The McKay Correspondence and Orbifold Riemann–Roch

Rebecca Catherine Leng

A thesis submitted to the University of Warwick
for the degree of Doctor of Philosophy

Mathematics Institute
University of Warwick
May 2002

Contents

Acknowledgments **vi**

Declaration **viii**

Introduction **ix**

1 Preliminaries **1**

1.1 Background to McKay correspondence 1

1.2 Introduction to orbifolds 7

1.3 An introduction to G-sheaves 9

2 Explicit construction of G-Hilb $\mathbb{C}^2$ for binary dihedral groups. **11**

2.1 Representation theory 11

2.2 G-graphs 12

2.3 Preliminaries on G-clusters and G-graphs 14

2.4 List of possible G-graphs 19

2.4.1 Case A 19

2.4.2 Case B 20

2.4.3 Case C_k 21

2.4.4 Case D 22

2.4.5 Case E 24

2.4.6 Conclusion 25
2.5 Construction of open sets contained in G-Hilb $\mathbb{C}^2$ 26
2.5.1 Case A 26
2.5.2 Case B 34
2.5.3 Case C_k 38
2.5.4 Case D 40
2.5.5 Case E 43
2.5.6 Conclusion 46

3 Explicit construction of G-Hilb $\mathbb{C}^3$ for trihedral groups 48
3.1 Classification of trihedral groups 48
3.1.1 Representation theory of G 52
3.2 G-graphs 53
3.3 Classification of G-graphs for the group $\left\langle \frac{1}{31}(1,5,25) \ltimes \mathbb{Z}_3 \right\rangle$ . . 65
3.3.1 Step 0 68
3.3.2 Step 1 69
3.3.3 Step 2 72
3.4 Syzygies 78
3.4.1 Using syzygies 79
3.5 Proof of Conjecture 3.21 82
3.6 Construction of possible sets S 83
3.6.1 Step Zero 83
3.6.2 Step One 84
3.6.3 Step Two 85
3.6.4 Step $i+1$ 87
3.7 Proof that each of these sets is a G-set 89
3.8 The ideals 94
3.9 Conclusion of the proof 97

3.10 Conclusion 98

4 Equivariant and Orbifold Riemann–Roch 100

4.1 Introduction 100

4.1.1 Notation 100

4.1.2 Background to the equivariant Riemann–Roch theorem 101

4.1.3 Technical preliminaries on higher direct image sheaves 102

4.1.4 Roots of unity 106

4.2 Results on G-sheaves 106

4.3 Proof of Theorem 4.1 on a smooth projective curve 108

4.3.1 Proof of Theorem 4.1 for the trivial G-sheaf when the action is coprime. 110

4.3.2 Proof of Theorem 4.1 for any invertible G-sheaf 112

4.3.3 Proof of Theorem 4.1 when the action is not coprime . 114

4.4 Proof of Theorem 4.1 on a smooth projective surface. 115

4.4.1 Notation 116

4.4.2 Proof of Theorem 4.1 for the trivial G-sheaf when the action is coprime 117

4.4.3 Proof of Theorem 4.1 for any invertible G-sheaf 122

4.4.4 Proof of Theorem 4.1 when the action is not coprime . 124

4.5 Orbifold Riemann–Roch 127

Bibliography 132

List of Figures

1.1 The triangulation of the junior simplex 5

1.2 The shape of a G-graph when G is an Abelian subgroup of $\mathrm{SL}(3,\mathbb{C})$ 7

3.1 Decomposition of the monomials in T 67

3.2 The set obtained by choosing x^5y^4 in S 70

3.3 If $x^2y^6 \in S$ and $x^{19} \notin S$ 73

3.4 If $x^5y, x^4y^3 \in S$ 74

3.5 The monomials in T 83

3.6 $S(b_1)$ 84

3.7 $S(b_i, c_i)$ 85

3.8 $S(a_i, b_i)$ 88

3.9 $S(\mathrm{end})$ 89

3.10 $S(b_i, c_i)$ 91

3.11 $S(a_i, b_i)$ 93

4.1 Exceptional locus for the blowup of Q 118

4.2 Exceptional locus for the blowup of Q 126

Acknowledgments

I am indebted to my supervisor Professor Miles Reid for his continuing encouragement and enthusiasm throughout my PhD. Miles gave of his time generously and I thank him for his support and friendship.

