathrm{irr}}$*, where D_{irr} is the irrational part of D; that is, the approximation concerns only the irrational multiplicities of D;*
- $\|D - Q_m\| < \varepsilon/m^{1+r}$.

Moreover, if D is not a $\mathbb{Q}$-divisor, then there exists an $\mathbb{R}$-divisor D', and a sequence of $\mathbb{Q}$-divisors Q_m on X as above for D' such that:

- $\operatorname{Supp}(D - D') \subset \operatorname{Supp} D_{\mathrm{irr}}$*;*
- $D' \le D$*; but*
- $\operatorname{mult}_{P_i} Q_m > \operatorname{mult}_{P_i} D$ *at least in one prime divisor P_i on X; and*
- $\operatorname{mult}_{P_i} Q_m = \operatorname{mult}_{P_i} D$ *for the rational multiplicities* $\operatorname{mult}_{P_i} D$*.*

Proof To apply Lemma 5.15 we include D into a rational polyhedron $\mathfrak{Q}$.

Since D is semiample/Z,

(NBH) each divisor in a *neighborhood* of D is (nef and) semiample/Z;

moreover, (BND) holds in it. Indeed, $D \sim_{\mathbb{R}} g^*H$, where

- $g\colon X \to T/Z$ is a contraction; and
- H is a numerically ample/Z $\mathbb{R}$-divisor on T.

In other words, $D = g^*H + \sum r_i(f_i)$, where $r_i \in \mathbb{R}$ and $0 \ne f_i \in k(X)$. Let F be a reduced divisor on X that contains the supports of divisors (f_i) and g^*H. Take any rational polyhedral cone $\mathfrak{H}$ of numerically ample/Z $\mathbb{R}$-divisors on T with $H \in \mathfrak{H}$, and with $\operatorname{Supp} g^*\mathfrak{H} \subset F$. Then the rational polyhedral cone

$$\mathfrak{D} = g^*\mathfrak{H} \oplus \left(\bigoplus \mathbb{R}(f_i)\right) \subset \mathfrak{D}_F$$

is generated by semiample divisors, because each numerically ample/Z divisor is semiample/Z by the Kleiman criterion, and each principal divisor is also semiample/Z. By Example 5.18 it satisfies (BND) with some natural number M. (In addition, we can suppose that $\mathfrak{H}$ and $\mathfrak{D}$ are respectively neighborhoods of H and D respectively in their linear spans/$\mathbb{Q}$.) By construction $D \in \mathfrak{D}$.

To preserve the rational multiplicities (coordinates) under the approximation, set $\mathfrak{Q} = \mathfrak{D} \cap Q$, where Q is the subspace of $\mathfrak{D}_F$ given by the system of affine equations/$\mathbb{Q}$: $\operatorname{mult}_{P_i} X = \operatorname{mult}_{P_i} D$, provided that the latter multiplicity is rational. By construction

- $\mathfrak{Q}$ is a rational polyhedron; and

- $D \in \mathfrak{Q}$.

Then Lemma 5.15 gives real numbers r and ε, and a sequence of $\mathbb{Q}$-divisors $Q_m \in \mathfrak{Q}$. They satisfy the corollary except for $|mQ_m|$ free. However mQ_m is *integral* divisor, and $|MmQ_m|$ has no base points by Example 5.18. Thus as in Step 2 in the proof of Lemma 5.15, we can replace Q_m by $Q_{Mm} := Q_m$ and take $\varepsilon := M^{1+r}\varepsilon$. $\square$

However, to apply the corollary to b-divisors we need the following result.

Lemma 5.20 *Let $\mathcal{C}$ and $\mathcal{D}$ be two b-divisors of X, F a reduced divisor, and $\alpha \leq 1, \tau$ positive real numbers such that:*

- *F is $\mathbb{Q}$-Cartier and*
$$\operatorname{mult}_{P_i} \alpha\mathcal{C} > -\alpha + \tau \operatorname{mult}_{P_i} F; \tag{$*$}$$
in each prime b-divisor P_i,

- *the descent data $\mathcal{E}$ of $\mathcal{D}$ over X is bounded by $(1-\alpha)\mathcal{C}$,* that is,
$$\mathcal{E} \leq (1-\alpha)\mathcal{C}/X; \tag{$**$}$$
and

- *$\mathcal{D}_X = D_{\mathrm{bf}} + D_{\mathrm{fr}}$, where $|D_{\mathrm{bf}}|$ is free,* $\operatorname{Supp} D_{\mathrm{fr}} \subset F$, *and $\|D_{\mathrm{fr}}\| < \tau$.*

Then the following estimate for $\operatorname{Mov} \lceil \mathcal{D} + \mathcal{C} \rceil / Z$ *holds: on any model W/X*
$$\operatorname{Mov} \lceil \mathcal{D} + \mathcal{C} \rceil_W \geq (\overline{D}_{\mathrm{bf}})_W;$$
in particular, the nonvanishing $|\lceil \mathcal{D} + \mathcal{C} \rceil_W| \neq \emptyset / Z$ holds.

Note that $\mathcal{D}_X$ is $\mathbb{R}$-Cartier since its descent data is bounded (cf. Definition 5.7).

Proof By definition of descent data, $\mathcal{D} = \overline{\mathcal{D}_X} - \mathcal{E}$. Then by the decomposition of $\mathcal{D}_X$,
$$\mathcal{D} = \overline{D_{\mathrm{bf}} + D_{\mathrm{fr}}} - \mathcal{E} = \overline{D_{\mathrm{bf}}} + \overline{D_{\mathrm{fr}}} - \mathcal{E}.$$
Hence $\mathcal{D} + \mathcal{C} = \overline{D_{\mathrm{bf}}} + \overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E}$. Since $\overline{D_{\mathrm{bf}}}$ is integral, it is enough to verify that $\lceil \overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E} \rceil \geq 0$ (cf. the proof of Proposition 4.42), or $\overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E}$ is almost effective (cf. (AEF) in 5.9, Choice of Y in the asymptotic descent).

We make it into two steps. First, on X. Since $0 < \alpha \le 1$, $(*)$ implies

$$\lceil \mathcal{C}_X - \tau F \rceil \ge 0. \qquad (***)$$

Indeed, $(*)$ times α^{-1} gives $(***)$ with τ/α instead of τ. Since $F \ge 0$ and $1/\alpha \ge 1$ we can replace τ/α by τ.

Since $\operatorname{Supp} D_{\mathrm{fr}} \subset F$, F is reduced, and $\|D_{\mathrm{fr}}\| < \tau$, then $F = \sum P_i$ and $D_{\mathrm{fr}} = \sum d_{\mathrm{fr},i} P_i$ with each $|d_{\mathrm{fr},i}| < \tau$. Thus each $d_{\mathrm{fr},i} > -\tau = -\tau \operatorname{mult}_{D_i} F$ for P_i in F. In other words, $D_{\mathrm{fr}} \ge -\tau F$. Therefore $(\overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E})_X = D_{\mathrm{fr}} + \mathcal{C}_X \ge \mathcal{C}_X - \tau F$, and by $(***)$, $\lceil \overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E} \rceil_X \ge 0$ on X.

Second, $/X$. Note that $D_{\mathrm{fr}} \ge -\tau F$ implies $\overline{D_{\mathrm{fr}}} \ge -\tau \overline{F}$. Therefore by $(**)$,

$$\overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E} \ge \mathcal{C} - \tau F - (1-\alpha)\mathcal{C} = \alpha C - \tau F / X.$$

Hence by $(*)$ $\lceil \overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E} \rceil \ge 0$ over X because $-\alpha \ge -1$.

Finally, the nonvanishing follows since $|D_{\mathrm{bf}}|$ is free. $\square$

Proof of Theorem 5.12 After a truncation of $\mathcal{D}_\bullet$ we can assume that the saturation index is $I = 1$ and all other assumptions hold for each i literally. The conditions of the theorem are preserved by Remarks 5.8, (4), 5.11, (4), 5.6, (1), and 4.30, (7). To prove strict asymptotic descent, we can do it after any truncation.

First, we verify that $D := \mathcal{D}_Y$ is a $\mathbb{Q}$-divisor. Thus suppose that D is not a $\mathbb{Q}$-divisor. Then we find i and j such that: on any model $W/Y/Z$ of X/Z

$$\operatorname{Mov} \lceil j\mathcal{D}_i + \mathcal{C} \rceil_W \not\le j\mathcal{D}_W,$$

and the mobile part is finite, that is, a *nonvanishing*$/Z$: $|\lceil j\mathcal{D}_i + \mathcal{C} \rceil_W| \ne \emptyset$, holds. This contradicts asymptotic $\mathcal{C}$-saturation. Indeed, by (MXD) this implies the mobile part is $\not\le j(\mathcal{D}_j)_W$ for such i and j on any sufficiently high model $W = X_{\mathrm{hr}}$.

A estimate for the mobile part can be done by Lemma 5.20 for $\mathcal{D} := j\mathcal{D}_i$ and F in the prediction with some j and $i \gg 0$ on any W/Y.

By Corollary 5.19 and (SAM), for $D := \mathcal{D}_Y$, there exist positive real numbers r and ε, an $\mathbb{R}$-divisor D' on Y, an infinite set of natural numbers m and a sequence of $\mathbb{Q}$-divisors Q_m on Y such that:

- each $|mQ_m|$ is free;

- each $\mathrm{Supp}\,(mD' - mQ_m) \subset \mathrm{Supp}\,(mD_{\mathrm{irr}}) \subset F$ by (UAD) for the prediction;
- $\|mD' - mQ_m\| < \varepsilon/m^r$.
- $D' \le \mathcal{D}_Y$; but
- $mQ_m \not\le m\mathcal{D}_Y$; and
- $mQ_m = mD' = m\mathcal{D}_Y$ in the rational multiplicities of $\mathcal{D}_Y$.

It is enough to verify that, for some $j = m$ and $i \gg 0$, on any model $W/Y/Z$ of X/Z

$$\mathrm{Mov}\,\lceil j\mathcal{D}'_i + \mathcal{C}\rceil_W \ge m(\overline{Q_m})_W,$$

where $\mathcal{D}'_i = \mathcal{D}_i - \overline{E}$, and $E = \mathcal{D}_Y - D' \ge 0$ is an effective divisor on Y by construction. Since $\overline{E}$ is also effective, then, for the same j and $i \gg 0$, on any model $W/Y/Z$ of X/Z

$$\mathrm{Mov}\,\lceil j\mathcal{D}_i + \mathcal{C}\rceil_W = \mathrm{Mov}\,\left\lceil j\mathcal{D}'_i + j\overline{E} + \mathcal{C}\right\rceil_W \ge \mathrm{Mov}\,\lceil j\mathcal{D}'_i + \mathcal{C}\rceil_W \ge m(\overline{Q_m})_W.$$

On the other hand, $m(\overline{Q_m})_W \not\le m\mathcal{D}_W = j\mathcal{D}_W$ even on $W = Y$.

Note that the b-divisors $\mathcal{D}'_i$ share most of their properties with b-divisors $\mathcal{D}_i$: e.g.,

- (FDS) and (UAD) hold for $\mathcal{D}'_\bullet$; moreover, each $\mathrm{Supp}\,(j\mathcal{D}'_i - j\mathcal{D}')_Y = \mathrm{Supp}\,(\mathcal{D}_i - \mathcal{D})_Y \subset F$, where $\mathcal{D}' = \mathcal{D} - \overline{E}$; usually, in (FDS) we need a reduced divisor $\ge F$;
- $\lim_{i\to\infty} \mathcal{D}'_i = \mathcal{D}'$, whereas $\mathcal{D}'_Y = D'$; and
- (SAB) holds for $\mathcal{D}'_\bullet$ with the same i, because $\mathcal{E}(\overline{E}) = 0$ and $\mathcal{E}(\mathcal{D}'_i) = \mathcal{E}(\mathcal{D}_i - \overline{E}) = \mathcal{E}(\mathcal{D}_i)/Y$ by (EFF) and (ADD) in Proposition 5.4

Now take any positive $\alpha < 1$, and set $\tau = \alpha\gamma$. Then (LGR) for the prediction (multiplied by α) implies $(*)$ in Lemma 5.20. Take any $j = m \gg 0$ such that $\varepsilon/m^r \le \tau$. Then $\mathcal{D}_Y = mD' = D_{\mathrm{bf}} + D_{\mathrm{fr}}$ satisfies the assumptions for the decomposition in the lemma where $\mathcal{D} := m\mathcal{D}'$, $D_{\mathrm{bf}} = mQ_m$ and $D_{\mathrm{fr}} = mD' - mQ_m$. Indeed, by construction, $\|D_{\mathrm{fr}}\| < \varepsilon/m^r \le \tau$ and $\mathrm{Supp}\, D_{\mathrm{fr}} \subset F$.

By (FDS) the limit $\lim_{i\to\infty}(\mathcal{D}'_i)_Y = \mathcal{D}'_Y = D'$ takes place on Y in the norm $\|\ \|$. Thus by (UAD), for such $j = m$ and *any* $i \gg 0$, the decomposition

$\mathcal{D}_Y = j(\mathcal{D}'_i)_Y = D_{\mathrm{bf}} + D_{\mathrm{fr}}$ satisfies the assumptions of Lemma 5.20, where $D_{\mathrm{bf}} = mQ_m$ is still the same, but $\mathcal{D} := j\mathcal{D}'_i$, and $D_{\mathrm{fr}} := j(\mathcal{D}'_i)_Y - mQ_m = j(\mathcal{D}_i)_Y - jE - mQ_m$ with $\|D_{\mathrm{fr}}\| < \tau$ and $\operatorname{Supp} D_{\mathrm{fr}} \subset F$.

The boundedness $(**)$ in the lemma holds for *some* $i \gg 0$. More precisely, take any $i \gg 0$ with $m/r_i \leq (1-\alpha)$; the latter is > 0. Then $(**)$ holds for the i. Indeed, by (EEF) in the choice of Y, $m\mathcal{C}/r_i \leq (1-\alpha)\mathcal{C}/Y$. Thus by (ADD) in Proposition 5.4 and by the definition of the boundedness,

$$\mathcal{E}(\mathcal{D}) = \mathcal{E}(j\mathcal{D}'_i) = m\mathcal{E}(\mathcal{D}'_i) = m\mathcal{E}(\mathcal{D}_i) = m\mathcal{E}_i \leq m\mathcal{C}/r_i \leq (1-\alpha)\mathcal{C}/Y.$$

Hence by Lemma 5.20

$$\operatorname{Mov} \lceil j\mathcal{D}'_i + \mathcal{C} \rceil_W = \operatorname{Mov} \lceil \mathcal{D} + \mathcal{C} \rceil_W \geq (\overline{D_{\mathrm{bf}}})_W = m(\overline{Q_m})_W.$$

This completes the proof that $\mathcal{D}_Y$ is a $\mathbb{Q}$-divisor.

Second, we check that the limit $\lim_{i\to\infty} \mathcal{D}_i = \mathcal{D}$ stabilizes on Y. Using Corollary 5.19 again, but now in the trivial case that $D := \mathcal{D}_Y$ is a $\mathbb{Q}$-divisor, and by above arguments, this time we can't derive a contradiction. But by the asymptotic saturation we get that, for some $j = m$ (with integral $m\mathcal{D}_Y$) and some $i \gg 0$, on any $W/Y/X$

$$j(\mathcal{D}_j)_W \geq \operatorname{Mov} \lceil j\mathcal{D}_i + \mathcal{C} \rceil_W \geq m(\overline{Q_m})_W,$$

whereas $m(\overline{Q_m})_Y = mQ_m = mD = m\mathcal{D}_Y$ because the approximation is trivial. Hence, for such $j = m \gg 0$, $\overline{Q_m} \leq \mathcal{D}_j$, and, on Y, $\mathcal{D}_Y = Q_m \leq (\mathcal{D}_j)_Y$. Thus (MXD) gives the stabilization of the limit on Y, namely, $(\mathcal{D}_j)_Y = \mathcal{D}_Y$.

Third, we assume (BNF) and prove the stabilization in Addendum 5.12.1. It is enough to verify that $\mathcal{E}(\mathcal{D}) = 0/Y$. Indeed, then, for some $j = m \gg 0$, $\overline{Q_m} = \mathcal{D}$ and $\mathcal{D}_j \geq \overline{Q_m} = \mathcal{D}$. Hence (MXD) gives stabilization, namely, $\mathcal{D}_j = \mathcal{D}$.

By Remarks 5.8, (1–2), under (FDS) and (SAB),

$$\mathcal{E}(\mathcal{D}) = \mathcal{E}(\lim_{i\to\infty} \mathcal{D}_i) = \lim_{i\to\infty} \mathcal{E}(\mathcal{D}_i) \leq \lim_{i\to\infty} \mathcal{C}/r_i = \mathcal{C}/\lim_{i\to\infty} r_i = 0/Y.$$

On the other hand, under (BNF), $\mathcal{E}(\mathcal{D}) \geq 0/Y$ again by Remarks 5.8, (1–2). Therefore $\mathcal{E}(\mathcal{D}) = 0/Y$.

Finally, we assume also (CRG) and verify Addendum 5.12.2, i.e., $D := \mathcal{D}_Y$ contracts every prime divisor P_i on Y with $\operatorname{mult}_{P_i} \mathcal{C} > 0$. (Equivalently, $D_{|P_i}$ is semiample but is not big on every such P_i. For b-divisors P_i exceptional on

Y, the same follows from the above descent.) Here we use the arguments of Reid [6, Proposition 1.2]. By our assumptions, $\delta P_i \le \mathcal{C}$ for some $\delta > 0$. By asymptotic descent Addendum 5.12.1 and by Reid (or by Addendum 3.15.1), if $\operatorname{mult}_{P_i} \mathcal{C} > 0$, on any model $W/Y/Z$ of X/Z, and for all $j = m \gg 0$ such that $|mD|$ is free, we have the following lower bound:

$$\operatorname{Mov}\left(j(\mathcal{D}_j)_W + P_i\right) = \operatorname{Mov}\left(j\mathcal{D}_W + P_i\right) = j\mathcal{D}_W + P_i = j(\mathcal{D}_j)_W + P_i.$$

This contradicts asymptotic $\mathcal{C}$-saturation whenever the upper bound

$$\operatorname{Mov}\lceil j\mathcal{D}_j + \mathcal{C}\rceil_W \le j(\mathcal{D}_j)_W$$

with $i := j$ holds in asymptotic saturation over some W (cf. Remarks 4.30, (2) and (5)) that does not depend on j. More precisely, the estimate holds on any X_{hr}/W and any $j = m$ as above. Indeed, since $j\mathcal{D}_j = j\mathcal{D}$ is integral,

$$j(\mathcal{D}_j)_W + P_i \le \lceil j(\mathcal{D}_j)_W + \delta P_i\rceil \le \lceil j\mathcal{D}_j + \mathcal{C}\rceil_W$$

and

$$j(\mathcal{D}_j)_W + P_i = \operatorname{Mov}\left(j(\mathcal{D}_j)_W + P_i\right) \le \operatorname{Mov}\lceil j\mathcal{D}_j + \mathcal{C}\rceil_W,$$

which contradicts asymptotic $\mathcal{C}$-saturation.

The independence of the upper estimate on X_{hr}/W follows from Proposition 4.42 for $\mathcal{D} := j\mathcal{D}_j$ and $\mathcal{E} := 0$. Note only that the latter for some $j = m$ implies the same with the same W for any other natural number $j := m'$ under our assumptions.[5.3] Indeed, we can assume that $m' \ge m$ and $|(m' - m)D|$ is free. Then

$$\lceil m'\mathcal{D}_{m'} + \mathcal{C}\rceil = \lceil m'\mathcal{D} + \mathcal{C}\rceil = (m' - m)\mathcal{D} + \lceil m\mathcal{D}_m + \mathcal{C}\rceil.$$

Thus by the projection formula if

$$\mathcal{O}_W(\lceil m\mathcal{D}_m + \mathcal{C}\rceil_W) = \mathcal{O}_W(\lceil m\mathcal{D}_m + \mathcal{C}\rceil)$$

then the same holds for other natural numbers m' on W, because $(m-m')\mathcal{D} = \overline{(m' - m)D}$ is a Cartier b-divisor/Y (cf. b-free in Example 4.12). □

Now we are ready to explain what one needs to prove (FGA) and how to apply it to flips (cf. Corollary 1.5).

[5.3] I give up. I have absolutely no idea what "the latter" or "the same" refers to.

Example 5.21 Let $\mathcal{L}$ be an algebra of type (FGA) over $(X/T, B)$ (see Definition 4.34 above), and $\mathcal{D}_\bullet$ its characteristic system. We expect that such $\mathcal{L}$ is always f.g. By Theorem 4.26, this is equivalent to an affirmative solution of the problem of asymptotic descent for the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ on some model $Y/\operatorname{Proj}_T \mathcal{L}/T$ of X/T, where $\mathcal{D}$ is b-semiample/T and is a semiample/Y/T $\mathbb{Q}$-b-divisor. This last condition means that $D = \mathcal{D}_Y$ is a $\mathbb{Q}$-Cartier divisor that is semiample/T, and $\mathcal{D} = \overline{D}$.

By Addendum 4.20.1 and Proposition 4.13, (1) respectively, the system $\mathcal{D}_\bullet$ satisfies the minimal and additional assumptions in Theorem 5.12. Indeed, (BSD) in Addendum 4.20.1 implies (FDS). Moreover, by (FGA) of Definition 4.34, asymptotic saturation holds for $\mathcal{D}_\bullet$ with $\mathcal{C} = \mathcal{A} = \mathcal{A}(X, B)$, that is, lca saturation holds. By Example 4.43, (CGR) holds for the chosen $\mathcal{C} = \mathcal{A}$.

Thus the main obstacle to apply Theorem 5.12 is to choose a prediction $(Y/Z = T, \mathcal{C}, F, \gamma)$. Our choice of $\mathcal{C} = \mathcal{A}$ is canonical. On the other hand, because (X, B) is Klt, on any model Y/T of X/T and for any $\mathbb{Q}$-Cartier divisor F on Y, (LGD) holds for some $\gamma > 0$ (cf. Example 8.21 below). In particular, (FDS) implies (UAD) on any Y/T for some F (cf. Remarks 5.6, (3) and 5.11, (3) above).

However to satisfy the effectiveness (EEF) we need to take a model Y/T on which every prime b-divisor P_i with nonpositive discrepancy $a_i = d_i = d(P_i, B, X) = \operatorname{mult}_{P_i} \mathcal{A}$ [12, Example 1.1.4] is blown up, that is, P_i is a divisor on Y. Again since (X, B) is Klt, such models exists by [12, Lemma 1.6]. In addition, we can assume that Y is $\mathbb{Q}$-factorial.

Using LMMP we can make a more accurate choice of Y, where infinitely many $D_i = (\mathcal{D}_i)_Y$ are nef/T and even semiample/T. This implies (SAM) for $D = \mathcal{D}_Y = \lim_{i\to\infty} D_i$ on such Y/T (cf. Remark 5.6, (4)), and indicates where one can choose a prediction (cf. triples in (MOD) of Conjecture 6.14 and Corollary 6.40 below).

Note that D as the above limit D is only nef/T in particular $\mathbb{R}$-Cartier by Remark 5.6, (4). But, in general, nef/T does not implies semiample/T. This is the difference between numerical and linear geometry. However, by the log semiampleness conjecture [12, Conjecture 2.6] (proved in dimension ≤ 3 [12, Theorem 2.7]), nef is equivalent to the semiample when

(0LP) X/T is a 0-log pair for some boundary B as in Remark 3.26, (2); in particular, if $(X/T, B)$ has an $\mathbb{R}$-complement (cf. [13, (EC) in Conjecture 1.3]) for example, by [9, Proposition 5.5], this holds if

(WLF) $(X/T, B)$ is a weak log Fano contraction as in Proposition 4.38.

Moreover, under (WLF) we do not need the log semiampleness conjecture, and (WLF) implies

(RPF) the nef cone $\overline{\mathrm{NE}}\, X/T$ is *rationally polyhedral* and has *contractible* and *flippable birational* faces/T (cf. [13, (RPC) in Inductive Theorem 2.3]).

In addition, the (WLF) for some boundary and (RPF) are preserved under modifications in any face of $\overline{\mathrm{NE}}\, X/T$, that is, under the D-flips of any birational contraction of X/T with respect to which D is negative.

Indeed, let $X \dashrightarrow X^+/T$ be such a modification, H an ample/T divisor on X^+, and H^- its birational transform on X. Then there is some $\varepsilon > 0$ and a boundary $B' \geq B$ such that $(X/T, B' + \varepsilon H^-)$ is a 0-log pair. Then the modification is a flop of such pair, $(X^+/T, (B')^+ + \varepsilon H)$ is again a 0-log pair, and $(X^+/T, (B')^+)$ is a weak log Fano contraction. Therefore it again satisfies (RPF).

Since $(X/T, B)$ satisfies (WLF), under LMMP, we can apply this to modify X/T making a divisor semiample (cf. Theorem 3.29). For any effective $\mathbb{R}$-Cartier divisor D (even up to $\sim_{\mathbb{R}}$, cf. the effectiveness of bss divisors explained before Proposition 3.2), there is a model Y/T of X/T, where the birational transform (composition of above modifications in faces) of D is nef/T, and so by (RPF) is semiample/T. Moreover, if

(SA1) D is semiample in *codimension* 1, that is, for any effective divisor E on X, there is effective $D' \sim_{\mathbb{R}} D$ with disjoint $\mathrm{Supp}\, E$ and $\mathrm{Supp}\, D'$; hence such D is nef on general curves/T of any divisor in X,

then the model Y/T can be taken isomorphic to X/T in codimension 1.

Indeed, if D is nef, we take $Y = X$ and we are done. Otherwise by (RPF) there is a contraction $X \to Z/T$ on which D is negative. It is birational because $D \geq 0$. Moreover, it is small under (SA1). Thus we can modify X/T by a D-flips, and the D-MMP gives a required model Y/T. The modification is small under (SA1). The required termination is as in the proof of Theorem 3.29, that is, by LMMP for $(X/T, B' + \varepsilon D)$, where $(X/T, B')$ is a 0-log pair.

Note that the use of LMMP is justified by our approach (cf. Corollary 1.5 and the remark at the end of Conjecture 4.35).

Since f.g. dependends on $\mathcal{L}$ only up to quasi-isomorphism, or equivalently, stabilization dependends on $\mathcal{D}_\bullet$ only up to similarity, we can assume that all $\mathcal{D}_i$ are effective. Moreover, by (LBF) in Conjecture 4.35 we can assume (SA1) for $D := D_i := (\mathcal{D}_i)_X$. Thus if D_i is $\mathbb{Q}$-factorial, we can construct a model Y_i/T such that:

- Y_i is isomorphic to X in codimension 1; and
- D_i is nef, and even semiample$/T$ on Y_i by (RPF).

After taking a complement $(X/T, B')$, by the Borisov–Alexeev conjecture[5.4], we expect that there only a finite number of models Y_i/T satisfying the above conditions. The same would follow from Kawamata's conjecture[5.5] on finiteness of log minimal models. However, for our purposes, it is sufficient to have (LMMP by (RPF) in the proof of) the Second Main Theorem [12, Theorem 6.20] with prime b-divisors $D_i := P_i$ of $\operatorname{Supp} F$ in (FDS) and exceptional on X P_i with $a_i = \operatorname{mult}_{P_i} \mathcal{A}(X, B) \le 0$. Then the equivalence of weakly log canonical models $(Y_i/T, B' + \varepsilon D_i)$ means that the divisors D_i are nef on both models. This gives a model Y/T with (SAM) and infinitely many semiample $D_i = (\mathcal{D}_i)_Y$.

Finally, to secure the other properties of a prediction, e.g., (EEF), we need to make a crepant terminal (in codimension 2) resolution of $(Y/T, B')$, or of $(X/T, B')$, or even of (X, B) before the modifications. We can assume also that the resolution is $\mathbb{Q}$-factorial. The existence of such a resolution follows from [12, Theorem 3.1] again up to the LMMP. (Apply it to (X, B') with $B' \ge B$ and such that the new discrepancies $a_i := \operatorname{mult}_{P_i} \mathcal{A}(X, B')$ are negative for all exceptional P_i with old $a_i := \operatorname{mult}_{P_i} \mathcal{A}(X, B) \le 0$.)

This satisfies all the requirements for a prediction, except for boundedness (SAB)! This may actually fail (cf. Addendum 6.26.2 and Example 6.38). However we expect that it holds for Y/T provided that $\mathcal{D}$ is big (cf. Conjecture 6.14, and Proposition 6.26). Moreover, this is the only possible choice by Addendum 5.12.2 and Example 6.39.

Thus the main difficulty in constructing pl flips is boundedness (SAB). We start to attack it in the next section where we propose a conjecture to

[5.4] See footnote 4.12 on p. 67

[5.5] Reference?

resolve the difficulty and solve it in dimension 2. The latter is enough for 3-folds log flips.

6 Canonical bounds and saturation

We start with another interpretation of (asymptotic) boundedness in the *canonical* case, where $\mathcal{C} = \mathcal{A} = \mathcal{A}(X, B)$ is the discrepancy b-divisor (cf. Example 5.21, Definition 6.4 and Proposition 6.8 below).

Lemma 6.1 *Let (X, B) be a log pair, $\mathcal{D}$ an $\mathbb{R}$-b-divisor, and $\{\eta\}$ a set of (scheme theoretic) points of X such that:*

- *the divisors $K + B$ and $\mathcal{D}_X$ of X are $\mathbb{R}$-Cartier;*
- *the codimension of each point η is ≥ 2 in X; and*
- *$\mathcal{D} \geq 0$ over each η,* that is, $\operatorname{mult}_{E_i} \mathcal{D} \geq 0$ in each prime b-divisor E_i with $\operatorname{center}_X E_i = \eta$.

Then for any real number $c \geq 0$ such that $(X, B + c\mathcal{D}_X)$ is canonical in each η, we have

$$c\mathcal{E} \leq \mathcal{A}$$

over each η, where $\mathcal{E}$ is the descent data of $\mathcal{D}$ over X, and $\mathcal{A} = \mathcal{A}(X, B)$ is the discrepancy b-divisor of (X, B).

The inequalities *hold*$/X$ if $\{\eta\}$ is the set of all codimension ≥ 2 points in X (cf. (EEF) in 5.9 Choice of prediction). For example, for curves, namely, when $\dim X = 1$, the inequalities always hold because $\{\eta\} = \emptyset$ – there no exceptional divisors.

Remark 6.2 It is easy to generalize the lemma to the situation when

- the codimension of η is arbitrary; and
- canonical η is replaced by ε-log canonical or ε-log terminal (where ε may depend on η).

However the inequality $c\mathcal{E} \leq \mathcal{A}$ should be replaced by

$$\operatorname{mult}_{P_i} \mathcal{A} \geq \varepsilon - 1 + c \operatorname{mult}_{P_i} \mathcal{E}$$

for each prime b-divisor P_i with $\operatorname{center}_X P_i = \eta$ in the ε-log canonical case (respectively $>$ in ε-log terminal case, cf. (LGD) in 5.9, Choice of prediction), or replace $\mathcal{A}$ by its ε-log version if it is defined as a *b-divisor*.

In the lemma $\varepsilon = 1$. The log canonical and log terminal cases have $\varepsilon = 0$ (cf. Example 8.21).

The converse of the lemma usually fails, essentially because $\mathcal{A}$ and $\mathcal{E}$ are independent of $\mathcal{D}$ up to $\sim_{\mathbb{R}}$ by (DEP) in Proposition 5.4, whereas canonical *depends* on $\mathcal{D}$.

Example 6.3 (cf. Addendum 6.40.1 below) Let D be an $\mathbb{R}$-Cartier divisor on X and $\mathcal{D} = \overline{D}$. Then $\mathcal{D}_X = D$, $\mathcal{E} = 0$, and for any c, $0 = c\mathcal{E} \leq \mathcal{A}/X$ if and only if (X, B) is canonical in codimension 2. Of course, this does not imply that $(X, B + c\mathcal{D}_X)$ is always canonical in codimension 2: e.g., when $c > 1$, D is effective Cartier passing,[6.1] and $\dim X \geq 2$ (cf. Example 6.7, (3)).

However, if $\mathcal{D} = D_X$ then by the proof of Lemma 6.1, $c\mathcal{E} \leq \mathcal{A}$ over each η if and only if $K + B + c\mathcal{D}_X$ is canonical in each η of Lemma 6.1. Indeed, then $\mathcal{K} = \mathcal{K} + c\mathcal{D}$ over each η.

Proof of Lemma 6.1 By definition $\mathcal{K} = \overline{K + B} + \mathcal{A}$ and $\mathcal{D} = \overline{\mathcal{D}_X} - \mathcal{E}$ (cf. the proof of Proposition 4.44). Thus for each η, since $\mathcal{D} \geq 0/\eta$,

$$\mathcal{K} \leq \mathcal{K} + c\mathcal{D} = \overline{K + B + c\mathcal{D}_X} + \mathcal{A} - c\mathcal{E}/\eta$$

and $\mathcal{K} - \overline{K + B + c\mathcal{D}_X} \leq \mathcal{A} - c\mathcal{E}/\eta$. Therefore if $K + B + c\mathcal{D}_X$ is canonical over η, then by definition $\mathcal{A}(X, B + c\mathcal{D}_X) = \mathcal{K} - \overline{K + B + c\mathcal{D}_X} \geq 0/\eta$ as the discrepancy b-divisor. Hence $\mathcal{A} - c\mathcal{E} \geq 0$ and $c\mathcal{E} \leq \mathcal{A}/\eta$. $\square$

Definition 6.4 Let $\mathfrak{D}$ be a set of (effective) $\mathbb{R}$-b-divisors and (X, B) a log pair such that:

- B may not be a boundary, but
- $K + B$ and $\mathcal{D}_X$ for every $\mathcal{D} \in \mathfrak{D}$ are $\mathbb{R}$-Cartier.

[6.1] I don't understand "passing": maybe "passing through η."

We say that $\mathfrak{D}$ has *canonically bounded singularities* on (X, B), or simply that it is *canonically bounded* on (X, B) if there exists a real number $c > 0$ such that each pair $(X, B + c\mathcal{D}_X)$ is canonical in codimension 2. More precisely, the family is *bounded* by c (from *below*).

If we replace (X, B) by a family (X_i, B_i) of log pairs, with each X_i a model of X, $\mathfrak{D}$ bounded *on the family* means that, for each $\mathcal{D} \in \mathfrak{D}$, there is a pair (X_i, B_i) on which $\mathcal{D}$ is bounded by the *same* c. Thus c is *universal*.

Remark 6.5 (1) Similarly, we can define the ε-log canonical (terminal) boundedness of $\mathfrak{D}$ on (X, B) in a point η or family of points (cf. Remark 6.2). Thus in the case $\varepsilon = 1$, and for η of codimension ≥ 2, we get again the canonical (terminal) boundedness. For $\varepsilon = 0$, we get log canonical (terminal) boundedness (cf. Example 8.21).

(2) If $\mathfrak{D}$ is bounded, we can define its *threshold* as $\sup\{c\}$. This is a more precise characteristic of the singularities of $\mathfrak{D}$.

Definition 6.6 A set $\mathfrak{S}$ of reduced divisors $S = \sum P_i$ on X is *bounded* if its elements are bounded as reduced subvarieties of X (cf. the proof of Theorem 8.4 and Fixed restriction 7.2). More generally, a *bounded set of divisors* is[6.2] a set of divisors $\mathfrak{D}$ such that all $S = \operatorname{Supp} D$ of $D \in \mathfrak{D}$ are *bounded*, that is, belong to a bounded family of reduced divisors $\mathfrak{S}$. In this case we say that $\mathfrak{D}$ is *bounded* by $\mathfrak{S}$, and D is bounded by S respectively.

We add[6.3] to the boundedness with *multiplicities* on Y, whenever it includes the boundedness of the multiplicities for each $\mathcal{D}_Y$, where Y is a model of X.

Example 6.7 (Trivial) (1) If $\mathfrak{D}$ is a finite or bounded set of b-divisors $\mathcal{D}$, including *multiplicities* on X, and (X, B) is terminal in codimension 2, then $\mathfrak{D}$ is bounded on (X, B).

(2) If each b-divisor $\mathcal{D}$ of $\mathfrak{D}$ is the Cartier closure $\mathcal{D} = \overline{D}$ of a sufficiently general element in a free linear system, and (X, B) is terminal in codimension 2, then $\mathfrak{D}$ is also bounded on (X, B).

[6.2] As terminology, this has the disadvantage that it already has two other ordinary meanings (family having bounded moduli, finite), which are quite different, and both of which you use later in the paper. I think here you should say a *set of divisors with bounded singularities*.

[6.3] I can't make any sense of this sentence.

(3) If each b-divisor of $\mathfrak{D}$ is sufficiently general and b-free, that is, as in (2) on some model Y/X of X, and (X, B) is Klt, then $\mathfrak{D}$ is bounded on the family (X_i, B_i) which consists of the crepant models $(X_i/X, B_i)$ terminal in codimension 2. Note that $c = 1 + \min \mathcal{A}$ is the minimal *log* discrepancy of (X, B) (cf. Example 6.25). In particular, $c = 1$ when (X, B) is canonical.

Now we show how to apply canonical boundedness to obtain asymptotic boundedness.

Proposition 6.8 *Under the assumptions of Definition 6.4, suppose also that*

(CMD) *each model $(X_i/Z, B_i)$ is a* crepant model *of $(X/Z, B)$ (cf.* Example 6.7, (3)*); and*

(EFF) *each $\mathcal{D} \in \mathfrak{D}$ is effective.*

Then $\mathfrak{D}$ canonically bounded by c implies that, for each $\mathcal{D} \in \mathfrak{D}$, there exists a pair (X_i, B_i) such that the descent data $c\mathcal{E}$ of $c\mathcal{D}$ over X_i is bounded by $\mathcal{A}/X_i$, where $\mathcal{A} = \mathcal{A}(X, B) = \mathcal{A}(X_i, B_i)$ is the discrepancy b-divisor.

Addendum 6.8.1 *Let $\mathcal{D}_i$ be a sequence of b-divisors, and p_i a sequence of positive real numbers respectively such that:*

- $\lim_{i\to\infty} p_i = +\infty$*; and*
- *each $p_i\mathcal{D}_i \sim_{\mathbb{R}} \mathcal{M}_i \in \mathfrak{D}$ (or even just numerically equivalent), where*
- *all the $\mathcal{M}_i$ are canonically bounded by c* on a common model ($Y = X_j$, *$B_Y = B_j$) in the family.*

Then the descent data for the sequence $\mathcal{D}_i$ is asymptotically bounded by $\mathcal{A}/Y$.

