TUTORIAL ON TOM AND JERRY: THE TWO SMOOTHINGS OF THE ANTICANONICAL CONE OVER $\mathbb{P}(1,2,3)$

GAVIN BROWN, MILES REID, AND JAN STEVENS

ABSTRACT. This is a first introduction to unprojection methods, and more specifically to Tom and Jerry unprojections. These two harmless tricks deserve to be better known, since they answer many practical questions about constructing codimension 4 Gorenstein subschemes. In particular, we discuss here the two smoothing components of the anticanonical cone over $\mathbb{P}(1,2,3)$.

Section 2 treats the "6×6 extrasymmetric format", that describes the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and some of its degenerations. One can view this as just algebraic manipulations, or as a typical case of Tom unprojection. In a similar vein, Section 3 treats the "Double Jerry construction", that describes the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and some of its degenerations. In Section 4 we put these two unprojection constructions together as a versal deformation of the anticanonical cone over $\mathbb{P}(1,2,3)$ over a reducible base, with the obstructions also controlled by the matrix format. We conclude with some general remarks, mnemonics, slogans, and FAQ. We do not pretend any generality, or any theoretical treatment of Gorenstein codimension 4 (compare [G4]).

1. THE ANTICANONICAL CONE OVER $\mathbb{P}(1,2,3)$

Let $X \subset \mathbb{A}^7$ be the anticanonical cone over $\mathbb{P}(1,2,3)_{\langle u,v,w \rangle}$; this is also the quotient by the group action $\frac{1}{6}(1,2,3)$ on $\mathbb{A}^3_{\langle u,v,w \rangle}$. We set out its 7 coordinate monomials as the Newton polygon

$$
\begin{array}{llll}
u^6 & u^4v & u^2v^2 & v^3 \\
u^3w & uvw & & \\
w^2 & & &
\end{array}
\quad = \quad
\begin{array}{llll}
a & b & c & x \\
d & e & & \\
f & & &
\end{array}
\tag{1}
$$

The somewhat idiosyncratic choice of coordinates on $\mathbb{A}^7$ relates to the extrasymmetric format of Section 2.

One finds the equations defining X without difficulty. The semigroup ideal of internal monomials of the Newton polygon is generated by the single monomial $e = uvw$. There are tag relations between any three consecutive boundary monomials, that involve e if we turn a corner:

$$ ac - b^2, \quad xb - c^2, \quad cf - e^2, \quad xdf - e^3, \quad af - d^2, \quad bd - ae. \tag{2} $$

1

Note in particular the equations $cf = e^2$ (that is, the tag at x is 0) and $xd = f^{-1}e^3$ or $xdf = e^3$ (the tag at f is -1).

These equations define the toric variety X in the complement of the coordinate hyperplanes, where e is invertible. The remaining generators of I_X come by coloning out e: for example, $cf - e^2$ and $xdf - e^3$ give $\big(c(xdf - e^3) - xd(cf - e^2)\big)e^{-2} = xd - ce$ where e is invertible, so that the ideal is generated by the 9 binomials:

$$ac - b^2, \quad xb - c^2, \quad cf - e^2, \quad af - d^2, \quad bd - ae$$
$$xd - ce, \quad bf - de, \quad dc - be, \quad xa - bc. \tag{3}$$

Another way to view the equations is that they describe a singular del Pezzo surface S of degree 6, embedded by plane cubics having flex line $u = 0$ at $(0 : 0 : 1)$:

$$\begin{array}{llll} u^3 & u^2 v & uv^2 & v^3 \\ u^2 w & uvw & & \\ uw^2 & & & \end{array} \tag{4}$$

It is an amusing exercise to recover from this the A_1 singularity $cf = e^2$ at P_x and the A_2 singularity $xdf = e^3$ at P_f.

2. Extrasymmetric format

2.1. Extrasymmetric format.
Tom unprojections frequently lead to equations in extrasymmetric format. Consider for example the 6×6 skew matrix[1]

$$N = \begin{pmatrix} z & y & a & b & d \\ & x & b & c & e \\ & & d & e & f \\ & & & \lambda z & \lambda y \\ & & & & \lambda x \end{pmatrix} = \begin{pmatrix} B & A \\ -A & \lambda B \end{pmatrix} \tag{5}$$

A matrix of this shape is *extrasymmetric* (the term also covers slightly more general cases, see [TJ], 9.1). It is made up of 3×3 blocks, where the top right block A is symmetric, the top left block B is skew, and the bottom right block λB repeats the information contained in the top left block, in this case with a scalar factor λ.