I would also like to thank Alastair King and Alessio Corti for lots of good advice and assistance.

I would like to acknowledge the financial assistance which EPSRC provided during this research.

Part of this work was carried out during a trip to RIMS, Kyoto University. I thank RIMS both for their financial support and their hospitality. In particular I would like to thank Professors Miyaoka and Mori for their invitation and for making the trip possible. I would also like to express gratitude to the British Council for additional sponsorship during this trip.

I am grateful to many people at Warwick for their help, friendship and encouragement, including Gavin Brown, Anita Buckley, Alastair Craw, Adrian Langer, Jorge Neves, Stavros Papadakis, Simone Pavanelli, Daniel Ryder, Balázs Szendrői and Nikos Tziolas. I would particularly like to thank Ali and Balázs for their comments on an earlier draft of this thesis and Dan for answering any number of questions over the last four years.

Finally I thank my family, friends and church for their support and for helping me to keep everything in perspective. I especially thank Richard

and my parents for always believing in me and Richard for looking after me during the last few months.

Declaration

Chapter 1 is purely expository. Otherwise I declare that to the best of my knowledge, the material contained in this thesis is my original work except when explicitly stated otherwise.

Introduction

The classical statement of the McKay correspondence relates the irreducible representations of a finite subgroup $G \subset \mathrm{SL}(2,\mathbb{C})$ with a basis for the cohomology of the unique minimal resolution Y of the orbifold $\mathbb{C}^2/G$. Gonzalez-Sprinberg and Verdier [GSV83] construct this correspondence as an isomorphism on K-theory. This is because the G-equivariant K-theory of $\mathbb{C}^2$ is equal to the representation ring of G.

The exceptional locus of Y consists of a chain of -2 curves intersecting in the way shown by the Dynkin diagram for G. Ito and Nakamura [IN99] show that $Y = G\text{-Hilb } \mathbb{C}^2$, the fine moduli space parametrising G-clusters, and compute the clusters along the exceptional locus of Y. In Chapter 2, I compute explicitly an affine cover of $G\text{-Hilb } \mathbb{C}^2$ when G is a binary dihedral group which shows that $G\text{-Hilb } \mathbb{C}^2$ is a nonsingular rational variety. Unlike the case when G is an Abelian group the natural cover I construct does not consist of copies of $\mathbb{C}^2$ but smooth surfaces in $\mathbb{C}^4$.

When G is a finite subgroup of $\mathrm{SL}(3,\mathbb{C})$, Bridgeland, King and Reid [BKR01] prove that $G\text{-Hilb } \mathbb{C}^3$ is a crepant resolution of the orbifold $\mathbb{C}^3/G$. In Chapter 3, I consider the case when G is a trihedral group. I begin by classifying the possibilities for the maximal normal Abelian subgroup of G and show that it must be of the form $\mathbb{Z}_r \times \mathbb{Z}_R$ where $r \mid R$, $\mathbb{Z}_r$ has generator $\frac{1}{r}(1,-1,0)$ and $\mathbb{Z}_R$ has generator $\frac{1}{R}(a,b,c)$.

I then compute explicitly an affine cover of G-Hilb $\mathbb{C}^3$ using the same methods as in Chapter 2. Again G-Hilb $\mathbb{C}^3$ is nonsingular and rational, but the natural affine cover we construct does not consist of copies of $\mathbb{C}^3$.

In Chapter 4, I state and prove a Riemann–Roch style formula for an orbifold having only isolated quotient singularities. I begin by giving an explicit proof of Atiyah and Singer's equivariant Riemann–Roch theorem for invertible G-sheaves in dimensions 1 and 2.