Proof The bound $c\mathcal{E} \leq \mathcal{A}/X_i$ follows from Lemma 6.1 for $(X, B) := (X_i, B_i)$, for the set $\{\eta\}$ of points of codimension ≥ 2, and for c the given canonical bound, where (X_i, B_i) itself corresponds to $\mathcal{D}$ under the canonical bound. By (HOM) of Proposition 5.4, $\mathcal{E}(c\mathcal{D}) = c\mathcal{E}(\mathcal{D})/X := X_i$.

Since $c > 0$, each $r_i = cp_i > 0$, and also $\lim_{i\to\infty} cp_i = +\infty$. On the other hand, by Proposition 5.4, (DEP) and (HOM) $r_i\mathcal{E}_i = cp_i\mathcal{E}_i = c\mathcal{E}(p_i\mathcal{D}_i) =$

$c\mathcal{E}(\mathcal{M}_i)/Y$. Hence the above boundedness[6.4] for $\mathfrak{D}$ implies $r_i\mathcal{E}_i = c\mathcal{E}(\mathcal{M}_i) \leq \mathcal{A}/Y$, which means asymptotic boundedness. $\square$

We define desirable pairs (X_i, B_i) of bounding families in terms of triples.[6.5]

Definition 6.9 A *triple* $(X/T/Z, B, \mathfrak{F})$ includes

- a log pair $(X/Z, B)$, for which $K + B$ is $\mathbb{R}$-Cartier, but B may not be a boundary;
- a contraction $g\colon X \to T/Z$; and
- a bounded set $\mathfrak{F}$ of reduced divisors on X.

Let $(X/Z, B)$ be a weak log Fano contraction (see (WLF) in Proposition 4.38). A *desirable triple* for $(X/Z, B)$ is a triple $(Y/T/Z, B_Y, \mathfrak{F})$ such that:

(CRP) $(Y/Z, B_Y)$ is a crepant model of $(X/Z, B)$;

(QFC) Y is $\mathbb{Q}$-factorial;

(TER) (Y, B_Y) is terminal in codimension 2; and

(RPC) the nef cone $\overline{\mathrm{NE}}\,T/Z$ is *rationally polyhedral* and has *contractible* faces$/T$ (cf. [13, Inductive Theorem 2.3]).

Let $\mathcal{M}$ be a b-free b-divisor. A triple $(Y/T/Z, B_Y, \mathfrak{F})$ is a *desirable triple* for $\mathcal{M}$ if in addition[6.6]

- $\mathcal{M}_Y = g^*M$, where M is nef and big on T/Z.

Remark 6.10 (1) T/Z is a contraction if X/Z is.

[6.4] Does "above boundedness" just mean the given canonical bound for $\mathfrak{D}$?

[6.5] Slava: You can't have "wanted triples" – you can sometimes use a past participle as an adjective, but not of an auxiliary verb such as "want". "desirable" is unambiguous and probably translates the intention of the Russian, although it has a slightly humorous sound. It's put in by a macro \w. You could have "ideal triples" or "target triples" if you don't like my solution.

[6.6] I change the wording of the definition, since you clearly want to include all of the above conditions.

(2) Sometimes, even for desirable triples, we need to weaken the assumptions: e.g., X/T may be a incomplete morphism of normal varieties, cutting log singularities, or (X, B) may not be Klt.

(3) It might be better to spread the boundedness of $\mathfrak{F}$ into boundedness of the family of *simple* triples $(X/T/Z, B, F)$ having a 1-element set $\{F\} \subset \mathfrak{F}$ for $F \in \mathfrak{F}$. The *boundedness* of a family of triples means that their moduli are bounded.

(4) (CRP) in the definition means that $\mathcal{A}(Y, B_Y) = \mathcal{A}(X, B) = \mathcal{A}$ (cf. Caution 4.33).

(5) We could possibly drop (QFC), but then we can only bound singularities of a divisor $\mathcal{D}$ having $\mathbb{R}$-Cartier restriction $\mathcal{D}_Y$ (cf. Lemma 6.1 vs Lemma 8.12). This holds in many applications. For example, $\mathcal{M}_Y$ is $\mathbb{Q}$-Cartier on a desirable triple for $\mathcal{M}$ by definition.

(6) In (TER), the divisor B_Y need not be a boundary. However, since (X, B) is Klt, B_Y is a *Klt subboundary*: that is, each of its multiplicity $b_i < 1$.

(7) (RPC) is intermediate among the following conditions on T/Z, that we list in order of increasing strength:

    (LSA) the limit of any semiample sequence of $\mathbb{R}$-divisors on T/Z is a semiample $\mathbb{R}$-divisor on T/Z, and we assume that M is *semiample* in the definition of desirable triples (cf. Remark 5.6, (4));

    (NSA) each divisor M on T that is nef$/Z$ is semiample;

    (0LP) T/Z is a 0-log pair for some boundary B_T as in Remark 3.26, (2); in particular, (NSA) then holds for each effective divisor, and we can assume that $\mathcal{M}$ is effective, and any limit of effective divisors is also effective under (MXD);

    (RPC) as in Definition 6.9 (cf. Lemma 8.12, (2)); and

    (WLF) (T, B_T) is a weak log Fano contraction for some boundary B_T.

For the main application of (CBS) below (cf. Theorem 6.19), (LSA) is enough. But for approximations we need at least (RPC) (cf. Theorem 8.15). In addition, we prefer[6.7] to work with numerical properties rather than linear properties.

The reader may wonder what role the set of divisors $\mathfrak{F}$ plays in triples. In fact, we need very special sets $\mathfrak{F}$ that we call *standard*. First, the standard families are bounded in typical examples, such as the fixed and fractional parts that appear in Corollary 6.40 and in Proposition 9.15. Second, they generalize the canonical bound conjecture (cf. (GCB) in Conjecture 6.14). Finally, they are crucial in our construction of 4-fold flips (cf. Theorem 6.44).

Definition 6.11 Let $(X/Z, B)$ be a log pair. We say that a set $\mathfrak{F}$ is *standard* on $(X/Z, B)$ if it contains only standard divisors. A *standard divisor* S on $(X/Z, B)$ is a reduced divisor $S = \operatorname{Supp} \mathcal{D}_X$, where $\mathcal{D}$ is an *effective* $\mathbb{R}$-b-divisor such that:

(SEF) $\mathcal{D}$ is *strictly effective*$/X$; that is, it has $\operatorname{mult}_{E_i} \mathcal{D} \geq 0$ in each exceptional$/X$ prime b-divisor E_i and > 0 *over* $\operatorname{Supp} \mathcal{D}_X$, where the support is considered as a *divisorial subvariety* (cf. (EEF) in 5.9, Choice of prediction); and

(STD) 0 is saturated with respect to $\mathcal{A} + \mathcal{D}$ on any sufficiently high model X_{hr}/Z of X/Z.

The *standard set* $\mathfrak{S}$ is the set of all standard divisors. In general, it may be unbounded. However by (SSB) in Conjecture 6.14, we expect that this is true[6.8] when $(X/Z, B)$ is a weak log Fano contraction. In particular, then $(X/X/Z, B, \mathfrak{S})$ is a triple. For any other triple $(Y/T/Z, B_Y, \mathfrak{S}_Y)$, the *induced standard set* $\mathfrak{S}_Y$ is defined as the log birational transform of the set $\mathfrak{S}$ of X.

A desirable triple for $(X/Z, B)$ with the *induced set* $\mathfrak{S}_Y$ is a desirable triple $(Y/T/Z, B_Y, \mathfrak{S}_Y)$ with the induced standard set.

Caution 6.12 It may happen that $\mathfrak{S}_Y$ itself is actually *nonstandard* (cf. Remark 6.13, (2)).

[6.7] Maybe it would be better to say "we prefer to work with notions defined up to numerical equivalence rather than up to linear equivalence".

[6.8] You mean the opposite, namely "We expect that $\mathfrak{S}$ is bounded."

Remark 6.13 (1) If D is an effective $\mathbb{R}$-Cartier divisor on X then $\overline{D}$ satisfies (SEF).

(2) The induced set $\mathfrak{S}_Y$ is well defined, namely, it is bounded when $\mathfrak{S}$ is bounded. Indeed, the log birational transform of $S \in \mathfrak{S}$ is $g^{-1}S+\sum E_i$, where each E_i is a prime divisor on Y that is exceptional on X. Note that $\sum E_i$ is finite and fixed for any such transform.

(3) In general, $\mathfrak{S}_Y$ is not standard even up to the additive log contribution $\sum E_i$. Let X be a del Pezzo surface having two -1-curves E_1, E_2 intersecting transversally in a single point P, for example, two intersecting lines on a cubic surface in $\mathbb{P}^3$. Then E_1 and E_2 belongs to the standard set $\mathfrak{S}(X/Z = \text{pt.}, 0)$, but $E_1 + E_2$ does not (cf. Example 6.38). For the blowup $g\colon Y \to X$ in P, the blowup E_3 of P (the log transform of 0) is standard. The birational transform $g^{-1}E_i$ of each E_i is also standard (in addition to their log transforms $g^{-1}E_i+E_3$). But now $g^{-1}E_1+g^{-1}E_2$ is also standard since $g^{-1}E_1 + g^{-1}E_2 + a(E_3)E_3 = g^{-1}E_1 + g^{-1}E_2 + E_3$ is not mobile on Y.

Nonetheless the difference between $\mathfrak{S}_Y$ and $\mathfrak{S}(Y/Z, B_Y)$ is not great (cf. (GFC) in Conjecture 6.14 below).

Other facts about standard sets and equivalent forms of the standard property (SEF)+(STD) are given below in Proposition 6.34. In general, a bounding family (X_i, B_i) may itself be unbounded (cf. Example 6.7, (3)), and an infinite subset in $\mathfrak{D}$ may not be bounded on any bounded subfamily of (X_i, B_i). However we need and expect the following:

Conjecture 6.14 (Canonical bounds) *Let $(X/Z, B)$ be a weak log Fano contraction with a contraction X/Z. We expect boundedness of the following sets. First,*

(SSB) the standard set $\mathfrak{S} = \mathfrak{S}(X/Z, B)$ of $(X/Z, B)$ is bounded.

In particular, this includes boundedness of prime components of the standard divisors. Moreover, we expect

(PRM) *the* prime standard set $\mathfrak{P} = \mathfrak{P}(X/Z, B)$ is bounded, *where $\mathfrak{P}$ is the set of* prime standard divisors $P = \operatorname{Supp} \mathcal{D}_X$ *on X with $\mathcal{D} \geq 0$ and $\mathcal{D}$ satisfying saturation (STD) of Definition 6.11.*

The third set is the sat set $\mathfrak{D} = \mathfrak{D}(X/Z, B)$ *of* effective $\mathbb{R}$-*b-divisors* $\mathcal{D}$ *under saturation*

(SAT) $\mathcal{D}$ *is* log canonically saturated$/(X, B)$ *on any sufficiently high model* X_{hr}/Z *of* X/Z *(cf. Definition 4.32); that is, it is* $\mathcal{A}$-*saturated, where* $\mathcal{A} = \mathcal{A}(X, B)$ *is the discrepancy b-divisor.*

In most cases, it is unbounded even on X*. However we expect that*

(PFC) *the set of* fixed prime components*, that is, the set of prime components of* $(\mathrm{Fix}\,\mathcal{D})_X$ *for* $\mathcal{D} \in \mathfrak{D}$ *on* X *(cf. Remark 6.15, (4)) is* bounded*.*

(PFC) is equivalent to (PRM). Actually, the set is equal to $\mathfrak{P}$ *(cf. Theorem 6.19, (1) and its proof). By definition, (PRM) and (PFC) hold on any desirable triple* $(Y/T/Z, B_Y, \mathfrak{F})$ *if they hold on* $(X/Z, B)$*. Indeed,* $\mathfrak{P}(Y/Z, B_Y)$ *is the birational transform of the* $\mathfrak{P}$ *plus some* E_i *on* Y *that are exceptional on* X*. Moreover, we expect that*

(GFC) *boundedness (PRM), (PFC) and* (SSB) *each holds on any* general *log Fano contraction* $(X/Z, B)$ *(see* (GLF) *in Proposition 4.44) with* Klt *subboundary* B *for* (SSB).

Each $\mathcal{D} \in \mathfrak{D}$ *has the form* $\mathcal{D} = \mathcal{M} + \mathcal{F}$ *with b-free* $\mathcal{M} \geq 0, \mathcal{M} \in |\mathrm{Mov}\,\mathcal{D}|$*, and fixed* $\mathcal{F} = \mathrm{Fix}\,\mathcal{D} \geq 0$*. This splits* $\mathfrak{D}$ *as a* partial *sum (cf. Remark 6.15, (6))*

$$\mathfrak{D}(X/Z, B) \subset \mathfrak{M}(X/Z, B) \oplus \mathfrak{F}(X/Z, B)$$

where $\mathfrak{M} = \mathfrak{M}(X/Z, B)$ *and* $\mathfrak{F} = \mathfrak{F}(X/Z, B)$ *are the* subsets *of b-free and fixed (with* $\mathcal{M} = 0$*) b-divisors in* $\mathfrak{D}$*. Indeed, under Klt,* (SAT) *is equivalent to*

(SAF) $\mathcal{M}$ *is* saturated *with respect to* $\mathcal{A} + \mathcal{F}$ *on any sufficiently high model* X_{hr}/Z *of* X/Z *by Proposition 6.34, (2);*

and in turn by Proposition 6.34, (3), (SAF) is equivalent to

(STD) *as in Definition 6.11 for* fixed $\mathcal{D} := \mathcal{F}$*.*

Hence $\mathcal{F}$ also satisfies (SAT)*, and by Lemma 4.40 with $\mathcal{C}_1 = \mathcal{A}+\mathcal{F}$ and $\mathcal{C}_2 = \mathcal{A}$, so does $\mathcal{M}$. We use the splitting $\mathcal{D} = \mathcal{M} + \mathcal{F}$ to bound the singularities of the sum in terms of its two summands. Moreover, we can consider any such sum (cf. Remark 6.15, (6)).*

Indeed, $\mathcal{M}$ and $\mathcal{F} \geq 0$, and, on each desirable triple $(Y/T/Z, B_Y, \mathfrak{F})$ of $(X/Z, B)$, $\mathcal{D}_Y \sim \mathcal{M}_Y + \mathfrak{F}_Y$ with a $\mathbb{Q}$-Cartier Weil divisor $\mathcal{M}_Y \geq 0$ and an $\mathbb{R}$-Cartier $\mathcal{F}_Y \geq 0$. Thus because the canonical property is convex (cf. [9, 1.3.1]), it is enough to prove the latter[6.9] *for each summand. If $\mathcal{F}$ is bounded by c_f from below, and $\mathcal{M}$ by c_m then $\mathcal{D}$ is bounded by any weighted combination of them: e.g., half the sum $c = (c_m + c_f)/2$.*

On any desirable triple $(Y/T/Z, B_Y, \mathfrak{F})$, by (PRM) and Example 6.7, (1), each irreducible component f_iP_i of $\mathcal{F}_Y$ (with prime P_i) has canonically bounded singularities if the multiplicities f_i are bounded. For the whole $\mathcal{F}$, we need more, namely, boundedness of the number of irreducible components, which is equivalent to (SSB) *and holds for $\mathcal{D} := \mathcal{F}$ under condition (SEF)/Y of Definition 6.11, or for the log transforms in $\mathfrak{S}_Y$ from $(X/Z, B)$.*

The mobile part $\mathcal{M}$ behaves worse (cf. Remark 6.15, (7)):

- *canonical boundedness from below holds for* some *effective $\mathcal{M}$ (usually meaning* sufficiently general*); and*
- *on some triples.*

With a certain dose of optimism *we expect*

(CBS) *there is a* algebraic *(throughout the paper; cf. Remark 8.16)* bounded *family of desirable triples $(Y/W/T, B_Y, \mathfrak{F})$ on which $\mathfrak{M} = \mathfrak{M}(X/Z, B)$ has* canonically bounded singularities *up to $\sim$, where $\mathfrak{M}$ is the set of b-free b-divisors satisfying log saturation* (SAT)*. More precisely, for each $\mathcal{M} \in \mathfrak{M}$ there is $\mathcal{D} \in |\mathcal{M}|$ and a desirable triple $(Y/W/T, B_Y, \mathfrak{F})$ for $\mathcal{M}$ in the family such that $\mathcal{D}$ is canonically bounded on (Y, B_Y).*

Thus jointly

(GCB) canonical boundedness *for some $\mathcal{D} + \mathcal{F} \sim \mathcal{M} + \mathcal{F}$ over the family*

[6.9] What? possibly "enough to prove that $\mathcal{D}_Y \sim \mathcal{M}_Y + \mathfrak{F}_Y$ is canonical, or is canonically bounded."

is equivalent to (CBS)+(SSB)*. But the latter needs extra assumptions on $\mathcal{F}$ as (SEF). Without them (GCB) means (CBS)+(PRM).*

Finally, we expect (cf. Remark 6.15, (8)) that

(BIG) *for* big $\mathcal{M}$*, the subfamily of desirable triples is finite; in particular,*

(BIR) *for* birational X/Z*, the whole family of desirable triples can be taken to be finite; and, more generally,*

(MOD) *among* birationally equivalent $\mathcal{M}$*, that is,* $\mathcal{M}$ *that define birationally the same contractions* $Y \to T/Z$ *(see Definition 6.20, the subfamily of desirable triples* $\mathcal{M}$ *is finite; in addition,* $\dim T/Z = \kappa(X/Z, \mathcal{M})$ *is the Iitaka dimension of* $\mathcal{M}/Z$*.*

However to be more realistic *we can restrict ourself to very special subsets in* $\mathfrak{M}$ *and impose further conditions. We consider mobile systems* $\mathcal{M}_\bullet \subset \mathfrak{M}$ *at least under* asymptotic saturation (LCA)*: e.g.,*

(CBS)(fga) *(CBS) for* $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L}$ *for each* [6.10] *(separate) functional algebra* $\mathcal{L}$ *in Conjecture 4.35; and*

(CBS)(rfa) *(CBS) for each algebra in Definition 3.43.*

(CBS)(rfa,bir) is what we really need (see the notation below, and Conjecture 3.44).

Remark 6.15 (1) We can expect generalizations of the conjectures when X/Z is only a proper morphism of normal varieties but possible not a contraction, or X/Z has other extra structures: e.g., a group action, a morphism X/W, etc. The latter relates in particular to the nonlocal case X/W with a proper morphism $W \to Z$ and *local*$/Z$, as we always assume.

(2) (PRM) drops (SEF) in Definition 6.11 but assumes that $S = P$ is prime.

(3) We can drop the effective condition of $\mathcal{D} \in \mathfrak{D}$ in (PFC). Indeed, if $|\mathcal{D}| = \emptyset$, the prime component of the base locus is a single prime variety

[6.10] Possibly you mean "condition for each individual functional algebra".

X that we can add to $\mathfrak{P}$. Otherwise, $D \geq 0$ up to $\sim$ that preserves the saturation (SAT) and the fixed components. Perhaps, we can add as well other prime maximal fixed centers.

Again, in (CBS) and (GBC), we need the effectiveness of $\mathcal{M}$ and $\mathcal{M}+\mathcal{F}$ respectively up to $\sim$. Otherwise $\mathcal{M} = -\infty$ is canonically bounded on any model. Thus we can also drop effectiveness.

(4) Any $\mathbb{R}$-b-divisor $\mathcal{D}$ has a decomposition $\mathcal{D} = \mathcal{M} + \mathcal{F}$ with b-free *mobile part* $\mathcal{M} = \operatorname{Mov} \mathcal{D}$ and *fixed part* $\mathcal{F} = \operatorname{Fix} \mathcal{D} \geq 0$. We explained this in Proposition 4.13, (1) and (3) when $\mathcal{O}_Z(\mathcal{D})$ is coherent. In general, there is the *maximal coherent* subsheaf

$$\mathcal{O}_Z(\mathcal{M}) = \mathcal{O}_Z(\mathcal{D})^{ch} \subset \mathcal{O}_Z(\mathcal{D}) \subset \mathcal{O}_Z(\mathcal{D}_X) \subset k(X)$$

(because the coherent sheaf $\mathcal{O}_Z(\mathcal{D}_X)$ is Noetherian). Note that in the *global* case with $Z = \text{pt.}$, each $\mathcal{O}_Z(\mathcal{D})$ is coherent.

An $\mathbb{R}$-b-divisor $\mathcal{D} \geq 0$ up to $\sim$ if and only if $\mathcal{M} \neq -\infty$, or equivalently, $|\mathcal{M}| \neq \emptyset$, or $\mathcal{O}_Z(\mathcal{M}) = \mathcal{O}_Z(\mathcal{D})^{ch} \neq 0$. If $\mathcal{D} \geq 0$ then $\mathcal{D} \geq \mathcal{F} \geq 0$. In the same way, $\mathcal{D} \geq 0$ is *mobile* if and only if $|\mathcal{M}| \neq |0|$, or equivalently, $\mathcal{O}_Z(\mathcal{M}) = \mathcal{O}_Z(\mathcal{D})^{ch}$ has > 1 generator (is nonprincipal).

(5) For general log Fano contractions in (GLF) of Proposition 4.44, we drop any properties of B. In general, (SSB), (PRM) and (PFC) do not hold for contractions that are not general log Fano contractions, even for rational varieties X. For example, let $(X/\text{pt.}, 0)$ be a sufficiently high resolution of $\mathbb{P}^2$: e.g., such that the set of exceptional (= contractible) curves on Y is infinite.[6.11] Then $\mathfrak{S}$ and $\mathfrak{P}$ include at least the exceptional curves C (take $\mathcal{D} = \overline{C}$ for (SSB)). We expect that (PRM) and (PFC) hold on any general log Fano contraction but for (SSB) we need Klt; in particular, B is a subboundary (cf. bad singularities in Remark (9) below).

(6) We say that $\mathfrak{D}$ splits only as a *partial sum* because for rather powerful $\mathcal{M}$, $\operatorname{Mov} \lceil \mathcal{M} + \mathcal{F} + \mathcal{A} \rceil$ can certainly be $> \mathcal{M}$, which gives a *different* splitting.[6.12]

[6.11] "unbounded" contradicts your earlier usage "bounded set of divisors", and I've replaced it by infinite, which means the same.

[6.12] "Powerful" is not defined. Presumably it means "very big" linear system. The remark is too cryptic, and much too far away from the statement in Conjecture 6.14 to which it refers.

(7) Write $l = \mathrm{lld}(X, B)$ for the least log discrepancy of (X, B). If some element $\mathcal{D} \in |\mathcal{M}|$ is canonically bounded by c with $0 \le c \le l$, then so is the general $\mathcal{D}' \in |\mathcal{M}|$. Hence (CBS) for some $\mathcal{D}$ implies (CBS) for general $\mathcal{D}$, in fact with the same c. Indeed, $\mathcal{D}$ canonically bounded on (X, B) means that $(X, B+c\mathcal{D}_X)$ has canonical singularities in codimension 2. Equivalently, on *any* crepant model $(Y/X/Z, B)$, the crepant subboundary

$$B_Y := \mathcal{B}(X, B + c\mathcal{D}_X)_Y = \mathcal{B}(X, B)_Y + (\overline{c\mathcal{D}_X})_Y$$

has only nonpositive multiplicities in divisors that are exceptional on X, and (Y, B_Y) is canonical in codimension 2. Note that, for any other $\mathcal{D}'$, if $B'_Y := \mathcal{B}(X, B+c\mathcal{D}'_X)_Y$ is a subboundary on Y such that $B'_Y = cL+F$, where $F \le B_Y$ in the divisors that are exceptional on X, and $\mathrm{Bs}\,|L| = \emptyset$, then by monotonicity (cf. [9, 1.3.3]), for sufficiently general L (that is, $\mathcal{D}$), $(X, B + c\mathcal{D}'_X)$ is also canonical in codimension 2. This holds on *some* good model Y/Z, where $|\mathcal{D}'| = \overline{|\mathcal{D}'_Y|}$, with $\mathrm{Bs}\,|\mathcal{D}'_Y| = \emptyset$, and $L = \mathcal{D}'_Y = 0/X$. Indeed, for such fixed model and for a general $\mathcal{D}'$,

$$B'_Y = \mathcal{B}(X, B)_Y + (\overline{c\mathcal{D}'_X})_Y \quad \text{and}$$
$$(\overline{c\mathcal{D}'_X})_Y = c(\overline{\mathcal{D}'_X})_Y \le c(\overline{\mathcal{D}_X})_Y = (\overline{c\mathcal{D}_X})_Y/X.$$

This can also be established by inversion of adjunction (cf. [9, after Proposition 5.13]).

Klt implies that $\mathrm{lld}(X, B) > 0$ (least log discrepancy). On the other hand, $c \le \mathrm{lld}$ on $\mathfrak{M}$ by Example 6.7, (3).

(8) We could perhaps expect more for the triples $(Y/T/Z, B_Y, \mathfrak{F})$ that are required for the conjecture: e.g.,

- each model Y/Z is a contraction over a crepant terminal resolution of $(X/Z, B)$, and is thus a rational 1-contraction over X/Z; and
- in (BIG), Y/Z is isomorphic in codimension 1 to a crepant terminal resolution of $(X/Z, B)$.

The discussion in Example 5.21 sheds light on the finiteness in (BIG). In fact, each of the finiteness assertions in (BIG), (BIR), (MOD) and (CBS)(fga), (CBS)(rfa) amounts to a *single* triple.

To prove (CBS) in general, it is enough to consider a *desirable* triple $(X/T/Z, B, \mathfrak{F})$ with the subset $\mathfrak{N} = \mathfrak{N}(X/T/Z, B)$ of $\mathcal{M} \in \mathfrak{M}$ satisfying the following conditions:

(NEF) $\mathcal{M}_X$ is nef/Z and

(SFB) $\mathcal{M}_X$ is *supported in fibres* of X/T of *Iitaka dimension*/Z $\kappa(X, \mathcal{M}_X) = \dim T/Z$.

In particular, both hold when the triple is desirable for $\mathcal{M}$ (conversely, $\mathcal{M}$ is desirable in codimension 1 on T, so if T is $\mathbb{Q}$-factorial this always holds). Then to prove that $\mathfrak{N}$ is *canonically bounded* on (X, B) modulo $\sim$, we can assume that moreover

(IRR) $\mathcal{M}_X$ is *irreducible*

except for pencils[6.13] when any boundedness of singularities is known. Since $\mathcal{M}_X$ is integral, and by the cone property (RPC), the divisors for which the triple is desirable, correspond modulo $\sim$ to integral points in a rational cone dual to the Kleiman–Mori cone $\overline{\mathrm{NE}}\, T/Z$. As in Truncation Principle 4.5, it is enough to establish the canonical bound:

- for a *truncated* subcone: for $\mathcal{M}$ with a rather high *index of height* on X, namely, for some natural number I, every such $\mathcal{M}_X \sim \sum h_i g^* H_i$, where each H_i is nef and big on T/Z, and $I \mid \sum h_i$, and
- for *bounded* heights $\sum h_i$.

We hope that, for some $I \gg 0$, the former $\mathcal{M}$ are *free* over X, that is, $\mathcal{M} = \overline{\mathcal{M}_X}$ and $\mathrm{Bs}\,|\mathcal{M}_X| = \emptyset$. Perhaps, in the latter case, bounded heights implies bounded singularities by reduction to lower dimension of T. Finally, for general elements in $\mathfrak{N}$ we can use the birational case after a localization over non$\mathbb{Q}$-factorial points of T, or to assume that T is $\mathbb{Q}$-factorial.

Example 5.21, explains how to find such triples and to secure their finiteness if $\mathcal{M}$ is big and $T = X$. In particular, this reveals why we expect Conjecture 6.14 to hold in full. But for (FGA) we only need the boundedness of triples (cf. of Theorem 6.19, (3)).

[6.13] I don't understand this. Possibly "Except for the case of a pencil, when all the conditions of bounded singularities are known."

Canonical boundedness really requires the terminal property (TER) of desirable triples (cf. Remark (11) below). For example, if $\mathcal{M}_i = i\overline{H}$ over a terminal resolution Y/X of $(X/Z, B)$ with ample H/Z, then $\mathcal{M}_\bullet \subset \mathcal{M}$, but $\mathcal{M}_\bullet$ has bounded singularities on X only when $X = Y$.

Note also that we need fibred triples, with X/T not birational (cf. Example 6.38 below).

(9) Possible important generalizations concern bad singularities of (X, B): e.g., we could weaken the Klt condition, as for general log Fano varieties (GLF), but still assume that $B \geq 0$. But then we need extra assumptions to ensure good behavior on the bad singularities, namely:

(GEN) On a universal crepant model $(Y/Z, B_Y)$ of $(X/Z, B)$), each $\mathcal{D}_Y$ is *free* or even *(very) ample*$/Z$ on LCS (Y, B_Y).

Therefore in this case, we do care about canonical singularities outside LCS (Y, B_Y).

(10) Another generalization concerns the numerical conditions of Fano type (GLF) that can be replaced by

(ADJ) $\mathcal{D}_X \equiv (K + B + D)/T$, where D is a sum $D = F + H$ of *effective* F plus *nef and big* H/T.

(11) We expect similar results in the ε-log category, for example, ε-log canonical boundedness. But this requires appropriate restrictions: e.g., ε-log saturation with respect to ε-log discrepancy. For triples, (TER) can be replaced by ε-log terminal in codimension 2.

Question 6.16 Could we replace the condition that $\mathfrak{M}$ is b-free by the b-nef assertion of Lemma 4.21?

Philosophy 6.17 Saturation (SAT) improves properties of any $\mathcal{D}$ up to $\sim$, even if it is not b-free. (By analogy with regularizing solutions of elliptic differential equations.) On the other hand, by (DEP) in Proposition 5.4 and by Proposition 6.8, we can bound the descent data for $\mathcal{M}$ in terms of the canonical bound of an improved divisor. The same applies when $p\mathcal{D} \sim_{\mathbb{R}} \mathcal{M}$ with real number $p > 0$ and $\mathcal{M}$ has good canonical singularities (cf. Addendum 6.8.1), for example, by Conjecture 6.14.

Notation 6.18 We use the following specifications:

- $(-)_n$ means $(-)$ with $\dim X = n$;
- $(-)(X/Z, B)$ means $(-)$ for $(X/Z, B)$;
- $(-)$(big) means that $(-)$ concerns only big/Z b-divisors;
- $(-)$(bir) means that X/Z is birational in $(-)$;
- $(-)$(fga) means that we consider $(-)$ only on a subset $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L}$ for an algebra $\mathcal{L}$ of type (FGA) in Definition 4.34;
- $(-)$(gl) means that $Z = $ pt., that is, X is global in $(-)$; and
- $(-)$(rfa) means that we consider $(-)$ only on a subset $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L}$ for an algebra $\mathcal{L}$ of type (RFA) in Definition 3.43.

We apply these mainly to (CBS). For example,

- $(\mathrm{CBS})_n$ means (CBS) for all $(X/T, B)$ with $\dim X = n$;
- $(\mathrm{CBS})_n(\mathrm{fga}, \mathrm{bir})$ means (CBS) for any system $\mathcal{M}_\bullet$ of type (FGA) with birational X/Z and with $\dim X = n$; and
- $(\mathrm{CBS})_n(\mathrm{bir}, \mathrm{gl}) = \emptyset$ for $n \geq 1$.

The main result of this section is

Theorem 6.19 *(1)* (PFC) = (PRM) *(equality of sets).*

(2) $(\mathrm{CBS})_n$ *implies the same with any specifications* (big), (bir), (gl), (fga) *and* (rfa).

(3) $(\mathrm{CBS})(X/Z, B)$ *implies* $(\mathrm{FGA})(X/Z, B)$*, and* $(\mathrm{CBS})(\mathrm{fga})(X/Z, B)$ *is equivalent to* $(\mathrm{FGA})(X/Z, B)$*.*

(4) $(\mathrm{CBS})_n(\mathrm{fga})$ *is equivalent to* $(\mathrm{FGA})_n$*.*

We start by clarifying (MOD) of Conjecture 6.14.

Definition 6.20 Two contractions $X_1 \to Y_1$ and $X_2 \to Y_2$ are *birationally equivalent*, if they are isomorphic over an open nonempty subsets $U_i \subset Y_i$. In the definition we can assume that $X_1 = X_2$ after some resolution.

Example 6.21 Suppose that X_1 and X_2 are birationally equivalent. Then

(1) any two big contractions are birationally equivalent;

(2) any two contractions to pt. are birationally equivalent; and

(3) any two contractions over a curve are birationally equivalent if they define the same pencil composed of their fibres.

Lemma 6.22 *Let $\mathcal{D}_1 \geq \mathcal{D}_2$ be two $\mathbb{R}$-b-divisors on X/Z. Then*

(1) the Iitaka dimensions satisfy $\kappa(X/Z, \mathcal{D}_1) \geq \kappa(X/Z, \mathcal{D}_2)$, with equality if and only if they are birationally equivalent;

(2) if both $\mathcal{D}_1$ and $\mathcal{D}_2$ are b-nef, then $\nu(\mathcal{D}_1/Z) \geq \nu(\mathcal{D}_2/Z)$ with equality if and only if $\nu(\mathcal{D}_2{}_{|(\mathcal{D}_1-\mathcal{D}_2)}/Z) \leq \nu(\mathcal{D}_2) - 1$; and

(3) if $\mathcal{D}_1$ and $\mathcal{D}_2$ are both b-semiample, $\nu(\mathcal{D}_1/Z) \geq \nu(\mathcal{D}_2/Z)$ with equality if and only if they are birationally equivalent.

Proof (1) Immediate from the definitions.

(2) Taking an appropriate model of X/Z, we can assume that each $\mathcal{D}_i = \overline{D_i}$ for a nef/Z divisor $D_i = (\mathcal{D}_i)_X$. By definition each $\nu(\mathcal{D}_i) = \nu(D_i)$. Since $D_1 \geq D_2$, it is enough to verify that $\nu(D_1) \geq \nu(D_2)$. Indeed, $D_1 = D_2 + F$ with $F \geq 0$. Taking a general point of Z, we can assume that $Z =$ pt. and each $\nu(D_i/Z) = \nu(D_i)$.

It is enough to consider the case $\nu(D_1) \leq \nu = \nu(D_2)$. Thus for some cycle C_ν of dimension ν, $C_\nu D_2^\nu > 0$. Equivalently, the same holds for mobile cycles $H^{n-\nu}$, where $n = \dim X$ and H is a hyperplane section of X, because $H^{n-\nu}$ is rationally equivalent to C_ν plus an effective cycle. Taking a hyperplane section gives an induction on the dimension $n \geq \nu$ with divisors $D_i{}_{|H}$. Hence we can assume that $\nu = n$. Then

$$\begin{aligned} D_1^n &= D_2 D_1^{n-1} + F D_1^{n-1} \\ &\geq D_2 D_1^{n-1} \geq \cdots \geq D_2^{n-1} D_1 \\ &= D_2^n + D_2^{n-1} F \geq D_2^n > 0. \end{aligned}$$

This means that also $\nu(D_1) = \nu$. In addition,

$$\nu(D_2{}_{|(D_1-D_2)}) = \nu(D_2{}_{|F}) \leq \dim F \leq n-1 = \nu - 1.$$

(3) is immediate by (1) because $\nu(\mathcal{D}/Z) = \kappa(\mathcal{D}/Z)$ for any b-semiample divisor $\mathcal{D}$. $\square$

Proof of Theorem 6.19 (1) Compare the proof of Proposition 6.34, (3). For $\mathcal{D} \geq 0$, Klt implies that

$$\operatorname{Mov} \lceil \mathcal{A} + \mathcal{D} \rceil \geq \operatorname{Mov} \mathcal{D} \geq 0.$$

Hence (PRM) for $P = \mathcal{D}_X$ implies (PFC) for the $\mathcal{D}$ with prime $P = (\operatorname{Fix} \mathcal{D})_X$, because $\operatorname{Fix} \mathcal{D} = \mathcal{D}$. Conversely, for any $\mathcal{D} = \mathcal{M} + \mathcal{F} \geq 0$ under (SAT) and with $\mathcal{M} = \operatorname{Mov} \mathcal{D}$ and $\mathcal{F} = \operatorname{Fix} \mathcal{D}$,

$$\begin{aligned} \mathcal{M} + \operatorname{Mov} \lceil \mathcal{A} + \mathcal{F} \rceil &\leq \operatorname{Mov} (\mathcal{M} + \lceil \mathcal{A} + \mathcal{F} \rceil) \\ &= \operatorname{Mov} \lceil \mathcal{D} + \mathcal{A} \rceil \leq \operatorname{Mov} \mathcal{D} = \mathcal{M}, \end{aligned}$$

since it is $\leq \mathcal{D}$. Hence $\operatorname{Mov} \lceil \mathcal{A} + \mathcal{F} \rceil \leq 0$, and actually $= 0$, by Klt again. This means (STD) of Definition 6.11 for $\mathcal{D} := \mathcal{F}$. By Lemma 4.40. The same holds for each prime component of $P = \mathcal{D}_X := \mathcal{F}_X$, that is, $P \in \mathfrak{P}$.

(2) Immediate from the definitions, except for (fga) and (rfa). In these cases we only need to verify that $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L} \subset \mathcal{M}$. More precisely, this concerns only effective $\mathcal{M}_i$ up to $\sim$. Indeed, each $\mathcal{M}_i$ satisfies (SAT) by (LCA) and Remark 4.30, (5).

(3) Let $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L}$ be a mobile system on $(X/Z, B)$ for an algebra $\mathcal{L}$ of type (FGA). By Limiting Criterion 4.26, the f.g. of $\mathcal{L}$ is equivalent to stabilization of its characteristic system, that is, of the limit $\mathcal{D} = \lim_{i \to \infty} \mathcal{D}_i$ with $\mathcal{D}_i = \mathcal{M}_i / i$. In turn stabilization is equivalent to the asymptotic descent problem. To solve the problem we apply Theorem 5.12.

First, we choose a prediction (Definition 5.10). By Lemma 6.22, (3) and the arithmetic monotonicity in Lemma 4.22, the numerical dimension $\nu(\mathcal{D}_i/Z)$ stabilizes. After a truncation, we can assume that each $\nu(\mathcal{D}_i/Z) = \nu$, the Iitaka dimension of $\mathcal{L}$. Moreover, up to a similarity, corresponding to a quasi-isomorphism of algebras by Proposition 4.13, (8), we can assume that $\nu \geq 0$ and entire $\mathcal{M}_\bullet \subset \mathfrak{M}$. Otherwise $\mathcal{L} = k \oplus 0_\bullet$ and is f.g. Besides the effectiveness of each $\mathcal{M}_i$ we use the saturation (LCA) of (FGA) as in (2) above.