The 4×4 Pfaffians of N generate the ideal of the Segre embedding

$$\text{Segre}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8_{\langle a,b,c,d,e,f,x,y,z \rangle}.$$

More precisely, the extrasymmetry means that the 15 upper-triangular entries of N consist of 9 independent entries and 6 repeats. The same is true of the 4×4 Pfaffians of N, which give 9 relations and 6 repeats. The resulting 9 equations define a variety in $\mathbb{A}^9_{\langle x,y,z,a,b,c,d,e,f \rangle}$ that, for $\lambda \neq 0$, is a linear transformation away from the affine cone over $\text{Segre}(\mathbb{P}^2 \times \mathbb{P}^2)$.

---

[1]We omit the diagonal terms (which are zero) and the $m_{ji} = -m_{ij}$ with $i < j$.

The linear transformation involves taking $\sqrt{-\lambda}$; swapping the signs of the square root interchanges the two copies of $\mathbb{P}^2 \times \mathbb{P}^2$. We leave the calculations as entertainment.

A more banal way to define $\mathrm{Segre}(\mathbb{P}^2 \times \mathbb{P}^2)$ is $\bigwedge^2 M = 0$ with M a generic 3×3 matrix. If we write $M = A + \sqrt{-\lambda}B$ with A symmetric and B skew, the ideal of 2×2 minors of M equals the ideal of 4×4 Pfaffians of the extrasymmetric matrix $N = \left(\begin{smallmatrix} B & A \\ -A & \lambda B \end{smallmatrix} \right)$.

More geometrically, this format displays $\mathbb{P}^2 \times \mathbb{P}^2$ as a nongeneric linear section of $\mathrm{Grass}(2, 6)$.

2.2. **Specialise to $v_6(\mathbb{P}(1,2,3))$.** Now we consider λ as a variable and specialise by setting $\lambda = 0$, $z = 0$ and $y = c$; the Pfaffians equations specialise to the same as (3). That is, the anticanonical cone X over $\mathbb{P}(1,2,3)$ is the particular section $\lambda = 0$, $z = 0$ and $y = c$ of a degeneration of the cone over $\mathbb{P}^2 \times \mathbb{P}^2$. Wiggling the section gives one of the smoothing components of the deformations of X.

2.3. **The same viewed as a Tom unprojection.** As we said, the extrasymmetric matrix N in (5) has 6 repeated entries. The entries that are not repeated are the three diagonal entries a, c, f of the top right 3×3 block A. They correspond to the three coordinate points of $\mathbb{P}^2 \times \mathbb{P}^2$ such as $P_a = (1 : 0 : 0;\ 1 : 0 : 0)$, etc. Here again λ is a nonzero scalar.

Now project from P_a, and view the original equations as the result of undoing this projection. A practical point of view on unprojection is that it groups the 9 equations according to how they involve a. Because of the format of (5), a only appears linearly in 4 equations

$$ac = \cdots , \quad ae = \cdots , \quad af = \cdots , \quad ax = \cdots ,$$

and the remaining 5 equations not involving a are the Pfaffians of

$$N_{\widehat{4}} = \begin{pmatrix} z & y & b & d \\ & x & c & e \\ & & e & f \\ & & & \lambda x \end{pmatrix}$$

(delete row and column 4 from N of (5)). What makes this a Tom_1 matrix is that the 6 entries not in row and column 1 are in the codimension 4 complete intersection ideal (x, c, e, f). The coincidences $m_{25} = m_{34} = e$ and $m_{45} = \lambda x = \lambda m_{23}$ that bring this about are remnants of the extrasymmetry of N.

Geometrically, these are the equations of the projection of $\mathbb{P}^2 \times \mathbb{P}^2$ from P_a. It is a 4-fold section of $\mathrm{Grass}(2, 5)$ containing the 3-plane $\mathbb{P}^3_{\langle b,d,z,y \rangle}$ defined by the ideal (x, c, e, f).