Theorem 0.1 (Equivariant Riemann–Roch [AS68]) *Let Z be a non-singular projective variety, V a holomorphic vector bundle on Z and G a finite group of automorphisms of the pair (Z, V). For $g \in G$, let Z^g denote the fixed locus of g on Z. Then*

$$\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p\left(Z, \mathcal{O}(V)\right)\right) \;=\; \int_{Z^g} \frac{\operatorname{ch}(V_{Z^g})(g)\cdot \operatorname{td}(Z^g)}{\operatorname{ch}\lambda_{-1}\left((\mathcal{N}^g)^*\right)(g)}. \qquad (1)$$

I show that this theorem is an elementary consequence of Hirzebruch Riemann–Roch and an eigensheaf decomposition of sheaves on the quotient variety. I then use this formula to generate a Riemann–Roch type expression for an orbifold having only isolated cyclic quotient singularities.

The structure of the proof is as follows. First prove the theorem when Z is a curve. This is done by constructing eigensheaves $\mathcal{L}_i = \mathcal{H}om(V_i, \pi_*\mathcal{F})^G$ where $\pi\colon Z \to X = Z/\langle g\rangle$ is the quotient morphism. The theorem is then proved by computing their degrees. When Z is a surface, choose a resolution $f\colon Y \to X$ and find locally free sheaves $\mathcal{M}_i$ on Y such that $f_*\mathcal{M}_i = \mathcal{L}_i$. Then compute $\chi(Y, \mathcal{M}_i)$.

For any smooth n dimensional projective variety Z, we show that the contribution an isolated fixed point, P, makes to Equation 1 depends only on how g acts on a local analytic neighbourhood P. Thus the contribution

P would make to

$$\operatorname{Tr}\left(g\colon \bigoplus_i (-1)^i \operatorname{H}^i(Z, \mathcal{F})\right)$$

is the same as the contribution a point $P' \in C_1 \times \cdots \times C_n$ makes to

$$\operatorname{Tr}\left(g'\colon \bigoplus_i (-1)^i \operatorname{H}^i(C_1 \times \cdots \times C_n, \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_n)\right)$$

if a local analytic neighourhood of P' in $C_1 \times \cdots \times C_n$ is isomorphic to a local analytic neighourhood of P in Z.

We can then use this theorem to give an expression for $\chi(X, \mathcal{L})$ when $X = Z/\langle g\rangle$ and $L = (\pi_* \mathcal{F})^{\langle g\rangle}$ for $\mathcal{F}$ an invertible G-sheaf on Z. The contribution that a isolated singularity $P \in X$ makes to this theorem depends only on the local analytic neighbourhood of P. Thus we can use this to give an orbifold Riemann–Roch style formula for $\chi(X, \mathcal{L})$ when X is only locally a quotient of a group, that is, an orbifold.

Chapter 1

Preliminaries

1.1 Background to McKay correspondence

Throughout this thesis we will use the notation $g = \frac{1}{r}(a_1, \ldots, a_n)$ to mean that $g = \operatorname{diag}(\varepsilon^{a_1}, \ldots, \varepsilon^{a_n}) \in \mathrm{GL}(n, \mathbb{C})$, where ε is a primitive rth root of unity.

Many papers have been written on the McKay correspondence over the last few years. Good introductions include the first two chapters of Alastair Craw's thesis [Cra01] or Miles Reid's Bourbaki talk [Rei02].

Several people have worked on adapting the classical statement of the McKay correspondence to a more general setting. Let M be a smooth quasiprojective variety of dimension n and G a finite group of automorphisms of M such that the stabiliser subgroup acts on the tangent space $T_P M$ as a subgroup of $\mathrm{SL}(T_P M)$ for every point $P \in M$. Let $\pi \colon M \to X = M/G$ be the finite quotient morphism and $f \colon Y \to X$ a crepant resolution (if one

exists).

$$\begin{array}{ccc} & & M \\ & & \downarrow \pi \\ Y & \xrightarrow{f} & X \end{array}$$

A generalisation of the McKay correspondence is that there should be a natural isomorphism between the G-equivariant K-theory of M and the ordinary K-theory of Y. See [BKR01].