Since by Lemma 6.22, (3), the b-divisors $\mathcal{M}_i/Z$ are birationally equivalent, by (MOD) there is a finite family of desirable triples $(Y/T/Z, B_Y, \mathfrak{F})$ *for* them. (Actually, it can be replaced by a single triple; cf. the remark below.) Moreover, this can be in the *strict* form: each of the triple has an *infinite* subset of $\mathcal{M}_i$ for itself, and the same hold after any truncation. Indeed, we can discard all triples in our finite family that does not satisfy the former infiniteness and replace $\mathcal{M}_\bullet$ by a truncation for which the corresponding $\mathcal{M}_i$

was discarded with triples. Then we take a truncation such that some triple does not satisfy the same infiniteness, etc. Finally, we get a family with the strict property.

Now any remaining triple $(Y/T/Z, B_Y, \mathfrak{F})$ predicts $(Y/Z, \mathcal{C} = \mathcal{A}, F, \gamma)$ for the asymptotic descent problem $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$. If we consider only i for which this is a desirable triple then (SAM) for $\mathcal{D}_Y = \lim_{i\to\infty}(\mathcal{D}_i)_Y$ follows from the cone property (RPC) because, by our choice of b-divisors $\mathcal{D}_i$, each $(\mathcal{D}_i)_Y$ is semiample$/Z$ (cf. (LSA) in Remark 6.10, (7) and the discussion in Example 5.21). The effectiveness (EEF) for the b-discrepancy $\mathcal{C} = \mathcal{A}$ means (TER) for the triples. The boundedness of algebra $\mathcal{L}$ and (BSD) of Addendum 4.20.1 imply that there is a reduced divisor F that contains the support of each $(\mathcal{D}_i)_Y$. Thus (UAD) holds for F (cf. Remark 5.11, (3)). On the other hand, the growth (LGD) for F and some $\gamma > 0$ follows from (QFC), (CRP) and (X, B) Klt.

The harder property of predictions is boundedness (SAB); by Addendum 6.8.1, this follows from (CBS)(fga)$(X/Z, B)$ with the common model $(X_i/Z, B_i) = (Y/Z, B_Y)$, for $\mathcal{D}_i$ with i for which this is a desirable triple, and with $p_i = i$. Indeed, the descent data for $\mathcal{D}_\bullet$ is asymptotically bounded by $\mathcal{A}/Y$. This bound is strict because each truncation leaves infinitely many b-divisors $\mathcal{D}_i$ for the required triple in the descent problem depending on the choice of the triple.

Secondly, $\mathcal{D}_\bullet$ satisfies the minimal assumptions (FDS) by (BSD) and (MXD) by Addendum 4.20.1, and the additional assumption (BNF) because each $\mathcal{M}_i$ is b-free.

Thirdly, as we know, (LCA) in (FGA) implies the required saturation. Thus by Addendum 5.12.1, the asymptotic descent problem has a solution on Y.

Finally, (CGR) of Proposition 4.42 holds for $\mathcal{C} = \mathcal{A}$ by Example 4.43. Hence by Addendum 5.12.2 $\mathcal{A}_{Xst} \le 0$ on the stable model Xst/Z, which proves Remark 4.36, (7). Indeed, by Klt, the crepant boundary Bst is actually a boundary and (Xst, Bst) is Klt. On the other hand, by the improved (BIG) in Remark 6.15, (8) it is dominated by a rational 1-contraction of a crepant weak log Fano contraction for $(X/Z, B)$. Thus $-(K_{Xst} + Bst)/Z$ is big and b-semiample. Hence by LMMP, (Xst, Bst) is isomorphic in codimension 1 to a weak log Fano contraction, with the same boundary, on which $-(K_{Xst} + Bst)$ is semiample = nef and big (cf. Example 5.21).

Remark 6.23 We can use the boundedness of triples instead of the finiteness in (MOD) in the descent of Theorem 5.12. Indeed, an irreducible family of birational models is birationally a single model over a function field, that can have infinitely many special points corresponding to elements of the family and also for any truncation. The canonical bound at any special point implies the same over the generic point by inversion of adjunction. The saturation (SAT) also holds over the generic point, because it holds over each point as a birational property over a *sufficiently high* model. The latter need just a good resolution as in the proof of Proposition 4.42.

Conversely, the stabilization in Limiting Criterion 4.26 gives (CBS) for a truncation of $\mathcal{M}_\bullet$. Then by Truncation Principle 4.5, the same holds for any system $\mathcal{M}_\bullet$ (cf. Remark 6.15, (7)).

(4) Immediate from (3). $\square$

The following examples illustrate Conjecture 6.14.

Example 6.24 (Global curve) Let $(X/Z = \text{pt.}, B)$ be a complete log curve X with arbitrary B, reduced but possibly reducible. Then it is terminal in codimension 2. The b-divisors are divisors $\mathcal{D}$ on the normalization of X, that is the space of a triple of the required form. Conjecture 6.14 holds in this case. Note that $\mathcal{D}_X$ is well defined for nonsingular divisors D (with $\operatorname{Supp} D \subset \operatorname{NonSing} X$), and $\mathcal{D} = \mathcal{D}_X = D$. Set $D = M + F \geq 0$ with $M = \operatorname{Mov} D$ and $F = \operatorname{Fix} D$. Then M satisfies (CBS) with $c = 1$, which does not need any saturation. (Even (SAT) with $\deg B \gg 0$ can exclude some M.) But the boundedness of F follows from the saturation (SAT) and RR. In addition, the multiplicities of F are bounded. For example, if $B = 0$, then $\lceil F \rceil \leq g - 1$, where $g = g(X)$ is the arithmetic genus of X.

Example 6.25 (Two dimensional birational case) Let $f\colon X \to Z$ be a birational contraction of a normal surface X. As we will soon see (cf. the proof of (FGA) in the Main Theorem at the end of this section), this is the key point in our construction of 3-fold log flips. Except for (CBS), Conjecture 6.14 holds for *any* birational contraction with finite $\mathfrak{S}, \mathfrak{P}$ and $\operatorname{Supp} \mathfrak{F}_X$ supported[6.14] in divisors of X that are exceptional on Z, and in the fractional part of B.

Suppose first that $B = 0$, X is nonsingular, and X/Z is cohomologically rational, that is, Z has rational singularities. We claim that $(X/X/Z, 0, \mathfrak{F})$

[6.14] I guess that $\mathfrak{S}$ is the standard set of divisors of Definition 6.11 and $\mathfrak{P}$ is the standard prime set of Conjecture 6.14. Who is $\mathfrak{F}$? In the next sentence, who is $\mathfrak{M}$?

is a desirable triple for any $\mathcal{M} \in \mathfrak{M} = \mathfrak{M}(X/Z, B)$.

Indeed, $\mathcal{A} = \mathcal{A}(X, 0) = \lceil \mathcal{A} \rceil \geq \mathcal{E} \geq 0$, where $\mathcal{E}$ is the reduced b-divisor of all the prime b-divisors that are exceptional on X, the *support of the exceptional locus*. By the rationality of X/Z, on any model Y/Z of X/Z, each Cartier divisor M that is nef/Z is free/X. In particular, this holds for any integral $M \geq 0$ for which $\operatorname{Supp} M$ has no exceptional components of Y/Z. In addition, if C is a -1-curve on Y/X and $M \cdot C \geq 1$ (intersects any M modulo $\sim$), then $M + C$ is also free, and

$$\begin{aligned}\operatorname{Mov} \lceil M + \mathcal{A}_Y \rceil &= \operatorname{Mov} (M + C + \mathcal{A}_Y - C) \\ &\geq \operatorname{Mov} (M + C) + \operatorname{Mov} (\mathcal{A}_Y - C) \\ &\geq M + C > M,\end{aligned}$$

because $\mathcal{A}_Y - C \geq 0$. Hence no b-free $\mathcal{M} = \overline{M} \in \mathfrak{M}$, since it does not satisfy (SAT). Conversely, by (SAT) every $\mathcal{M} \in \mathfrak{M}$ descends to X as a free divisor/Z. Thus we can take $c = 1$ in (CBS) of Conjecture 6.14. This explains the role of saturation.

Indeed, if Y/X is a minimal resolution of the base locus for $|\mathcal{M}|_X$, then a -1-curve C as above exists over the locus. If $Y := X_{\mathrm{hr}}$ is also a sufficiently high resolution for (SAT) and $M = \mathcal{M}_Y$ is sufficiently general in $|\mathcal{M}_Y| = |\mathcal{M}|_Y$, then we replace C by $\overline{C}_Y \leq \mathcal{A}_Y$. This again contradicts (SAT). Thus $\mathcal{M}$ is free on X/Z.

Moreover, the same holds for $(X/Z, B)$ under (TER). But in this case we may have $c < 1$, namely, $c = \min\{1 - b_i\}$.

If $(X/Z, B)$ is Klt, then (CBS) holds on a crepant terminal resolution $(Y/X, B_Y)$. Since $(X/Z, B)$ is not assumed to be a weak log Fano contraction, Remark 6.15, (10) holds, even without (ADJ).

Another approach that we discuss below reduces our problem to freedom in dimension 1 (cf. Example 6.24 above). It is not so effective, and moreover has some unpleasant new features (see Remark 6.28.(2)), but this finally leads to 4-fold flips.

The main techniques to establish Conjecture 6.14 in dimension 2 are contained in the following proposition.

Proposition 6.26 (General two dimensional case) *Let $(X/Z, B)$ be a log pair with a surface X, and $\mathcal{D} \geq 0$ an* integral *b-divisor such that:*

(GLF) $(X/Z, B)$ *is a general log Fano contraction as in Proposition 4.44 with arbitrary* B, *not even assumed to be a boundary; and*

(RIR) *on any model* Y/Z *of* X/Z, *the general* $\mathcal{D}_Y \in |\mathcal{D}|_Y$ *is* reduced and irreducible*; and*

(SAT) $\mathcal{D} \in \mathfrak{D} = \mathfrak{D}(X/Z, B)$*;* that is, $\mathcal{D}$ is saturated with respect to $\mathcal{A} = \mathcal{A}(X, B)$.

Then except for a bounded set of complete divisors $\mathcal{D}_X$, $|\mathcal{D}|_X$ *is free/*Z *outside* $\mathrm{CS}\,(X/Z, B)$ *on* X, *where* $\mathrm{CS}\,(X/Z, B)$ *denotes the* locus of canonical singularities. *In other words,* $(X/Z, B)$ *satisfies (TER) outside* $\mathrm{CS}\,(X/Z, B)$.

Commentary: We only need two essential cases:

- $\mathcal{D}$ is a b-free nonpencil (or an irreducible member of pencil); or
- $|\mathcal{D}| = \{P\}$ is a fixed prime b-divisor.

By (RIR), $|\mathcal{D}|_X$ free/Z somewhere on X means that $\mathcal{D}$ has a descent $\mathcal{D}_X$ over the locus. (In the proposition, outside CS.) From this point of view "sufficiently general" $\mathcal{D}_Y$ means in particular that $\mathcal{D}_Y$ has minimal multiplicities in every divisor that is exceptional on X: e.g., 0 for b-free $\mathcal{D}$.

Addendum 6.26.1 *If the assumptions* (RIR) *and* (SAT) *in the proposition are replaced respectively by*

(RED) *on any model* Y/Z *of* X/Z, *a general* $\mathcal{D}_Y \in |\mathcal{D}|_Y$ *is* reduced*; and*

(SA′T) $\mathcal{D}$ *is saturated with respect to* $\mathcal{A}' = \mathcal{A}(X, B) + \sum E_i$, *where b-divisors* E_i *are the exceptional on* X *prime over* integral components of $\mathcal{A}$

then any two prime components (or even two branches) $D_1 \neq D_2$ *of general elements* $\mathcal{D}_X \in |\mathcal{D}|_X$ *intersect each other only in points* $Q \in X$ *at which* $K + B$ *is not Cartier or in* $\mathrm{LCS}\,(X, B)$, *with a finite set of exceptions for* D_1 or D_2. *More precisely, any set of exceptions to this can only include* D_i *with* $(K + B) \cdot D_i = 0$ *(in particular,* D_i *is complete, so lies over* P*). Thus these exceptions* D_i *belong to a bounded family of reduced divisors* $\mathfrak{F}$.

Moreover, we can replace (SA′T) *by a weaker condition* (cf. Proposition 6.34, (4)).

(Sa′T) *as in* (SA′T)*, but assume that the primes* E_i *are over* $\operatorname{Bs}|\mathcal{D}|_X$*.*

Addendum 6.26.2 *Suppose in addition that*

- (X, B) *is Klt and*
- $\mathcal{D}$ *is b-free.*

Then $|\mathcal{D}|_X$ *has at most one base point on* X *outside* $\mathrm{CS}\,(X, B)$*; in particular, it has at most one base point on a terminal resolution of* $(X/Z, B)$*. In addition,* $|\mathcal{D}|_X$ *is free in* CS *after the point is blown up. Moreover, any pencil* $\mathcal{D}$ *is elliptic, that is, its general member is a curve of geometric genus* 0.

Corollary 6.27 *Locally/*Z*, under the assumptions of* Proposition 6.26*, the singularities of a general* $\mathcal{D}_X \in |\mathcal{D}|_X$ *are bounded outside* CS*; in particular, outside* $\mathrm{LCS}\,(X, B)$ *on a terminal resolution.*

Proof The singularities of $\mathcal{D}_X$ are bounded for any free $|\mathcal{D}|_X$. The same holds for any bounded set of divisors $\mathcal{D}_X$ (Example 6.7, (1)). □

Remark 6.28 (1) (RIR) and (1–2) in the proof of Proposition 6.26 below hold when $\mathcal{D}$ is b-free and big. For surfaces, the latter means that $\mathcal{D} \neq 0$ and is not a pencil.

(2) Sometimes, $\operatorname{Bs}|\mathcal{D}|_X \neq \emptyset$, and requires $c < 1$ even if (X, B) is terminal in codimension 2 (see Example 6.38 below).

(3) However, for b-free $\mathcal{D}$, we expect only a *finite* set of linear systems $|\mathcal{D}|$ having base points outside CS.

For example, one verifies that this finiteness holds when $(X/Z = \mathrm{pt.}, B)$ is a del Pezzo surface with a standard boundary B and of nonexceptional type.

Lemma 6.29 (Lifting $\mathbb{R}$-Cartier divisors) f^*D *is defined up to* $\sim$ *under the following conditions:*

- *if* D *is an* $\mathbb{R}$*-Cartier divisor defined up to* $\sim$*, or*
- *for any morphism* $f\colon X \to Y$ *such that* $f(X)$ *is not contained in the fractional prime components of* D*.*

Addendum 6.29.1 (Functoriality) *For $g \circ f \colon X \to Y \to Z$,*

$$(g \circ f)^* D = f^*(g^* D).$$

Proof Since D is given up to $\sim$, we can assume that f and D are in *general position*, i.e., $\operatorname{Supp} f(X) \cap \operatorname{Supp} D$ is proper in each irreducible component of $f(X)$. $\square$

Lemma 6.30 (Invariance of integral part under $\sim$) *If $D \sim D'$ then*

$$\lfloor D \rfloor \sim \lfloor D' \rfloor \quad \textit{and} \quad \lceil D \rceil \sim \lceil D' \rceil .$$

Thus the same holds for the fractional parts.

Proof $F = D - D'$ is principal, in particular integral, and

$$\lfloor D \rfloor - \lfloor D' \rfloor = \lfloor F + D' \rfloor - \lfloor D' \rfloor = F = \lceil D \rceil - \lceil D' \rceil$$

is also principal. $\square$

Lemma 6.31 *Let D be a divisor on a complete nonsingular curve C with* $\deg D \geq d$. *Then* $\deg \lceil D \rceil \geq \lceil d \rceil$.

Proof We need to verify that if $\{d_i\}$ is a set of real numbers such that $\sum d_i \geq d$, then $\sum \lceil d_i \rceil \geq \lceil d \rceil$. We can apply this to the set of multiplicities of divisor $D = \sum d_i P_i$ on C.

Indeed, each $\lceil d_i \rceil \geq d_i$ (and $=$ holds only if d_i is integral). The sum of these gives

$$\sum \lceil d_i \rceil \geq \sum d_i \geq d$$

and the required inequality, because $\sum \lceil d_i \rceil$ is integral, and any integer $\geq d$ is $\geq \lceil d \rceil$. $\square$

Proof of Proposition 6.26 We restrict to $D = \mathcal{D}_X$ after a resolution. Up to a bounded (even finite) set of divisors D we can assume that

(1) $\operatorname{Supp} D \cap \operatorname{Supp} B$ is *small*,

that is, there no divisors in the intersection.

We consider a sufficiently high resolution $g\colon Y \to X$, where we have the saturation (SAT). Since $\sim$ preserves (SAT) (cf. Remark 4.30, (7)) we can assume by (RIR) that $D := \mathcal{D}_Y$ is general, irreducible and reduced. Let $g_D\colon D \to X$ be the induced morphism. After an additional resolution we can assume also that Y is a log resolution for $(X, B + D)$. In particular, D is nonsingular.

We have a surjective restriction$/Z$

$$|D + \lceil \mathcal{A}_Y \rceil| \dashrightarrow \left| (D + \lceil \mathcal{A}_Y \rceil)_{|D} \right| = |K_D + \lceil -g_D^*(K+B) \rceil| .$$

Indeed, here we can use Lemma 6.29, because D is reduced by (RIR), and $K + B$ is integral on D by (1). Note also that $\mathcal{A}_Y \sim K_Y - g^*(K+B)$, whereas $\lceil \mathcal{A}_Y \rceil \sim K_Y + \lceil -g^*(K+B) \rceil$ by Lemma 6.30. Thus $D + \lceil \mathcal{A}_Y \rceil \sim D + K_Y + \lceil -g^*(K+B) \rceil$ and by adjunction, normal crossings on Y and Addendum 6.29.1, we have

$$(D + \lceil \mathcal{A}_Y \rceil)_{|D} \sim (D + K_Y)_{|D} + \lceil -g_D^*(K+B) \rceil = K_D + \lceil -g_D^*(K+B) \rceil .$$

Also we use Kawamata–Viehweg vanishing: $R^1 h_* \mathcal{O}_Y(\lceil \mathcal{A}_Y \rceil) = 0$ with $h = f \circ g\colon Y \to Z$, because $\lceil \mathcal{A}_Y \rceil \sim K_Y + \lceil -g^*(K+B) \rceil$ and (GLF).

If $|\mathcal{D}|_X$ has a base point $Q \in X/Z$ outside CS, then by (SAT)

$$\operatorname{Mov}(D + \lceil \mathcal{A}_Y \rceil) = \operatorname{Mov} \lceil \mathcal{D} + \mathcal{A} \rceil_Y \leq \mathcal{D}_Y = D.$$

Thus $\operatorname{Fix}(D + \lceil \mathcal{A}_Y \rceil) \geq E_i$ for some exceptional divisor E_i with discrepancy $a_i = a(X, B, E_i) > 0$ and such that E_i intersects D on Y. Hence $\operatorname{Fix}((D + \lceil \mathcal{A}_Y \rceil)_{|D}) \geq E_{i|D} > 0$ and $\operatorname{Bs}\left| (D + \lceil \mathcal{A}_Y \rceil)_{|D} \right| \neq \emptyset$ – the restriction is not free by the above surjection. Hence D is complete. Otherwise $\left| (D + \lceil \mathcal{A}_Y \rceil)_{|D} \right|$ is always free.

Thus D is complete and $\left| (D + \mathcal{A}_Y)_{|D} \right| = |K_D + \lceil -g_D^*(K+B) \rceil|$ must have base points or be $\emptyset$. This is impossible if $\deg B' > 1$, where $B' = -g_D^*(K+B)$ because then by the Lemma 6.31 $\deg \lceil B' \rceil > 1$ and so ≥ 2.

But for such a divisor B', the linear system $|K_D + \lceil B' \rceil|$ is free. Hence each D having fixed points on X outside CS should have bounded degree on X/Z, namely $-(K+B) \cdot D \le 1$. Such divisors are bounded. Indeed, we need to consider only complete divisors D/Z. In particular, they include the contracted divisors D of X:

(2) $-(K+B)$ is not big$/Z$ on D

this adds a finite set of divisors with discrepancy 0 to $\operatorname{Supp} B$; cf. (GLF) above. $\square$

Proposition 6.32 *If $D_1 \ge D_2$, then D_1 C-saturated implies D_2 C-saturated over prime divisors at which $D_1 = D_2$. The same holds in the b-divisorial versions.*

Proof–Explanation Saturation over the prime divisors at which $D_1 = D_2$ (or over any other set of divisors), means that $\operatorname{Mov} \lceil D_2 + C \rceil \le D_2$ over this set of divisors, in other words, the inequality concerns only the prime divisors P_i in the set: e.g., in our situation, $\operatorname{mult}_{P_i} D_1 = \operatorname{mult}_{P_i} D_2$. Thus the proposition is immediate from the definition. $\square$

Proof of Addendum 6.26.1 Suppose that $K+B$ is Cartier and canonical at a point $Q \in D_1 \cap D_2$. Then $K+B$ Cartier implies that each D_i lies over an integral component of B. Again by (1) in the proof of Proposition 6.26, we assume either that $(K+B) \cdot D_i < 0$ for $i = 1, 2$, or that D_i is *not complete*. Then we verify that D is not $\mathcal{A}'$-saturated on any sufficiently high Y/X. Since $-(K+B)$ is nef and big$/Z$, there exists only a finite number of complete D_i with $(K+B) \cdot D_i = 0$. Thus up to finitely many exceptions, the contradiction with $\mathcal{A}'$-saturation means that Q is not Cartier or not canonical.

By (RED) and Remark 4.30, (1) (cf. the proof of Proposition 6.34, (ii)) we can replace D by its components that are not exceptional on X, or in other words, omit the divisors that are exceptional on X (because they are fixed in $|\mathcal{D}|_Y$). Then by (RED) and Proposition 6.32 it is enough to verify that the *new* $D := D_1 + D_2$ is not $\mathcal{A}'$-saturated on any sufficiently high Y/X *over* $D_1 \cup D_2$ and some exceptional prime b-divisors E_i/Q. We assume also that each $g(D_i)$ passes through Q and that either $(K+B) \cdot D_i < 0$, or D_i is not complete. We derive a contradiction with $\mathcal{A}'$-saturation on any log resolution $g\colon Y \to X/Z$ for $B + D$.

On Y we put together a chain of curves (chain maybe after additional blowups) $C = D_1 \cup (\bigcup E_i) \cup D_2$ with only nodal singularities, where E_i are the exceptional curves over Q and D_i the end curves. We verify that $|C + \lceil \mathcal{A}_Y \rceil| = |\lceil D + \mathcal{A}_Y + \sum E_i \rceil|$ has no base points in a neighborhood of C on Y/Z. Since $\mathcal{A}_Y + \sum E_i \le \mathcal{A}'_Y$ even under (Sa$'$T), then by Lemma 4.40 this contradicts $\mathcal{A}'$-saturation for D on Y/X:

$$\operatorname{Mov} \left\lceil D + \mathcal{A}_Y + \sum E_i \right\rceil \ge D + \sum E_i > D$$

over each E_i. Note that $\mathcal{A}_Y \ge 0$ and integral over Q, because Q is canonical for $K + B$.

As in the proof of Proposition 6.26, for the restriction to D, we get a surjective restriction$/Z$ on C:

$$|C + \lceil \mathcal{A}_Y \rceil| \dashrightarrow \left| (C + \lceil \mathcal{A}_Y \rceil)_{|C} \right| = |K_C + \lceil -g_C^*(K + B) \rceil|$$

with $g_C \colon C \to X/T$. Indeed, here we can use Lemma 6.29, because C is reduced and $K + B_Y$ is integral on C (since $K + B$ is Cartier at Q). As above, by Lemma 6.30,

$$C + \lceil \mathcal{A}_Y \rceil \sim C + K_Y + \lceil -g^*(K + B) \rceil$$

Thus by Lemma 6.30, adjunction, normal crossings on Y together with Addendum 6.29.1, we get

$$(C + \lceil \mathcal{A}_Y \rceil)_{|C} \sim (C + K_Y)_{|C} + \lceil -g_C^*(K + B) \rceil = K_C + \lceil -g_C^*(K + B) \rceil .$$

Now, as before, Kawamata–Viehweg vanishing gives $R^1 h_* \mathcal{O}_Y(\lceil \mathcal{A}_Y \rceil) = 0$.

The construction gives a Cartier divisor $L = (C + \lceil \mathcal{A}_Y \rceil)_{|C} \sim K_C + M/Z$ such that

- M is nef on C/Z;
- each $M_{|E_i} \sim 0$; and
- for each D_i, either $\deg M_{|D_i} \ge 1$ or D_i is not complete.

Indeed, $M = \lceil g^*(-K - B) \rceil_{|C} = \lceil -g_C^*(K + B) \rceil$ by the normal crossings and by Addendum 6.29.1, and we have:

- $-(K+B)$ is nef on X/Z;
- each $K+B$ is Cartier in Q; and
- for each D_i, either $(K+B)\cdot D_i < 0$ or D_i is not complete.

Then $\mathrm{Bs}\,|L| = \emptyset$ and this implies that $|C + \lceil \mathcal{A}_Y \rceil|$ is free near C on Y/T. Indeed, by (2) we can consider a Cartier divisor L' on a nodal curve C' with a single node, that is a result of a contraction $h\colon C \to C'$ of all E_i and $L = h^*L'$. Note that C' has ≥ 2 components D_i, and $\deg(K_{C'} + L')_{|D_i} \geq 1$ on each complete D_i. Thus $\mathrm{Bs}\,|L'| = \emptyset$ by[6.15]

(In the case of two branches, this gives a nonvanishing and a contradiction to the $\mathcal{A}'$-saturation near $\bigcup E_i$.) $\square$

Question 6.33 State and prove a semi log canonical version of the last freedom result in higher dimensions.

Proof of Addendum 6.26.2 Since $K+B$ is Klt, $\lceil \mathcal{A}_Y \rceil \geq 0$. Thus because D is b-free, in the proof of Proposition 6.26 we have an effective Cartier divisor $L \in |K_D + \lceil B' \rceil|$, this time with $\deg B' > 0$. Thus L has at most a single base point Q of multiplicity 1. Moreover, L has trivial mobile part (0) and has Q as a base point only if $K_D \equiv 0$ and $\deg L = 1$, that is, when D belongs to an elliptic pencil. Since $\deg L = 1$ we only need one blowup outside CS to resolve the base point, namely, the blowup in Q. $\square$

Proposition 6.34 *Let $(X/Z, B)$ be a Klt pair. Then the following holds.*

(i) For any crepant birational contraction $g\colon X \to Y/Z$ is a birational contraction, $\mathfrak{S}_Y = g(\mathfrak{S}) = \{g(S), S \in \mathfrak{S}\} \supset \mathfrak{S}(Y/Z, g(B) = B_Y)$.

(ii) (SAT) = (SAF) *in Conjecture 6.14 with the fixed $\mathcal{F}$.*

(iii) (SAF)=(STD) *in Definition 6.11 for any $\mathcal{F} \geq 0$.*

(iv) The standard set $\mathfrak{S} = \mathfrak{S}(X/T, B)$ has the following equivalent description, up to a component over the fractional part of B (see below),

[6.15] Something missing.

(Sa′F) $S = \mathrm{Supp}\,(\mathrm{Fix}\,\mathcal{D})_X$ *for some* $\mathcal{D} \geq 0$ *satisfying* (Sa′T) *of Addendum 6.26.1; and includes*

(SA′F) *a reduced divisor* F *is contained in the* substandard *set* $\mathfrak{S}' = \mathfrak{S}'(X/Z, B)$ *if* $\mathrm{Supp}\, F$ *only lies over integral components of* B*, and some b-free b-divisor* $\mathcal{M}$ *is saturated for* $\mathcal{A}' + F$ *on* some *sufficiently high model* X_{hr}/X.

$\mathcal{A}' = \mathcal{A} + \sum E_i$ *with the exceptional divisors* E_i *on* X *over integral components of* $\mathcal{A}$.

Remark 6.35 (1) *Up to a component over the fractional part of* B means that each $S \in \mathfrak{S}$ is $S' + F$, whereas S' under (Sa′F) and F is over the fractional part. Thus bounds on the two sets are equal.

(2) F is considered as a b-divisor.

Lemma 6.36 *Let* $\mathcal{C}$ *and* $\mathcal{M}$ *be b-divisors such that:*

- $\lceil \mathcal{C} \rceil \geq 0$, *and*
- $\mathcal{M}$ *is b-free.*

Then $\mathcal{C}$*-saturation of* $\mathcal{M}$ *on* some *sufficiently high model of* X/Z *is equivalent to* $\mathcal{C}$*-saturation on* every *sufficiently high model.*

Proof Some sufficiently high model Y/Z means any model of X/Z on which $\mathcal{M} = \overline{\mathcal{M}_Y}$ is free/Z, that is, $\mathrm{Bs}\,|\mathcal{M}_Y| = \emptyset$. Then $\mathcal{C}$-saturation of $\mathcal{M}$ on any model $X_{\mathrm{hr}}/Y/Z$ follows from that on Y. Indeed, since $\lceil \mathcal{C} \rceil \geq 0$, and $\mathcal{M}$ is free/Y/Z, $\mathcal{C}$-saturation on Y implies that

$$\mathcal{O}_Z(\lceil \mathcal{M} + \mathcal{C} \rceil) = \mathcal{O}_Z^{ch}(\lceil \mathcal{M} + \mathcal{C} \rceil) = \mathcal{O}_Z(\mathcal{M}) = f_*\mathcal{O}_Y(\mathcal{M}_Y) = f_*\mathcal{O}_{X_{\mathrm{hr}}}(\mathcal{M}_{X_{\mathrm{hr}}}).$$

This is the $\mathcal{C}$-saturation on X_{hr} (cf. Remarks 4.30, (1–2)). $\square$

Proof of Proposition 6.34

(i) For each $s \in \mathfrak{S}(Y/Z, B_Y)$, the log transform S of s is an element of $\mathfrak{S}(X/Z, B)$. Thus $s = g(S)$.

(ii) By definition $\operatorname{Mov}\lceil \mathcal{M}+\mathcal{F}+\mathcal{A}\rceil \le \mathcal{M}$. This means (SAF). Conversely, assuming Klt,

$$\lceil \mathcal{A}+\mathcal{F}\rceil \ge 0, \quad \operatorname{Mov}\lceil \mathcal{M}+\mathcal{F}+\mathcal{A}\rceil \ge \mathcal{M},$$

and $= \mathcal{M}$ by (SAF). On the other hand, since $\mathcal{F} \ge 0$, $\mathcal{M} \le \mathcal{M}+\mathcal{F}$, and this gives (SAT) for $\mathcal{M}+\mathcal{F}$.

(iii) $\mathcal{L} = \operatorname{Mov}\lceil \mathcal{A}+\mathcal{F}\rceil$. Again $\lceil \mathcal{A}+\mathcal{F}\rceil \ge 0$, and $\mathcal{L} \ge 0$. Since $\mathcal{M}$ is b-free, by (SAF),

$$\mathcal{M} \ge \operatorname{Mov}\lceil \mathcal{M}+\mathcal{F}+\mathcal{A}\rceil \ge \operatorname{Mov}\mathcal{M} + \operatorname{Mov}\lceil \mathcal{F}+\mathcal{A}\rceil = \mathcal{M}+\mathcal{L},$$

and $\mathcal{L} \le 0$. Thus $\mathcal{L} = 0$, which means (STD) for $\mathcal{D} = \mathcal{F}$. Conversely, (STD) for $\mathcal{D}$ means (SAF) for $\mathcal{M} = 0$ and $\mathcal{F} = \mathcal{D}$.

(iv) If $S = \operatorname{Supp}\mathcal{D}_X \in \mathfrak{S}$, then by Lemma 4.40, (STD) implies that 0 is saturated with respect to $\lceil S'+\mathcal{A}'\rceil \le \lceil \mathcal{A}+\mathcal{D}\rceil$, where S is fixed and S' is as in Remark 6.35, (1). The inequality follows from (SEF), since $\operatorname{Bs}|S'|_X = S' \subset S$. Thus (Sa$'$F) holds for $\mathcal{D} = S'$.

Conversely, if $\mathcal{D}$ satisfies (Sa$'$F) then, as in (ii–iii) above, $\mathcal{F}+\sum E_i + E'$ satisfies (STD) and (SEF), where $\mathcal{F} = \operatorname{Fix}\mathcal{D}$ and where $E' = \sum \varepsilon_i E_i$ with $0 < \varepsilon_i \ll 1$ for the exceptional divisors over the *fractional* components of $\mathcal{A}$. Note that E' does not affect (Sa$'$F) (cf. Addendum 4.40.1):

$$\left\lceil \mathcal{F}+\mathcal{A}+\sum E_i + E'\right\rceil = \lceil \mathcal{F}+\mathcal{A}'\rceil,$$

but (SEF) holds because $\mathcal{F}_X \subset \operatorname{Bs}|\mathcal{D}|_X$. Thus $\operatorname{Supp}\mathcal{F}_X \in \mathfrak{S}$.

By Lemma 4.40, (SA$'$F) implies (Sa$'$F) for $\mathcal{D} = \mathcal{M}+F$, where $\operatorname{Fix}\mathcal{D} = F$ and $\operatorname{Supp}F = F \in \mathfrak{S}$. In (SA$'$F), we can replace “for some model” by “for every model” by Lemma 6.36 with $\mathcal{C} = \mathcal{A}'$. $\square$

Corollary 6.37 *Let $(X/T, B)$ be as in Proposition 6.26, and suppose that it is Klt (but B may only be a subboundary). Then the standard set $\mathfrak{S} = \mathfrak{S}(X/Z, B)$ is bounded.*

In particular, $\mathfrak{S}'$ is also bounded in (SA$'$F).

Proof By Proposition 6.34, (3–4) and Remark 6.35, (1), it is enough to establish the boundedness for a *fixed reduced* $D = \sum D_i = \mathcal{D}_X = \mathcal{D}$ that satisfies (Sa′F), and in particular, (SA′F) with $\mathcal{L} = 0$ and $F = D$; we can also assume that each component $D_i \not\subset \operatorname{Supp} B$.

Since F is fixed and satisfies (Sa′F), by Lemma 4.40 each component D_i is $\mathcal{A}'$-saturated, and even $\mathcal{A}$-saturated (cf. Proposition 6.34, (3)). The same holds for any reduced $0 \le D' \le D$. In particular, these components are complete by Proposition 6.26 applied to a terminal resolution of $(X/T, B)$. Moreover, the prime components D_i belong to a bounded set. Thus we need to verify that the number of components in D is bounded. It is enough to consider only the global case with $Z = \mathrm{pt}$.

The boundedness of D_i implies that they have bounded intersections. Thus any two components can be disconnected by a bounded number of resolutions. Suppose that the set of D is unbounded. We derive from this a contradiction with the boundedness of resolutions to disconnect two components D_i and D_j. (Equivalently, the intersection numbers $D_i \cdot D_j$ are bounded.)

Since the number of components is unbounded, there exist a set of divisors D containing unboundedly many algebraically equivalent irreducible components D_i; clearly, $D_i^2 \ge 0$, for otherwise there can only be one such D_i. If $D_i^2 = 0$, the divisors D_i are all fibres of a fibration $X \to C$ over a curve C. Then for sufficiently many such components, there is an unbounded number of nonsingular fibres $0 < D' = \sum D_i \le D$ and $\sum D_i$ is free, which contradicts $\mathcal{A}$-saturation. Moreover, $D := D'$ is never sufficiently general on any model Y/Z of X/Z, that is, general in the free linear system $|D|$ on Y/Z. Note that two b-free divisors have the same 0 multiplicities in any fixed finite set of prime b-divisors. Indeed, then $K + B_Y + D$ is again Klt. Hence $\overline{D} \le \lceil D + \mathcal{A} \rceil$; this holds at each prime P_i on Y, because D is integral and $\lceil \mathcal{A}_Y \rceil \ge 0$ by Klt. If P_i is exceptional on Y, each $\operatorname{mult}_{P_i}(\mathcal{A} - \overline{D}) > -1$ and so each $\operatorname{mult}_{P_i} \mathcal{A} > \operatorname{mult}_{P_i} \overline{D} - 1$. Taking $\lceil\ \rceil$ we get $\operatorname{mult}_{P_i} \lceil \mathcal{A} \rceil \ge \operatorname{mult}_{P_i} \overline{D}$, because D is Cartier on Y with only integral multiplicities. Thus on any fixed sufficiently high model X_{hr}/Y, by the $\mathcal{A}$-saturation for $D := D'$ as b-divisor, $D = D_{X_{\mathrm{hr}}} \ge \operatorname{Mov} \lceil D + \mathcal{A} \rceil_{X_{\mathrm{hr}}} \ge \overline{D}_{X_{\mathrm{hr}}}$, which contradicts D fixed. (Compare the proof of Proposition 6.26.)

Hence $D_i^2 > 0$ and the divisors D_i intersect each other. By Addendum 6.26.1, any of these D_i intersects any other $D_j \cong D_i$ in a finite number of points, namely, in $D_i \cap (\operatorname{Supp} B \cup \mathrm{CS})$, where CS denotes non-Gorenstein or noncanonical points. Therefore, an unbounded number of prime components

D_i pass through the same point Q. This point depends on D. We can make a bounded resolution in such a points Q, e.g., the minimal resolution if Q is singular, or the usual blowup if Q is nonsingular on X. Then we replace (X, B) by its crepant transform.

The $\mathcal{A}$-saturation is birational, and the intersection points of D_i lie only over the log transform of $D_i \cap (\operatorname{Supp} B \cup \mathrm{CS})$. We consider the proper birational transform of D and of the prime components D_i. We again have an unbounded family of algebraically equivalent D_i, and again $D_i^2 > 0$. Again we have some (new) point Q through which an unbounded number of components D_i are passing. Again we make a bounded resolution, etc.

But this process must terminate, since any two components are disconnected by a bounded resolution. Or we can see that the new self-intersection $D_i^2 \leq$ the old self-intersection $D_i^2 -$ const., where const. $= 1$ for the usual blowup of nonsingular points, and some positive numbers at other points. $\square$

Example 6.38 Let $(X/\mathrm{pt.}, B = 0)$ be a (terminal) nonsingular del Pezzo surface. Then in Proposition 6.26, every $D = \mathcal{D}_X$ is either free or is a -1-curve, except for del Pezzo surfaces of degree 1 with $D \in |-K|$. The -1-curves D give fixed D of degree 1: $-K \cdot D = 1$. The standard sets are the disjoint sums of these -1-curves.