2.4. **Finding the Tom format from** $v_6(\mathbb{P}(1,2,3))$**.** We can start from the other end, dividing the 9 equations (3) of $v_6(\mathbb{P}(1,2,3))$ into 4 that are linear in a and 5 not involving a. One gets $af = d^2$, $ae = bd$, $ac = b^2$ and $ax = bc$ together with the five Pfaffians of

$$\begin{pmatrix} 0 & c & b & d \\ & x & c & e \\ & & e & f \\ & & & 0 \end{pmatrix}. \tag{6}$$

If we hope to describe the set of all 9 equations as Pfaffians of a special 6×6 skew matrix, we must put a where it multiplies x, c, e, f and *not* b, d, so put it as m_{16}.

3. Double Jerry format

3.1. **Double Jerry.** A neat starting point [TJ, 9.2] is to view Double Jerry as a theorem saying that a codimension 2 complete intersection $m_1 = m_2 = 0$ that contains two different codimension 3 complete intersections (x_1, x_2, x_3) and (y_1, y_2, y_3) is defined by two bilinear forms

$$m_1(x_1, x_2, x_3;\ y_1, y_2, y_3) \quad \text{and} \quad m_2(x_1, x_2, x_3;\ y_1, y_2, y_3). \tag{7}$$

We can then introduce two parallel sets of unprojection equations

$$s \cdot (x_1, x_2, x_3) = \cdots \quad \text{and} \quad t \cdot (y_1, y_2, y_3) = \cdots, \tag{8}$$

each taking us to codimension 3, together with a long equation $st = \cdots$. Each unprojection separately is given by Cramer's rule, leading to a 5×5 Pfaffian Jerry matrix, but the long equation is an intriguing and surprisingly complicated function of m_1, m_2, x_i, y_i. A particular case is worked out in Brown and Georgiadis [BG].

3.2. **Our particular case.** Rather than rework the general material of [TJ, 9.2], consider only the case of the Newton polygon (1).

As before, a only appears linearly in 4 equations, so can be eliminated or "projected out", expressing the variety as an unprojection; as before, note that a multiplies x, c, e, f:

$$af = d^2, \quad ae = bd, \quad ac = b^2 \quad \text{and} \quad ax = bc; \tag{9}$$

the 5 equations not involving a are

$$xb = c^2, \quad xd = ce, \quad fc = e^2, \quad bf = de \quad \text{and} \quad be = dc. \tag{10}$$

These are the Pfaffians of the Jerry$_{45}$ matrix

$$\begin{pmatrix} b & d & c & 0 \\ & 0 & e & c \\ & & f & e \\ & & & x \end{pmatrix}. \tag{11}$$

The matrix is a permutation of (6). What makes it a Jerry$_{45}$ is that the 7 entries in the 4th and 5th rows and columns are combinations of

the regular sequence x, c, e, f. What makes it a *double* Jerry is that the pivot $m_{45} = x$ is one of the variables on the nose, rather than a linear combination.

The matrix

$$\begin{pmatrix} b & d & c - \nu f & \mu f \\ & g & e & c \\ & & f & e \\ & & & x \end{pmatrix}, \tag{12}$$

is a deformation respecting the Jerry$_{45}$ requirements just described. Here g is a new indeterminate of degree 1 and μ and ν scalars. Putting back a as unprojection variable defines a family of del Pezzo 3-folds

$$W_{\mu,\nu} \subset \mathbb{P}^7_{\langle a,b,c,d,e,f,g,x \rangle}. \tag{13}$$

We recover $v_6(\mathbb{P}(1, 2, 3))$ on setting $\mu = \nu = 0$ and taking the hyperplane section $g = 0$.

For general μ, ν, the 3-fold $W_{\mu,\nu}$ is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ up to projective equivalence. However, to make the coordinate change requires an S_3 Galois field extension. This corresponds to permuting the 3 copies of $\mathbb{P}^1$. At the same time, it reflects the Weyl group $W(A_2) = S_3$ symmetry behind the deformation theory of the A_2 singularity.

In the deformation (12), the tag equation $xdf = e^3$ of (1) deforms to

$$x(df + eg) = e^3 - \nu e f^2 - \mu f^3 = \Phi(e, f). \tag{14}$$

The required field extension is the splitting field of Φ. When the discriminant of Φ does not vanish, the 3-fold $W_{\mu,\nu}$ is projectively equivalent to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ over this field. The projective equivalence involves the roots of Φ.