In the case $M = \mathbb{C}^3$ Nakamura [Nak01] conjectured, and proved when G is Abelian, that G-Hilb $\mathbb{C}^3$, the Hilbert scheme parametrising G-clusters, is a crepant resolution of $\mathbb{C}^3/G$. Bridgeland, King and Reid [BKR01] prove that G-Hilb $\mathbb{C}^n$ is irreducible and a crepant resolution of $\mathbb{C}^n/G$ when $n \leq 3$.

Let $\operatorname{Irr} G$ be the set of irreducible representations of G. The regular representation, $\mathbb{C}[G]$, is isomorphic to $\bigoplus_{\rho \in \operatorname{Irr} G} \rho^{\oplus \deg \rho}$. Recall the following definition.

Definition 1.1 A *G-cluster* is a subscheme $Z \subset \mathbb{C}^n$ with defining ideal $I_Z \subset \mathbb{C}[x_1, \ldots, x_n]$, and structure sheaf $\mathcal{O}_Z$ such that

- Z is zero dimensional, that is a cluster,
- Z is G-invariant and
- $\mathcal{O}_Z = \mathbb{C}[x_1, \ldots, x_n]/I_Z \cong \mathbb{C}[G]$ as G-modules.

The fine moduli space parametrising G-clusters is called G-Hilb $\mathbb{C}^n$.

Thus a point in G-Hilb $\mathbb{C}^n$ is an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ such that the quotient $\mathbb{C}[x_1, \ldots, x_n]/I \cong \mathbb{C}[G]$.

Definition 1.2 Let G be a finite diagonal Abelian subgroup of $\mathrm{SL}(n, \mathbb{C})$. A *$G$-graph* is a set of monomials $\Gamma \subset \mathbb{C}[x_1, \ldots, x_n]$ satisfying the following two conditions.

1. There exists one monomial in each representation space of G

2. If $x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \Gamma$ then $x_1^{i_1} \cdots x_n^{i_n} \in \Gamma$ for all $0 \leq i_j \leq \lambda_j$.

This is the same as Nakamura's definition of a G-graph [Nak01].

Let G be a finite diagonal Abelian subgroup of $\mathrm{SL}(n, \mathbb{C})$ and I an ideal defining a G-cluster. Now $\mathbb{C}[x_1, \ldots, x_n]/I \cong \mathbb{C}[G] \cong \bigoplus_i V_i$ where each V_i is a 1 dimensional representation. Since G is diagonal each monomial belongs to a well defined representation space. Therefore we can choose a set of monomials $\Gamma(I)$ with one monomial belonging to each V_i which forms a basis for the complex vector space $\mathbb{C}[x_1, \ldots, x_n]/I$. If a monomial $x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \Gamma(I)$ then $x_1^{\lambda_1} \cdots x_n^{\lambda_n} \notin I$. Therefore $x_1^{i_1} \cdots x_n^{i_n} \notin I$ for all $0 \leq i_j \leq \lambda_j$ so we may assume that $x_1^{i_1} \cdots x_n^{i_n} \in \Gamma(I)$. Thus we may choose a basis $\Gamma(I)$ for $\mathbb{C}[x_1, \ldots, x_n]/I$ which is a G-graph. Clearly this basis will rarely be unique.

Example 1.3 Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be generated by $\frac{1}{r}(1, -1)$ then every ideal defining a G-cluster has a G-graph of the form

$$\begin{array}{cccccc} [\,\cdot\,] & & & & & \\ y^{r-i-1} & & & & & \\ \vdots & & & & & \\ y & [\,\cdot\,] & & & & \\ 1 & x & \cdots & x^i & [\,\cdot\,] & \end{array}$$

for some $i = 0, \ldots, r-1$, where $[\,\cdot\,]$ means we do not take this monomial. Then polynomials of the form

$$A := x^{i+1} - ay^{r-i-1}, \quad B := y^{r-i} - bx^i, \quad C := xy - c$$

belong to any ideal I which defines a G-cluster and has this G-graph, for

some $a, b, c \in \mathbb{C}$. Notice that

$$yA - x^i C + aB = abx^i - cx^i$$

so $(ab - c)x^i \in I$. Since $x^i \notin I$ it follows that $ab = c$. Therefore a and b are the algebraic coordinates on this affine piece of G-Hilb $\mathbb{C}^2$. The relation $yA - x^i C + aB = 0$ is called a syzygy. See for example [Rei97] for more details.