Then (FGA)$(X/\mathrm{pt.}, 0)$ gives the models $Xst = X$, except for $= \mathbb{P}^1$ in the case of a pencil, and $=$ pt. (cf. Example 9.11).

Pencils and irreducibility in (RIR) of Proposition 6.26 There is no pencil of elliptic curves every element of which is tangent to the 1-complement boundary C, because the restriction of the pencil on C is an isomorphism for a general divisor. Thus we have only a single element in the pencil with a single intersection (and tangent) point. However for higher genus these are possible.

Example 6.39 For (CBS) on X/Z or on a triple Y/Z in the 3-fold or higher dimensional case (even the local case), we need the numerical condition that $\mathcal{M}_Y/Z$ is *nef* (cf. (NEF) in Remark 6.15, (8)). In the one dimensional case, it is implied because $\mathcal{D}(X/Z, B)$ is effective in Conjecture 6.14; in the two dimensional case, because $\mathcal{M}$ is b-free or $|\mathcal{M}|_Y$ is free in codimension 1. But nef is important in dimension ≥ 3. Indeed, let $f\colon X \to Z$ be a small extremal contraction that is negative with respect to D. Then any infinite set of b-divisors $\mathcal{D}$ of the form $\mathcal{D}_X = nD$ for a natural number n is not canonically bounded, or equivalently, not log canonically bounded. Otherwise, we have a very negative curve C contracted by f for $K+cD_X$ with C as a LCS (center of log canonical singularities), and log threshold c along C for all D_X. This is impossible by anticanonical boundedness [11, Theorem]. However we can take another model Y/Z (for example, the D-flip) of X/Z, where $\{\mathcal{D}_Y\}$ is bounded (if the flip conjecture holds).

Corollary 6.40 *Conjecture 6.14 holds in dimension 2: for any weak log Fano contraction $(X/Z, B)$ with $\dim X = 2$, there exists a bounded family of desirable triples $(Y/T/Z, B_Y, \mathfrak{F})$ with induced standard $\mathfrak{F} = \mathfrak{S}_Y$ such that, for each b-free $\mathcal{M} \in \mathfrak{M}(X, B)$, the singularities of $\mathcal{M}_Y$ are bounded* for some desirable Y/T. *In particular, they are canonically bounded with respect to B_Y.*

Addendum 6.40.1 *In general, the only estimate is $0 < c \leq \mathrm{lld}(X, B)$.*

Remark 6.41 In applications, we can extend the induced standard set $\mathfrak{S}_Y$ to some $\mathfrak{F}$ by adding some fixed reduced divisors, e.g., $\operatorname{Supp} B_Y$, or even bounded sets (cf. proof of Proposition 9.15). However, we then lose (SA′F), for example, on triples other than $(X/X, B, \mathfrak{S})$. Otherwise to preserve (SA′F), we need to take $\mathfrak{S}_Y$ as an integral log transform: with $\sum E_i$, where E_i lie over integral components of $\mathcal{A}_Y$. Thus we need (SA′F) only as a condition to establish that $\mathfrak{F}$ is bounded.

The integral condition on F in (SA′F) of Proposition 6.34 can be replaced by $\operatorname{Supp} F \cap \operatorname{Supp} B_Y = \emptyset$ when B_Y is a boundary and (X, B) is Klt.

Proof We assume that the surface X is complete. This is important for the applications below, and nontrivial.

We associate desirable triples with respect to the bigness of $\mathcal{M}$, which proves (MOD). As in (CRP), each $(Y/T, B_Y)$ is a crepant model over (X, B) and satisfies (QFC). The corresponding set $\mathfrak{F}$ on Y is the log birational transform of the standard set $\mathfrak{S} = \mathfrak{S}(X, B)$. This is bounded by Corollary 6.37. We need to satisfy (TER) and on T (RPC), choose a triple *for each* $\mathcal{M} \in \mathfrak{M}(X, B)$.

Thus for big $\mathcal{M}$, we take a terminal resolution $(Y/T, B_Y)$ as a desirable triple with the identity $Y/Y = T$. In particular, it is a weak log Fano contraction satisfying (TER) and (RPC). In addition, $\mathcal{M}_Y = id^* M$, where $M = \mathcal{M}_Y$ is big. Since $\dim X = 2$ and $\mathcal{M}$ is b-free, M is nef. On any model, all such sufficiently general $\mathcal{M}_Y$ are reduced and irreducible. Hence by Proposition 6.26, all such sufficiently general divisors $\mathcal{M}_Y$ have bounded singularities everywhere, because (Y, B_Y) is terminal in points. We can take just one such triple.

The next case is when $\mathcal{M}$ gives a 1-dimension image, that is, there exists a rational contraction $X \dashrightarrow C$ onto a curve C with $\mathcal{M}_X$ in the fibres. In classical terminology, $|\mathcal{M}|_X$ is *composed of a pencil*. In this case, we define Y/T as the regular contraction defined by this pencil. In particular, $T = C$, and $\mathcal{M}_Y = g^*M$ for some divisor $M > 0$. Hence M is nef and big, and by construction, all such sufficiently general $\mathcal{M}_Y$ have bounded singularities everywhere. This family of triples is bounded, because the intersection of the elements of the pencil are bounded by Proposition 6.26. More precisely, by Addendum 6.26.2, every such pencil is elliptic and has a regularization in one blowup of a terminal resolution. In this case $C = T = \mathbb{P}^1$ by the rational connectedness of X, satisfies (RPC), and (Y, B_Y) satisfies (TER).

Finally, let the dimension of $|\mathcal{M}|_X$ be 0, that is, the effective divisor $\mathcal{M} = 0$. Take Y/T to be $X \to$ pt.; then $M = 0$. This triple is unique. $\square$

Proof of (FGA) **in the Main Theorem** Immediate by Theorem 6.19, (3) and Corollary 6.40. The birational case was done in Example 6.25. $\square$

Corollary 6.42 *Any* $(\mathrm{FGA})_n(\mathrm{bir})$ *algebra* $\mathcal{L} = \mathcal{R}_{X/T}\, D$*, is f.g., up to codimension* 2 *over* T*,* that is, over points of codimension ≤ 2.

Proof Immediate by $(\mathrm{FGA})_n$ with $n \leq 2$ in the Main Theorem. $\square$

Corollary 6.43 $(\mathrm{FGA})_3^{pl}$ *of* Example 4.45 *holds.*

Moreover, we can drop the condition that $\mathcal{L}$ is bounded by D and satisfies Main Lemma 3.39 (cf. Remark 11.8).

Proof Immediate by $(\mathrm{FGA})_d(\mathrm{bir})$ with $d \leq 2$, which was essentially done in Example 6.25, and Example 4.45. $\square$

Of course, by Corollary 4.46 (CBS) also implies (RFA), but we can prove more.

Theorem 6.44 *LMMP in dimensions* $d \le n-1$ *implies* $(\mathrm{CBS})^*_{d-1}$ *and* $(\mathrm{SSB})_{d-1}(\mathrm{gl})$*, where* * *means a modification in dimensions* $\le d-1$ *including log singularities.*

Moreover, we can drop $(\mathrm{CBS})^*_{d-1}$ *and* $(\mathrm{SSB})_{d-1}(\mathrm{gl})$ *with* $d \le 3$*, i.e, for* $n \le 4$*.* $(\mathrm{RFA})_{n,m}(\mathrm{bir})$ *holds with* $m \le n-1$*.*

Essentially, we prove the theorem for $n = 4$, that is, $(\mathrm{RFA})_{4,m}(\mathrm{bir})$ in Section 11; the * modification of (CBS) is explained in the proof of Corollary 10.13. The proof of the theorem is based on the stabilization of Theorem 9.9 in Section 9 and destabilization results in Section 10. However before we need to clarify restrictions of b-divisors in Section 7, and develop approximations in Section 8.

7 Restrictions of b-divisors

The[7.1] new tools we introduce here are different flavors of b-restriction from X to a b-divisor E. The proof of Theorem 9.9 uses systematically restrictions of b-divisors (cf. also Proposition 4.44). There are essentially the two following types, or some mixture of them.

7.1 Mobile restriction

Let E be a prime b-divisor on X and $\mathcal{D}$ a b-free divisor$/Z$. The *mobile restriction* of $\mathcal{D}$ to E is[7.2] the b-free divisor $\mathcal{M} = \mathcal{D}_{|E}$ on E defined by the formula

$$\mathcal{M}_Y = \mathcal{D}_{Y|E}$$

where Y/X is a model such that E is a normal divisorial subvariety of Y and $\mathcal{D}$ is free on Y/Z. Thus $\mathcal{M}_Y$ is also free on E/Z, and $\mathcal{M} = \overline{\mathcal{M}_Y}$. To cut out the restriction as a Weil divisor it is enough to take a normal model Y/X, where E is a divisorial variety,[7.3] nonsingular in X in divisors of E. Similarly,

[7.1] You could explain that this notion corresponds to the restriction of mobile linear systems.

[7.2] The broken bar for b-restriction is given by the macro \frestbar, and I may redesign it to give better line spacing.

[7.3] I think this means: It is enough to take a model Y/X on which E appears as a prime divisor, and such that Y is nonsingular at every prime divisor of E.

we can define a Cartier b-divisor $\mathcal{D}_{\shortmid E}$ for any Cartier b-divisor $\mathcal{D}$ on X (cf. (CAR) in the additional assumptions of 5.5). Thus mobile restriction is well defined, because the divisor has only 0 multiplicities up to linear equivalence. mobile restriction is *defined up to linear equivalence*.

7.2 Fixed restriction

When we define a fixed restriction $D_{|E}$, we usually require D to be Cartier and $E \not\subset \operatorname{Supp} D$. Moreover, by Lemma 6.29, we can assume that D is K-Cartier (where $K = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$); then $D_{|E}$ is also K-Cartier and restriction preserves linear equivalence, since it extends it. (Locally, each K-Cartier divisor is K-principal; cf. Definition 3.22. The restriction of any K-principal divisor is also K-principal.) In the same way we can define the *fixed restriction* $\mathcal{D}_{\shortmid E}$ of an K-Cartier b-divisor $\mathcal{D}$ whenever $\operatorname{mult}_E \mathcal{D} = 0$. Again we take

$$\mathcal{M}_Y = \mathcal{D}_{Y|E},$$

where Y/X is a model such that E is normal divisorial subvariety of Y and $\mathcal{D}$ is K-Cartier$/Y$, that is, $\mathcal{D} = \overline{\mathcal{D}_Y}$. The b-divisor $\mathcal{M} = \overline{\mathcal{M}_Y}$ is well defined a K-Cartier divisor$/E$. In this case some multiplicities grow on subsequent blowups. This restriction also preserves *linear equivalence*.

7.3 Mixed restriction

If $\mathcal{D}$ is a K-Cartier b-divisor having $\operatorname{mult}_E \mathcal{D} \in K' \subset K$, we can define the *mixed restriction* $\mathcal{M} = \mathcal{D}_{\shortmid E}$ by setting

$$\mathcal{M}_Y = \big(\mathcal{D}_Y - (\operatorname{mult}_E \mathcal{D})E\big)_{|E} + (\operatorname{mult}_E \mathcal{D})E'_{|E},$$

where Y/X is a model such that E is a normal divisorial subvariety of Y, $\mathcal{D}$ is K-Cartier$/Y$, and $E' \sim E$ on Y, but $E \not\subset \operatorname{Supp} E'$. This is again a K-Cartier b-divisor.

In other words, if E is $\mathbb{Q}$-Cartier on Y, K has characteristic 0, we obtain

$$\mathcal{M} = \big(\mathcal{D} - (\operatorname{mult}_E \mathcal{D})\overline{E}\big)_{\shortmid E} + (\operatorname{mult}_E \mathcal{D})\overline{E'}_{\shortmid E}.$$

Of course, this divisor depends on the choice of model Y/X and on the equivalence $E' \sim E$. Nonetheless this restriction is compatible with K'-*linear equivalence*, in particular, with $\sim$ for $K' = \mathbb{Z}$, by Lemma 6.29 and because $E' \sim E$ implies $\overline{E'} \sim \overline{E}$.

Caution 7.4 In general E' is not effective or prime. Note also, that E is automatically $\mathbb{Q}$-Cartier, provided that Y is $\mathbb{Q}$-factorial.

Where is the support of $\mathcal{D}_{|E}$? For saturations, we need to take integral parts of fractional divisors, and to estimate their fractional parts, in particular, for restrictions. For this purpose we now describe the support of restrictions on a case-by-case basis. In addition, for a birational contraction $f\colon X \to T$, we consider the case (df) that recalls discrepancy.[7.4]

Definition 7.5 A b-divisor $\mathcal{D}$ on X is K-Cartier at $p \in X$ if $\mathcal{D} = \overline{D'}$ locally near p, for some K-Cartier divisor D'. Note that in this case $D' = \mathcal{D}_X$ near p.

Example 7.6 Suppose that $\mathcal{D}$ is an *effective* b-free divisor/Z, or a K-linear combination $\sum k_i \mathcal{D}_i$ of such (for example, a difference). We assume that the $\mathcal{D}_i$ are in general position on X, and in particular, $\operatorname{Supp}\bigl(\sum k_i \mathcal{D}_i\bigr)_X = \bigcup \operatorname{Supp}(\mathcal{D}_i)_X$. This condition is equivalent to requiring that $\mathcal{D}$ is a K-Cartier b-divisor up to $\sim$. Then $\mathcal{D}$ is *Cartier* outside $\operatorname{Supp}(\mathcal{D}_X)$ (a divisorial subvariety on X). Indeed, $\mathcal{D} \sim 0$ outside the base locus of the linear system of $|\mathcal{D}|_X$. The same holds for K-Cartier divisors $\mathcal{D}$ on X. They are K-Cartier and even *Cartier* outside $\operatorname{Supp}(\mathcal{D}_Y)$ on any model Y/Z of X/Z whenever D_Y has above presentation with $\mathcal{D}_i$ in general position on Y. Moreover, we can replace $\operatorname{Supp}\mathcal{D}_Y$ by the intersection

$$\bigcap \operatorname{Supp}(\mathcal{D}'_Y),$$

where $\mathcal{D}'$ runs through the b-divisors $\mathcal{D}' \sim_K \mathcal{D}$ at least locally/Y, and D' is a K-linear combination as above, also general on Y, although not necessary effective.

For example, let $X \to Y = \mathbb{P}^2$ be the blowup of $P \in \mathbb{P}^2$, with -1-curve E over P, and let L, L_P be distinct lines through P. Then $\overline{E} = \mathcal{L} - \mathcal{L}_P$, where $\mathcal{L} = \overline{L}$ and $|\mathcal{L}_P| = |\mathcal{L} - P|$ are b-free. Thus $\overline{E}$ is Cartier outside

[7.4] I omit "such that $f_*(\mathcal{D}_1) = f_*(\mathcal{D}_2)$," because it doesn't seem to make sense without any $\mathcal{D}_1$ and $\mathcal{D}_2$ defined.

$\operatorname{Supp}\big((\mathcal{L}-\mathcal{L}_P)_Y\big) = L \cup L_P$. Note also that $\mathcal{L}_P \sim \mathcal{L} - \overline{E} \not\geq 0$ if $P \notin L$. However, $\mathcal{L}_P$ is not Cartier at P outside $\operatorname{Supp}\big((\mathcal{L}-\overline{E})_Y\big) = L$.

Proposition 7.7 *Let X be a model where E is a (normal) divisorial subvariety, and let $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2$ be K-Cartier b-divisors that are K-Cartier outside* $\operatorname{Supp}\big((\mathcal{D}_*)_X\big)$. *Then there are just two possible cases*[7.5] *for* $\operatorname{Supp}(\mathcal{D}_{|E})_E$*:*

(mv) $\operatorname{Supp}(\mathcal{D}_{|E})_E \sim \emptyset$ *if $\mathcal{D}$ is either Cartier or b-free over X; in other words, a Cartier b-divisor on X restricts to a Cartier b-divisor on E;*

(fx) $\operatorname{Supp}(\mathcal{D}_{|E})_E = \bigcap \operatorname{Supp} \mathcal{D}_X$ *if* $\operatorname{mult}_E \mathcal{D} = 0$.

Suppose in addition that E is $\mathbb{Q}$-Cartier. Write $\mathcal{D}_*$ for any of $\mathcal{D}$ or $\mathcal{D}_i$, and set $\mathcal{D}'_* \sim \mathcal{D}_* - (\operatorname{mult}_E \mathcal{D}_*)\overline{E}$, so that $\operatorname{mult}_E \mathcal{D}'_* = 0$. Suppose that each $\mathcal{D}_* - (\operatorname{mult}_E \mathcal{D}_*)\overline{E}$ is K-Cartier outside $\operatorname{Supp}(\mathcal{D}'_*)_X$. *Then, modulo $\sim$, there are just two possible cases for* $\operatorname{Supp}(\mathcal{D}_{|E})_E$*:*

(mx) $\operatorname{Supp}(\mathcal{D}_{|E})_E$ *equals the union of*

$$E \cap \operatorname{Supp}(\mathcal{D}_X - (\operatorname{mult}_E \mathcal{D})E) = E \cap \operatorname{Supp} \mathcal{D}'_X$$

and $E \cap \operatorname{Supp} E'$*; or*

(df) $\operatorname{Supp}(\mathcal{D}_{|E})_E$ *is contained in the union of*

$$E \cap \operatorname{Supp}((\mathcal{D}_1)_X - (\operatorname{mult}_E \mathcal{D}_1)E) = E \cap \operatorname{Supp}(\mathcal{D}'_1)_X$$

and

$$E \cap \operatorname{Supp}((\mathcal{D}_2)_X - (\operatorname{mult}_E \mathcal{D}_2)E) = E \cap \operatorname{Supp}(\mathcal{D}'_2)_X$$

for $\mathcal{D} := \mathcal{D}_1 - \mathcal{D}_2 - \operatorname{mult}_E(\mathcal{D}_1 - \mathcal{D}_2)\overline{E}$.

Remark 7.8 If E is $\mathbb{Q}$-Cartier, by Example 7.6, the condition that a b-divisor $\mathcal{D}$ is K-Cartier away from $\operatorname{Supp} \mathcal{D}$ holds if $\mathcal{D}$ is b-free or a K-linear combination of such b-divisors in general position on X.

[7.5] I can't figure out if this means $\operatorname{Supp}\Big((\mathcal{D}_{|E})_E\Big)$ or $\Big(\operatorname{Supp}(\mathcal{D}_{|E})\Big)_E$. All of this would be easier to read if we put appropriate sized brackets $\Big(\big(()\big)\Big)$ around the arguments, although that is time-consuming.

Proof (mv) and (fx) hold by definition (cf. Example 7.6 above). (fx) implies (mx) and (df). Indeed, $\mathcal{D} - (\operatorname{mult}_E \mathcal{D})\overline{E}$ is nontrivial and not K-Cartier$/E$ only over

$$E \cap \operatorname{Supp}\bigl(\mathcal{D} - (\operatorname{mult}_E \mathcal{D})\overline{E}\bigr)_X = E \cap \operatorname{Supp}\bigl(\mathcal{D}_X - (\operatorname{mult}_E \mathcal{D})E\bigr),$$

that is, modulo $\sim$, only over $E \cap \operatorname{Supp} \mathcal{D}'_X$. In (df) we take $\mathcal{D}' = \mathcal{D}'_1 - \mathcal{D}'_2$. Note that E' is $\mathbb{Q}$-Cartier everywhere on X together with E. $\square$

Corollary 7.9 *Suppose that*

- *K is a field;*
- *$a_2\mathcal{D}_2$ is b-free$/Z$, for some $a_2 \in K$;*
- *X/T is a birational contraction$/Z$, and $(\mathcal{D}_1)_T = (\mathcal{D}_2)_T$ on T.*

Then modulo $\sim$, we can replace the union in (df) *by the union*

$$\Bigl(E \cap \operatorname{Supp}(\mathcal{D}_1)_X\Bigr) \cup \Bigl(E \cap \operatorname{Bs}|a_2\mathcal{D}_2|_X\Bigr) \cup \left(E \cap \begin{pmatrix} \textit{other exceptional} \\ \textit{divisors of } X/T \end{pmatrix}\right).$$

In particular, for $a_1 = 1$, the non-Cartier points *on E and the* fractional part *of $(\mathcal{D}_{|E})_E$ are all contained in $E \cap \operatorname{Bs}|\mathcal{D}_1|_X$, $E \cap \operatorname{Bs}|a_2\mathcal{D}_2|_X$, and $E \cap$ other exceptional divisors of X/T.*

Addendum 7.9.1 *In the local case over $P \in Z$ (which is more general), modify the above assumptions to*

- *K is a field;*
- *$a_1\mathcal{D}_1$ is sufficiently general effective b-free$/Z$ for some $a_1 \in K$;*
- *$a_2\mathcal{D}_2$ is b-free$/Z$, for some $a_2 \in K$;*
- *X/T is a birational contraction$/Z$, and $(\mathcal{D}_1)_T = (\mathcal{D}_2)_T$ outside $f^{-1}P$.*

Then modulo $\sim$, the union in (df) *is contained in*

$$\Bigl(E \cap \operatorname{Supp}(\mathcal{D}_1)_X\Bigr) \cup \Bigl(E \cap \operatorname{Bs}|a_2\mathcal{D}_2|_X\Bigr) \cup \Bigl(E \cap (\textit{divisors of } X/P)\Bigr)$$

In particular, for $a_1 = 1$, the non-Cartier points *on E and the* fractional part *of $(\mathcal{D}_{|E})_E$ are all contained in $E \cap \operatorname{Bs}|\mathcal{D}_1|_X$, $E \cap \operatorname{Bs}|a_2\mathcal{D}_2|_X$, and $E \cap$ divisors of X/P).*

Proof $\mathcal{D}_1 - \mathcal{D}_2$ is exceptional/T (and respectively, in divisors/P). The same holds over X up to exceptional divisors of X/T. More precisely, by (df) (with $\mathcal{D}'_1 = \mathcal{D}_1$, when $a_1\mathcal{D}_1$ is sufficiently general effective b-free) and because $(\mathcal{D}_2)_X = (\mathcal{D}_1)_X$ up to exceptional divisors (and respectively, divisors/P), $\mathcal{D}_{|E}$ is nontrivial modulo $\sim$ only over $E\cap$ the exceptional divisors ($\neq E$) of X/T (or respectively $E\cap$ divisors/P), $E\cap \operatorname{Supp}(\mathcal{D}_1)_X$ or $E\cap \operatorname{Supp}(\mathcal{D}_2^g)_X$, where $\mathcal{D}_2^g \sim a_2\mathcal{D}_2$ is sufficiently general effective. The case with $\operatorname{mult}_E \mathcal{D}_1 \neq 0$ is trivial: $E \subset \operatorname{Supp}(\mathcal{D}_1)_X$.

Indeed, $\mathcal{D}_X = 0$ near any point $Q \in E$ outside $E\cap$ exceptional divisors ($\neq E$) of X/T (and respectively in $E\cap$ divisors of X/P) and $E\cap\operatorname{Supp}(\mathcal{D}_1)_X$. Hence $(\mathcal{D}_2)_X = eE$ near Q with $e = \operatorname{mult}_E \mathcal{D}_2$. Since $\mathcal{D}_2$ is b-free/X (near Q, in particular), then $\mathcal{D}_{|E}$ is nontrivial over Q, only when $\mathcal{D} \neq e\overline{E}$ near Q and therefore, by Example 7.6 above, only when all $(D_2^g)_X$ pass through Q.

Taking the intersection of $E\cap\operatorname{Supp}(\mathcal{D}_2^g)_X$ for all effective $\mathcal{D}_2^g \sim a_2\mathcal{D}_2$ we obtain that $Q \in E\cap \operatorname{Bs}(a_2\mathcal{D}_2)_X$, whenever $\mathcal{D}_{|E}$ is nontrivial/Q. $\square$

8 Approximations

Definition 8.1 Let $F \subset X$ be a reduced divisor.[8.1] A *set of (effectively) bounded semiample/Z divisors* (or an *bnd set/Z*) is defined to be a set $\mathfrak{N}_{\mathrm{bnd}}(F) = \mathfrak{N}_{\mathrm{bnd}}(X/Z, F)$ of $\mathbb{R}$-divisors of X that are semiample/Z and satisfy the following conditions:

(BFP) the *fractional part* of D is *bounded* by F: $D = D_{\mathrm{int}} + D_{\mathrm{fr}}$, where D_{int} is integral, and $\operatorname{Supp} D_{\mathrm{fr}} \subset F$;

(CFG) $\mathfrak{N}_{\mathrm{bnd}}(F)$ has a set of generators that is *compact plus finite*: there is a compact rational polyhedron $\mathfrak{N}_c$ and a (finite) set of integral divisors D_i in $\mathfrak{N}_{\mathrm{bnd}}(F)$ such that each $D \in \mathfrak{N}_{\mathrm{bnd}}(F)$ has a decomposition $D = D_c + \sum n_i D_i$, or even *the same up to* $\sim$, where $D_c \in \mathfrak{N}_c$ and every $n_i \in \mathbb{N}$; and

(BND) the same as condition (BND) in Definition 5.17 for $\mathfrak{N}_{\mathrm{bnd}}(F)$, or equivalently, the same holds for $\mathfrak{N}_c$.

[8.1] This material is very similar to that of Section 5, especially Lemma 5.15 and Corollary 5.19. Maybe it would be better to organize them together.

Taking a bigger F we can assume that the polyhedron of (CFG) is contained in the $\mathbb{R}$-space $\mathfrak{D}_F$ (for the notation, see Section 5, in proof of Corollary 5.13). Note that (CFG) implies (BND) when $\{D_i\}$ is finite.

The *uniform neighbourhood* of $\mathfrak{N}_{\text{bnd}}(F)$ with *diameter* $\delta \in \mathbb{R}$ is the set of Weil $\mathbb{R}$-divisors

$$\begin{aligned}\mathfrak{U}_{\text{bnd},\delta}(X/Z,F) &= \mathfrak{U}_{\text{bnd},\delta}(F)\\ &= \left\{D = \sum d_i P_i \;\middle|\; d_i \in \mathbb{R} \text{ and } \|D - \mathfrak{N}_{\text{bnd}}(F)\| < \delta\right\},\end{aligned}$$

assuming that $\| \ \|$ is taken over $\operatorname{Supp}(D - \mathfrak{N}_{\text{bnd}}(F)) \subset F$. In other words, the following two equivalent conditions holds:

(i) $D = D_{\text{int}} + D_{\text{fr}}$ such that D_{fr} is supported in F and there exists $D' = D_{\text{int}} + D'_{\text{fr}} \in \mathfrak{N}_{\text{bnd}}(F)$ with $\|D - D'\| = \|D_{\text{fr}} - D'_{\text{fr}}\| < \delta$;

(ii) there exists $D' \in \mathfrak{N}_{\text{bnd}}(F)$ such that $D - D'$ is supported in F and $\|D - D'\| < \delta$.

In particular, in either case $\|D - \mathfrak{N}_{\text{bnd}}(F)\| \leq \|D - D'\| < \delta$. (i) implies (ii), because $D - D' = D_{\text{fr}} - D'_{\text{fr}}$ is supported in F. (ii) implies (i) for $D_{\text{fr}} = D - D' + D'_{\text{fr}}$, because then $D = D_{\text{fr}} + D' - D'_{\text{fr}} = D_{\text{fr}} + D_{\text{int}}$ and $D - D' = D_{\text{fr}} - D'_{\text{fr}}$.

Example 8.2 (1) Any rational polyhedral cone generated by a finite set of semiample/Z divisors D_i is a *bnd cone*/Z. We can assume that each D_i is integral. Then we can take

- F as the common support of all divisors D_i; and
- $\mathfrak{N}_c = \left\{\sum r_i D_i \;\middle|\; r_i \in [0,1]\right\}$.

(2) Let $X \to T/Z$ be a contraction. The most important example of a bnd set is the Abelian semigroup $\mathfrak{N}(F) = \mathfrak{N}(X/T/Z, F)$ of divisors D that are nef/Z and $\sim_{\mathbb{R}} 0/T$, and have fractional part bounded by F. (In particular, D is assumed to be an $\mathbb{R}$-Cartier divisor.) In this case, a *uniform* neighbourhood $\mathfrak{U}_\delta(F) = \mathfrak{U}_\delta(X/T/Z, F)$ of $\mathfrak{N}(F)$ with *diameter* δ is also defined as in Definition 8.1.

In fact, $\mathfrak{N}(F)$ is a bnd set, and is even semiample in several cases which, although very special, are crucial for our purposes:

(WLF) $(X/Z, B)$ is a weak log Fano contraction for some boundary B and $T = X$;

(0LP) $(X/Z, B)$ is a 0-log pair for some boundary B (cf. Remark 3.26, (2)), and $T = Z$; and

(TRP) triples $(X/T/Z, B, \mathfrak{F})$ with $F \in \mathfrak{F}$.

In (0LP), the bound (BND) does not hold in general: torsion points of Abelian varieties contradict (BND). To satisfy (CGF) and (BND), we need to impose irregularity $= 0$, so that the Picard group is finitely generated. Thus integral points of $\mathfrak{F}(X/Z/Z, F)$ are finitely generated modulo $\sim$, and $\mathfrak{N}_c$ is a cube with bounded multiplicities as in (1) above. In particular, this is true if X/Z is a weak log Fano contraction. Moreover, the bnd property for (WLF) follows from (TRP) in the case of desirable triples (see Lemma 8.12, (5) below).

Definition 8.3 (Freedom with tolerance $\tau > 0$) We fix a reduced divisor F; then $D \sim D_{\mathrm{bf}} + D_{\mathrm{fr}}$, where $\operatorname{Bs}|D_{\mathrm{bf}}| = \emptyset$. In particular, D_{bf} is Cartier, and D_{fr} is supported in F with $\|D_{\mathrm{fr}}\| < \tau$ (cf. Lemma 5.20). We denote this by $\operatorname{Bs}|D| = \emptyset \mod \tau$, and pronounce the linear system to be free *with tolerance* τ.

Nonvanishing with tolerance $\tau > 0$ is similar: $D \sim D_{\mathrm{int}} + D_{\mathrm{fr}}$, where D_{int} is effective integral and D_{fr} is supported in F with $\|D_{\mathrm{fr}}\| < \tau$. We denote this by $|D| \neq \emptyset \mod \tau$ and pronounce the linear system to be *nonempty with tolerance* τ.

The main result of the section is

Theorem 8.4 *Suppose given:*

- *a bounded family $(X_u/Z_u, F_u)$ for $u \in U$ of pairs with reduced divisor F_u on X_u;*
- *a* family $\mathfrak{N}_{nbf}(X_u/Z_u, F_u)$ *of bnd sets/U of divisors* (see (7) and (7′) in the proof below); *and*
- *a tolerance $\tau > 0$.*

Then there exists a natural number M (depending on the family $(X_u/Z_u, F_u)$ and τ) such that, for any $u \in U$ and any $D \in \mathfrak{N}_{\mathrm{bnd}}(X_u/Z_u, F_u)$, $\operatorname{Bs}|mD| = \emptyset \mod \tau$*, in particular, $|mD| \neq \emptyset \mod \tau$, for some $1 \leq m \leq M$ (depending on D).*

Addendum 8.4.1 *Moreover, there is a real number $\delta > 0$ (depending on the family $(X_u/Z_u, F)$ and τ), such that the same freedom and nonvanishing hold for any $u \in U$ and any $D \in \mathfrak{U}_{\mathrm{bnd},\delta}(X_u/Z_u, F_u)$.*

Remark 8.5 Both choices of M and δ are very ineffective, as in Kronecker's theorem.

We start with properties of freedom with tolerance.

Proposition 8.6 *(1) Freedom with tolerance τ is invariant under linear equivalence: if $D \sim D'$, then $\operatorname{Bs}|D| = \emptyset \mod \tau$ implies $\operatorname{Bs}|D'| = \emptyset \mod \tau$.*

(2) Freedom with tolerance is an open property*: if $\operatorname{Bs}|D| = \emptyset \mod \tau$, then there is an $\varepsilon > 0$ such that, for any D' with $D - D'$ supported in F and $\|D - D'\| < \varepsilon$, also $\operatorname{Bs}|D'| = \emptyset \mod \tau$.*

(3) If $D = D_{\mathrm{bf}} + D'$, where $\operatorname{Bs}|D_{\mathrm{bf}}| = \emptyset$ and $\operatorname{Bs}|D'| = \emptyset \mod \tau$, then $\operatorname{Bs}|D| = \emptyset \mod \tau$.

The same holds for nonvanishing with tolerance τ.

Proof (1) holds by definition. In (2), we can take any $0 < \varepsilon \le \tau - \|D_{\mathrm{fr}}\|$, where $D \sim D_{\mathrm{bf}} + D_{\mathrm{fr}}$ as in Definition 8.3 and $D' \sim D_{\mathrm{bf}} + D'_{\mathrm{fr}}$. (3) holds because the sum of two free linear systems is also free. □

Proof of Theorem 8.4 Let $(X/Z, F)$ be a point of the family, possibly nonclosed (= prime subfamily). Let $F = \sum P_i$ and $\mathfrak{D}_F = \left\{\sum d_i P_i \;\middle|\; d_i \in \mathbb{R}\right\}$ be the set of divisors generated over F.

For $\mathfrak{N}_{\mathrm{bnd}}(F)$, suppose that we have a compact subspace $\mathfrak{N}_c \subset \mathfrak{N}_{\mathrm{nef}}(F)$, a (finite) set of Cartier divisors $T_i \in \mathfrak{N}_{\mathrm{bnd}}(F)$ and a finite set of $\mathbb{Q}$-Cartier divisors $N_i \in \mathfrak{N}_c$ with open neighbourhoods U_i of $N_i \in \mathfrak{D}_F$, such that

(1) each T_i is free$/Z$;

(2) each N_i has a *freedom index* $n_i > 0$ for which $\operatorname{Bs}|n_i N_i| = \emptyset$;

(3) in addition, for each $D \in U_i$, $\operatorname{Bs}|n_i D| = \emptyset \mod \tau$; more precisely, $n_i D = n_i N_i + D_{\mathrm{fr}}$ with D_{fr} supported in F with $\|D_{\mathrm{fr}}\| < \tau$;

(4) $\mathfrak{N}_c$ is a compact rational polyhedron with generators N_i;

(5) each $D \in \mathfrak{N}_{\mathrm{bnd}}(F)$ decomposes as $D \sim D_c + \sum t_i T_i$ with natural numbers t_i and $D_c \in \mathfrak{N}_c$; and in turn

(6) $\mathfrak{N}_c \subset \bigcup U_i$; and

(7) each T_i is defined and is free$/Z_u$ for each specialization of $(X/Z, F)$ and for any general point of $(X/Z, F)$. That is, for every u in some nonempty Zariski open subset U, $\mathfrak{N}_{\mathrm{bnd}}(F)$ specializes isomorphically to $\mathfrak{N}_{\mathrm{bnd}}(F_u)$ preserving the structures T_i, N_i and $\mathfrak{N}_c$ in (1–4) above.

Then Theorem 8.4 holds for $\mathfrak{N}_{\mathrm{bnd}}(F)$, and there exists $\delta > 0$ such that Addendum 8.4.1 holds for $\mathfrak{U}_{\mathrm{bnd},\delta}(F)$. Moreover, the theorem and addendum both hold for every u in a nonempty $V \subset U$. In particular, in the proof of the theorem and Addendum 8.4.1, we can use Noetherian induction; that is, it is enough to verify both on a single element $(X/Z, F)$ of the family. Thus if there exist a natural number $M_V = M_F$ and real number $\delta_V = \delta_F$ giving the theorem and addendum at $(X/Z, F)$, and numbers $M_{(X/Z,F)\setminus V}$ and $\delta_{(X/Z,F)\setminus V}$ that give the same over the closed proper subfamily $(X/Z, F) \setminus V$, then both hold over all points (= specializations) of $(X/Z, F)$ with $M = \max\{M_V, M_{(X/Z,F)\setminus V}\}$ and $\delta = \min\{\delta_V, \delta_{(X/Z,F)\setminus V}\}$.

Now (1–6) holds over some V. (1) holds by our assumption in (7). (2) and (5) define a nontrivial affine open subset V over which all $n_i N_i$ are free, and restriction (= specialization) of $\sim$ on X_u/Z_u is correct (= in general position; cf. (GPN) of Proposition 4.44). To have good specialization of $\sim$, it is enough to consider X_u/Z_u that are smooth at the generic point of X_u (= multiplicity 1), or just on the moduli space by Lemma 6.29. Thus (2) and (5) hold in each $u \in V$. The inequality in (3), the generators (4), and the inclusion in (6) concern multiplicities of divisors P_i and are uniform over the connected component of F in U, because we assume that F_U is reduced everywhere over U by (7′) below. This establishes the Noetherian induction by (7).

Now derive the theorem and the addendum for a fixed $(X/Z, F)$ from (1–6), and then verify these together with (7). We take $M_F = \max\{n_i\}$ for the indexes in (2); δ_F is chosen later.

Indeed, Proposition 8.6, (1) and (3) and the above (1) and (5) imply that it is enough to verify the theorem and the addendum respectively for $\mathfrak{N}_c$ and for

$$\mathfrak{U}_{c,\delta} = \{D \mid D - D' \in \mathfrak{D}_F,\ D' \in \mathfrak{N}_c \text{ and } \|D - \mathfrak{N}_c\| \le \|D - D'\| < \delta\}.$$

In turn both the theorem and the addendum follow from (2–3) and (6). For the addendum we need to choose such $\delta = \delta_F$ that $\mathfrak{U}_{c,\delta_F} \subset \bigcup U_i$. Such δ_F exists since $\bigcup U_i$ is open.

Finally, we verify (1–7). Note for this that the boundedness of the family means

(7′) on the total space of the family, there are a reduced divisor F, integral semiample divisors D_i, and semiample $\mathbb{Q}$-divisors V_i such that, for each u,

- $F_u = F_{|X_u}$ is reduced;
- $V_{i|X_u}$ are vertices of $\mathfrak{N}_c(F_u)$ (= a simplicial structure with the vertexes);
- $D_{i|X_u}$ and $\mathfrak{N}_c(F_u)$ generate $\mathfrak{N}_{\mathrm{bnd}}(F_u)$ as in (CFG); and, finally,
- (BND) holds *uniformly* for the family.