3.3. A_2 **symmetry.**

The little exercise in A_2 symmetry is fun and not quite obvious: in (12), the three equations involving x are

$$xb = c^2 - \mu ef - \nu cf$$
$$-xd = \mu f^2 - ce + \nu ef$$
$$xg = e^2 - cf$$

We write s, t, u for the roots of Φ, so that $s + t + u = 0$,

$$\nu = -(st + ut + su) = s^2 + st + t^2, \quad \mu = ust = -st(s + t),$$
$$\text{and} \quad \Phi(e, f) = (e - sf)(e - tf)(e - uf),$$

Now set y_0, y_1, y_2 and z_0, z_1, z_2 to be the following linear combinations of $(b, -d, g)$ and (c, f, e):

$$\begin{aligned} z_0 &= b - s\,d + tu\,g, & y_0 &= c + s\,e + tu\,f, \\ z_1 &= b - t\,d + su\,g, & \text{and} \quad y_1 &= c + t\,e + su\,f, \\ z_2 &= b - u\,d + st\,g, & y_2 &= c + u\,e + st\,f. \end{aligned}$$

We find

$$xz_i = y_j y_k,$$
$$ay_i = z_j z_k, \qquad \text{for} \quad \{i, j, k\} = \{0, 1, 2\}.$$
$$xa = y_i z_i,$$

These are the standard equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as the 2×2 minors of the 3-cube.

3.4. Interpretation as double Jerry.

In (12), the two equations not involving x are

$$bf - de + g(c - \nu f) = 0 \quad \text{and} \quad be - dc + \mu g f = 0 \qquad (15)$$

This is a codimension 2 complete intersection containing the codimension 3 c.i. $f = e = c = 0$. Adjoining x as the unprojection variable of this ideal gives the 5 Pfaffians of (12). This works as follows. Write the equations (15) as

$$\begin{pmatrix} g & -d & b \end{pmatrix} \begin{pmatrix} c - \nu f & \mu f \\ e & c \\ f & e \end{pmatrix} = 0.$$

By Cramer's rule the vector $(g, -d, b)$ is proportional to the minors of the 3×2-matrix. The correct proportionality constant follows from the unperturbed equations (10). However, (15) also contains the codimension 3 c.i. $b = d = g = 0$, and can be written as

$$\begin{pmatrix} f & e & c \end{pmatrix} \begin{pmatrix} b - \nu g & \mu g \\ -d & b \\ g & -d \end{pmatrix} = 0.$$

Note that this is also the result of interchanging (c, e, f) and $(b, -d, g)$ in (15), so the equations are invariant under the involution $(x, b, -d, g) \leftrightarrow (a, c, e, f)$.

Adjoining a as the unprojection variable gives the other half of the double Jerry:

$$\begin{pmatrix} c & -e & b - \nu g & \mu g \\ & f & -d & b \\ & & g & -d \\ & & & a \end{pmatrix}.$$

It remains to find the long equation for ax. This can be done using the other equations: start for example with $gx = e^2 - fc$, multiply by a and rewrite the right hand side until it is divisible by g. The result is

$$ax = (b - \nu g)(c - \nu f) + \mu g e - \mu f d.$$

4. Unprojection and deformations

4.1. **Unprojection.** The general theory of unprojection was initiated by Kustin and Miller [KM] and developed in the present form by Papadakis and Reid, see [Ki, P, PR].

Let $P \in D \subset X$ be a singular point of a Gorenstein scheme, lying on a Gorenstein codimension 1 subscheme D. Consider the adjunction sequence

$$0 \to \omega_X \to \operatorname{Hom}(\mathcal{I}_D, \omega_X) \to \omega_D \to 0.$$

By [PR, Lemma 1.1], the $\mathcal{O}_X$-module $\operatorname{Hom}(\mathcal{I}_D, \omega_X)$ is generated by two elements; we can take one of these as an injective map $s \colon \mathcal{I}_D \hookrightarrow \omega_X \cong \mathcal{O}_X$ that projects to a basis element of $\omega_D \cong \mathcal{O}_D$. The *unprojection* Y of D in X is the spectrum of the $\mathcal{O}_X$-algebra $\mathcal{O}_X[S]/(Sf_i - s(f_i))$, where the f_i generate the ideal $\mathcal{I}_D \subset \mathcal{O}_X$. The scheme Y is again Gorenstein.

As X is Gorenstein, $\operatorname{Hom}(\mathcal{I}_D, \omega_X) \cong \operatorname{Hom}(\mathcal{I}_D, \mathcal{O}_X)$. We calculate generators of $\operatorname{Hom}(\mathcal{I}_D, \mathcal{O}_X)$ in concrete cases by computer algebra, cf. [BP]. This construction also applies in a relative situation, over a base space T. The most general T is the base of a versal deformation of the inclusion map $i \colon D \hookrightarrow X$.