Let I' be the ideal generated by A, B and C such that $ab = c$. These polynomials express every monomial in $\mathbb{C}[x, y]$ as a linear combination of monomials in Γ. They imply no relations between the monomials in Γ. Therefore Γ forms a basis for $\mathbb{C}[x, y]/I'$ and so I' defines a G-cluster.

Several explicit descriptions of G-Hilb $\mathbb{C}^3$ have been given when G is Abelian. Craw and Reid [CR99] calculated G-Hilb $\mathbb{C}^3$ explicitly by choosing a triangulation of the junior simplex $\Delta_1 \subset \mathbb{R}^3$ in a lattice L. Each triangle corresponds to an affine piece of G-Hilb $\mathbb{C}^3$ whose defining ideals can have the same G-graph. They prove that the ideals have at most seven generators and the affine piece requires exactly three algebraic parameters. Thus they show that G-Hilb $\mathbb{C}^3$ is a smooth rational variety. Craw [Cra01] then goes on to show how to construct a bijection between the irreducible representations of G and a basis for the cohomology of G-Hilb $\mathbb{C}^3$.

Let G be a cyclic subgroup of $\mathrm{SL}(3, \mathbb{C})$ generated by $\frac{1}{r}(a, b, c)$. Then Craw and Reid define the lattice L to be $L := \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(a, b, c)$ and the junior simplex to be $\Delta_1 := \left\{(r_1, r_2, r_3) \in L \otimes \mathbb{R} \;\middle|\; r_i \geq 0, \sum r_i = 1\right\}$. Any triangulation of Δ_1 into basic triangles will give a crepant resolution of $\mathbb{C}^3/G$. In the next example we give an example of the triangulation corresponding to G-Hilb $\mathbb{C}^3$.

Example 1.4 Let G be generated by $\frac{1}{31}(1, 5, 25)$. The triangulation of Δ_1

corresponding to G-Hilb $\mathbb{C}^3$ is shown in Figure 1.1. Each triangle corresponds