(BND) means that there is natural number $I > 0$ such that, for any semiample integral divisor D, $\mathrm{Bs}\,|ID| = \emptyset$. To satisfy (1) and (7), we take $T_i = ID_i$. We set $\mathfrak{N}_c = I\mathfrak{N}_c(F)$, and $N_i = IV_{i|X}$. This implies (2) with some n_i for each N_i, also by (BND). More precisely, $n_i \le Id_i$, where d_i is the minimal natural number such that d_iN_i is integral. Thus (BFP) and Proposition 8.6, (2–3) implies (3) for some U_i. By definition we can take U_i as an open disc with the center N_i in a rational translate of $\mathfrak{D}_F$ and of the radius $r_i = \tau/n_i \ge \tau/Id_i$.

(4) and (7), with U as a normal points in the subfamily given by point $(X/Z, F)$, holds by (7′).

(5) holds by (CFG) for $(X/Z, F)$ in (7′).

To satisfy (6) we need to add $\mathbb{Q}$-divisors N_i from $\mathfrak{N}_c$ that does not affect (4). Since each such N_i is semiample, as above we can find n_i and U_i for (2–3). If it is an open covering then by the compactness of $\mathfrak{N}_c$ we can take a finite subset of N_i that satisfies (6) and still (2–4). Actually, the discs form a covering. To verify this we can assume that $\mathfrak{N}_c$ is a simplex, and all its faces are covered by an induction.

Indeed, the $\mathbb{Q}$-points of the rational simplex $\mathfrak{N}_c$ are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, so these are covered.

On the other hand, by simultaneous approximation, Cassels [2, Chap. 1, Theorem VII], for any $\varepsilon > 0$, each point of the affine span $\mathfrak{L}$ of $\mathfrak{N}_c$, in particular, each point $D \in \mathfrak{N}_c$, has a rational approximation N_i in $\mathfrak{L}$ such that $\|D - N_i\| < \varepsilon/d_i$. Hence $N_i \in \mathfrak{N}_c$ and $D \in U_i$ respectively whenever $\varepsilon/d_i \leq \varepsilon <$ the distance from $D \in \mathfrak{N}_c$ to the complement $\mathfrak{L} \setminus \mathfrak{N}_c$ and $\varepsilon < \tau/I$. Thus the internal points of $\mathfrak{N}_c$ are covered by U_i. This completes the induction and the proof of the theorem. $\square$

Freedom with tolerance sometimes implies the usual nonvanishing, but typically this is still far from freedom.

Corollary 8.7 *Let C_u be a divisor on X_u with* $\operatorname{mult}_{P_i} C_u > -1 + \tau$ *for all prime divisors P_i of X. Then, for any $D \in \mathfrak{U}_{\mathrm{bnd},\delta}(F_u)$, there exists some natural number $1 \leq m \leq M$ such that $|\lceil mD + C_u \rceil| \neq \emptyset$.*

Addendum 8.7.1 *If* $\operatorname{mult}_{P_i} F_u = 0$ *we can assume just* $\operatorname{mult}_{P_i} C_u > -1$.

Proof In the proof below "all ..." means for all prime divisors P_i of X. We drop the subscripts $_u$.

Take m in Addendum 8.4.1 such that $|mD| \neq \emptyset \mod \tau$, that is, $mD \sim D_{\mathrm{int}} + D_{\mathrm{fr}}$, where $D_{\mathrm{fr}} = \sum d_{\mathrm{fr},i} P_i$ with all $|d_{\mathrm{fr},i}| < \tau$, in particular $d_{\mathrm{fr},i} > -\tau$. Hence

$$\operatorname{mult}_{P_i}(D_{\mathrm{fr}} + C) = d_{\mathrm{fr},i} + \operatorname{mult}_{P_i} C > \operatorname{mult}_{P_i} C - \tau > -1$$

and all $\operatorname{mult}_{P_i} \lceil D_{\mathrm{fr}} + C \rceil = \lceil \operatorname{mult}_{P_i}(D_{\mathrm{fr}} + C) \rceil \geq 0$. Thus $\lceil D_{\mathrm{fr}} + C \rceil \geq 0$ and $\lceil D_{\mathrm{int}} + D_{\mathrm{fr}} + C \rceil = D_{\mathrm{int}} + \lceil D_{\mathrm{fr}} + C \rceil \geq 0$. This proves the required nonvanishing: $\lceil mD + C \rceil \sim \lceil D_{\mathrm{int}} + D_{\mathrm{fr}} + C \rceil \geq 0$ by Lemma 6.30, since $mD + C \sim D_{\mathrm{int}} + D_{\mathrm{fr}} + C$.

Since $d_{\mathrm{fr},i} = 0$ when $\operatorname{mult}_{P_i} F = 0$, we get Addendum 8.7.1, because then again $\operatorname{mult}_{P_i}(D_{\mathrm{fr}} + C) = d_{\mathrm{fr},i} + \operatorname{mult}_{D_i} C > 0 - 1 > -1$. $\square$

We can use Corollary 8.7 as a restriction on τ for which the former[8.2] holds.

Example 8.8 1. Any $\tau < 1$, when $C \geq 0$.

[8.2] I don't know what "the former" refers to.

2. Let $C = \mathcal{A}_E$, the discrepancy divisor. Then all

$$\tau < \min\{1 - b_i\} \leq \operatorname{lld}(X, B).$$

In particular, there exists $\tau > 0$ provided that all $b_i < 1$, that is, (X, B) is Klt in divisors.

Corollary 8.9 *Suppose that each F_u is $\mathbb{Q}$-Cartier on X_u. Let $\alpha < 1$ and τ be positive real numbers, and $\mathcal{C}_u$ an $\mathbb{R}$-b-divisor of X_u such that $\mathcal{C}_u \geq 0/X_u$ and*

$$\operatorname{mult}_{P_i} \alpha\mathcal{C}_u > -\alpha + \tau \operatorname{mult}_{P_i} F_u \quad \textit{for all } P_i. \tag{$*$}$$

Then there exists a natural number M and positive real numbers β, δ giving the following nonvanishing$/Z_u$ on any Y_u/X_u (uniformly in u). For any u, and for all b-divisors $\mathcal{D}$ such that the descent data $\mathcal{E}$ of $\mathcal{D}$ over X_u is bounded by $\beta\mathcal{C}_u$, that is, $\mathcal{E} \leq \beta\mathcal{C}_u/X_u$, and $\mathcal{D}_X \in \mathfrak{U}_{\mathrm{bnd},\delta}(F_u)$, we have

$$|\lceil m\mathcal{D}_Y + (\mathcal{C}_u)_Y \rceil| \neq \emptyset \quad \textit{for some } 1 \leq m \leq M.$$

Proof Again we drop the subscripts $_u$. By Addendum 8.4.1, we can choose a natural number M and $\delta > 0$, depending on the family $(X/Z, F)$ and τ. Now take any

$$\beta \in \mathbb{R} \quad \text{with} \quad 0 \leq \beta \leq \frac{1-\alpha}{M}.$$

Take now any b-divisor $\mathcal{D}$ satisfying the nonvanishing conditions. Let m be a natural number in $[1, M]$ for which $\operatorname{Bs}|m\mathcal{D}_X| = \emptyset \mod \tau$. We verify that $|\lceil m\mathcal{D}_Y + \mathcal{C}_Y \rceil| \neq \emptyset$ for arbitrary Y/X; this follows from Lemma 5.20 with $\mathcal{D} := m\mathcal{D}$. Assumption $(**)$ of the lemma follows from the boundedness of the descent data of the original $\mathcal{D}$. Since $\mathcal{C} \geq 0/X$ and by (HOM) of Proposition 5.4, for any natural number $m \leq M$ (or even for any positive real number), the descent data $\mathcal{E}$ of $\mathcal{D} := m\mathcal{D}$ is bounded by $m\beta\mathcal{C} \leq M\beta\mathcal{C} \leq (1-\alpha)\mathcal{C}/X$, that is,

$$\mathcal{E} \leq (1-\alpha)\mathcal{C}/X. \quad \square \tag{$**$}$$

Remark 8.10 For a given family $(X_u/Z_u, F_u)$ with $\mathcal{C}_u$, we find constants $\alpha, \beta, \delta, \tau$ and M in the following order. First, we take any positive $\alpha < 1$. Next, we choose an appropriate positive τ (cf. Example 8.21 below). Next, by Theorem 8.4 and Addendum 8.4.1 we can find δ and M. Finally, we take β as in the preceding proof. Note that to find τ we need to assume that each F_u is $\mathbb{Q}$-Cartier. For this, we usually assume that X is $\mathbb{Q}$-factorial. In applications we only need $\beta, \delta > 0$ and a natural number M, and so can drop α, τ (cf. Corollary 8.26 below).

Remark 8.11 If $\mathcal{D}_X$ is $\mathbb{R}$-Cartier, the descent data $\mathcal{E}$ exists by (EXI) in Proposition 5.4; hence it exists for all $\mathcal{D}$ if X is $\mathbb{Q}$-factorial.

Now we apply the results obtained above to triples $(X/T/Z, B, \mathfrak{F})$ (see Definition 6.9, and cf. Corollary 6.40). A *bounded* family of triples is a family with bounded moduli; for the details, see the proof of Theorem 8.15 below.

The next preliminary result explains also the role of the conditions (CRP) and (RPC) in Definition 6.9 (cf. Example 8.18 below). We first fix notation: Let

$$\mathrm{CDiv}^0_{\mathbb{R}}(X/T/Z) = \{D \in \mathrm{CDiv}_{\mathbb{R}}(X/Z) \mid D \sim_{\mathbb{R}} 0/T\}.$$

Equivalently, every such divisor $D \sim_{\mathbb{R}} g^*M/Z$ for some $\mathbb{R}$-Cartier divisor $M \in \mathrm{CDiv}_{\mathbb{R}}(T/Z)$. Moreover, M is defined up to $\sim_{\mathbb{R}} /Z$ (cf. Lemma 3.24), and $\mathbb{R}$-linear equivalence $\sim_{\mathbb{R}} /Z$ and numerical equivalence $\equiv /Z$ are the same on T/Z (cf. Lemma 8.12, (1) below). Thus we have an $\mathbb{R}$-linear projection

$$g_*\colon\ \mathrm{CDiv}^0_{\mathbb{R}}(X/T/Z) \to \mathrm{CDiv}_{\mathbb{R}}(T/Z)/\equiv .$$

This map is defined over $\mathbb{Q}$, but not over $\mathbb{Z}$ in general, since there may be multiple fibres. Thus on each $\mathbb{R}$-linear subspace $L \subset \mathrm{CDiv}^0_{\mathbb{R}}(X/T/Z)$ we have an *induced* splitting

$$L = L^0 \oplus L^1,$$

where $L^0 = \ker g_{*|L}$ and L^1 is isomorphic to g_*L. Note that L^0 is a subspace of $\ker g_* = \mathrm{CDiv}^0_{\mathbb{R}}(X/Z/Z)$. Note also that L^1 is not defined uniquely. Nonetheless, we use it to identify the second summand, and denote L^1 by g_*L. If L is defined/$\mathbb{Q}$, the splitting can also be defined over $\mathbb{Q}$. For example, if

$$L = \mathfrak{C} = \mathfrak{C}(X/T/Z, F) = \{D = \sum d_i P_i \mid d_i \in \mathbb{R} \text{ and } D \sim_{\mathbb{R}} 0/T\} \subset \mathfrak{D}_F$$

with $F = \sum P_i$, we obtain a splitting

$$\mathfrak{C} = \mathfrak{C}^0 \oplus g_*\mathfrak{C}, \quad \text{where}$$
$$\mathfrak{C}^0 = \mathfrak{C}^0(X/T/Z, F) = \{D = \sum d_i P_i \mid d_i \in \mathbb{R} \text{ and } D \sim_{\mathbb{R}} 0/Z\}.$$

Lemma 8.12 *Let $(X/T/Z, B, \mathfrak{F})$ be a desirable triple for a weak log Fano contraction and $F = \sum P_i \in \mathfrak{F}$ a reduced divisor. Then*

(1) g_* induces a splitting/$\mathbb{Q}$

$$\mathfrak{C} = \mathfrak{C}^0 \oplus g_*\mathfrak{C}, \quad \textit{where now}$$
$$\mathfrak{C}^0 = \mathfrak{C}^0(X/T/Z, F) = \{D = \sum d_i P_i \mid d_i \in \mathbb{R} \textit{ and } D \equiv 0/Z\}.$$

(2) Cone theorem: the nef or even semiample cone

$$\begin{aligned}\mathfrak{N}_{\mathrm{nef}} &= \mathfrak{N}_{\mathrm{nef}}(X/T/Z, F) \\ &= \{D = \sum d_i P_i \mid d_i \in \mathbb{R}, D \textit{ is nef and } \sim_{\mathbb{R}} 0/T\}\end{aligned}$$

is a rational polyhedral cone in $\mathfrak{C}$.

(3) $g_*\mathfrak{C}$ is isomorphic/$\mathbb{Q}$ to $\mathrm{CDiv}_{\mathbb{R}}(T/Z)/\equiv$*, whenever the components P_i generate the group of Weil divisors of X / $\sim$.*

(4) There exists a uniform constant, the freedom index I*, which is a natural number such that for each nef integral $\mathbb{Q}$-Cartier $D \in \mathrm{CDiv}^0_{\mathbb{R}}(X/T/Z)$, the multiple ID is free on X/Z.*

(5) $\mathfrak{N}(X/T/Z, F)$ is a bnd set (cf. (TRP) in Example 8.2).

Proof (1) Since the triple is a desirable triple for a weak log Fano contraction $(X'/Z, B')$, we have a diagram of contractions

$$\begin{array}{ccccc} & & W & & \\ & {}^{i}\swarrow & & \searrow^{h} & \\ X' & & & & X \xrightarrow{g} T/Z \end{array}$$

where $i \circ h^{-1} \colon X \to X'$ is a birational map to a weak log Fano contraction X'. We need to prove that each $\mathbb{R}$-Cartier divisor $D \equiv 0$ on X/Z is $\sim_{\mathbb{R}}$ $/Z$. It is enough to verify this for h^*D on W/Z. Since X' has rational singularities, the problem descends to X'/Z up to $\sim_{\mathbb{R}}$. However each $\mathbb{R}$-Cartier divisor $\equiv 0/Z$ is $\sim_{\mathbb{R}} 0/Z$ on any weak log Fano contraction$/Z$.

In addition, if D is a Cartier divisor and $D \equiv 0$, then $D \sim 0$ essentially by the Contraction Theorem.

Remark 8.13 Apart from the final statement, the above holds if we assume only that X' has rational singularities and irregularity $0/Z$.

(2) follows from the numerical description of the splitting in (1):

$$\mathfrak{N}_{\mathrm{nef}} = \mathfrak{C}^0 \oplus g_*\mathfrak{N}_{\mathrm{nef}}.$$

By (1) and (RPC), this means that each nef$/Z$ divisor in $\mathrm{CDiv}^0_{\mathbb{R}}(X/T/Z)$ is also semiample$/Z$, that is, the cone is also semiample$/\sim_{\mathbb{R}}$. Since

$$g_*\mathfrak{N}_{\mathrm{nef}} = g_*\mathfrak{C} \cap \text{the nef cone in } \mathrm{CDiv}_{\mathbb{R}}(T/Z)/\equiv,$$

the nef cone $g_*\mathfrak{N}_{\mathrm{nef}}$ is rational polyhedral by (RPC) again. Hence $\mathfrak{N}_{\mathrm{nef}}$ is also rational polyhedral.

(3) follows from the surjectivity of the projection g_*, since the integral Cartier divisors are also generated by the components D_i.

To prove (4), as in (3), we can assume that an integral Weil divisor D/Z that is nef and $\mathbb{Q}$-Cartier is contained in $\mathfrak{C}$ up to $\sim$. By (2) we can decompose D into $D_0 + D_1$, where $D_0 \in \mathfrak{C}^0$ and $D_1 \in g_*\mathfrak{N}_{\mathrm{nef}}$. This is a decomposition$/\mathbb{Q}$ but with bounded denominators, that we include in I; thus we can assume that both D_0 and D_1 are integral. Moreover, (4) for D_1 follows from a similar decomposition by the cone property (RPC). Now D_0 is free provided it is Cartier, as we remarked in the proof of (1). Hence we can take I as Cartier index for D_0: ID_0 is Cartier. This exists because $\mathfrak{C}^0$ is finite dimensional.

(5) Each divisor in $\mathfrak{N}(X/T/Z, F)$ is semiample by (2). (BND) follows from (4). If F is big as in (3), the (CFG) follows from (2) and (3) as in Example 8.2, (1). For any subset of F we need to cut $\mathfrak{N}_c$ by equations $\mathrm{mult}_{P_i} D = i$ for $i \in \mathbb{Z}$, to drop P_i, and[8.3] keep integral generators D_i. In each compact rational polyhedron $\mathfrak{N}_c$, this also cuts out a rational compact polyhedron. $\square$

[8.3] Is this right? Maybe $\mathrm{mult}_{P_i} D = d_i$ for $i \in \mathbb{Z}$ for all i, then to omit some of the P_i implicit in the argument?

Notation 8.14 For a triple $(X/Z, B, \mathfrak{F})$ with a bounded family $\mathfrak{F}$, we set

$$\mathfrak{N} = \mathfrak{N}(X/T/Z, \mathfrak{F}) = \bigcup_{F \in \mathfrak{F}} \mathfrak{N}(F)$$

(cf. Example 8.2, (2)), and $\mathfrak{N} = \mathfrak{N}(X/Z, \mathfrak{F}) = \mathfrak{N}(X/X/Z, \mathfrak{F})$ if $T = X$ (cf. Theorem 8.23). In other words, each $D \in \mathfrak{N}$ is nef/Z, $\sim_{\mathbb{R}} 0/T$, and $D = D_{\text{int}} + D_{\text{fr}}$, where D_{int} is integral, and $\operatorname{Supp} D_{\text{fr}}$ is in $\mathfrak{F}$. We set

$$\mathfrak{U}_\delta = \mathfrak{U}_\delta(X/T/Z, \mathfrak{F}) = \bigcup_{F \in \mathfrak{F}} \mathfrak{U}_\delta(F),$$

and $\mathfrak{U}_\delta = \mathfrak{U}_\delta(X/Z, \mathfrak{F}) = \mathfrak{U}_\delta(X/X/Z, \mathfrak{F})$.

Theorem 8.15 *Let $(X_u/T_u/Z_u, B_u, \mathfrak{F}_u)$ be a bounded family of triples that are desirable for some weak log Fano contractions* (see the proof); *and let $\tau > 0$ be a tolerance.*

Then there exists a natural number M (depending on the family and τ) such that, for any $(X_u/T_u, B_u, \mathfrak{F}_u)$ in the family and for each divisor $D \in \mathfrak{N} = \mathfrak{N}(X_u/T_u/Z_u, \mathfrak{F}_u)$, we have $\operatorname{Bs}|mD| = \emptyset \mod \tau$ *for some $1 \le m \le M$ (depending on D), and in particular $|mD| \neq \emptyset \mod \tau$.*

Moreover, there exists some $\delta > 0$ such that the same nonvanishing and freedom results hold uniformly in the neighbourhood $\mathfrak{U}_\delta(X_u/T_u/Z_u, \mathfrak{F}_u)$.

Proof We reduce to the case of bnd sets in Theorem 8.4 and Addendum 8.4.1 using Lemma 8.12, (5). To apply this to a bounded family of triples we need to verify $(7')$ in the proof of Theorem 8.4.

Note that a bounded family of triples $(X_u/T_u/Z_u, B_u, \mathfrak{F}_u)$ is equivalent to a bounded family of simple triples $(X_u/T_u/Z_u, B_u, F_u)$ for $u \in U$, where the reduced divisor $F_u = \sum P_{u,i}$ is considered as a 1-element set $\mathfrak{F}_u = \{F_u\}$ (cf. spreading in Remark 6.10, (3)). This means that we have a bounded family $X/T/Z/U$, of projective morphisms/U and horizontal divisors $F = \sum P_i$ and B on X, such that each triple $(X_u/T_u, B_u, F_u)$ of our family can be given as a specialization $(X/T, B, F)$ for some $u \in U$.

In particular, the pairs $(X_u/Z_u, F_u)$ belong to a bounded family $(X/Z, F)$. To satisfy $(7')$ we need to transform the family: e.g., change the base U and split it into an open subset and a closed complement, etc. By Noetherian induction, this finally gives a family satisfying $(7')$.

Each $F_u = F_{|X_u}$ is reduced by definition. After a base change, we can assume that each prime component P_i of F specializes to a prime $P_{u,i}$. Thus the finite dimensional $\mathbb{R}$-vector space $\mathfrak{D}_F$ (generated by F) is locally constant (with respect to the basis P_i) over U, that is, specializes isomorphically to $\mathfrak{D}_{F,u}$ from its generic point. By Lemma 8.12, (1) with $T = Z$, the equivalence $\sim_{\mathbb{R}} 0/T$ is defined geometrically, namely, by the contraction $g\colon X \to T/Z$ or by $\equiv 0/T$. Thus for some nonempty open subset V of U, $\mathfrak{C} = \mathfrak{C}(X/T/Z, F)$ is a constant/V subspace of $\mathfrak{D}_F$ that specializes isomorphically to $\mathfrak{C}_u = \mathfrak{C}(X_u/T_u/Z_u, F_u)$ for each $u \in V$.

Similarly by Lemma 8.12, (2) we can assume that over V the rational polyhedral cone $\mathfrak{N}_{\mathrm{nef}} = \mathfrak{N}_{\mathrm{nef}}(X/T/Z, F)$ specializes isomorphically to the rational polyhedral cone $\mathfrak{N}_{\mathrm{nef},u} = \mathfrak{N}_{\mathrm{nef}}(X_u/T_u/Z_u, F_u)$ for each $u \in V$. If the former is given by inequalities $D \cdot C_i \geq 0$ for a finite set of (bounded) curves C_i/Z, then the latter is given by inequalities $D \cdot C_{u,i} \geq 0$ for their specializations $C_{u,i}$. (Here it is better to assume that $k = \mathbb{C}$. Since each T_u/Z_u satisfies (RPC), it also satisfies (RPC) with generating curves of bounded degree. Otherwise, the complement to the union stratified by the cones that are generated by curves of bounded degree gives a very generic point where (RPC) does not hold. Indeed, each subfamily (a priori nonalgebraic) that corresponds to cones $\mathfrak{N}_{\mathrm{nef},u}$ generated by curves of bounded degree, is *constructive* (see Remark 8.16 below.)

To find the decomposition, in (CFG) we can add some prime components P_i to F such that together they generate the group of Weil divisors of $X/\sim$ and the same after specialization. In a *big* case such as this with extended F we get (CFG) and then for original F as in the proof of Lemma 8.12, (5) (cf. Example 8.2, (1)).

Thus the compact rational polyhedron $\mathfrak{N}_c$ specializes isomorphically to $\mathfrak{N}_{c,u}$ for $(X_u/Z_u, F_u)$ and D_i specializes to $D_{u,i} = D_{i|X_u}$. Since we have a finite set of generators D_i, (CFG) implies (BND), uniformly for the family. Again over some open V of U. Then we repeat the same for a generic point of the closed complement $U \setminus V$, etc. $\square$

Remark 8.16 As explained in the proof, the boundedness of generators for $\overline{\mathrm{NE}}\,(T_u/Z_u)$ is crucial. For weak log Fano contractions $(X_u/Z_u, B_u)$ this follows from anticanonical boundedness [11]. This is not enough for nonalgebraic families: nonalgebraic subsets in algebraic families (cf. Example 8.18). However, for triples any subfamily that corresponds to the bounded generators C_i is constructive. Indeed, we can assume that U is irreducible, and the subfamily is *maximal*, that is, is not in a proper Zariski subset. Then we need to verify that $\mathfrak{N}_{\mathrm{nef}}$ is given by inequalities $D \cdot C_i \geq 0$. By maximality each $D \in \mathfrak{N}_{\mathrm{nef}}$ is nef$/Z_u$ and semiample$/Z_u$ for very general $u \in U$. Exclude u with $D_{|X_u} \cdot C' < 0$ for some curve C'/Z. To prove that $\mathfrak{N}_{\mathrm{nef}}$ is given by the above inequalities over the generic point it is enough to verify that each $D \in \mathfrak{N}_{\mathrm{nef}}$ is semiample over the generic point. We can assume that D is $\mathbb{Q}$-Cartier and even Cartier. Then it is very semiample (free) over the very general point $u \in U$ by the standard properties of $f_*\mathcal{O}_X(D)$. This implies that it is also very semiample over the generic point of U.

As above we derive results similar to Corollaries 8.7, 8.9 and a new one 8.20.

Corollary 8.17 *For some triple $(X_u/T_u/Z_u, B_u, \mathfrak{F}_u)$, let C_u be a divisor on X_u such that all $\operatorname{mult}_{P_i} C_u > -1 + \tau$. Then, for any $D \in \mathfrak{U}_\delta(X_u/T_u/Z_u, \mathfrak{F}_u)$, there exists some natural number $1 \leq m \leq M$ such that $|\lceil mD + C_u \rceil| \neq \emptyset$.*

Addendum 8.17.1 *For $D \in \mathfrak{U}_\delta(X_u/T_u/Z_u, F_u)$, if $\operatorname{mult}_{P_i} F_u = 0$ we can assume just $\operatorname{mult}_{P_i} C_u > -1$.*

Proof The proof of Corollary 8.7 with Theorem 8.15 instead of Addendum 8.4.1. $\square$

Example 8.18 Let $(X/T = \mathrm{pt.}/Z = \mathrm{pt.}, B = 0, \mathfrak{F})$ be a triple such that X is a complete elliptic curve, and $\mathfrak{F} = \{p+q \mid p \neq q \in E\}$. Then Theorem 8.15 does not hold for this triple. (Quiz time: why not applied?)

Otherwise, by Corollary 8.17 with $C = 0$ there exists a natural number M such that, for any $D \in \mathfrak{N} = \mathfrak{N}(X/\mathrm{pt.}/\mathrm{pt.}, \mathfrak{F})$, $\lceil mD \rceil \neq \emptyset$ for some $1 \leq m \leq M$. Thus if $D = p - q$ is a torsion of the elliptic curve, then $D \sim_{\mathbb{R}} 0$ and $D \in \mathfrak{N}$. But then $m(p-q) = mD = \lceil mD \rceil \sim 0$, or all torsions are bounded. This gives a contradiction.

Corollary 8.19 *Let $\alpha < 1, \tau$ be positive real numbers and $\mathcal{C}_u$ an $\mathbb{R}$-b-divisor of X_u such that $\mathcal{C}_u \geq 0/X_u$ and*

$$\text{all } \operatorname{mult}_{P_i} \alpha\mathcal{C}_u > -\alpha + \tau \operatorname{mult}_{P_i} F_u \text{ for any } F_u \in \mathfrak{F}_u. \qquad (*)$$

Then there exists a natural number M, and positive real numbers β, δ which give the following nonvanishing$/Z_u$ on any Y_u/X_u (uniformly in u). For all b-divisors $\mathcal{D}$ such that the descent data $\mathcal{E}$ of $\mathcal{D}$ over X_u is bounded by $\beta\mathcal{C}_u$, that is, $\mathcal{E} \leq \beta\mathcal{C}_u/X_u$, and $\mathcal{D}_X \in \mathfrak{U}_\delta(X_u/T_u/Z_u, \mathfrak{F}_u)$, we have $|\lceil m\mathcal{D}_Y + (\mathcal{C}_u)_Y\rceil| \neq \emptyset$ for some $1 \leq m \leq M$.

Proof The proof of Corollary 8.9 with Theorem 8.15 instead of Addendum 8.4.1. Note that each $F_u \in \mathfrak{F}_u$ is $\mathbb{Q}$-Cartier on X_u by (QFC) of desirable triples. $\square$

Corollary 8.20 *There are real numbers $\beta, \delta > 0$ and a natural number M giving the nonvanishing as in* Corollary 8.19 *for the discrepancy b-divisors $\mathcal{C}_u = \mathcal{A}_u = \mathcal{A}(X_u, B_u)$.*

Example 8.21 Let (X, B) be a Klt pair with a subboundary B, $\mathbb{Q}$-factorial X, and $\mathfrak{F}$ a *bounded* family of reduced divisors on X. Then there exists a real number $\gamma > 0$ such that $(X, B + \gamma F)$ is Klt for any $F \in \mathfrak{F}$. Therefore all

$$\operatorname{mult}_{P_i} \mathcal{A} - \gamma \operatorname{mult}_{P_i} F = \operatorname{mult}_{P_i} (\mathcal{A} - \gamma \overline{F}) = a(X, B + \gamma F, P_i) > -1$$

(cf. the proof of Lemma 6.1, and Example 5.21). Thus for any real $\alpha > 0$, we have $\alpha \operatorname{mult}_{P_i} \mathcal{A} > -\alpha + \tau \operatorname{mult}_{P_i} F$ for $\tau = \alpha\gamma$. This gives $(*)$ and the required τ in Corollary 8.19 for $\mathcal{C}_u = \mathcal{A}_u$.

If in addition (X, B) is canonical in codimension 2, and $\alpha < 1$ is a positive real number (for example, $\alpha = 1/2$), then there exists $\tau > 0$ that satisfies the inequalities, including $(*)$, in Corollary 8.19 for $\mathcal{C}_u = \mathcal{A}_u$. Note that $\mathcal{A}_u \geq 0/X$, because (X, B) is canonical in codimension 2.

Lemma 8.22 *Let $\mathcal{C}_u = \mathcal{A}_u$ be discrepancy b-divisors on a bounded family of desirable triples for weak log Fano contractions* (for example, as in Corollary 8.20*), and let $\alpha \in (0, 1)$. Then there exists τ such that all $\mathcal{C}_u$ satisfy the conditions of* Corollary 8.19*, in particular, $(*)$.*

Proof Example 8.21 proves the lemma for each triple. Then we can use a Noetherian induction over U as in the proof of Theorem 8.15. Note that Klt is open in families. For any desirable triple $(X_u/T_u/Z_u, B_u, \mathfrak{F}_u)$, each X_u is $\mathbb{Q}$-factorial by (QFC) of Definition 6.9, and each (X_u, B_u) is terminal in codimension 2 by (CRP) and (TER). $\square$

Proof of Corollary 8.20 We can find $\beta, \delta > 0$ and M by Lemma 8.22 and then use Corollary 8.19 (cf. Remarks 8.10 and 8.11 above). $\square$

An explanation on bounded nonvanishing: in fact, in applications, we can weaken the nef and big condition on M in the definition of desirable triples to just nef. Thus we are interested in two types of desirable triples corresponding to cases when $\mathcal{M}$ is big or a pencil in the proof of Corollary 6.40: (1) (weak) log del Pezzos $(X_t/\text{pt.}, B_t)$ with only terminal closed points (= terminal resolution) corresponding to triples $(X_t/X_t/\text{pt.}, B_t, \mathfrak{F})$ with crepant B_t, and (2) elliptic fibrations$/T_u = \mathbb{P}^1$, corresponding to triples $(X_u/T_u/\text{pt.}, B_u, \mathfrak{F})$ with an elliptic fibration $X_u \to T_u = \mathbb{P}^1$, and crepant B_u. In either case, $\mathfrak{F}$ is obtained from the standard family on $(X/\text{pt.}, B)$ by log birational transform.

As usual, the simplest form of freedom is on log Fano contractions.

Theorem 8.23
- *Let $(X/Z, B)$ be a weak log Fano contraction;*
- *let $\mathfrak{F}$ be a bounded family of reduced divisors; and*
- *let $\tau > 0$ be a tolerance.*

Then there exists a natural number M (depending on $(X/Z, B)$, $\mathfrak{F}$ and τ) such that, for each divisor $D \in \mathfrak{N} = \mathfrak{N}(X/Z, \mathfrak{F})$, we have $\operatorname{Bs}|mD| = \emptyset \mod \tau$*, in particular,* $|mD| \neq \emptyset \mod \tau$ *for some $1 \leq m \leq M$ (depending on D).*

Addendum 8.23.1 *Moreover, there exists a real number $\delta > 0$ (depending on $(X/Z, B)$, $\mathfrak{F}$ and τ) such that the same nonvanishing holds for all $D \in \mathfrak{U}_\delta = \mathfrak{U}_\delta(X/Z, \mathfrak{F})$.*

Proof Immediate by Theorem 8.15. We consider the weak log Fano contraction $(X/Z, B)$ with bounded $\mathfrak{F}$ as a triple $(X/X/Z, B, \mathfrak{F})$. $\square$

Corollary 8.24 *Let C be a divisor such that all* $\operatorname{mult}_{P_i} C > -1 + \tau$*. Then, for any $D \in \mathfrak{U}_\delta$, there exists some natural number $1 \leq m \leq M$, such that $|\lceil mD + C \rceil| \neq \emptyset$.*

Addendum 8.24.1 *If* $\operatorname{mult}_{P_i} F = 0$ *we can assume just* $\operatorname{mult}_{P_i} C > -1$.

Proof Immediate by Corollary 8.17. $\square$

Corollary 8.25 *Suppose that each $F \in \mathfrak{F}$ is $\mathbb{Q}$-Cartier. Let $\alpha < 1, \tau$ be positive real numbers and $\mathcal{C}$ a b-divisor such that $\mathcal{C} \geq 0/X$ and all*

$$\operatorname{mult}_{D_i} \alpha\mathcal{C} > -\alpha + \tau \operatorname{mult}_{D_i} F \quad \textit{for any } F \in \mathfrak{F}. \qquad (*)$$

Then there exists a natural number M, and positive real numbers β, δ giving the following nonvanishing/Z on any Y/X (uniformly in F). For all b-divisors $\mathcal{D}$ such that the descent data $\mathcal{E}$ of $\mathcal{D}$ over X is bounded by $\beta\mathcal{C}$, that is, $\mathcal{E} \leq \beta\mathcal{C}/X$, and $\mathcal{D}_X \in \mathfrak{U}_\delta$, $|\lceil m\mathcal{D}_Y + \mathcal{C}_Y\rceil| \neq \emptyset$ for some $1 \leq m \leq M$.

Proof Immediate by Corollary 8.19. $\square$

Corollary 8.26 *Let $(X/Z, B)$ be a weak log Fano contraction, with X $\mathbb{Q}$-factorial and with (X, B) canonical in codimension 2. Then, for any given bounded family $\mathfrak{F}$ of reduced divisors, there are real numbers $\beta, \delta > 0$ and a natural number M giving the nonvanishing/Z as in* Corollary 8.25 *for $\mathcal{C} = \mathcal{A}$.*

Proof Immediate by Corollary 8.20. $\square$

9 Stabilization in divisor

We are now ready to establish the stabilization of a central multiplicity. We first describe what we mean by an inductive model and its central divisor E_c. Then we state under what condition we expect a limit of b-divisors to stabilize near E_c. Finally, we prove this for 3-folds. This means a nonvanishing result on E_c for the limiting divisor. Afterwards, we extend this stabilization to the freedom result of Corollary 10.15 under Klt and in the realistic situation (CBS)(rfa) (cf. (CBS)(fga) in Conjecture 6.14). This is a freedom result on a Klt model that is terminal in codimension 2, and this leads to the proof of Theorem 6.44 in Section 11.

9.1 Conventions

In this section we assume LMMP in $n = \dim X$ whenever we state "under LMMP". We disregard this for $n \leq 3$. By Corollary 6.40 we can drop (CBS), (MOD) and (SSB) respectively for $n \leq 2$.

Definition 9.2 We say that a limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ of $\mathbb{R}$-b-divisors *stabilizes near a set of prime b-divisors* E_i if there exists a model Y/X on which the limit stabilizes near the centers of these b-divisors (viewed as closed subvarieties). That is, $\mathcal{D}_i = \mathcal{D}$ for some $i \gg 0$ over a neighborhood[9.1] of these centers *in* Y.

Definition 9.3 [cf. Prokhorov and Shokurov [5]] An *inductive model* of a log pair $(X/Z, B)$ (respectively, *strict inductive model*) is a log model $(Y/Z, B_Y)$ of $(X/Z, B)$ for which

(1) $(X, B_X)/(Y, B_Y)$ is *log proper*, that is, equivalently

$$\mathcal{B}^Y = \mathcal{B}(Y, B_Y) \geq \mathcal{B}^X = \mathcal{B} = \mathcal{B}(X, B), \quad \text{or}$$
$$\mathcal{A}^Y = \mathcal{A}(Y, B_Y) \leq \mathcal{A}^X = \mathcal{A} = \mathcal{A}(X, B).$$

(2) (Y, B_Y) is *exceptional* in the sense that there is a single divisor E_c of Y with $\operatorname{mult}_E B_Y = 1$, and (Y, B_Y) is *purely* log terminal, respectively *strictly* log terminal; here "strictly" means that Y is $\mathbb{Q}$-factorial ([9, p. 99], see also [4], Definition 2.13).

(3) $-(K_Y + B_Y)$ is nef and big$/Z$ (big is automatic if X/Z is birational, and $-(K_Y + B_Y)$ is nef$/Z$) (respectively, ample$/Z$). And

(4) $-(K_Y + B_Y)$ is nef and big on E_c/Z (respectively, ample on E_c/Z).

We say that E_c is the *central b-divisor* of the inductive model. For a b-divisor $\mathcal{D}$, the multiplicity $d_c = \operatorname{mult}_{E_c} \mathcal{D}$ is referred to as *central*. Note that (2–3) mean (WLF) except with Klt replaced by purely log terminal (cf. (PFN) in Lemma 9.7 below); but it is still not a general log Fano contraction.

Caution 9.4 In this definition, we do not assume that Y is$/X$.

Definition 9.5 A *local weak log Fano contraction* $(X/T, B)$ means (WLF) of Proposition 4.38) in the local situation, that is, when

- X/T is *local* contraction, that is, $\dim T \geq 1$ and $f\colon X \to T$ is onto T.

Lemma 9.6 (cf. [5]) Under LMMP. *Let $(X/T, B)$ be a local weak log Fano contractin. Then locally$/T$ there exists an inductive model. Moreover, we can assume for this model that*

[9.1] I think this means as divisors on Y, not as b-divisors.

- *B_Y is a $\mathbb{Q}$-divisor;*
- *E_c is complete* (that is, E_c/P).

Addendum 9.6.1 *Thus we have a weak log Fano contraction (E_c, B_c) such that $K_{E_c}+B_c = (K_Y+B_Y)_{|E_c}$. We call it the* central model. *We can assume that B_c is $\mathbb{Q}$-Cartier.*

Addendum 9.6.2 *We can upgrade any inductive model to a strict inductive model.*

Proof We consider here only the case that X/T is birational, which is all that we need. The general case can be found in Prokhorov and Shokurov [5, 3.2]. After taking a complement, we assume that $K + B \equiv 0/T$. We can suppose in addition that B is a $\mathbb{Q}$-divisor. This increases $\mathcal{B}$, which is important for (1) in the definition. Hence after a making a contraction we can assume that $X = T$.