4.2. **Our case.** The first order infinitesimal deformations of $i \colon D \hookrightarrow X$ are described as the Pfaffian perturbations of the equations contained in the ideal (x, c, e, f). The trivial deformations are given by vector fields $\operatorname{Der}(-\log D)$ preserving D. For deformations of weight -1, this means that we make the matrix as general as possible, with no coordinate transformations of x, c, e and f allowed. The result is

$$\begin{pmatrix} z & c+y & b & d \\ & x & c & e \\ & & e & f \\ & & & -g \end{pmatrix}$$

We choose the minus sign to conform with the deformation (12).

For deformations of weight ≥ 0 a short computation in Singular [DGPS] shows that the above deformations generate the module of deformations: we can replace y and z with polynomials in f, and g with a polynomial in x having deformation variables as coefficients. Since our singularity is nonisolated some care is needed with the meaning of infinite dimensional versal deformation. We restrict ourselves here to deformations of nonpositive weight, that globalise to deformations of the projective cone. Then the first order infinitesimal deformations are given by

$$\begin{pmatrix} z+\mu f & c+y-\nu f & b & d \\ & x & c & e \\ & & e & f \\ & & & -g+\lambda x \end{pmatrix} \tag{16}$$

For higher order deformations, the equations are the Pfaffians of the matrix (16), as the deformation is in particular a deformation of X. The obstruction is that they must lie in the ideal (x, c, e, f). Hence setting these variables to zero in (16) we find $gy = gz = 0$ as the equations of the base space.

We compute $\mathrm{Hom}(\mathcal{I}_D, \mathcal{O}_X)$ using SINGULAR [DGPS] to determine the unprojection, obtaining the equations

$$ac - b(b - \nu g) - \lambda(z + \mu f)^2 - \mu dg - \lambda \nu c(c + y - \nu f),$$
$$ae - (b - \nu g)d - \lambda(c + y)(z + \mu f) - \mu g^2 - \lambda \nu xd + \lambda \mu xg,$$
$$af - d^2 + bg - \lambda(c + y)^2 + \lambda \nu(c + y)f,$$
$$ax - (b - \nu g - \lambda \nu x)(c + y - \nu f) + d(z + \mu f) - \mu eg.$$

We find two components, with total spaces isomorphic up to a smooth factor with the Tom and Jerry formats of Sections 2 and 3.4. We replace y by $y + c$ in the Tom equations, to obtain the cone as section $\lambda = 0$, $z = 0$ and $y = 0$. The coordinate transformations needed are $a \mapsto a + \lambda \nu(c + y)$, $y \mapsto y + \nu f$ and $z \mapsto z - \mu f$ for the Tom component and $a \mapsto a + \lambda \mu e$, $g \mapsto g + \lambda x$ for Jerry. Note that these coordinate transformations mix the deformation and the space variables.

4.3. The versal deformation of the cone over $v_6(\mathbb{P}(1,2,3))$.
Altmann [A, Table 5.1] records the result of our computation of the infinite dimensional versal deformation. What we have actually computed is the part in nonpositive weight, giving the (embedded) versal deformation of the projective cone. After a simple coordinate transformation and translation to our present coordinates, the formulas there give exactly the same ideal as computed above in terms of unprojection.

4.4. The cone over an elliptic curve of degree 6.
The versal deformation of the cone over an elliptic normal curve of degree 6 is described without equations by Mérindol [Me]. The base space is the product of the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ with the germ of an appropriate modular curve.

Deformations of negative weight can be described by Pinkham's construction of "sweeping out the cone". More precisely, the total space over a line in the base space is the cone over the anticanonical model of an almost del Pezzo surface of degree 6, with the given elliptic curve E as hyperplane section. Such a surface is obtained by blowing up three points on the curve, embedded in the plane by a linear system of degree 3. Mérindol's construction starts with a family of such surfaces over an Abelian variety A, which is the hypersurface in $\mathrm{Pic}^3 \times E^3$ given by $3H - (P_1 + P_2 + P_3) = 6O$. The Weyl group $W = A_1 \times A_2$ acts on

this: A_2 permutes the three points, and A_1 acts by

$$(H; P_1, P_2, P_3) \mapsto$$
$$(2H - P_1 - P_2 - P_3; H - P_2 - P_3; H - P_1 - P_3, H - P_1 - P_2).$$

Thus the base space of the versal deformation in negative weight is the cone over $A/W \cong \mathbb{P}^1 \times \mathbb{P}^2$.