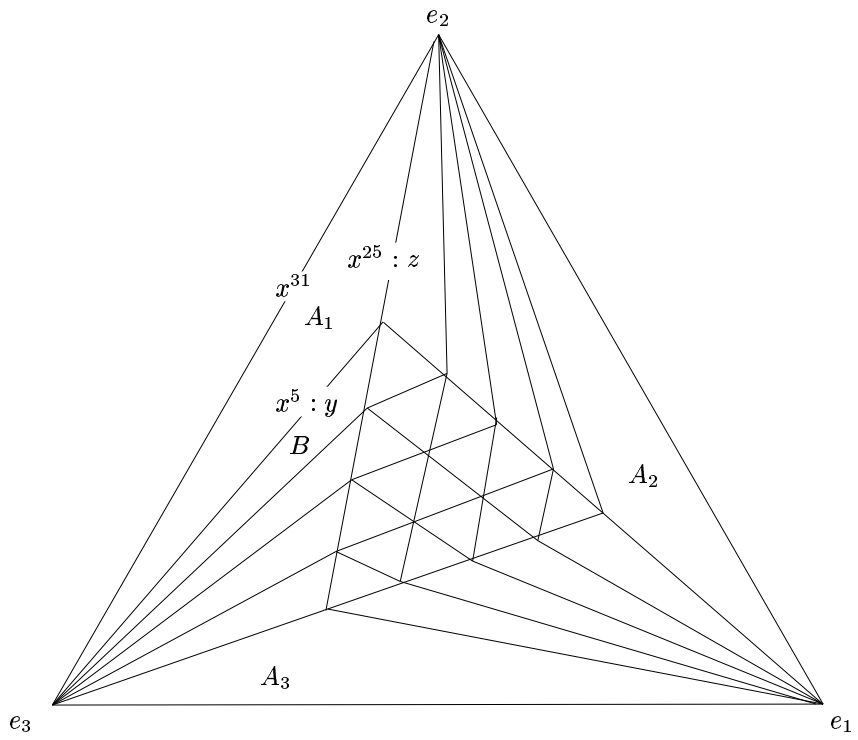


Figure 1.1: The triangulation of the junior simplex

to an affine piece of G-Hilb $\mathbb{C}^3$. Given one such affine piece U of G-Hilb $\mathbb{C}^3$ for any point $P \in U$ it is possible to choose the same G-graph as a basis for $\mathbb{C}[x, y, z]/I_P$ where I_P is the defining ideal for P.

The triangle marked A_1 corresponds to ideals having the G-graph

$$\Gamma_{A_1} = \begin{array}{ccccc} 1 & x & \cdots & x^{30} & [\,\cdot\,]. \end{array}$$

and so the ideals are of the form

$$I_{\xi,\eta,\zeta} = \left\langle x^{31} - \xi, y - \eta x^5, z - \zeta x^{25} \right\rangle.$$

The coordinates ξ, η, ζ are the algebraic coordinates on this affine piece of G-Hilb $\mathbb{C}^3$.

The triangle marked with B corresponds to ideals having the G-graph

$$\Gamma_B = \begin{array}{ccccc} [\cdot] & & & & \\ y^6 & [\cdot] & & & \\ y^5 & xy^5 & \cdots & x^4y^5 & \\ \vdots & \vdots & & \vdots & \\ y & xy & \cdots & x^4y & \\ 1 & x & \cdots & x^4 & [\cdot] \end{array}$$

and so the ideals are of the form

$$I = \left\langle A := x^5 - \xi y, B := y^7 - \eta x^4, C := z - \zeta y^5, D := xy^6 - d \right\rangle .$$

Consider the relation or syzygy

$$xB - yD + \eta A = -\xi\eta y + dy.$$

This syzygy shows that $d = \xi\eta$ and so ξ, η, ζ are the algebraic coordinates on this affine piece of G-Hilb $\mathbb{C}^3$.

Definition 1.5 Let M be a module over a local ring (A, m) then the *socle* is the annihilator in M of the maximal ideal m. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ define a G-cluster then write $\mathrm{Soc}(I)$ for the socle of $\mathbb{C}[x_1, \ldots, x_n]/I$ viewed as a module over $\mathbb{C}[x_1, \ldots, x_n]$ localised at the origin.

It follows immediately that $\mathrm{Soc}(I)$ is zero if and only if I supported at the origin

Let G be an Abelian subgroup of $SL(3, \mathbb{C})$ then [Nak01] and [CR99] prove that if an ideal I defines a point in G-Hilb $\mathbb{C}^3$ then I has a G-graph of the shape shown in Figure 1.2. This shows that the ideal has at most seven generators and $\mathrm{Soc}(I) \leq 6$.

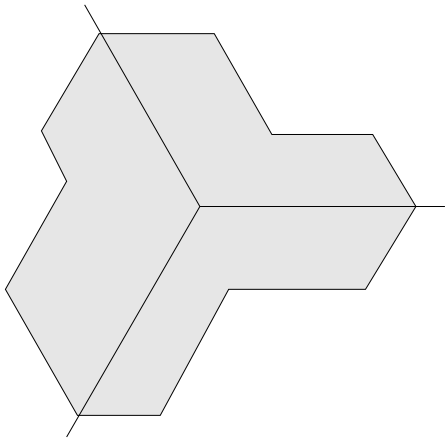


Figure 1.2: The shape of a G-graph when G is an Abelian subgroup of $\mathrm{SL}(3,\mathbb{C})$

Definition 1.6 A *trihedral group* is a finite subgroup of $\mathrm{SL}(3,\mathbb{C})$ of the form $G = A \rtimes \mathbb{Z}_3$, where A is a diagonal Abelian subgroup and $\mathbb{Z}_3$ is generated by

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Ito [Ito95] shows that a crepant resolution of $\mathbb{C}^3/G$ exists when G is a trihedral group.