First, we can build a log canonical singularity by adding εH for an ample divisor through $P \in X$. Moreover, after a perturbation of H, we can assume that P is exceptionally log canonical for $B+H$ with some effective $\mathbb{Q}$-Cartier divisor H. Again $\mathcal{B}$ was increased, and strictly over P. By the exceptional property we have the required central divisor E_c/P as a unique b-divisor with log discrepancy 0.

Now take a strict log terminal resolution $(Y/X, B_Y)$ resolving E_c. The boundary B_Y is given by the crepant modification of $(X, B+H)$. In addition, we can assume that $g^{-1}H$ is nef$/X$; otherwise, we can apply LMMP to $B_Y + \varepsilon g^{-1}H$.

Finally, this give the required model if we replace B_Y by $B_Y - \varepsilon g^{-1}H$. Properties (1–3) follow from the construction. For (4), we need to check that $g^{-1}H$ is big on E_c. Otherwise, because $g^{-1}H$ is nef$/X$ and strictly effective near E_c it defines a fibre contraction $Y \to Z/X$, where Z/X is small nontrivial and H is positive on Z/X. Since H is a $\mathbb{Q}$-Cartier divisor this is impossible (by the projection formula).

Addendum 9.6.1 follows from the adjunction formula [9, 3.1 and 3.2.3], and the fact that E_c is normal [9, Lemma 3.6]. Addendum 9.6.2 follows directly from the following result. $\square$

Lemma 9.7 (Under LMMP; cf. remark at the end of the proof.)

(PFN) *Let* $(X/T, B)$ *be a* weak purely log terminal *Fano contraction,* that is, (GLF) of Proposition 4.44 with purely log terminal (X, B) and with a boundary B

such that X *is* $\mathbb{Q}$*-factorial, and suppose that* $-(K+B)$ *is nef and big on each reduced component of* B *(there are* at most two*). Then we can find another boundary* B^+ *such that:*

- $(X/T, B^+)$ *still satisfies* (PFN) *with ample* $-(K + B^+)/T$*;*
- $B^+ \geq B$*; and*
- B *and* B^+ *have the same reduced components.*

Proof We can contract all 0-curves C with respect to $K + B$, that is, $C \cdot (K + B) = 0$, by a contraction $X \to Y/T$. The contraction is given by $-(K + B)$. (PFN) is preserved because the reduced divisors of B is not contracted. There exists an effective mobile divisor D on X which is negative/Y. In particular, for a general such D, the reduced divisors of B are not in $\operatorname{Supp} D$. Thus we can replace B by the boundary $B^+ = B + \varepsilon D$ for some $0 < \varepsilon \ll 1$. Then $-(K + B^+)/T$ is ample, and $(X/T, B^+)$ satisfies all the required properties. For ampleness, we use that the Kleiman–Mori cone of $(X/T, B)$ is polyhedral. □

Remark 9.8 We don't need the full strength of LMMP for Lemma 9.7; it is enough to know the cone and the contraction theorems, which work under our conditions.

Now we are ready to state the stabilization of the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ *at* some central divisor E_c:

$$d_c = d_{c,j}$$

for some $j \gg 0$, where $d_{c,j} = \operatorname{mult}_{E_c} \mathcal{D}_j$. This holds in the following situation.

Theorem 9.9 We assume LMMP and $(\mathrm{CBS})_{n-1}(\mathrm{gl})$, $(\mathrm{SSB})_{n-1}(\mathrm{gl})$. *Suppose that* $(X/T, B)$ *is a log pair, and let* $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ *be a limit of b-divisors such that:*

(LWF) $(X/T, B)$ *is a local weak log Fano contraction;*

(LBF) linear b-freedom*: $i\mathcal{D}_i \sim \mathcal{M}_i/T$ for all i, where $\mathcal{M}_i$ is b-free;*

(LCA) *$\mathcal{D}_\bullet$ is lca saturated;*

(MXD) maximality of the limit: *each $\mathcal{D}_i \le \mathcal{D}$; and*

(BED) *each $\mathcal{D}_i = \mathcal{D}$ outside $f^{-1}P$ over T* (cf. Proposition 4.47).

Then there exists an inductive model with central divisor E_c such that the limit $d_c = \lim_{i\to\infty} d_{c,i}$ is a rational number, and the limit stabilizes; that is, $d_c = d_{c,j}$ for infinitely many j.

Corollary 9.21 below improves this result slightly.

Remark 9.10 Under (LWF), *linear b-freedom* (LBF) is equivalent to *numerical freedom $i\mathcal{D}_i \equiv \mathcal{M}_i/T$*, provided that $i\mathcal{D}_i$ is Cartier, or just integral. It is enough to establish this for $\overline{i(\mathcal{D}_i)_Y} = i\mathcal{D}_i \equiv 0/T$. Then the required linear equivalence follows from descent of Cartier divisors for their contractions on rational singularities, and from stable freedom on $Y = X$ as in (LWF) (cf. Remark 8.13 in the proof of Lemma 8.12, (1)).

The main steps in the proof of Theorem 9.9 are as follows. First, we establish the slightly more general rationality result, $d_c \in \mathbb{Q}$, and a stabilization result in Proposition 9.13. Then using Proposition 9.15 on bounded presentation of the fixed part of linear systems on the central b-divisor, we reduce Theorem 9.9 to Proposition 9.13. Proposition 9.13 itself and its proof are similar to the proof of the rationality theorem in the Kleiman–Mori cone and the nonvanishing theorem. However this time, we need a birational version of nonvanishing: e.g., Corollary 8.20 and Theorem 8.23. We interpret this nonvanishing as the stabilization of $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ in E_c. Some preparatory results in Lemmas 9.6–9.7 above and in Lemma 9.18 and Corollary 9.19 are needed to construct an appropriate inductive model.

The following example illustrating the use of an inductive model is a paradigm for Theorem 9.9 and for other results in this section and in the paper.

Example 9.11 (Projective space) Let $(X/Z, B) = (\mathbb{P}^n/\mathrm{pt.}, 0)$ be projective space. Then (FGA), together with (SSB), (PRM), (PFC), (CBS) and (MOD) of Conjecture 6.14 hold on it. Moreover, $\mathfrak{S} = \mathfrak{P} = \emptyset$, and in (CBS) $\mathrm{Bs}\,|\mathcal{M}|_{\mathbb{P}^n} = \emptyset$ for each $\mathcal{M} \in \mathfrak{M}$; so, $c = 1$! Moreover, for any $\mathbb{R}$-b-divisor *under saturation* (SAT), $\operatorname{Bs}|\mathcal{D}|_{\mathbb{P}^n} = \emptyset$. (FGA) gives as the stable models only $\mathbb{P}^n$ itself or pt.

It is possible to prove this using restriction to $D = \mathcal{D}_Y$, as in the proof of Proposition 6.26. But this needs the Fujita estimate on freedom, that is still not established. Another *realistic* approach is to use inductive models. The first one is $(Y/\text{pt.}, E = P^{n-1})$, where $Y \to \mathbb{P}^n$ is the blowup in a closed point $p \in \mathcal{D}_{\mathbb{P}^n} \subset \mathbb{P}^n$ and E is the exceptional locus over p. If $\mathcal{D} = \mathcal{D}_Y$ is fixed (prime) we use saturation (STD) for $\mathcal{D}_{|E}$ for generic $P \in \mathcal{D}$; the saturation is preserved (cf. Lemma 9.16 below). Thus by induction on n, $\mathcal{D}$ is mobile.

If $\mathcal{D}$ is mobile, for example, $\mathcal{D} = \mathcal{M} \in \mathfrak{M}$, we can also use the models $(\mathbb{P}^n/\text{pt.}, H = \mathbb{P}^{n-1})$ with (general) hyperplane sections H. Again by induction $|\mathcal{D}|_{\mathbb{P}^n}$ only has *closed* base points p (cf. Proposition 4.47). For such $p \in \operatorname{Bs}|\mathcal{D}|_{\mathbb{P}^n}$, we use induction. Namely, $|\mathcal{M}|_{Y_{|E}}$ is free where $\mathcal{M} = \operatorname{Mov}\mathcal{D}$ is *general* in the linear system on Y with $\mathcal{M}_{\mathbb{P}^n} = \mathcal{D}_{\mathbb{P}^n}$ (cf. Proposition 9.15). Hence

$$\operatorname{Mov}\lceil\mathcal{D} + \mathcal{A}\rceil \geq \operatorname{Mov}(\mathcal{M} + \overline{E}) = \overline{(\mathcal{M})_Y} + \overline{E},$$

which contradicts saturation of $\mathcal{D}$ with respect to $\mathcal{A} = \mathcal{A}(\mathbb{P}^n, 0)$.

The same works for any other nonsingular Fano variety such that:

- each blowup in a closed point,[9.2] and
- there are ladders of smooth log Fano varieties (down to surfaces),

both of which give inductive models. For example, this is true for nonsingular quadrics. All of this is aesthetically appealing, but unfortunately not deep.

The following result generalizes the Rationality Theorem for Cones.

Notation 9.12 In the following proposition, $\mathcal{A}' = \mathcal{A}'(X, B) = \mathcal{A} + E_c$ is an *adjusted (truncated)* discrepancy where (X, B) is *exceptionally* log terminal with a single E_c having $\operatorname{mult}_{E_c}\mathcal{A} = -1$.

Proposition 9.13 *Let $(X/T, B)$ be a strict inductive model with central divisor E_c, $E' \sim E_c$ a linear equivalence with $E_c \not\subset \operatorname{Supp} E'$, and $\mathcal{D}_\bullet$ a system of b-divisors $\mathcal{D}_i$ such that $d_c = \lim_{i\to\infty} d_{c,i}$. Then $d_c \in \mathbb{Q}$ under the following conditions:*

[9.2] Something missing? e.g., each blowup in a closed points is again a Fano variety?

(BNF) *each* $\mathcal{D}_i$ *is* b-nef$/T$ *in the sense of Lemma 4.21;*

(εA′S) integral asymptotic saturation over E_c (cf. Definition 4.39) *holds for* $\mathcal{D}_\bullet$ *with respect to* $\mathcal{C} = \mathcal{A}' + \varepsilon \overline{E_c}$*, for some* $\varepsilon > 0$*;*

(MXC) maximality of the limit: *each* $d_{c,i} \leq d_c$*;*

(BWQ) *there exists a bounded family of triples* $(E_u/T_u/\mathrm{pt.}, B_u, \mathfrak{F}_u)$ *as required*[9.3] *for* $(E_c/\mathrm{pt.}, B_c)$*,* (cf. Theorem 8.15) *such that: each* E_u/E_c*, each* $(\mathcal{D}_{i}{}_{\,|E_c})_{E_i} \in \mathfrak{N}(E_i/T_i/\mathrm{pt.}, \mathfrak{F}_i)$ (the mixed restriction for $K = \mathbb{R}$, cf. (mx) in Proposition 7.7)*, and the descent data of the restrictions* $\mathcal{D}_{i}{}_{\,|E_c}$ *are asymptotically bounded with respect to* $\mathcal{A}_c = \mathcal{A}(E_c, B_c)$ *over the* corresponding models E_i*;*

(BRE′) Supp $(\overline{E'_{|E_c}})_{E_u} \leq F \in \mathfrak{F}_u$ *for each* $F \in \mathfrak{F}_u$ *on each model* E_u *of triples in* (BWQ)*.*

Moreover, the limit stabilizes: $d_c = d_{c,j}$ *for infinitely many* j*.*

Remark 9.14 $E' \sim E_c$ *induces* $E'' \sim E = g^{-1}E_c$ on any other model $g\colon Y \to X$ with $E \not\subset \operatorname{Supp} E''$. Take $E'' = g^*E' + E - g^*E_c$. Then the restriction (mx) defined by E'' on Y is independent of Y/X: in fact, since $\operatorname{mult}_{E_c} \mathcal{D} = \operatorname{mult}_E \mathcal{D} = d$, we have

$$\begin{aligned}
\mathcal{D}_{|E_c} &= (\mathcal{D} - d\overline{E_c})_{|E_c} + d\overline{E'}_{|E_c} = (\mathcal{D} - d\overline{E_c})_{|E_c} + d\overline{E'_{|E_c}} \\
&= (\mathcal{D} - d\overline{E_c})_{|E} + d\overline{g^*E'_{|E}} = (\mathcal{D} - d\overline{E_c})_{|E} + d\overline{(E'' + g^*E_c - E)}_{|E} \\
&= (\mathcal{D} - d(\overline{E_c} - \overline{g^*E_c} + \overline{E}))_{|E} - d\overline{E''}_{|E} = (\mathcal{D} - d\overline{E})_{|E} - d\overline{E''}_{|E},
\end{aligned}$$

because $g^*E' = E'' + g^*E_c - E$ and $\overline{E_c} = \overline{g^*E_c}$.

Proof By the monotonicity in Lemma 4.40 and because $(K+B)/T$ is negative, we can replace ε by a smaller positive value so that the saturation (εA′S) still holds, and at the same time $-(K+B) + \varepsilon E_c$ is nef and big$/T$. Moreover, we can assume that the same holds for any smaller $\varepsilon \geq 0$.

[9.3] I don't know what this means. Possibly desirable triples?

In particular, for any $0 \le d \le \varepsilon$, for any real number $j \ge 0$, and for any natural number i, on a sufficiently high log resolution $g\colon Y \to X/T$ of $(X/T, B)$, we have the following vanishing

$$R^1 g_* \mathcal{O}(\lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil) = 0.$$

Note that the resolution Y depends only on i. Indeed, we take a log resolution *over* which $(\mathcal{D}_i)_Y$ is nef/T and $\mathbb{R}$-Cartier. Then

$$\begin{aligned}\lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil &= K_Y + \lceil -g^*(K+B) + dg^*E_c + j(\mathcal{D}_i)_Y \rceil \\ &= K_Y + \lceil g^*(-(K+B) + dE_c) + j(\mathcal{D}_i)_Y \rceil,\end{aligned}$$

where $-(K+B) + dE_c$ is nef and big/T and $(\mathcal{D}_i)_Y$ is nef/T. It follows that $g^*(-(K+B) + dE_c) + j(\mathcal{D}_i)_Y)$ is nef and big/T, and Kawamata–Viehweg gives the required vanishing.

However to apply this vanishing to a restriction to the birational image E of E_c in Y, we need to assume that $d\overline{E_c} + j\mathcal{D}_i$ has integral multiplicity in E_c, by integral saturation in (εA$'$S), or equivalently, that the $d + jd_{c,i}$ are integers. More precisely, then we have a surjective map of linear systems

$$\left| E + \lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil \right| \dashrightarrow \left| (E + \lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil)_{|E} \right|.$$

In addition, because the $d + jd_{c,i}$ are integers, normal crossing, by Lemmas 6.29-.6.30 and adjunction, the linear system

$$\begin{aligned}&\left| (E + \lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil)_{|E} \right| = \\ &\qquad = \left| \left\lceil (E + \mathcal{A}_Y)_{|E} + (d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y)_{|E} \right\rceil \right| \\ &\qquad = \left| \left\lceil (\mathcal{A}_c)_E + ((d\overline{E_c} + j\mathcal{D}_i)_{\,\rceil E_c})_E \right\rceil \right| \\ &\qquad = \left| \left\lceil (\mathcal{A}_c)_E + (((d + ju_{c,i})\overline{E'} + j\mathcal{D}'_i)_{\,\rceil E_c})_E \right\rceil \right| \\ &\qquad = \left| \left\lceil (\mathcal{A}_c)_E + (((d + jd_{c,i})\overline{E''} + d(\overline{E_c} - \overline{E}) + j\mathcal{D}''_i)_{\,\rceil E})_E \right\rceil \right|,\end{aligned}$$

for a sufficiently high resolution Y (over which $\mathcal{D}_i$ is nef $\mathbb{R}$-Cartier/X/T), is defined on E by the birational restriction $\left(d\overline{E_c} + j\mathcal{D}_i\right)_{\,\rceil E_c}$, that is, by

$$\left((d + jd_{c,i})\overline{E'} + j\mathcal{D}'_i\right)_{\,\rceil E_c} = \left((d + jd_{c,i})\overline{E''} + d(\overline{E_c} - \overline{E}) + j\mathcal{D}''_i\right)_{\,\rceil E_c}$$

(as in Remark 9.14), where E is a sufficiently high resolution of E_c and $E' \sim E_c$ on X/T with $E_c \not\subset \operatorname{Supp} E'$ and where $E'' \sim E$ is induced on Y/T with $E \not\subset \operatorname{Supp} E''$, $\mathcal{D}'_i = \mathcal{D}_i - d_{c,i}\overline{E_c}$ and $\mathcal{D}''_i = \mathcal{D}_i - d_{c,i}\overline{E}$. This last resolution only depends on i. Hence if we have nonvanishing for the linear system restricted to E, the corresponding linear system

$$\left|E + \lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil\right| = \left|\lceil \mathcal{A}'_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil\right|$$

is free on E.

In the present situation this is impossible, because it contradicts (MXC), the maximality of d_c. Namely,

(NBF) $\left|\lceil \mathcal{A}'_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil\right|$ cannot be free on E for j a positive integer and integers $d + jd_{c,i} > jd_c$.

Indeed, then the multiplicity of the mobile part for the linear system is

$$\operatorname{mult}_E \lceil \mathcal{A}'_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil = \operatorname{mult}_{E_c}(d\overline{E_c} + j\mathcal{D}_i) = d + jd_{c,i}.$$

On the other hand, for any natural number i, by asymptotic saturation (see Definition 4.29), the multiplicity is $\le \operatorname{mult}_{E_c} j\mathcal{D}_j = jd_{c,j}$, that is, $d + jd_{c,i} \le jd_{c,j}$. Hence $d_{c,j} \ge d_{c,i} + d/j > d_c$ by the inequality in (NBF). This contradicts (MXC).

By the above surjectivity, (NBF) is equivalent to the following vanishing:

(VN) on a sufficiently high resolution E/E_c,

$$\left|\left\lceil (\mathcal{A}_c)_E + (d + jd_{c,i})\overline{E'_{|E_c}} + j\mathcal{D}'_{i\,|E_c})_E \right\rceil\right| = \emptyset,$$

whenever j is a positive integer and integer $d + jd_{c,i} > jd_c$.

However if $d_c \notin \mathbb{Q}$, using a bounded nonvanishing we can disprove this for some $i \gg 0$.

For triples $(E_u/T_u/\mathrm{pt.}, B_u, \mathfrak{F}_u)$ in (BWQ), by Corollary 8.20 there exist real numbers $\beta, \delta > 0$ and a natural number M such that for any b-divisors $\mathcal{D}$ with the descent data bounded by $\beta\mathcal{A}_c$ over E_u and $\mathcal{D}_{E_u} \in \mathfrak{U}_\delta$, we have bounded nonvanishing: $|\lceil m\mathcal{D}_{E_{\mathrm{hr}}} + (\mathcal{A}_c)_{E_{\mathrm{hr}}} \rceil| \ne \emptyset$ on any E_{hr}/E_u, in particular, for some rather high E_{hr}/E_c and for some $1 \le m \le M$.

We apply this to the b-divisor

$$\mathcal{D} = q(d/j + d_{c,i})\overline{E'_{|E_c}} + q\mathcal{D}'_{i\,|E_c}$$

for some natural number $q > 0$ and satisfying the nonvanishing conditions. Equivalently, $\mathcal{D} = p\overline{E'_{|E_c}} + q\mathcal{D}'_{i\,|\,E_c}$, where now $j = mq$ and $p = d/m + qd_{c,i}$ under the following conditions:

(1) $m > 0, p$ are integral, and

(2) $p/q > d_c$.

These imply that $mq = j > 0$ and $mp = md/m + mqd_{c,i} = d + jd_{c,i}$ are integral and $mp = d + jd_{c,i} > jd_c$ as in (VN).

Now we rewrite the nonvanishing conditions for $\mathcal{D}$:

(3) $\mathcal{D} \in \mathfrak{U}_\delta(E_i/T_i/\text{pt.}, \mathfrak{F}_i)$, and

(4) the descent data of $\mathcal{D}$ is bounded by $\beta\mathcal{A}_c/E_i$.

Since

$$\mathcal{D} = \left(p - qd_{c,i}\right)\overline{E'_{|E_c}} + q\left(d_{c,i}\overline{E'_{|E_c}} + \mathcal{D}'_{i\,|\,E_c}\right),$$

we have $q(d_{c,i}\overline{E'_{|E_c}} + \mathcal{D}'_{i\,|\,E_c}) = q\mathcal{D}_{i\,|\,E_c}$ (this (mx) restriction is defined up to $\sim_{\mathbb{R}}$), $(\mathcal{D}_{i\,|\,E_c})_{E_i} \in \mathfrak{N}(E_i/T_i/\text{pt.}, F_i)$ for some $F_i \in \mathfrak{F}_i$, and $\operatorname{Supp}(E'|E_c)_{E_i} \leq F_i$ by (BWQ) and (BRE$'$), then, for a *natural number* q, (3) holds for $\mathcal{D} = p\overline{E'_{|E_c}} + q\mathcal{D}'_{i\,|\,E_c}$, as long as we ensure that $\|(p - qd_{c,i})(\overline{E'_{|E_c}})_{E_i}\| < \delta$. (For a natural number q, note that $\mathcal{D} \in \mathfrak{N}(E_i/T_i/\text{pt.}, F_i)$ implies that $q\mathcal{D} \in \mathfrak{N}(E_i/T_i/\text{pt.}, F_i)$ because $\mathfrak{N}(E_i/T_i/\text{pt.}, F_i)$ is an Abelian semigroup, although it is not a convex body.) This last inequality holds provided that

(3$'$) $0 \leq p - qd_{c,i} < \delta/N$, where N is $\geq$ the *maximal* absolute multiplicity of $(\overline{E'_{|E_c}})_{E_u}$ over all $F \in \mathfrak{F}_u$,

because $\operatorname{Supp}(E'|E_c)_{E_i} \leq F_i \in \mathfrak{F}_i$ by (BRE$'$). N is bounded under (BWQ). The left inequality in (3$'$) has the stricter form $0 < p - qd_{c,i}$, which follows from (2) and (MXC).

Since E_i/E_c, the descent data for $(\overline{E'_{|E_c}})$ is $0/E_i$, and the descent data for $\mathcal{D}$ is the same as for $q\mathcal{D}_{i\,|\,E_c}$ by (ADD) in Proposition 5.4. Hence for fixed q, the descent data for $\mathcal{D}$ is asymptotically bounded with respect to $\mathcal{A}_c/E_i$ (cf. Remark 5.8, (4); the only difference with Definition 5.7 is that we consider a sequence of models E_i). In particular, for some $i \gg 0$ the descent data of

$\mathcal{D}$ is bounded by $\beta\mathcal{A}_c$, because by definition $\mathcal{A}_c \geq 0/E_i$. This gives (4) for some $i \gg 0$.

Thus we need to find integers $q > 0$ and p satisfying (2) and (3$'$). This gives

$$m\mathcal{D} = mp\overline{E'}_{|E_c} + mq\mathcal{D}'_{i\,|E_c} = (d + jd_{c,i})\overline{E'}_{|E_c} + j\mathcal{D}'_{i\,|E_c},$$

which contradicts (VN) for $j = mq$ and $d = m(p - qd_{c,i})$ with some natural number $1 \leq m \leq M$ by Corollary 8.20.

Suppose that $d_c \notin \mathbb{Q}$ then, for any real numbers N and $\varepsilon > 0$, there exists a rational (chain) approximation $r = p/q$ with integers $q > 0$ and p such that

- $r > d_c$, and
- $r - d_c < \varepsilon/Nq$.

We apply this result for certain real numbers: $0 < \varepsilon \leq \delta$, where ε is $\leq$ that at the start of the proof, and $N \geq M$ as in (3$'$). Thus we get required integers $q > 0$ and p satisfying (2) and (3$'$) for all $i \gg 0$. Indeed, $p/q = r > d_c$ implies (2) and the left inequality in (3$'$) by (MXC). On the other hand, since $d_c = \lim_{i\to\infty} d_{c,i}$, then for all $i \gg 0$, we have

$$p/q - d_{c,i} = r - d_{c,i} < \varepsilon/Nq \leq \delta/Nq \quad \text{or} \quad p - qd_{c,i} < \delta/N.$$

We also need to verify that $0 \leq d \leq \varepsilon$: since $d = m(p - qd_{c,i})$ and we get above more than (3$'$) $0 < p - qd_{c,i} < \varepsilon/N$, we get $0 < d < m\varepsilon/N \leq M\varepsilon/N \leq \varepsilon$, because $M \leq N$. This leads to the promised contradiction. Therefore $d_c \in \mathbb{Q}$.

Finally, we verify the stabilization of the limit $d_c = \lim_{i\to\infty} d_{c,i}$. We use the same arguments, but now the contradiction turns into an honest nonvanishing and stabilization. In particular, we replace (2) by the equation

(2$'$) $p/q = d_c$.

Nonetheless, we get (3$'$) and the nonvanishing of the linear system in (VN) for some $i \gg 0$ by (3) and (4), in particular, for some $i \geq n$. As explained above this implies that $\left|\lceil \mathcal{A}'_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y \rceil\right|$ is free on E. This time, by asymptotic saturation, we get inequalities $d_{c,j} \geq d_{c,i} + d/j = d_c$. Thus by (MXC) $d_c = d_{c,j}$ for $j = mq$. We can find infinitely many j if we replace p and q by lp and lq respectively for any natural number $l > 0$. (They may not be proportional to the first j since m depend on $\mathcal{D}$.) $\square$

To apply Proposition 9.13 in the proof of Theorem 9.9, we need to obtain bounded presentations on the central divisor E_c of the restrictions $(\mathcal{D}_{i|E_c})_{E_i}$ in (BWQ) for an appropriate inductive model of $(X/T, B)$. For this we use Corollary 7.9 in conjunction with the following result, for a certain set of b-divisors $\mathcal{M}$, on the boundedness of their fixed components on E_c, that is, the boundedness of the divisorial components of $\operatorname{Bs}|\mathcal{M}|_X \cap E_c$.

Proposition 9.15 Under $(\mathrm{CBS})_{n-1}(\mathrm{gl})$ and $(\mathrm{SSB})_{n-1}(\mathrm{gl})$. *Let $(X/T, B)$ be a strict inductive model with central divisor E,* Bs $\subset X$ *a proper reduced subvariety, and $\{\mathcal{M}\}$ a set of (sufficiently general) b-free b-divisors $\mathcal{M}$ such that:*

(BSP) $\operatorname{Bs}|\mathcal{M}|_X \subset \operatorname{Bs}$ *; and*

(SA′) *$\mathcal{M}$ is $\mathcal{A}'$-saturated on sufficiently high models X_{hr}/X* with $'$ over P and Bs, that is, $\mathcal{A}' = \mathcal{A} + \sum E_i$ with $\sum E_i$ *exactly* over the integral components of $\mathcal{A}$ over P and Bs, that is, with $f(\operatorname{center}_T E_i) = P$ or with $\operatorname{center}_X E_i \subset \operatorname{Bs}$;

then the set of b-divisors $\mathcal{M}$ satisfies

(1) boundedness of the fixed component *of each $|\mathcal{M}|_X$ on E, that is, the* whole *divisorial component of the intersection* $\operatorname{Bs}|\mathcal{M}|_X \cap E$ *belongs to a bounded set $\mathfrak{F}$ of reduced divisors on E;*

(2) in each linear system $\left|\mathcal{M}_{|E}\right|$, a general element *has bounded canonical singularities on a triple $(E_u/T_u/\mathrm{pt.}, B_u, \mathfrak{F}_u)$ with E_u/E, desirable for* $(E/\mathrm{pt.}, B)$ (and also just bounded if E is a surface); *and*

(3) the desirable triples in (2) belong to a bounded algebraic family.

Note that $\mathcal{M}$ is free on X outside $f^{-1}P$ if Bs $= \emptyset$ *there.*

Lemma 9.16 (cf. Proposition 4.44 and its proof) *Under the assumptions of* Proposition 9.15, (SA′) *for $\mathcal{M}$ implies* (SA′F) *of* Proposition 6.34, (4) *with $\mathcal{M} := \mathcal{M}_{|E}$ and* $F = \operatorname{Supp}\operatorname{Fix}(|\mathcal{M}|_{X|E})$, except over $\operatorname{Supp} B_E$.

Proof Suppose that $\mathcal{M}_X$ is sufficiently general in its linear system $|\mathcal{M}|_X$, and set $M = \mathcal{M}_X$.

Then, for any $0 < \varepsilon \ll 1$, on a sufficiently high log resolution $g\colon Y \to X/T$ of $(X/T, B)$, we have the following vanishing

$$R^1 g_* \mathcal{O}(\lceil \mathcal{A}_Y + \varepsilon \overline{M}_Y \rceil) = 0;$$

$\overline{M}$ is well defined because X is $\mathbb{Q}$-factorial. This resolution Y depends only on $\mathcal{M}$. We take a log resolution such that $\operatorname{Bs}|\mathcal{M}|_Y = \emptyset$. For general M, $g^{-1}M = \mathcal{M}_Y$ is also general in $|\mathcal{M}|_Y$. Then the resolution is also a log resolution for $B + \varepsilon M$, and

$$\begin{aligned}\lceil \mathcal{A}_Y + \varepsilon \overline{M}_Y \rceil &= K_Y + \lceil -g^*(K+B) + \varepsilon g^* M \rceil \\ &= K_Y + \lceil g^*(-(K+B) + \varepsilon M) \rceil,\end{aligned}$$

where $-(K+B) + \varepsilon M$ is ample$/T$ for all $0 < \varepsilon \ll 1$, since $-(K+B)/T$ is ample$/T$. Thus $g^*(-(K+B) + \varepsilon M)$ is nef and big$/T$, and Kawamata–Viehweg gives the required vanishing.

However, to apply this vanishing downstairs to the restriction to the birational image E_Y of E in Y, we need to assume that εM has rather small multiplicities e_i in the exceptional divisors E_i of Y/X. More precisely, each $e_i < 1 - \{a_i\}$ for each discrepancy $a_i = \operatorname{mult}_{E_i} \mathcal{A}$, where $\{a_i\}$ is the usual fractional part; in particular, $e_i < 1$ for integral a_i. Equivalently, $\lceil a_i + e_i \rceil = \lceil a_i^* \rceil$, where $a_i^* = \operatorname{mult}_{E_i} \mathcal{A}^* = a_i + 1$ for integral a_i, when E_i is over M and even exactly over $\operatorname{Bs}|\mathcal{M}|_X$ for the *general* M, and $a_i^* = a_i$ otherwise. Thus as compared to $'$, the operation * increases *exactly* by 1 those integral values a_i of components that are exceptional on X and lie over $\operatorname{Bs}|\mathcal{M}|_X$; in particular, * does not hold for E itself. (When $\mathcal{M}_X$ is free outside $f^{-1}P$ we can assume that * increases *only* values over P.)

Hence

$$\lceil \mathcal{A}_Y + \varepsilon \overline{M}_Y \rceil = \left\lceil \mathcal{A}_Y + \varepsilon \mathcal{M}_Y + \sum e_i E_i \right\rceil = \lceil \mathcal{A}_Y^* \rceil + \mathcal{M}_Y$$

and we get the vanishing $R^1 g_* \mathcal{O}(\lceil \mathcal{A}_Y^* \rceil + \mathcal{M}_Y) = 0$. This gives the surjective map of linear systems

$$\begin{aligned}&|E_Y + \lceil \mathcal{A}_Y^* \rceil + \mathcal{M}_Y| \dashrightarrow \\ &\qquad \left| \left(\lceil \mathcal{A}_Y^* + E_Y \rceil + \mathcal{M}_Y \right)_{|E_Y} \right| = \left| \left\lceil \left(\mathcal{A}_E^* \right)_{E_Y} \right\rceil + \left(\mathcal{M}_{|E} \right)_{E_Y} \right|.\end{aligned}$$

Indeed, by definition and our choice of Y, $\mathcal{M}_{Y}{}_{|E_Y} = (\mathcal{M}_{|E})_{E_Y}$. The same holds for *any* sufficiently high resolution X_{hr}/X and *some* sufficiently high resolution E_{hr}/E. In addition, by adjunction $(\mathcal{A}_Y + E_Y)_{|E_Y} = (\mathcal{A}_E)_{E_Y}$, where $\mathcal{A}_E = \mathcal{A}(E, B_E)$. Hence $(\mathcal{A}^*_Y + E_Y)_{|E_Y} = (\mathcal{A}^*_E)_{E_Y}$, where $(\mathcal{A}^*_E)_{E_Y} = (\mathcal{A}_E)_{E_Y} + \mathcal{F}_{E_Y}$ for $\mathcal{F}_{E_Y} = \sum F_i$ with $F_i = E_i \cap E_Y$ and E_i under *. This assumes that we can extend $\mathcal{F}_{E_Y}$ birationally to $\mathcal{F}$, or on (any) sufficiently high E_{hr}/E. Since $\overline{M}_{|E}$ has the same support on E as $M_{|E}$, then $F = \mathcal{F}_X$ is exactly the fixed divisorial component of $E \cap \operatorname{Bs}|\mathcal{M}|_X = \operatorname{Supp}\operatorname{Fix}(|\mathcal{M}|_{X|E})$ (in general, $\neq \operatorname{Fix}\left|\mathcal{M}_{|E}\right|_E = (\operatorname{Fix}\mathcal{M}_{|E})_E = 0$ for b-free $\mathcal{M}$!) over the locuse where B_E is integral, that is, *except over* $\operatorname{Supp} B_E$, because (E, B_E) is Klt. Since the resolution is divisorial, the whole $\mathcal{F}$ is exactly over the integral components of $\mathcal{A}_E$ and over F (cf. (Sa′F) in Proposition 6.34, (4)). Finally, by normal crossings, $\lceil \mathcal{A}^*_Y + E_Y \rceil_{|E_Y} = \lceil (\mathcal{A}^*_E)_{E_Y} \rceil$.

Now by Lemma 4.40, (SA′), and Remark 4.30, (1) (again because (E, B_E) is Klt), we get saturation on Y/T:

$$|E_Y + \lceil \mathcal{A}^*_Y \rceil + \mathcal{M}_Y| = |\mathcal{M}_Y| + E_Y + \lceil \mathcal{A}^*_Y \rceil,$$

where $|\mathcal{M}_Y|$ is free on Y. Indeed, $E_Y + \lceil \mathcal{A}^*_Y \rceil \geq 0$ and $\mathcal{M}_Y$ is integral; $\operatorname{mult}_{E_Y}(E_Y + \lceil \mathcal{A}^*_Y \rceil) = 0$. Hence, by the above surjectivity, on E_Y

$$\begin{aligned}\left|\left\lceil (\mathcal{A}^*_E)_{E_Y} + (\mathcal{M}_{|E})_{E_Y} \right\rceil\right| &= \left|\lceil (\mathcal{A}^*_E)_{E_Y} \rceil + (\mathcal{M}_{|E})_{E_Y}\right| \\ &= \left|(\mathcal{M}_{|E})_{E_Y}\right| + \lceil (\mathcal{A}^*_E)_{E_Y} \rceil,\end{aligned}$$

where $\left|(\mathcal{M}_{|E})_{E_Y}\right|$ is free on E_Y. Moreover, we can add to $\mathcal{F}$ exceptional$/X$ F_i over the integral components of $\mathcal{A}_E$, but not over F, preserving the mobile part $(\mathcal{M}_{|E})_{E_Y}$. This gives $\mathcal{A}^*_E = \mathcal{A}_E + \mathcal{F} = \mathcal{A}' + F$ with $F, \mathcal{A}'$ and with $\mathcal{M} := (\mathcal{M}_{|E})$ under the saturation (SA′F), where F is considered as a b-divisor. Then

$$\begin{aligned}\left|\left\lceil (\mathcal{M}_{|E})_{E_Y} + F + (\mathcal{A}'_E)_{E_Y} \right\rceil\right| &= \left|\left\lceil (\mathcal{A}^*_E)_{E_Y} + (\mathcal{M}_{|E})_{E_Y} \right\rceil\right| \\ &= \left|(\mathcal{M}_{|E})_{E_Y}\right| + F + \lceil (\mathcal{A}'_E)_{E_Y} \rceil,\end{aligned}$$

which means the saturation (SA′F). $\square$

The proof of the lemma used the invariance of saturations (SA′) and (SA′F) under $\sim$.

Proposition 9.17 (Invariance of saturations) *If $\mathcal{D} \sim \mathcal{D}'/Z$ then the $\mathcal{C}$-saturation of $\mathcal{D}$ is linearly equivalent to that of $\mathcal{D}'$. The same holds for $\mathbb{R}$-divisors.*

For asymptotic saturation, we can replace each $\mathcal{D}_i$ by $\mathcal{D}_i \sim \mathcal{D}'_i/Z$ uniformly *(that is, there exists a rational function $a \neq 0$ on X/T such that each $\mathcal{D}_i = \mathcal{D}'_i + \overline{(a)}$). The same holds for the similarity of* characteristic type *of $\mathcal{D}_\bullet$* (see Remark 4.30, (7)); *after a truncation by I', the index I of asymptotic saturation equals $I/\gcd(I, I')$.*

Proof It is enough to verify the divisorial version, when $D = \mathcal{D}_X \sim D' = \mathcal{D}'_X/Z$ and $C = \mathcal{C}_X$. Thus $D' = D + (a)$.

Indeed,

$$\begin{aligned}\operatorname{Mov} \lceil D' + C \rceil a &= \operatorname{Mov} \left(\lceil D + C \rceil + (a) \right) \\ &= \operatorname{Mov} \lceil D + C \rceil + (a) \leq D + (a) = D',\end{aligned}$$

because the fixed part is independent of $\sim$, and the mobile part changes by (a) under $\sim$.

For asymptotic saturation, on a sufficiently high model Y/X, we replace D' by $j(\mathcal{D}'_i)_Y = j(\mathcal{D}_i)_Y + j\overline{(a)}_Y$.