We find the elliptic curve as hyperplane section of the singular del Pezzo surface $v_6(\mathbb{P}(1, 2, 3))$. In affine coordinates related to (4) we take the curve $w^2 = v^3 + \gamma v^2 + v$, realising the cone as the hyperplane section $f - x - \gamma c - b = 0$. Thus the variable a does not appear in the equation.

For the deformations of negative weight, we perturb the matrix (6) (with $b = f - x - \gamma c$) with independent variables, subject to the resulting equations lying in the ideal (x, c, e, f). This means that the entries multiplied by $m_{1,5} = d$ are not perturbed, and moreover, no perturbation of x, c, e or f is absorbed by coordinate transformations. We take

$$\begin{pmatrix} z & c+y & b+u & d \\ & x & c & e+q \\ & & e & f+p \\ & & & s \end{pmatrix}. \tag{17}$$

The Pfaffians of this matrix with x, c, e and f (and therefore also b) equated to zero give the equations of the base space: the minors of

$$\begin{pmatrix} z & y & u \\ q & p & s \end{pmatrix}.$$

As for the space of deformations of weight zero, a computation with Singular shows that it has dimension two. One deformation is given by the modulus γ, but there is another, corresponding to the choice of point from which to project the curve.

The matrix (17) is neither a Tom nor a Jerry matrix. But it can written in these forms after a small resolution of the base space. We do this here for the Tom format. The cone over $\mathbb{P}^1 \times \mathbb{P}^2$ is resolved by $\mathbb{P}^1 \times \mathbb{A}^3$. We introduce an inhomogenous coordinate λ on $\mathbb{P}^1$ and set $q = \lambda z$, $s = \lambda y$ and $s = \lambda u$. Then we can make the matrix into a Tom_1 by row and column operations. After the coordinate transformation

$$(c, d, e, f, x, u, y, z) \mapsto (c - \lambda x, d - \gamma \lambda z + \lambda^2 z, e,$$
$$f + \lambda c + \gamma \lambda x - 2\lambda^2 x, x, u + b, y - c + \lambda x, z),$$

(so that $b = f - x - \gamma c + \lambda c - 2\lambda^2 x$), the matrix takes the form

$$\begin{pmatrix} z & y & u & & d \\ & x & c & & e \\ & & e & & f \\ & & & & (\lambda + \gamma \lambda^2 + \lambda^3)x \end{pmatrix}.$$

5. General remarks and FAQ

5.1. **Which is Tom, and which is Jerry?** We offer three answers as useful mnemonics. We do not assume any prior familiarity with the Hanna–Barbera characters.

(i) Tom is fatter. The ancestral Tom is the projective 4-fold $\mathbb{P}^2 \times \mathbb{P}^2$, whereas for Jerry it is the 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(ii) The Tom_i condition on a skew 5×5 matrix is that, deleting the ith row and column, the remaining 6 entries m_{jk} are in a codimension 4 c.i. ideal. In simple terms, this means two coincidences on the m_{jk}. On the other hand, the Jerry_{jk} condition is that the 7 elements $m_{ij} = -m_{ji}$ and $m_{ik} = -m_{ki}$ in the jth and kth rows and columns are in a codimension 4 c.i. ideal, which means 3 conditions.

(iii) Weight-for-weight, Jerry is more singular. Any point $P \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ lies on 3 lines, that the linear projection from P contracts to nodes. In contrast, if we take the 3-fold hyperplane section V of $\mathbb{P}^2 \times \mathbb{P}^2$ to get the flag variety of $\mathbb{P}^2$, the linear projection of V from P only has two nodes.

Trying to fit a Jerry unprojection into a 6×6 skew matrix format is invariably a waste of time.

5.2. **What's it all about?** A hypersurface or complete intersection is determined by the coefficients of its defining equations, so its deformations are unobstructed. The subtlety of the deformation theory in these cases is nothing to do with obstructions, but how to pass to the quotient by the appropriate equivalence relation, which involves dividing by the groupoid of local diffeomorphisms.