1.2 Introduction to orbifolds

Definition 1.7 By *orbifold* I mean a normal projective variety having at worst quotient singularities. I say X is a *cyclic orbifold* if for every singular point $P \in X$ there exists a neighbourhood U of P such that $U \cong V/\mathbb{Z}_r$ where V is smooth and $\mathbb{Z}_r$ acts on V.

Definition 1.8 Let X be a normal quasiprojective variety and $\mathcal{L}$ a coherent

sheaf of $\mathcal{O}_X$ modules. Then $\mathcal{L}$ is said to *divisorial* if $\mathcal{L} = j_*\mathcal{L}^0$ where $X^0 = \operatorname{NonSing} X \subset X$, $\mathcal{L}^0 = \mathcal{L}_{|X^0}$ is invertible and j denotes the inclusion.

The following results are both taken from [Bou65] see also [Rei80, Appendix 1]. Let X be a normal quasiprojective variety.

Proposition 1.9 (VII.4 Thm. 2 [Bou65]) *Let $\mathcal{L}$ be a coherent sheaf of $\mathcal{O}_X$-modules which is torsion free and of rank one. Then $\mathcal{L}$ is divisorial if and only if for every inclusion $\mathcal{L} \subset \mathcal{M}$, where $\mathcal{M}$ is a torsion free $\mathcal{O}_X$-module and $\operatorname{Supp}(\mathcal{M}/\mathcal{L})$ has codimension ≥ 2, $\mathcal{L} = \mathcal{M}$.*

Theorem 1.10 (VII.1 Thm. 2 [Bou65]) *The map $D \mapsto \mathcal{O}_X(D)$ defines a bijection*

$$\left\{ \begin{array}{c} \textit{Weil divisors} \\ \textit{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{divisorial subsheaves} \\ \mathcal{L} \in k(X) \end{array} \right\} /\Gamma(X, \mathcal{O}_X)^*.$$

Definition 1.11 A variety X is said to be $\mathbb{Q}$-factorial if for every $\mathbb{Q}$-divisor $D = \sum a_i D_i$ for $a_i \in \mathbb{Q}$ and D_i prime divisors there exists $0 \neq \lambda \in \mathbb{Z}$ such that λD is Cartier.

The remaining material of this section is taken from Kollár and Mori [KM98, Section 5.1].

Definition 1.12 A variety X is said to have *rational singularities* if every resolution of singularities $f\colon Y \to X$ satisfies the following properties:

- $f_*\mathcal{O}_Y = \mathcal{O}_X$ (equivalently X is normal);
- $\mathcal{R}^i f_*(\mathcal{O}_Y) = 0$ for every $i > 0$.

Proposition 1.13 (Prop 5.15 [KM98]) *If X is an orbifold then X has rational singularities and X is $\mathbb{Q}$-factorial.*

Definition 1.14 Let Y be a smooth variety. A $\mathbb{Q}$-Cartier divisor $D = \sum a_i D_i$ is called a *simple normal crossing* divisor (often abbreviated to snc) if each D_i is smooth and they intersect everywhere transversely. Let X be a variety and $D = \sum a_i D_i$ a $\mathbb{Q}$-Cartier divisor on X. A *log resolution* of the pair (X, D) is a proper birational morphism $f\colon Y \to X$ such that Y is smooth, the exceptional locus $\operatorname{Exc}(f)$ is a Cartier divisor and $\operatorname{Exc}(f) \cup f^* \operatorname{Supp}(D)$ is a snc divisor.

For any variety X over a field of characteristic zero the fact that there exists a log resolution of X follows from Hironaka's proof of resolution of singularities. See [KM98].