By the above, to prove invariance under similarity it is enough to consider a truncation $\mathcal{D}_\bullet^{[I']} = I'\mathcal{D}_{iI'}$. Then, by asymptotic $\mathcal{C}$-saturation of $\mathcal{D}_\bullet$ with index I, for any i and j, divisible by I,

$$\operatorname{Mov} \left\lceil j\mathcal{D}_i^{[I']} + \mathcal{C} \right\rceil = \operatorname{Mov} \lceil jI'\mathcal{D}_{iI'} + \mathcal{C} \rceil \leq jI'\mathcal{D}_{jI'} = j\mathcal{D}_j^{[I']},$$

that means that $\mathcal{D}_\bullet^{[I']}$ is asymptotically $\mathcal{C}$-saturated with the same index I. In fact, we can replace I by $I/\gcd(I, I')$. $\square$

Proof of Proposition 9.15 By Lemma 9.16, we obtain (SA$'$F), and this then applies in particular to the fixed component of $|\mathcal{M}|_X$ on E, because $\operatorname{Supp} B_E$ is fixed (cf. Remark 6.41). This gives (1) by $(\mathrm{SSB})_{n-1}(\mathrm{gl})$ and Proposition 6.34, (4). This gives also (2–3) by $(\mathrm{CBS})_{n-1}(\mathrm{gl})$ (cf. Corollary 6.40 for $n = 3$) because (SA$'$F) implies (SAT) for $\mathcal{D} = \mathcal{M}$ by Lemmas 4.40 and 6.36.

Finally, we can assume that each E_u is$/E$ after bounded blowups (e.g., resolutions) that preserves all the properties of desirable triples. This holds for (CBS) because any blowup decreases the descent data of b-free $\mathcal{M}$. Indeed, let $g\colon Y \to X/Z$ be birational, and $\mathcal{M}$ sufficiently general, namely, $\mathcal{M}_Y = \mathcal{M}_X$ as b-divisors. Then

$$\begin{aligned}\mathcal{E}(\mathcal{M})/X &= \overline{\mathcal{M}_X} - \mathcal{M}_X = \overline{g^*\mathcal{M}_X} - \mathcal{M}_Y = \overline{\mathcal{M}_Y} - \mathcal{M}_Y + \overline{E}\\ &\geq \overline{\mathcal{M}_Y} - \mathcal{M}_Y = \mathcal{E}(\mathcal{M})/Y,\end{aligned}$$

where $E = g^*\mathcal{M}_X - \mathcal{M}_X \geq 0$ by Lemma 4.21. $\square$

To secure the saturation (εA$'$S) we need to strength the inductive model.

Lemma 9.18 *Let $(X/T, B)$ be a weak log Fano contraction, and D any $\mathbb{R}$-Cartier divisor on X. Then there exists a boundary $B^+ \geq B$ such that*

- *$(X/T, B^+)$ is again a weak log Fano contraction; and*
- *$\mathcal{A}(X, B) \geq \mathcal{A}(X, B^+) + \varepsilon\overline{D}$ for any real number $0 < \varepsilon \ll 1$.*

Proof Since $(X/T, B)$ is a weak log Fano contraction, we can find an effective $\mathbb{R}$-Cartier divisor $D' \sim_{\mathbb{R}} -(K + B)$. Since this is big we can assume also that there is an *effective* $D^+ = ND' \geq D$ for some real number $N > 0$. Then, for any $0 < \delta \ll 1$, $(X, B^+ = B + \delta D^+)$ is again a weak log Fano contraction with the required properties. Indeed, $\mathcal{A}^+ = \mathcal{A}(X, B^+) = \mathcal{A} - \delta\overline{D}^+$ by definition. Thus for any $0 < \varepsilon \leq \delta$, we have

$$\mathcal{A} = \mathcal{A}^+ + \delta\overline{D}^+ \geq \mathcal{A}^+ + \varepsilon\overline{ND'} \geq \mathcal{A}^+ + \varepsilon\overline{D}. \quad \square$$

Corollary 9.19 Under the assumptions of Lemma 9.18*, saturation* (LCA) *implies*

(εAS) *asymptotic saturation for $\{\mathcal{D}_i\}$ with respect to $\mathcal{C} = \mathcal{A}^+ + \varepsilon\overline{D}$.*

Proof By Lemma 4.40 we need the inequality $\mathcal{A} \geq \mathcal{A}^+ + \varepsilon\overline{D}$, that we know by Lemma 9.18. $\square$

Proof of Theorem 9.9 To prove stabilization, we use Proposition 9.13. But before this application, we need to construct an appropriate inductive model of $(X/T,B)$. Note that the assumptions (LBF), (MXD), (BED) and (LCA) are birational/T. Thus they hold on any birational model of X/T. However (LCA) is sensitive to changes of the boundary B. This allows us to improve (LCA) on an inductive model of $(X/T,B)$ as follows.

Let D be an effective Cartier divisor that contains the special fibre $f^{-1}P$ of X/T. We can find such one in our local case. By Corollary 9.19 we can increase B so that $(X/T,B)$ is still a weak log Fano contraction (in particular, is still Klt), and (εAS) holds:

(εAS) asymptotic saturation holds for $D_\bullet$ with respect to $\mathcal{C}=\mathcal{A}+\varepsilon\overline{D}$ for some $\varepsilon>0$.

Now we modify our weak log Fano contraction $(X/T,B)$ to an inductive model $(X/T,B)$. Such a thing exists by Lemma 9.6. All our assumptions are preserved, including (εAS); this follows from monotonicity (1) in Definition 9.3 of inductive models and Lemma 4.40. After a $\mathbb{Q}$-factorialization we can assume that X is $\mathbb{Q}$-factorial. This time (εAS) is preserved since this modification is crepant.

Note that this final change affects not only the boundary B, but also the contraction X/T itself. Thus $\overline{D}$ in (εAS) is replaced by an effective Cartier b-divisor $\mathcal{D}$ that may not be Cartier on the new X itself. However it contains the special fibre $f^{-1}P$, that is, $\mathcal{D}\ge\overline{f^*H}$, where H is a hypersurface through P. (The latter is a birational invariant of modifications of X/T.) In particular, for any prime $\mathbb{Q}$-Cartier divisor F on X over P, $\mathcal{D}\ge\overline{F}$. By Lemma 4.40 we can replace $\mathcal{D}$ by $\overline{F}$ in (εAS). Taking $F=E_c$ we get an inductive model that satisfies (εAS) with $D=E_c$.

Moreover, by Addendum 4.40.1 we have (εA$'$S) for the integral *over* E_c asymptotic saturations with respect to $\mathcal{A}'+\varepsilon\overline{E_c}$ (cf. Example 4.41). Take $\mathcal{C}_1=\mathcal{A}+\varepsilon\overline{E_c}>\mathcal{C}_2=\mathcal{A}+d\overline{E_c}$ with $d<\varepsilon$ and then set $\varepsilon:=d$. Since $\mathrm{mult}_{E_c}\mathcal{A}=-1$ is integral for the inductive model $(X/Z,B)$, the integral property means that $j\mathcal{D}_i+\varepsilon\overline{E_c}$ also has integral multiplicity in E_c.

Now by Lemma 9.7, after an increasing of boundary B we can assume that $(X/T,B)$ is a purely log terminal Fano contraction. This gives the required strict inductive model of Proposition 9.13 that satisfies (BNF), (εA$'$S) and (MXC).

(εA$'$S) was proved above. And also (εAS), without the integral condition. (BNF) follows from (LBF). (MXC) follows from (MXD).

Fix a linear equivalence $E' \sim E_c$ with $E_c \not\subset \operatorname{Supp} E'$. Since the sets $\mathfrak{F}$ in (BWQ) are defined up to bounded addition of a fixed divisor (cf. Remark 6.35, (1)), and E' is fixed, we get (BRE$'$) whenever (BWQ) is established. Indeed, for a bounded family of models E of E_c, the divisors $(\overline{E'_{|E_c}})_E$ have bounded support, that is, *add* a bounded set (really a single element!) to each element of $\mathfrak{F}$. The sum is included in the log transform of that from E_c whenever E/E_c.

Hence to use Proposition 9.13, we want to verify (BWQ), after increasing B. This property is birational and it follows from

(1) boundedness of the fixed component of each $|\mathcal{M}_i|_X$ on E_c, that is, the divisorial component of $\operatorname{Bs}|\mathcal{M}_i|_X \cap E_c$ form a bounded family $\mathfrak{F}$;

(2) in each $\left|\mathcal{M}_{i\,|E_c}\right|$, a general element has bounded canonical singularities on a desirable triple of $(E_c/\mathrm{pt.}, B_c)$ for the element (also just bounded when E_c is a surface); and

(3) desirable triples in (2) form a bounded algebraic family.

Thus by (3) there exists a bounded family of triples $(E_u/T_u/\mathrm{pt.}, B_u, \mathfrak{F}_u)$ of the required form for $(E_c/\mathrm{pt.}, B_c)$ (cf. Corollary 6.40). On the other hand, by Proposition 9.15, we obtain (1–3), because $(X/T, B)$ is now a strict inductive model with central divisor E_c, and (εAS) for b-divisors $i\mathcal{D}_i$ implies (SA$'$) for the b-divisors $\mathcal{M}_i$. Indeed, by definition the asymptotic saturation (εAS) for $\mathcal{D}_i$ and $j = i$ means the saturation for $i\mathcal{D}_i$ with respect to $\mathcal{A} + \varepsilon\overline{E_c}$ (cf. Remark 4.30, (5)). Even increasing B, we can extend E_c to $E_c + M$ for any $0 < \varepsilon \ll 1$, where $M \geq M_1 = (\mathcal{M}_1)_X$ for a sufficiently general $\mathcal{M}_1$, and where $E_c + M$ contains the fibre $f^{-1}P$. Then, by the invariance of saturation in Proposition 9.17 and by (LBF), the saturation for $i\mathcal{D}_i$ with respect to $\mathcal{A} + \varepsilon\overline{E_c + M}$ implies the same saturation for $\mathcal{M}_i$. The latter implies (SA$'$) by Addendum 4.40.1 because each $\mathcal{M}_i$ is integral with $\operatorname{Bs} = \operatorname{Supp}(E_c + M)$ in (BSP) (cf. Example 4.41). Indeed, then, by (BED), for all $\mathcal{M}_i$, $\operatorname{Bs}(\mathcal{M}_i)_X \subset \operatorname{Supp} M_1 \cup f^{-1}P \subset \operatorname{Bs}$.

In fact, we proved (1–3) in a slightly more general settings: we can consider an arbitrary family of b-free b-divisors $\mathcal{M} = \mathcal{M}_i$ that satisfy (BSP) and the saturation (SA$'$).

Before applying Corollary 7.9, note that we can assume that $\mathcal{D}_1 \sim \mathcal{M}_1$ is sufficiently general. Otherwise we can replace $\mathcal{D}_1$ by general $\mathcal{D}_1^g \sim \mathcal{D}_1$, that is, by $\mathcal{D}_1^g = \mathcal{D}_1 + \overline{(a)}$, where $a \neq 0$ is a rational functions on X/T. Then we replace each $\mathcal{D}_i$ uniformly by $\mathcal{D}_i^g = \mathcal{D}_i + \overline{(a)} \sim \mathcal{D}_i$ (similarity!). Thus

$$\mathcal{D}^g = \lim_{i\to\infty} \mathcal{D}_i^g = \lim_{i\to\infty} \mathcal{D}_i + \overline{(a)} = \mathcal{D} + \overline{(a)}.$$

These changes preserve the conditions in the theorem. (LWF) is not affected. (LBF) continues to hold because $i\mathcal{D}_i^g = i\mathcal{D}_i + i\overline{(a)} \sim i\mathcal{D}_i$; (LCA) by Proposition 9.17; (MXD) because $\mathcal{D}^g = \mathcal{D} + \overline{(a)} \geq \mathcal{D}_i + \overline{(a)} = \mathcal{D}_i^g$; (BED) because, outside $f^{-1}P$, each $\mathcal{D}_i^g = \mathcal{D}_i + \overline{(a)} = \mathcal{D} + \overline{(a)} = \mathcal{D}^g$.

Finally, we derive (BWQ) from (1–3). By Corollary 7.9 with $K = \mathbb{Q}$, $a_1 = 1$, $a_2 = i$ and $\mathcal{D}_2 = \mathcal{D}_i$ by (BED), which gives the required condition in Corollary 7.9, and by (1), we obtain that each restriction $(\mathcal{D}_1 - \mathcal{D}_i)_{\mid E_c}$ has support bounded by some $F \in \mathfrak{F}$, where $\mathfrak{F}$ is a bounded set of reduced divisors. Since $\mathcal{D}_{1\mid E_c}$ is fixed, this last bound holds for each $\mathcal{D}_{i\mid E_c}$, and even on each E_u/T_u of our bounded family of triples. To be complete we include also $\overline{E'|_{E_c}}$ into consideration by definition of (mx) restrictions. Hence each $(\mathcal{D}_{i\mid E_c})_{E_i} \in \mathfrak{N}(E_i/T_i/\mathrm{pt.}, \mathfrak{F}_i)$ for a desirable triple $(X_i/T_i/\mathrm{pt.}, B_i, \mathfrak{F}_i)$, because by definition of such triple $(\mathcal{M}_{i\mid E_c})_{E_i} = g_i^* M_i$, where $g_i \colon E_i \to T_i/\mathrm{pt.}$ and M_i is nef and big on $T_i/\mathrm{pt.}$ In particular,

$$(\mathcal{D}_{i\mid E_c})_{E_i} \sim_{\mathbb{Q}} ((\mathcal{M}_i/i)_{\mid E_c})_{E_i} = g_i^* M_i/i \sim_{\mathbb{Q}} 0/T_i/\mathrm{pt.},$$

and is nef on E_i/pt.

By (LBF), $i\mathcal{D}_{i\mid E_c} \sim \mathcal{M}_{i\mid E_c}$. So by (2) and Addendum 6.8.1, the descent data of restrictions $\mathcal{D}_{i\mid E_c}$ are asymptotically bounded with respect to $\mathcal{A}_c$ over desirable models E_i/T_i for $\mathcal{M}_i$, because B_i is the crepant boundary for B_c as in the definition of the desirable triple, that is, $\mathcal{A}(E_i, B_i) = \mathcal{A}_c$, and $\lim_{i\to\infty} i = \infty$. Note that in Addendum 6.8.1, we replace a common model Y by the sequence $X_i/\mathrm{pt.}$ as in Proposition 6.8 itself. $\square$

Now we can prove stabilization of $\lim_{i\to\infty} \mathcal{D}_{i\mid E_c}$ on E_c. It is enough to prove this for a similar system $\mathcal{D}_\bullet$ (such as the characteristic system, cf. Remark 4.30, (7)).

Corollary 9.20 Under the assumptions of Theorem 9.9, *the system $\mathcal{D}_\bullet$ up to similarity of* characteristic type *satisfies the theorem with $d_{c,1} = d_c = 0$.*

In addition, we can assume that the new $\mathcal{D}_1 = \mathcal{M}_1$ *is sufficiently general; in particular, $\mathcal{D}_1 \geq 0$.*

Proof By the theorem there exists a natural number $j > 0$ such that $d_{c,j} = d_c$. Take sufficiently general $\mathcal{D}_1^{[j]} = \mathcal{M}_j \sim jD_j$. This data defines the required similarity.

Indeed, as in the proof of Theorem 9.9, we can verify that any similarity with $j = 1$ preserves the conditions of the theorem, and also its conclusions. The same holds for any truncation: e.g., for new $\mathcal{D}_i := \mathcal{D}_i^{[j]} = j\mathcal{D}_{ij}$. (LWF) is not affected by the change. (LBF) continues to hold because $i\mathcal{D}_i^{[j]} = ij\mathcal{D}_{ij} \sim \mathcal{M}_i^{[j]} = \mathcal{M}_{ij}$; (LCA) by Proposition 9.17; (MXD) because

$$\mathcal{D}^{[j]} = \lim_{i\to\infty} \mathcal{D}_i^{[j]} = \lim_{i\to\infty} j\mathcal{D}_{ij} = j\mathcal{D} \geq j\mathcal{D}_{ij} = \mathcal{D}_i^{[j]}$$

for any natural number i; (BED) because, outside $f^{-1}P$, each $\mathcal{D}_i^{[j]} = j\mathcal{D}_{ij} = j\mathcal{D} = \mathcal{D}^{[j]}$.

Finally, for general $\mathcal{M}_j$, $\operatorname{mult}_{E_c} \mathcal{M}_j = 0$ by (LBF) in Theorem 9.9. Hence by our choice of j, $d_c = d_{c,1} = \operatorname{mult}_{E_c} \mathcal{M}_j = 0$. Note that stabilization takes place for exactly the same divisors (but their indexes may be different and the values of multiplicities in E_c are j times the old ones).

The last sentence of Corollary 9.20 holds by our choice of *new* $\mathcal{D}_1$. $\square$

Corollary 9.21 Under the conditions of Theorem 9.9, *assume in addition*

(AMN) arithmetic monotonicity*: $\mathcal{D}_i \geq \mathcal{D}_j$ for any $j \mid i$; and*

(RRF) $d_{c,1} = d_c = 0$ *as in* Corollary 9.20.

Then $\lim_{i\to\infty} \mathcal{D}_{i\,|E_c}$ *stabilizes* (that is, $\mathcal{D}_{j\,|E_c} = \mathcal{D}_{|E_c}$ for some $j \gg 0$).

More precisely, there exist a natural number $N > 0$ and a b-free divisor $\mathcal{M}$ of E_c such that $\mathcal{M}$ satisfies (birational) saturation with respect to $\mathcal{A}_c = \mathcal{A}(E_c, B_c)$ (cf. (SAT) and Addendum 6.26.2)*, and if $N \mid j$, then*

- $d_{c,j} = d_c = 0$*; and*
- $j\mathcal{D}_{j\,|E_c}(= j\mathcal{D}_{|E_c}) = (j/N)\mathcal{M}.$

Thus each $\mathcal{M}_{j\,|E_c} \sim j\mathcal{D}_{j\,|E_c} = (j/N)\mathcal{M}$.

Remark 9.22 (1) (AMN) is preserved for any similarity in the same way as (MXD) of Theorem 9.9.

(2) (AMN) implies (MXD). For this we need that $\mathcal{D}_\bullet$ is bounded by a Cartier divisor that is nonzero on E_c, which follows from (BNF) and Lemma 4.21 (cf. Proposition 4.20).

(3) We can take a truncation of $\mathcal{D}_\bullet$ such that the corollary holds with $N = 1$.

(4) We *only* expect $\mathcal{M}_{j\,|E_c} = j\mathcal{D}_{|E_c}$ if stabilization holds in a neighborhood of some model (cf. Theorem 10.12 below).

Remark 9.23 If $d_{c,n} = 0$, $\mathcal{D}_{j\,|E_c}$ is the canonical *fixed* restriction as in (fx) of Proposition 7.7.

Lemma 9.24 *Let $(X/T, B)$ be an inductive model with central divisor E_c and $\mathcal{D}_\bullet$ a system of $\mathbb{R}$-b-divisors such that:*

(BSA) *each $\mathcal{D}_i$ is* b-semiample *as in Proposition 4.44, or just satisfies* (BNF)*;*

(ASA′) *the system $\mathcal{D}_\bullet$ is asymptotically saturated with respect to $\mathcal{C} = \mathcal{A}' = \mathcal{A}(X, B) + E_c$; and*

(FXR) *each $d_{c,i} = 0$;* in particular, each fixed restriction $\mathcal{D}_{i\,|E_c}$ is well defined.

Then the restricted system $\mathcal{D}_{\bullet\,|E_c}$ satisfies asymptotic saturation (LCA) *with respect to $\mathcal{A}_c = \mathcal{A}(E_c, B_c)$.*

Proof Immediate by Proposition 4.44 with $Y = E_c$. (GLF) and (LCC) follows from properties of inductive models. (FXR) ensures general position (GNP). Moreover, (BSA) can be replaced by (BNF) because b-nef is enough for Kawamata–Viehweg vanishing and saturation in the proof of the proposition. □

Lemma 9.25 *Let $\mathcal{D}_\bullet$ be a system on a log pair $(X/Z, B)$ such that:*

- *(X, B) is Klt;*
- *$\mathcal{D}_\bullet$ is bounded by a b-divisor, satisfies* (AMN), (LBF) *and saturation* (LCA).

Then up to a truncation it converges to a b-divisor.

Thus (AMN) works as the convexity of Lemma 4.22.

Proof (Compare the proof of Lemma 4.22.) Up to a similarity preserving the conditions, we can assume that $\mathcal{D}_1 \geq 0$ by (LBF), and $\mathcal{D}_\bullet$ satisfies (LCA) with index $I = 1$. Hence each $\mathcal{D}_i \geq 0$ by (AMN). Since the system is bounded and under (AMN), it has a convergent subsequence. In addition, for $i = jq+r$ with natural numbers i, j, q and r, (LCA) and (LBF) imply the estimate

$$\frac{jq}{jq+r}\mathcal{D}_j \leq \mathcal{D}_i,$$

that implies that the sequence has a limit in view of that for the subsequence $\mathcal{D}_j$ and the bound.

Indeed, by (LCA), (LBF) for $\mathcal{D}_j$, and since $\lceil r\mathcal{D}_j + \mathcal{A} \rceil \geq 0$ by Klt, we get the required inequality

$$\begin{aligned} jq\mathcal{D}_j &\leq qj\mathcal{D}_j + \operatorname{Mov}\lceil r\mathcal{D}_j + \mathcal{A}\rceil \\ &\leq i \operatorname{Mov}\lceil (jq+r)\mathcal{D}_j + \mathcal{A}\rceil \leq (jq+r)\mathcal{D}_i. \quad \square \end{aligned}$$

Proof of Corollary 9.21 First, by (AMN), (MXD) and (RRF), we have $d_{c,i} = d_c = 0$ for each natural number i. Thus the fixed restriction $\mathcal{D}_{i\,\rvert E_c}$ is well defined for each i.

Secondly, after restriction we preserve properties (LBF), (LCA), (MXD) and (AMN) of Theorem 9.9 and of the corollary, now for the new objects

$$(X/T, B) := (E_c/\mathrm{pt.}, B_c), \quad T = \mathrm{pt.}, \quad \mathcal{D} := \mathcal{D}_{\rvert E_c},$$
$$\mathcal{D}_i := \mathcal{D}_{i\,\rvert E_c}, \quad \mathcal{M}_i := \mathcal{M}_{i\,\rvert E_c} \quad \text{and} \quad \mathcal{A} := \mathcal{A}_c = \mathcal{A}(E_c, B_c).$$

Indeed, we get (LBF), (MXD) and (AMN) by definition of restrictions; (AMN) implies (MXD) by Remark 9.22, (2) above. For (AMN), note that a birational fixed restriction of an effective divisor is effective.

By Lemma, 9.24 (LCA) for $\mathcal{D}_{\bullet\,\rvert E_c}$ on $(E_c/\mathrm{pt.}, B_c)$ follows from (ASA$'$) of Lemma 9.24 for the inductive model in the proof of Theorem 9.9. In turn, (ASA$'$) follows from (εSA$'$) of Proposition 9.13 by Addendum 4.40.1, and (εSA$'$) itself follows from (LCA) of Theorem 9.9 by construction in its proof (cf. Example 4.45).

By Lemma 9.25 and the bound on $\mathcal{D}_{\bullet\,\rvert E_c}$ (cf. Remark 9.22, (2)), the limit $\lim_{i\to\infty} \mathcal{D}_{i\,\rvert E_c}$ exists up to a truncation, and it is a *candidate* for $\mathcal{D}_{\rvert E_c}$ (cf. Remark 9.22, (4)).

Now the corollary follows from the solution of the asymptotic descent problem in Addenda 5.12.1 and 5.13.1 for the limit $\mathcal{D} := \lim_{i\to\infty} \mathcal{D}_i$ with $\mathcal{D}_i := \mathcal{D}_{i}{}_{\restriction E_c}$. (FDS) follows from (MXD) for effective $\mathcal{D}_\bullet$. Asymptotic saturation holds for $\mathcal{C} = \mathcal{A}(E_c, B_c)$ by (LCA). (LBF) implies (BNF). A prediction exists by (CBS)$(E_c/\mathrm{pt.}, B_c)$, or (CBS)$_{n-1}$(gl). This can be done exactly as in the proof of Theorem 6.19, (3). The only difference between our current situation and (FGA) is that we do not have the convexity of Lemma 4.22. (However in our applications, this is satisfied.) But this is not at all required for Theorem 6.19, (3) and Addendum 5.12.1. We only need (AMN) under (LCA)! $\square$

Stabilization *near* E_c needs an opposite technique that we discuss in the next section.

10 Destabilization

We find a neighborhood of E_c on a 0-log pair, with a *certain* boundary, of Remark 3.26, (2) over which the system $\mathcal{D}_\bullet$ stabilizes. We also work with generalizations of such pairs (cf. (WLF) vs (GLF) and (PFN)).

Definition 10.1 Let $(X/T, B)$ be a log pair such that

- X/T is a projective contraction;
- $K + B \equiv 0/T$;
- $(X/T, B)$ is *exceptionally* log terminal; and, moreover,
- there exists *at most one* prime b-divisor E_c with log discrepancy 0 for $K + B$, and E_c/P.

If there exists such an E_c we say that $(X/T, B)$ is an 0-*exceptional log terminal pair* or 0-elt pair, and that the b-divisor E_c is *central*. Otherwise $(X/T, B)$ is Klt and is just a 0-*log pair*.

Example 10.2 Usually a 0-elt pair arises as a *complement* as follows. Suppose that $B = B_\eta + \eta C$ such that $(X/T, B_\eta)$ is a 0-log pair and $C \geq 0$ with $\eta = \mathrm{mult}_{E_c} C > 0$. Note that in this situation $C \sim_{\mathbb{R}} 0/T$, and it defines canonically a b-divisor $\mathcal{C}$ such that $C = \mathcal{C}_X$ and $\mathcal{C}$ is b-free over T whenever B_η and η are rational.

A typical example is building a log singularity. Let $(X/T, B)$ be a *birational* weak log Fano contraction, that is, X/T is birational. Then after taking a complement, we can assume that $K + B_\eta \equiv 0/T$. Thus after a contraction, X/T is the identity. Now we can find the required $C \geq 0$ passing through P such that, for some threshold $\eta > 0$, $K + B_\eta + \eta C$ has only one log discrepancy 0 in E_c and E_c/P. This is how we constructed our inductive model in Lemma 9.6.

To be more precise, we introduce neighbourhoods of E_c on some special and quite[10.1] (log) canonical models. They give an induction on the *rank* of non$\mathbb{Q}$-factoriality and the number of seminegative discrepancies (a kind of *difficulty*; cf. Lemma 10.11 below).

Key 10.3 For given b-free $\mathcal{L}$, we need to construct a birational model Y/T (later, a neighbourhood in such a model) on which there exists an effective $\mathbb{R}$-divisor $D = \sum d_i E_i$ such that:

- $d_i > 0$ precisely on the prime divisors E_i that are exceptional on T;
- $L + \sum d_i E_i$ is nef and even semiample$/T$, where $L = \mathcal{L}_Y$; and
- $\mathcal{L}$ is free over Y/T.

We call such a model a *destabilizing model* for $\mathcal{L}$ (destab model) because we can add to it (or to its multiple) a divisor D on Y that violates (destabilizes) D-saturation (for example, the exceptional saturation of Example 4.31). The divisor D is *destabilizing*, and it defines the b-divisor $\mathcal{D} = \overline{D}$.

Caution 10.4 The contraction Y/T is not necessarily divisorial.

Example 10.5 (1) If $L = \mathcal{L}_T = 0$, we can take $Y = T$. More generally, if $L = \mathcal{L}_T$ is $\mathbb{Q}$-factorial on T we can take $Y = T$ and also $D = 0$.

(2) If $\dim X = 2$ and L is a curve passing through P on T then for any Y/T we can find a divisor $D > 0$ that converts Y/X to a destab model. Indeed, we can find a numerically ample divisor $(D+L)/T$ (in the sense of Mumford) with $D > 0$ and $\operatorname{Supp} D = f^{-1}P$ on a $\mathbb{Q}$-factorial model (cf. Lemma 11.2).

[10.1] Something missing? Quite nice, quite bad? Or maybe you mean to say "completely canonical" or log canonical?

(3) In some cases in dimension ≥ 3 we can't construct a *divisorial* contraction Y/T: e.g., if X/T is a flopping 0-elt pair and $L = \mathcal{L}_X$ its hyperplane section. It means that in this case we can't destab $\mathcal{L}$ on any divisorial Y/T.

Probably a destab model does not exists for every $\mathcal{L}$ in dimensions ≥ 3. However in certain important situations it exists by LMMP.

Proposition 10.6 *Let $(X/T, B)$ be a local weak log Fano contraction with birational contraction X/T, and $\mathcal{L}$ a b-free divisor. Then LMMP in dimension $n = \dim X$ implies the existence of a destab model for $\mathcal{L}$.*

The same holds for a purely log terminal model whose reduced part is nonexceptional/T; moreover, if (X, B) has only log terminal singularities with reduced centers not over P and $\mathcal{L}$ is free/T over $X \setminus f^{-1}P$. Or we can assume log terminal singularities and that $\mathcal{L}$ is in rather general position, *that is, not passing through the log canonical centers of (X, B) and under* LSEPD *(see [9, 10.5] or [4, 2.30]).*

We only need flips as a noninductive new objects in LMMP *under the* LSEPD trick *[9].*

However sometimes *very* special flips are enough (cf. Example 10.9 and Theorem 10.12 below).

Addendum 10.6.1 *If we can construct such a model over a neighborhood U on Z/T, where Z is the model defined by $\mathcal{L}/T$, then destab holds over the neighbourhood. The construction in the proof is log canonical; thus it can be done locally/Z.*

10.7 Construction of a destab (destabilizing) model

The proof of Proposition 10.6 is completely effective.

(1) Increasing B, we can convert our model into a 0-elt pair: e.g., as in Example 10.2. Then we can identify $(X = T/T, B = B_T)$.

(2) We take a log resolution $(Y/T, B_Y)$ of $(T/T, B_T)$ such that:

- $B_Y = f^{-1}B_X + \sum E_i$ (a noncrepant boundary!), where $f\colon Y \to X = T$, and the divisors E_i are exceptional on T; and
- $L = \mathcal{L}_Y$ is free over Y, in particular, $\mathcal{L} = \overline{L}$, where $L = \mathcal{L}_Y$.

For plt and log terminal cases not resolving the LCS centers we proceed as follows:

(2′) We suppose that L is rather powerful: $L \equiv NH/T$, where H is Cartier (semiample$/T$): e.g., $N \geq 2\dim X + 2$ is enough. Otherwise we replace L by its multiple modulo $\equiv /T$. Finally, we get the same multiple for the destab divisor D.

(3) We apply LMMP to $(Y/T, B_Y + L)$. At each step, we contract an extremal ray R with $R \cdot (K_Y + B_Y + L) < 0$ by a birational contraction $g\colon Y \to Z/T$. Note: by (2′) and boundedness of negative contractions, $R \cdot L = 0$. Thus freedom of L and (2′) is preserved during the modifications. Finally, we obtain a model where $K_Y + B_Y + L$ is nef; it is always big because Y/T is birational, or by (2′) again.

(4) Then we make a semiample contraction for $K_Y + B_Y + L$. We obtain this because the divisor is always big, because Y/T is birational. We also need the LSEPD trick [9] and can do it if we secure it; in addition Addendum 10.6.1 holds over U. It also preserves the freedom of L, because again by (2′) if, for a curve C, $C \cdot (K_Y + B_Y + L) = 0$ then $C \cdot L = 0$. This is a destabilizing model.

(5) Finally, we take a destab divisor $D = l\mathcal{A}_Y$ (so, $\mathcal{D} = l\mathcal{A}(T, B_T)$) where l means that we take log discrepancies. By definition of log discrepancies

$$D = \sum d_i D_i = K_Y + B_Y - g^*(K + B) \equiv K_Y + B_Y,$$

for $g\colon Y \to T$, since $K + B \equiv 0/T$ for elt 0-models. Since it is Klt all $d_i > 0$. The same holds for purely log terminal models if their reduced part is not exceptional. For terminal models under (2).

$D + L \equiv K_Y + B_Y + L$ is ample$/T$ by construction. In particular, it is nef and semiample$/T$.

Finally, from LMMP we only need flips. Indeed, the termination is special because we have pl flips (cf. Example 10.9). Thus we get it by induction by Special Termination. However, the main problem is with flips. Really the model Y is over Z/T, so if we have flips for Y/Z, over some neighborhood $U \subset Z$, then we can construct a model Y/U and get the destab divisor over U/T, where $\equiv /U$ and $K_Y + B_Y$ is ample$/U$ now. Since the model is log canonical$/U$, it is *unique*$/U$ for given *choice* of B_Y, and can be constructed locally$/U$. $\square$

Lemma 10.8 *Let $\mathcal{D}$ be a b-divisor/X such that*

- $\mathcal{D}_X = 0$*;*
- effective: $\mathcal{D} \geq 0$*; and*
- *nef on general curves/X: on (some) sufficiently high models Y/X, the restriction $\mathcal{D}_Y$ is nef/X on curves covering the exceptional divisors of Y/X; in particular, $\mathcal{D}_Y$ is nef/X on such models.*

Then $\mathcal{D} = 0$.

Proof Immediate from the negativity of Lemma 3.18. □

Example–Construction 10.9 Working in dimension $n = \dim X$, when we do not yet know the existence of all flips, it would be very helpful to know in advance what types of flips are needed. Suppose that we have a 0-Klt pair $(X/T, B)$ (that is, replace elt by log terminal in Definition 10.1). For example, $(X/T, B)$ may have come from a purely log terminal Fano contraction by taking a complement. Suppose that $\mathcal{L} \sim_{\mathbb{R}} \mathcal{S} = S + \sum e_i E_i/T$ is b-free and that Z/T is a model for $\mathcal{L}$ over T such that:

- LSEPD is secured for B_Z (the log transform of B_T or B) on a model Z/T such that Z is nonsingular outside B_Z;
- $S = \sum s_i S_i$ is supported in the reduced part of B, considered as a b-divisor; and
- $\mathcal{S} \geq 0$ with the E_i exceptional on T.

Then over some neighborhood U of $\operatorname{Supp} \mathcal{S}_Z$, strictly log terminal and extremal flips of *type* (S$+-$) are sufficient: with $B_Y = S^- + S^+ + B'$, where $(Y/U, B_Y)$ is log terminal, B' has no reduced components, and $R \cdot S^- < 0$, $R \cdot S^+ \geq 0$.

Indeed, in the construction,

$$B_Y = g^{-1} B_T + \sum E_i \quad \text{and} \quad R \cdot (K_Y + B_Y + L) < 0 \quad \text{with } L = \mathcal{L}_Y.$$

Since R/Z, then $R \cdot L = 0$ and $R \cdot (K_Y + B_Y) < 0$. Thus the log discrepancy is supported in the exceptional/T locus:

$$D = l\mathcal{A}(X, B)_Y = K_Y + B_Y - g^*(K + B) \equiv K_Y + B_Y,$$

and there exists E_i with $R \cdot E_i \neq 0$. Moreover, $R \cdot E_i < 0$ since $D \geq 0$.

On the other hand, for $\mathcal{S}_Y = S + \sum e_i E_i$ on Y, $R \cdot (S + \sum e_i E_i) = R \cdot L = 0$. Hence if $R \cdot E_i \neq 0$ with $e_i \neq 0$, then there exists another divisor E_j or a component in S having opposite intersection with R. (In general, some e_i may be 0). R has a curve/ $\operatorname{Supp} \mathcal{S}_Z$ by Lemma 10.8 (cf. Example 7.6). Thus there is E_j that intersects the curve with $e_j \neq 0$, or a component of S does so. If this component has nonzero numerical intersection with R, we get a (S+−) flip by the above. Otherwise the intersection is $= 0$, and we again get a (S+−) flip with $S^- \geq E_i$.

Definition 10.10 Let $(X/T, B)$ be a 0-log pair and E a prime b-divisor. A (birational) *Klt neighbourhood* $(Y/T, B_Y)$ of $\operatorname{center}_Y E$ is a model $(Y/T, B)$, that is also a 0-log pair, and is a Zariski neighbourhood of $\operatorname{center}_Y E$ in it.

Lemma 10.11 *Under LMMP in dimension $n = \dim X$ and for birational X/T, Klt neighbourhoods terminate for inclusions.*

Proof On each model this is a Noetherian property. But there only exist a finite number of 0-log pair models (the finiteness of minimal models in the big case) [12]. □

We are now ready to establish stabilization in a birational neighbourhood of $E = E_c$.

Theorem 10.12 Under LMMP in dimensions $\leq n-1$. *Let $(X/T, B)$ be a log pair, E a prime b-divisor over a neighbourhood of P, and $\mathcal{L}_\bullet$ a system of b-divisors such that:*

(0LP) *$(X/T, B)$ is a birational 0-log pair with $\dim X = d \leq n-1$;*

(LBF) *each $\mathcal{L}_i$ is b-free; moreover,*

(STB) *the $\mathcal{L}_{i}{}_{|E}$ stabilize over E, and each $\mathcal{L}_{i}{}_{|E} = i\mathcal{M}$, where $\mathcal{M} = \mathcal{L}_{1}{}_{|E}$ is b-free; and*

(RFA) *the situation is a restriction of a pl contraction in dimension n,* that is, $\mathcal{L}_i$ is a restriction of the mobile system for some $(\mathrm{RFA})_{n,d}$ algebra (cf. Definition 3.43 and the proof below).

Suppose also that flips of type (S+−) *as in* Example 10.9 *exist in dimension n. Then there exists a Klt neighborhood* $(Y/T, B_Y)$ *of* $\operatorname{center}_Y E$ *on which all $\mathcal{L}_i$ with $i \gg 0$ are free, ample and $\mathcal{L}_i/i$ stabilizes* (over this neighbourhood all $\mathcal{L}_i/i = \mathcal{D}$, the limit). *Thus after a truncation, this holds for each $i > 0$, and each $\mathcal{L}_i$ gives this quasi-projective neighborhood on a contraction given by $\mathcal{L}_i$.*

Proof First, we recall and explain (RFA). There exists an extension

$$(X/T, B) \subset (X'/T', B'),$$

where $(X'/T', B'')$ is a pl contraction of dimension n, whereas X is the intersection of the reduced divisor in B'', and $B' \geq B''$. Such a complement always exists preserving (X', B'') log terminal, and we fix it. Then we define B by adjunction:

$$(K_{X'} + B')_{|X} = K + B.$$

We assume also that the system $\mathcal{L}_\bullet$ is the mobile restriction of a b-free system $\mathcal{L}'_\bullet$ (see 7.1), that is: $\mathcal{L}_i \sim \mathcal{L}'_{i\,|X}$, where $\mathcal{L}'_\bullet$ is the mobile system for the $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ algebra given by the pl contraction. Equivalently, the mobile system of the restriction for the algebra, where in either case we assume general position.