The Buchsbaum–Eisenbud theorem [BE] puts codimension 3 Gorenstein ideals in the same framework: the variety is given by a skew $(2k+1) \times (2k+1)$ matrix (most commonly 5×5), that encodes both the defining equations and the syzygies, so that the entries of the matrix can be freely deformed. In other words, the skew matrix is a given mold, into which one can simply pour functions on the ambient space in a liquid manner.

In contrast, one usually expects codimension 4 constructions to be obstructed. A typical case is the cone over dP_6, whose deformation theory has the 2 components we have mentioned many times.

The point of Tom and Jerry is that, in most commonly occurring cases, our variety admits a Gorenstein projection to codimension 3, with the projected variety given by the Pfaffians of a 5×5 skew matrix; that is, the projected variety is a regular pullback from $\text{Grass}(2,5)$ in its Plücker embedding, marked with an unprojection divisor that corresponds to a linear subspace of $\text{Grass}(2,5)$. Every geometer must have done the easy exercise of seeing that any linear subspace of $\text{Grass}(2,4)$ (the Klein quadric) either consists of lines of $\mathbb{P}^3$ passing through a point

P, or dually, of lines contained in a plane $\mathbb{P}^2 \subset \mathbb{P}^3$. The Tom and Jerry formats answer the same question for Grass$(2,5)$; see [TJ, 2.1].

5.3. **Do they do everything?** Unfortunately, no. Tom and Jerry provide two smooth components of the deformation theory, and for deformation problems entirely contained within one component or the other, they can be relied on to do everything. However, we know other cases in codimension 4 that appear not to have any useable structure of Kustin–Miller unprojection.

A general structure theorem for Gorenstein codimension 4 ideals is described in [G4]. It is rather complicated, as it should account for the singular total spaces of versal deformations, cf. the discussion of the cone over an elliptic curve of degree 6 above. Deformations of its hyperplane sections are even more complicated.

REFERENCES

[A]   Klaus Altmann, One parameter families containing three dimensional toric Gorenstein singularities, in Explicit birational geometry of 3-folds, LMS Lecture notes **281**, Cambridge Univ. Press, Cambridge, 2000, pp. 21–50

[BP]   Janko Böhm and Stavros Argyrios Papadakis, Implementing the Kustin–Miller complex construction. J. Softw. Algebra Geom. **4** (2012) 6–11

[TJ]   Gavin Brown, Michael Kerber and Miles Reid, Fano 3-folds in codimension 4, Tom and Jerry. Part I, Compositio Math. **148** (2012) 1171–1194

[BG]   Gavin Brown and Konstantinos Georgiadis, Polarized Calabi–Yau 3-folds in codimension 4, Math. Nachr. **290** (2017) 710–725

[BE]   David Buchsbaum and David Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. Amer. J. Math. **99** (1977) 447–485

[DGPS]   Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: SINGULAR 4-1-0 — A computer algebra system for polynomial computations (2016). http://www.singular.uni-kl.de

[KM]   Andrew Kustin and Matthew Miller, Constructing big Gorenstein ideals from small ones, J. Algebra **85** (1983) 303–322

[Me]   J.-Y. Mérindol, Les singularités simples elliptiques, leurs déformations, les surfaces de del Pezzo et les transformations quadratiques. Ann. Sci. École Norm. Sup. (4) **15** (1982) 17–44

[P]   Stavros Papadakis, Gorenstein rings and Kustin–Miller unprojection, Univ. of Warwick Ph.D. thesis, Aug 2001, vi + 72 pp., http://wrap.warwick.ac.uk/50146/

[PR]   Stavros Papadakis and Miles Reid, Kustin–Miller unprojection without complexes, J. Algebraic Geom. **13** (2004) 563–577, preprint math.AG/0011094, 18 pp.

[Ki]   Miles Reid, Graded rings and birational geometry, in Proc. of algebraic geometry symposium (Kinosaki, Oct 2000), K. Ohno (Ed.), 1–72. (Get from my website + 3-folds)

[G4]   Miles Reid, Gorenstein in codimension 4: the general structure theory. In: Algebraic geometry in East Asia–Taipei 2011, Adv. Stud. Pure Math., **65** (2015) pp. 201–227,

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK. COVENTRY CV4 7AL, ENGLAND
E-mail address: G.Brown@warwick.ac.uk
E-mail address: Miles.Reid@warwick.ac.uk

DEPARTMENT OF MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG. SE 412 96 GOTHENBURG, SWEDEN
E-mail address: stevens@chalmers.se