Fix some $i \gg 0$, and consider the model Z'/T' defined by $\mathcal{L}'_i$. By the construction of the (RFA) algebra, we can assume that

$$\mathcal{L}'_i \sim \mathcal{S}' = S' + \sum e_i E_i \geq 0,$$

as in Example 10.9. Note also that since X/T is birational, the model Z/T of X/T for $\mathcal{L}_i$ is in $\operatorname{Supp} \mathcal{S}'_{Z'}$ (and Z is normal by Addendum 4.44.1 and Example 4.45). It is completely contained in any neighborhood $U' \subset Z'$ of $\operatorname{Supp} \mathcal{S}'_{Z'}$. Thus the destab model$/T$ over Z/T is complete$/T$.

Now by Addendum 10.6.1 and Example 10.9, for each $\mathcal{L}'_i$, there exists a destab$/T'$ model $(Y'/U', B'_{Y'})$ over some neighborhood of $\operatorname{Supp} \mathcal{S}'_{Z'}$ in Z' (possibly not complete$/T'$). Then by exceptional saturation we can destabilize multiples of $\mathcal{L}'_i/U'$ over the destab divisor, whose support includes the exceptional$/T$ divisors. By (STB), $\operatorname{center}_{Y'} E$ does not intersect the destab divisor, that is, the exceptional divisorial locus of Y'/T' for the log canonical model for $(Y'/U', B' + NL')$ with $L' = (\mathcal{L}'_i)_{Y'}$. Note: $\operatorname{center}_{Y'} E$ is well defined for a sufficiently high resolution ($g'' : Y'' \to X'$ that is also regularly dominated over Y' by a Hironaka hut and E is a divisor on $(g'')^{-1}X$) for a blowup of E. In particular, $\operatorname{center}_{Y'} E \subset X_0$, where X_0 is the birational transform of X on Y'; and $\operatorname{center}_{Y'} E = \operatorname{center}_{X_0} E$

This gives the required neighbourhood. Indeed, if we apply adjunction of the subboundary $(B')^{Y'} = \mathcal{B}(X', B')_{Y'}$ on the normalization of X_0 (in fact, X_0 is normal) that is denoted also by X_0 we get an *effective* different divisor outside exceptional divisors of Y'/T'. In particular, the latter holds in a neighborhood of $\operatorname{center}_{Y'} E = \operatorname{center}_{X_0} E$. On the other hand, if we identify X' with T' as we did for 0-log pair, we get the adjunction $(K_{X'} + B')_{|X} = K + B$ with effective B. In addition, by the commutative diagram the above adjunction gives the subboundary $B^{X_0} = \mathcal{B}(X, B)_{X_0}$ as the adjunction for the subboundary that is effective on a neighbourhood $U \subset U' \cap Z$ of $\operatorname{center}_{X_0} E$. $K_{X_0} + B^{X_0}$ is numerically trivial$/P$. In particular, X_0/U is the identity. The discrepancies$/U$ for (X_0, B^{X_0}) are the same as for (X, B). Thus if we replace the subboundary B^{X_0} by a boundary $B_0 \geq B^{X_0}$ that is the same where the subboundary is effective and 0 (or even 1) where it is negative we obtain $K_{X_0} + B_0$ that is nef on U, that is, on a closure of each curve in U. If $K_{X_0} + B_0$ is not $\mathbb{R}$-Cartier outside U we can make it after some modification outside U/Z by the uniqueness as in Addendum 10.6.1. The discrepancies over U for (X_0, B_0) are the same as for (X, B). Now we can apply LMMP to $(X_0/T, B_0)$ the modification does not affect U because we increase a discrepancy that is the same as for $(X/T, B)$. However, at the end we obtain a 0-log pair of $(X/T, B)$ with the same discrepancies.

Finally, we can give $\mathcal{L}_i \sim \mathcal{L}'_{i\,|X}$ that is free and ample on U for $i \gg 0$. Indeed, by Proposition 10.6 it gives a multiple of $N\mathcal{L}'_i + \mathcal{D}$ since in the proposition the destab D is ample$/P$ and $N\mathcal{L}'_i$ is b-free$/P$. Since the neighborhood on Y' is not divisorial$/P$ it stabilizes (we can't add exceptional divisors by Lemma 10.8). (See also Addendum 4.44.1 and Example 4.45.) □

Now we can apply Theorem 10.12 to the inductive model in the proof of Theorem 9.9 to get a complete stabilization. This is the main result of this section.

Corollary 10.13 *If we assume LMMP in dimensions $d' \leq d = \dim X$, we obtain* $(\mathrm{CBS})^*_{d'-1}$ *and* $(\mathrm{SSB})_{d'-1}(\mathrm{gl})$. *(Moreover, we can omit the assumptions* $(\mathrm{CBS})^*_{d'-1}$ *and* $(\mathrm{SSB})_{d'-1}(\mathrm{gl})$ *for $d' \leq 3$, since they are known.)*

Assume also that flips of type (S+−) *exist in dimension n. Let $\mathcal{L}_\bullet$ be* *a mobile system of type* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *on* X/T. *Then there exists a* 0*-log pair* $(Y/T, B_Y)$ *over which each* $\mathcal{L}_i$ *with* $i \gg 0$ *is free and ample, up to a truncation.*

This gives a solution for $(\mathrm{CBS})_n(\mathrm{rfa}, \mathrm{bir})$ *under the assumptions.*

Lemma 10.14 Under LMMP in dimension $d = \dim X$.

(0LP) *Let* $(X/T, B)$ *be a birational* 0*-log pair;*

(NBU) *let* U *be a nonempty open subset in* X*, that is incomplete/*P*; and*

(CMP) *let* $(X/T, B+C)$ *be a complement* nonlog canonical 0*-log pair with* $C = \mathcal{C}_X$ *such that* $\mathrm{LCS}\,(X, B+C)$ *is completely contained inside* U (cf. Example 10.2).

Then there exists a different complement with a new $\mathcal{C}$ *such that the* new $\mathrm{LCS}\,(X, B+C)$ *is not completely contained in* U*, but all its* LCS *centers are in* U*, and* $(X, B+C)$ *is log canonical outside* U*. Thus the nonlog canonical part* X_s *of the* LCS *is inside* U, that is, does not intersect the complement of U. *In addition, there is a* transient LCS center E *such that* $(X, B+C)$ *is log canonical in its generic point, and* E *is not completely in* U*. After a perturbation of* C*, such* E *is unique, and is purely log terminal in its generic point and outside* U*.*

Moreover, we can find C^+ *such that* $(X/Z, B+C^+)$ *is a general log Fano contraction with the same support of* LCS *and the same purely log terminal center* E*. Thus adjunction on* E *gives a* general nonnormal *log Fano contraction* $(E/T_E, B_E, J_E)$*, where*

- $E/T_E = f(E)$ *is proper;*
- $(E^\nu/T_E, B_E)$ *is a normal general log Fano contraction with* $-(K_{E^\nu} + B_E) \equiv -(K + B + C^+)_{|E}$ *ample/*T_E *with normal* E^ν/E*; and*
- J_E *is the ideal sheaf of the* $\mathrm{LCS}\,(E, B_E, J_E)$ *on* E : the direct image of that on the normalization)*; it is supported in the intersection of* E *with the closure of* $\mathrm{LCS}\,(X, B+C^+) \setminus E$*. In particular,* (E, B_E) *is normal and Klt outside* $U_E = U \cap E$*, and* E/T_E *is a contraction where* U_E *is complete/*T_E*.*

Proof We can find a maximal $c > 1$ such that $K + B + cC$ is log canonical outside U; this is the required c. It exists by (NBU) because $\mathcal{C}_X = (\overline{\mathcal{C}_T})_X = f^*\mathcal{C}_T$ and $P \in \operatorname{Supp}\mathcal{C}_T$. Set $X_0 = X \setminus X_s$, that is, this is an open subset of X, where (X, B) is log canonical. Note that X_s is completely in U.

Indeed, we verify that the generic points of the log canonical centers are in U. Suppose that the exists such a center E_1 outside U, in particular, E_1 is completely in X_0. On the other hand, by (CMP), $c > 1$, there is a center E_n in X_s. By LMMP the LCS is connected. Hence we have a chain of LCS centers $E_1, E_2, \ldots, E_n$ with E_1 completely in X_0, E_2 in general in X_0, and E_n completely outside, that is, in X_s. Note that $K + B + cC \equiv 0/P$ on this chain. This is impossible. Now we consider only $d = 3$, that is, X is a 3-fold. Then E_i are at most of dimension 2. We check that each $E_i \subset X_0$, which gives a contradiction. If E_2 is completely in X_0 we can use induction on n. Thus suppose that E_2 intersects X_s. If E_2 is a curve then $E_2/f(E_2) = P$ is complete$/P$. Otherwise we drop E_2 from our chain. On the other hand, by adjunction [3], on the normalization E_2^ν, there is a boundary $B_2 \geq 0$, such that $K_{E_2^\nu} + B_2 \equiv 0$. In addition, E_2 has two points $\nu(P_1) \in E_1 \cap E_2$ and $\nu(P_2) \in E_2 \cap X_s$ with $\operatorname{mult}_{P_1} B_2 \geq 1$ and $\operatorname{mult}_{P_2} B_2 > 1$. That is impossible, because $\deg K_{E_2} \geq -2$. Thus E_2 is a surface that is in general in X_0. Now E/T_E is a complete surface, or a fibering of curves. Again adjunction on a normalization gives a boundary $B_2 \geq 0$ that has a reduced divisor P_1 in $E_2^0 = E_2 \cap X_0 \subset E_2^\nu$, and the nonlog canonical center P_m outside. On the other hand $K_{E_2} + B_2 \equiv 0/T_E$, so we have a chain of LCS centers for $(E_2^\nu/T_E, B_2)$ as above (if it is not a contraction$/T_E$ make a base change, or argue in the connected component of the fibre E/T_E). That gives a contradiction by an induction on the dimension of E.

For $d \geq 4$ we can do similarly, by LMMP.

Thus new $\mathcal{C} := c\mathcal{C}$. After a perturbation of $C_T = \mathcal{C}_T$ we can get a single transient center E. Perturbation: there is $H \geq 0$ a $\mathbb{Q}$-Cartier divisor on T with $P \in \operatorname{Supp} H$ such that on a (sufficiently high) log resolution $g\colon Y \to T$ of $(T, B_T + C_T + H)$, $g^*H \geq H'$, where H' is ample$/T$. Then we can perturb C_T and so $\mathcal{C} = \overline{C_T}$: replace by $(1-\varepsilon)C_T + (1-\delta)H''$, where $g^*H'' \geq H - H' + \sum e_i E_i$ with $0 < \varepsilon, \delta, e_i \ll 1$ and each E_i is exceptional$/T$ or is in the support of B_T and C_T.

Finally, before the complement we can improve $(X/T, B + C)$ to a Fano contraction$/T$ as in Lemma 9.18 (non$\mathbb{Q}$-factorial case). Again as in the perturbation $B^+ - B \geq 0$ should be $\ll$ than C.

The properties of $(C/T_C, B_C, J_C)$ are standard. $\square$

Proof of Corollary 10.13 By definition, the mobile system $\mathcal{L}_\bullet$ is defined on a log Fano contraction $(X/T, B)$. Its characteristic system is $\mathcal{D}_i = \mathcal{L}_i/i$. Using complements, we can convert the pair into a 0-log pair $(X/T, B)$ with a new fixed B. It is also a weak log Fano contraction because X/T is birational. Now Corollaries 9.20 and 9.21 applied to $\mathcal{D}_\bullet$ give the stabilization of $\mathcal{D}_\bullet$ on some prime b-divisor $E = E_c/P$, that is, (STB) of Theorem 10.12 for $\mathcal{L}_\bullet$. Indeed, (LWF) holds by construction. (LBF), (LCA) and (MXD) hold by Corollary 4.46 as for a (FGA) system. In addition, (AMN) also holds. Note that any complement preserves (LCA) by Lemma 4.40. (BED) for $\mathcal{D}_\bullet$ after a truncation follows from Proposition 4.47 and Theorem 6.19, (3–4) under the assumption $(\mathrm{CBS})_{d-1}$.

Therefore by Theorem 10.12 there is a Klt neighborhood $(Y/T, B_Y)$ of E on which after a truncation each $\mathcal{L}_i$ is ample and free, and $\mathcal{L}_i/i$ stabilizes. If U is complete, we are done (compare the next corollary). If not, we extend the neighborhood. This is eventually complete by Lemma 10.11 on Noetherian induction. (The final destab D does not intersect Y at all.)

Suppose that U is still not complete/T. Thus it satisfies (NBU). By Noetherian induction, $U \neq \emptyset$ (because it includes old centers), and satisfies (CMP). Hence Lemma 10.14 applied to $(X/T, B) := (Y/T, B_Y)$ with previous $\mathcal{C}$ gives a new transient center E and a new C and C^+. Therefore if $\mathcal{D}_{\bullet}{}_{|E}$ stabilizes/T_E or (STB) holds for $\mathcal{L}_{\bullet}{}_{|E}$, then again by Theorem 10.12 up to a truncation we get that each $\mathcal{L}_i$ is free, ample/T, and $\mathcal{D}_\bullet$ stabilizes in a Klt neighborhood that extends the old one by complete $\mathrm{center}_X E$! (Compare the proof of the theorem.)

Thus we need to verify (STB) on E. We do this in dimension $d = 3$. Thus $\dim E \leq 2$. (As we see later, this is enough for 4-folds flips.) For higher dimensions we only explain $(\mathrm{CBS})^*$.

Unfortunately, we can't apply Corollary 9.21 directly in the current situation because the general Fano contraction $(E/E_T, B_E, J_E)$ has singularities even worse than log canonical. However, Addendum 5.12.1 on descent and Corollary 5.13 in its proof can be modified as follows, which secures the proof.

Suppose that E is a surface; then by Proposition 4.44, together with the original $\mathcal{D}_\bullet$, the new $\mathcal{D}_\bullet := \mathcal{D}_{\bullet}{}_{|E}$ satisfies lca saturation (LCA). The only novelty is that we now have essentially negative components of $\mathcal{A} := \mathcal{A}(E^\nu, B_E)$ (that is, components with discrepancies ≤ -1). The negative components defined the ideal J_E. Since each $j\mathcal{D}_i$ is Cartier on the LCS, saturation is equivalent to the inclusion

(JLC) On any sufficiently high model $E_{\mathrm{hr}}/T := T_E$ of E/T that is the *identity* or *contractible* in the neighborhood $U := U_E$ of the LCS, we have

$$J_E((j\mathcal{D}_i + \lceil \mathcal{A} \rceil)_{E_{\mathrm{hr}}}) \subset \mathcal{O}_T((j\mathcal{D}_j)_{E_{\mathrm{hr}}}).$$

We construct a prediction $(E/T, \mathcal{C} = \mathcal{A}, F, \gamma)$ for the descent problem $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ on a modification of $(E/T, B := B, J := J_E)$ into a triple (we can even normalize it, so assume E is normal). The modification is *contractible* if it does not blow up exceptional divisors over U. However it can contract divisors: extend U and contract complete subvariety in the extension (see a case below). This gives (EEF) on any log terminal resolution of E.

The system $\mathcal{D}_\bullet$ is bounded, by Example 4.19. By construction each $\mathcal{D}_i$ is effective and $(\mathcal{D}_i)_U = \mathcal{D}_U$ is stable on U. The unstable part is over an effective divisor F supported in the complement $E \setminus U$. This gives (UAD) and (LGD) for some real number $\gamma > 0$ (cf. Example 8.21).

In the two dimensional case, $\mathcal{D}_E$ is nef = pseudo-nef$/T$ (that is, nef over general complete curves$/T$ in any prime divisor), because this holds for each $(\mathcal{D}_i)_E/T$. The latter follows from (LBF): each $\mathcal{M}_i := i\mathcal{D}_i = \mathcal{L}_{i\,|E}$ is b-free$/T$. Then (SAM) follows from generalized semiampleness (Florin Ambro): if D is an $\mathbb{R}$-Cartier divisor on E/T such that it is Cartier and ample on U/T and nef$/T$, then it is semiample$/T$. By the rationality of the nef cone for such divisors (variations only over F; LMMP*, contract all exceptional curves in F) it is enough to verify for $\mathbb{Q}$-divisors. This is essentially a consequence of the general Fano property. For more details,[10.2] see [1].

To verify (SAB) we can use Proposition 6.26. However to apply this to approximations we need more from the triple and the prediction: e.g., they should be strictly desirable (WAM) for $\mathcal{D}$, as explained below.

If the Kodaira dimension of $\mathcal{D}$ is 0, then each $\mathcal{D}_i = 0$ and we get (STB).

If the Kodaira dimension is 1, then, after a truncation, $|\mathcal{M}_i|_E$ is a pencil for each i. Its base point are only at good points, outside U. Resolving these gives a prediction in a form of a triple $(E/W/T, B, F)$, with crepant B, single F, and $g\colon E \to W/T$ given by the pencil. Moreover, each $(\mathcal{D}_i)_E = g^*M$ for some ample M on W/T. In general, this allows us to improve desirable triples for $\mathcal{M} = \mathcal{D}$ (not b-free) and $= \mathcal{M}_i$

[10.2] I'm guessing the missing reference – it's possibly in his thesis? The adjunction conjecture and its applications, Ph.D. dissertation, Johns Hopkins University 1999.

(WAM) a desirable triple for $\mathcal{M}$ is *strict* if M is ample on T/Z (see Definition 6.9); or at least $C \cdot M > 0$ on each complete curve$/T$ that intersects $g(\operatorname{Supp} J)$.

If U dominates W/P we get (STB) from W, that is induced in turn from U. Otherwise U and $\operatorname{Supp} J$ do not dominate W, so in the approximation methods in the proof of 5.12 we can disregard J in (JLC). This means that

(BFF) for any ample divisor H, or H in a certain cone of semi-ample divisors under the weak assumption in (WAM), and any proper subscheme given by ideal J, $|J(NH)|$ is free outside $\operatorname{Supp} J$ for $N \gg 0$.

We can also use this in general. Note that in both previous cases we get (SAB) without (CBS) (cf. Example 6.24). But we need Example 4.37.

Finally, suppose that $\mathcal{D}$ is big, then up to a truncation the same holds for each $\mathcal{D}_i$. By the saturation, for example, (JLB) with $i = j$, and Proposition 6.26, each $|\mathcal{M}_i|_E$ is free on a terminal resolution (up to a truncation again). That gives (CBS) and (SAB). The triple is the resolution $(E/E/T, B, F)$.

However to apply approximations of the proof of Theorem 5.12 we need (WAM) and (BFF) for $\mathcal{D}$: e.g., $D = \mathcal{D}_E$ should be ample$/T$. If this is not true, there is a complete curve C/T such that $C \cdot D = 0$. If C does not intersects U we can contract it because C is exceptional (as in LMMP* with good divisors on bad singularities). This gives birational $E \to W/T$. The triple $(E/W/T, B_T, F_W = g(F))$ is a general log Fano contraction with the same properties as $(E/E/T, B, F)$, except for possible *nonterminal* points near F but still Klt. After such contractions we get (WAM) in the weak form, that is, each C with $C \cdot D = 0, D := \mathcal{D}_W$, intersects U. Since $D \geq 0$, C does not intersect $\operatorname{Supp} D$ whenever $\mathcal{D}$ is taken general on U. Then (STB) holds on C, we can extend U by a neighborhood of C, and then contract C (together with all the curves in the same numerical class, to preserve the algebraic category). Note that now even in the *big* case we need to modify (BIG), (BIR), and (MOD), and consider E/W that are not the identity. After that the methods of the proof of 5.12 are applied to the prediction $(E/T, \mathcal{A}, F, \gamma)$ with approximations on W/T (cf. Example 5.18).

If E is a curve, the same arguments leads to a (rational) pencil of rational curves that do not intersect the LCS on a surface with log discrepancy 0 over the curve. Such a pencil is free, and we can treat it as the cases of Kodaira dimensions 0 and 1 above. Or we can use adjunction on E citeka.

In higher dimensions, it is better to use adjunction on E and over non-Klt strictly desirable triples of general log Fano contractions. Then $(\mathrm{CBS})^*_{d'-1}$ means the existence of such triple for each b-free M, and boundedness of the required triples. Above we verified that for $d' \le 3$, so we can drop it as the assumption. The next unknown case $d' = 4$: e.g., $(\mathrm{CBS})^*_3$ for a 3-fold E in a 4-fold X, which is unknown even without *. □

Corollary 10.15 Under the assumptions of Corollary 10.13. *The limit* $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ *of the characteristic system of any* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *algebra stabilizes. Thus f.g. for* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *holds under the assumptions.*

Proof Immediate from the proof of Corollary 10.13 or by Corollary 10.13 itself, Theorem 6.19, (3–4), and Limiting Criterion 4.26. □

Corollary 10.16 Under the assumptions of Corollary 10.13. *In dimension* n *there exist the pl flips.*

Proof Immediate by Main Lemma 3.39, Corollary 10.15 and Corollary 3.28. □

In the next section, we get rid of the assumption on the existence of flips of type (S+−) under $(\mathrm{CBS})^*$.

11 The main result

Theorem 11.1 Under LMMP in dimension $\le n-1$ and $(\mathrm{CBS})^*_{n-2}(\mathrm{bir})$. *Flips of type* (S+−) *exist in dimension* n. *We can moreover omit the assumption that the flips are extremal* ($\rho(X/T) = 1$; thus, X/T is just small), *and* $(\mathrm{CBS})^*_{n-2}$ *with* $n \le 4$.

Lemma 11.2 *Let* $\mathcal{L} = \overline{L}$ *be an ample b-divisor on* X/T, *and* E *and* E' *two prime b-divisors such that* E *intersects* E' *divisorially. Then there is an effective Cartier b-divisor* $\mathcal{E}'$ *supported over* E' *that* destabilizes $\mathcal{L}_{|E}$, that is, $(\mathrm{Mov}(\mathcal{E}' + N\mathcal{L}))_{|E} > N\mathcal{L}_{|E}$.

Proof Taking hyperplane sections of X/T by L reduces the lemma to the two dimensional case, with X a surface. Then we can use Example 10.5, (2). The main problem is that E' may not be $\mathbb{Q}$-Cartier. This can be resolved by a $\mathbb{Q}$-factorialization and perturbation in exceptional components to preserve numerical ampleness$/T$.

In higher dimensions, we close[11.1] a family of $\mathcal{E}'$, constructed in dimension 2. □

Lemma 11.3 Under LMMP in dimension n. *Let B be a subboundary on X with* $\dim X = n \geq 3$, *and S a prime divisor in X such that:*

- *$K + S + B$ is $\mathbb{R}$-Cartier,*
- *B has no reduced components,*
- *$(K + S + B)_{|S}$ is Klt, and*
- *the negative components of B intersect S only in points.*

Then there exists a weak $\mathbb{Q}$-factorialization, *that is, a model on which all the divisors that are exceptional$/X$ and the divisors that intersect S in a point, including the negative components of B, do not intersect S.*

Remark 11.4 (1) S is normal outside its intersection points with negative components of B [9, Lemma 3.6] because (X, B) is purely log terminal there.

(2) We expect a more perfect form of the lemma in any dimension n, with the intersections in codimension ≥ 3 rather than in points (cf. the remark at the end of Step 1 in the proof of Theorem 11.1).

Proof Take a strictly log terminal resolution of $(X, S + B)$ that we do with *reduced* exceptional divisors, intersecting S in a point, and negative components, that is, we set or change their multiplicities to 1. The reduced components of the resolution do not intersect new S because the construction give a log terminal resolution of (S, B_S), under the adjunction of original (X, B), and the resolution of (S, B_S) does not have reduced components. □

[11.1] What? We take the closure of, or we choose? I guess that you mean you fibre your higher dimensional variety in surfaces, construct the flip of the generic fibre, then take the closure.

Proof of Theorem 11.1 Since the flip is strictly log terminal, after slightly decreasing the other reduced components, we can assume that both S^+ and S^- are irreducible (cf. Example 10.9). Thus the boundary is $B + S^+ + S^-$ with $\lceil B \rceil = 0$.

This is a flip in the exceptional locus of dimension $\le n-2$. However, the contraction is birational on S^+, S^- and on their intersection $E = S^+ \cap S^-$. Indeed, if E is contractible, it contradicts the normal crossing of S^+ and S^- because then $S^+{}_{|S^-} = E$ is nef/T and exceptional on S^-. This last conclusion is impossible by [9, Negativity 1.1].

Let $\mathcal{L}_\bullet$ be the mobile system of a divisorial algebra $\mathcal{R}_{X/T}\, D$ with $D \sim S^-$, numerically negative/T. Then the algebra defines a flip if it is f.g. We have a translation t by Addendum 3.39.1, that corresponds to the inductive family with $s = t = 2$ and single $S_2 = S^-$ (cf. Example 3.36 with $S_1 = S^+$).

By Proposition 4.38 the system $\mathcal{L}_\bullet$ is of type $(\mathrm{FGA})_n(\mathrm{bir})$ over the pair $(X/T, B + (1-\varepsilon)(S^+ + S^-))$ for $0 < \varepsilon \ll 1$, because

(EXS) $\mathcal{L}_\bullet$ is exceptionally asymptotically saturated/T.

Unfortunately, we cannot preserve condition (EXS) when restricting, but the (FGA) is preserved for the birational restrictions $_{\mid}$ of $\mathcal{L}_\bullet$ on S^+, S^- and E by Example 4.41 and Proposition 4.44 (cf. Example 4.45).

In particular, $\mathcal{L}_{i\,\mid E}$ is lca saturated. Thus by our assumption, f.g. in $(\mathrm{FGA})_{n-2}(\mathrm{bir})$ follows from $(\mathrm{CBS})_{n-2}(\mathrm{bir})$ and Theorem 6.19, (3–4). Thus the characteristic system of $\mathcal{L}_{i\,\mid E}$ stabilizes. Since the translation t is preserved under the restriction, we get f.g. in (FGA) for $\mathcal{L}_{i\,\mid S^+}$ on S^+ by Main Lemma 3.39 with a single element inductive family $S^+ \cap S^-$ (cf. Corollary 6.43).

For the same reasons, the characteristic system of $\mathcal{L}_\bullet$ stabilizes over points (in general) of dimension 1. Thus Proposition 4.47 applied just to $\mathcal{L}_\bullet$ gives a stabilization for the characteristic system of $\mathcal{L}_\bullet$ over $T \setminus P$ with closed P with (BED) after a truncation.

11.5 Preamble

We also establish below f.g. in (FGA) for $\mathcal{L}_{i\,\mid S^-}$. Thus by Main Lemma 3.39 again, with a single element inductive family S^- on X we get f.g. in (FGA) for $\mathcal{L}_i$ itself and get the flip on X.

To prove f.g. on S^- we construct a 0-log pair $(S_m^-/T^- = f(S^-), B^-_{S_m^-})$ over which each $\mathcal{L}_{i}{}_{|S^-}$ with $i \gg 0$ is free and ample, up to a truncation (cf. Corollary 10.13). This gives (CBS) and f.g. by Theorem 6.19, (3–4) because we can secure lca saturation for the system $\mathcal{L}_{\bullet}{}_{|S^-}$ as above.

First, in Step 1 below, we construct a Klt neighborhood U_m of E (as E_c but now no longer$/P$) on which all $\mathcal{L}_{i}{}_{|S^-}$ with $i \gg 0$ are free, ample and $\mathcal{L}_i/i_{|S^-}$ stabilize *(over this neighbourhood all $\mathcal{L}_i/i = \mathcal{D}$, the limit).* Thus after a truncation this holds for each $i > 0$, and each $\mathcal{L}_i$ gives this quasi-projective neighborhood on a contraction given by $\mathcal{L}_i$. To compare with Theorem 10.12, now we can't us the (S+−) flips.

By Lemma 10.11 and LMMP assumption neighborhoods U_m of this form considerd under inclusions terminate. The maximal U_m is complete$/T^-$ and is the required 0-log pair.

Finally, in Step 2 we extend each incomplete U_m.

Step 1: Take $\mathcal{L} = \mathcal{L}_m$ for $m \gg 0$ for which the restricted linear systems $\mathcal{L}_{m}{}_{|E}$ and $\mathcal{L}_{m}{}_{|S_m^+}$ are very semiample and have compatible restrictions

$$\mathcal{L}_m/m_{|E} = \mathcal{D}_{|E} \quad \text{and} \quad \mathcal{L}_m/m_{|S_m^+} = \mathcal{D}.$$

Then they are stabilized. This follows from the normality of restricted algebras in Addendum 4.44.1. Then the linear system of $\mathcal{L}_m$ defines a flag of normal varieties $E_m \subset S_m^+ \subset X_m$, whereas $\mathcal{L} = \overline{L}$ for *just* ample but free $L = (\mathcal{L})_{X_m}/T$. This can be done by a normalization of the model given by $\mathcal{R}_{X/T}\, L$.

However, the birational transform of S^- in X_m may still be nonnormal. We denote its normalization by $\nu\colon S_m^- \to X_m$. We claim that U_m is a neighborhood of E_m in S_m^-. Note that E_m is also embedded in S_m^- by the above normality Addendum 4.44.1.

By (BED) and (EXC), or since $\mathcal{L}$ gives the flip over $T \setminus P$, the only exceptional divisors E_i of X_m/T are $/P$ (cf. Lemma 3.15).

In addition, no exceptional E_i/T intersects S_m^+ *divisorially*. For otherwise, an effective exceptional$/T$ b-divisor $\mathcal{E}'/E' = E_i$ destabilizes $\mathcal{L}_{|E}$ by Lemma 11.2. This is impossible by (EXS) on X and (MXD) on E since $\mathcal{L}/m_{|S_m^+} = \mathcal{L}_m/m_{|S_m^+} = \mathcal{D}$ stabilizes.

In particular, this implies that adjunction for the crepant subboundary B_{X_m} on S_m^+ has an *effective different* (as we expect due to the stabilization; cf.

Addendum 5.12.2). More precisely, by a perturbation, we convert $(X/T, B)$ into a 0-elt pair $(X/T, B^+)$ (actually, it is purely log terminal) with only reduced S^+. Thus S_m^+ is a 0-log pair for the adjunction $(K_T + B_T^+)_{|f(S_m^+)}$ on $T_m^+ = f(S_m^+)$ (which is normal). Hence by adjunction and the commutative diagram up to codimension 2, that is, for general surface sections, and near S_m^+, $K_{X_m} + B_{X_m}^+$ is log canonical with *birationally* transformed B^+. The adjunction can be extended on the whole of S_m^+ to a 0-log pair $(S_m^+/T_m^+, B_{S_m^+}^+)$ with

$$K_{S_m^+} + B_{S_m^+}^+ = (K_{X_m} + B_{X_m}^+)_{|S_m^+};$$

in particular, the pair is Klt everywhere on S_m^+ with $B_{S_m^+}^+ \geq 0$ (crepant, for $(S^+, B_{S^+}^+)$) by [9, 3.2.2].

Moreover, every exceptional E_i intersects S_m^+ in at most one point. Indeed, we can assume that the subvarieties in S_m^+ of dimension ≥ 1 in general are *weakly* $\mathbb{Q}$-factorial *for the exceptional divisors* E_i/T; that is, if the dimension of intersection is ≥ 1, these divisors intersect S_m^+ divisorially (in codimension 2 in X_m). Indeed, taking a hyperplane section up to the intersection with E_i by a point, by Lemma 11.3, we get weakly $\mathbb{Q}$-factorial for divisors E_i on a weak $\mathbb{Q}$-factorialization. Note that LMMP in the lemma holds under our assumptions of the theorem. On the other hand, by the numerical and geometric properties of S^+ and S^-, $af(S^+) + b(\mathcal{L}_m)_T$ is an $\mathbb{R}$-Cartier divisor on T for some $a > 0$ and $b \geq 0$; $b > 0$ when $R \cdot S^+ > 0$. Then

(PRS) on any model over T, $aS_m^+ + bL + \sum e_i E_i \equiv 0/T$ and, in particular, *locally* on X_m, $\mathbb{R}$-Cartier with multiplicities $e_i \geq 0$ in exceptional E_i; and $e_i > 0$ on E_i/P.

The latter is preserved on restricting to any general hyperplane section. This is impossible for some curve$/X_m$ (over a point in X_m) on a weak $\mathbb{Q}$-factorialization and, if the latter is nontrivial, its exceptional curves are on the modification of S_m^+, by the connectedness of the fibre $/X_m$. Thus as in the $\mathbb{Q}$-factorial case the divisorial part always intersects in a divisor if it intersect in a curve. (In the same way, we could eliminate intersections with S_m^+ in points if we knew the existence and termination of n-fold flips! Thus we expect this near S_m^+, and that S_m^- is normal by the argument below. But for now we just disregard such intersections in points.)

Therefore the *exceptional divisors E_i intersect S_m^+ at most in points*. In particular, the negative components in $B^+_{X_m}$ intersect S_m^+ at most in points. The same holds for $E_m \subset S_m^+$ with the adjoint subboundary B_E^+ given by the adjunction of a log terminal 0-log pair $(T/T, B_T^{+-})$. The pair has two reduced divisors S^+ and S^-, a *substantial* nonreduced part, that is, for some $C > 0$, and $(T/T, B_T^{+-} - C_T)$ is a 0-log pair. The second of these induces the log pair $(T^-/T^-, B^-_{T^-})$ which corresponds to a Klt neighborhood U_m under construction.

Now if we consider $E_m \subset S_m^-$, by the above, the negative components of crepant subboundary $B^{+-}_{X_m}$ intersects E_m in points. Thus we have an adjunction of (T, B_T^{+-}) on $(S_m^-, B^{+-}_{S_m^-})$ with the negative components in $B^{+-}_{S_m^-})$ intersecting E_m at most in points, and in turn an effective adjunction of $(S_m^-, B^{+-}_{S_m^-})$ on $(E_m, B^{+-}_{E_m})$ with $B_m \geq 0$. Hence again by Lemma 11.3 and the presentation (PRS) the intersections even in points on S_m^- is impossible, and $B^{+-}_{S_m^-}$ is a *boundary* near E_m. The same holds for $B^-_{S_m^-}$ on S_m^-. Moreover, the support of each restriction $E_{i|S_m^-}$ in the sense of Mumford (restricted to the normalization S_m^- and defined only divisorially) *does not intersect E_m as closed subvarieties*. This is a crucial point that *gives* a neighborhood U_m as the complement to the supports. Indeed, in the neighborhood $B^-_{S_m^-}$ is a boundary and $\mathcal{L}_{jm}/j_{|U_m}$ stabilizes for all $j \geq 1$! The boundary[11.2] by [9, 3.2.2]. Thus as at the end in the proof of Theorem 10.12 (based on the uniqueness in Addendum 10.6.1) we can convert (= embed) U_m into a Klt neighborhood (into a crepant log pair for $(T^-/T^-, B^-_{T^-})$). Stabilization holds by the following:

Trick 11.6 By (AMN) and (BED), and since the exceptional E_i in X_m are/P, $\mathcal{L}_{jm} = j\mathcal{L}_m + \mathcal{E}_{m,j}$, where each $\mathcal{E}_{m,j} \geq 0$ is b-free (in particular, b-Cartier) *over* X_m with $\operatorname{Supp}(\mathcal{E}_{m,j})_{X_m}$ in a union of exceptional E_i. Hence by Proposition 7.7 (fx) *divisorially* $\operatorname{Supp}(\mathcal{E}_{m,j\,|U_m})_{U_m} = 0$ (over X_m), or $(\mathcal{E}_{m,j\,|U_m})_{U_m} = 0$, and by Lemma 10.8 $\mathcal{E}_{m,j\,|U_m} = 0$. Then $\mathcal{L}_{jm}/j_{\,|U_m} = \mathcal{L}_{m\,|U_m}$. This implies the required stabilization on U_m.

Remark 11.7 The proof is essentially higher dimensional: i.e., it is better to assume $n \geq 4$. For $n \leq 2$, with only identical flips, no E_i, and complete U_m in Step 1. For $n = 3$, under LMMP in dimension 3, Lemma 11.3 and (PRS) imply that each E_i does not intersect E_m at all (cf. Remark 11.4, (2))

[11.2] Something missing.

and we have the same stabilization as in Step 1. However the existence of 3-fold log flips follows from $(\mathrm{FGA})_2$(bir) (see the proof of (FGA) in the Main Theorem at the end of Section 6), or from [9].

STEP 2: Using Lemma 10.14 we can extend the neighborhood U_m, whenever it is still incomplete, by a transient center E_t in a crepant model of $(T^-/T^-, B^-_{T^-})$, and establish a stabilization of the restricted system $\mathcal{L}_{jm}/j_{|E_t}$ as in the proof of Corollary 10.13 by $(\mathrm{CBS})^*_d$ with $d = \dim E_t \le n-2$ and the (FGA)* property (with singularities) of the restricted system.

Finally, we extend U_m to U_{jm} for $j \gg 0$ such that the restricted $\mathcal{L}_{jm}/j_{|E_t}$ reaches stabilization. Indeed, if $E_t \not\subset U_{jm}$ on S^-_{jm}, then we can make a destabilization as in the proof of Corollary 10.13. But now we replace Proposition 10.6 by Example 10.5, (1), because $\sum e_i E_i$ is $\mathbb{R}$-Cartier and the e_i are positive for E_i/P. This follows from ampleness of L/T (which is in particular Cartier), and because the $\sum e_i E_i{}_{|S^-_{jm}}$ are disjoint from $E_{jm} = S^+_{jm}{}_{|S^-_{jm}}$ (by the log terminal property of $(X_{jm}, B_{X^{+-}_{jm}})$, $B_{X^{+-}_{jm}}$ is a boundary in the generic point of $E_{jm} = S^+_{jm} \cap S^-_{jm}$, and [9, Corollary 3.11], because in general $E = S^+ \cap S^-$ goes birationally to E_{mj} and the exceptional/T prime b-divisors with log discrepancy 0 are only/ the generic point of E_{jm}). Indeed, by (PRS) $aS^+_{jm} + bL + \sum e_i E_i$ is $\mathbb{R}$-Cartier with $e_i > 0$, and $\equiv 0/T$.

Thus if we set new $m := jm$ we can return to Step 2 again. The abovementioned termination completes the proof. □

Proof of Theorem 6.44 Immediate by Corollary 10.16 and Theorem 11.1. □

Remark 11.8 During the proof of Theorem 6.44 we essentially established that each $(\mathrm{FGA})_3$(bir) algebra restricted from any *exceptionally* saturated algebra on a contraction of Example 3.36 is f.g. In general, this does not imply f.g. of the algebra under restriction. Perhaps, this affects all $(\mathrm{FGA})_3$(bir) algebras (cf. Remark 4.36, (6)).

Proof of (RFA) **in the Main Theorem** Immediate by Theorem 6.44, because for $n \le 4$ we can drop LMMP, (CBS)*, and (SSB)(gl). □

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