1.3 An introduction to G-sheaves

Definition 1.15 Let $\mathcal{F}$ be a coherent sheaf together with a homomorphism $\lambda_g^{\mathcal{F}}\colon \mathcal{F} \to g^*\mathcal{F}$ for each $g \in G$. If $\lambda_1^{\mathcal{F}} = \mathrm{id}_{\mathcal{F}}$ and $\lambda_{hg}^{\mathcal{F}} = g^*(\lambda_h^{\mathcal{F}})\lambda_g^{\mathcal{F}}$ then $\mathcal{F}$ is a *G-sheaf*. These compatibility conditions imply that $\lambda_g^{\mathcal{F}}$ is an isomorphism.

If $P \in Z^g$ this isomorphism induces an isomorphism λ_g on the stalk $\mathcal{F}_P \to \mathcal{F}_P$ and on the fibre $\mathcal{F}_{|P} \to \mathcal{F}_{|P}$. Since G acts on Z it follows that G induces an action on the constant sheaf $k(Z)$. Specifying that $\mathcal{F}$ is a subsheaf of $k(Z)$ determines an action of G on $\mathcal{F}$. Two isomorphic sheaves may have different actions of G because they have different embeddings into $k(Z)$.

Example 1.16 Let $Z = \mathbb{P}^1$ and $G = \mathbb{Z}_r$ generated by $\frac{1}{r}(0,1)$. Then 1 generates the global sections of $\mathcal{O}_Z$ and $g(1) = 1$. If $\mathcal{F} = \mathcal{O}(P_0 - P_1)$ then x_0/x_1 generates the global sections and $g(x_0/x_1) = \varepsilon^{-1}(x_0/x_1)$. Now $\mathcal{F} \cong \mathcal{O}_Z$ but they have different actions of G.

Definition 1.17 Let G be a finite group acting on a nonsingular variety Z.

The *trivial G-sheaf* is the sheaf $\mathcal{O}_Z$, where for every $g \in G$ the homomorphisms $\lambda_g^{\mathcal{O}_Z} : \mathcal{O}_Z \to g^*\mathcal{O}_Z$ are induced from the action of G on $k(Z)$.

A variety X is a *trivial G-variety* if $gx = x$ for all $x \in X$ and $g \in G$. The following proposition comes from Atiyah's paper on K-theory.

Proposition 1.18 (Prop 1.6.2 [Ati67]) *Let X be a trivial G-variety and $V_1, \ldots, V_n$ a complete set of irreducible representations. Then every locally free G-sheaf $\mathcal{F}$ on X is isomorphic to a direct sum $\bigoplus_i \mathcal{F}_i \otimes V_i$, where G acts trivially on $\mathcal{F}_i$.*

This is proved by letting $\mathcal{F}_i = \mathcal{H}om_{\mathcal{O}_X}(V_i, \mathcal{F})^G$. If X is a trivial G-variety then by the above proposition $K_G(X) = K(X) \otimes R(G)$, where $R(G)$ is the representation ring of G. Therefore $a \in K_G(X)$ can be written as $a = \sum u_i \otimes V_i$ where $u_i \in K(X)$. The Chern character $\mathrm{ch}(a)(g)$ of a is defined by the formula

$$\mathrm{ch}(a)(g) = \sum_i \mathrm{ch}(u_i)\chi_i(g)$$

where $\mathrm{ch}(u_i)$ is the ordinary Chern character of the sheaf u_i and χ_i is the character of the representation V_i.

Chapter 2

Explicit construction of G-Hilb $\mathbb{C}^2$ for binary dihedral groups.

2.1 Representation theory

Throughout this chapter $G = \mathrm{BD}_{4n} \subset \mathrm{SL}(2,\mathbb{C})$ is the binary dihedral group of order $4n$ generated by

$$g = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for ε a primitive $2n$th root of unity. The group has the presentation

$$\left\langle g, \tau \;\middle|\; g^{2n} = \tau^4 = 1,\ g^n = \tau^2,\ \tau g \tau^{-1} = g^{-1} \right\rangle.$$

It is well known that the ring of invariants is generated by

$$u = xy\left(x^{2n} - y^{2n}\right), \quad v = x^{2n} + y^{2n} \quad \text{and} \quad w = (xy)^2$$

where x, y are the coordinates on $\mathbb{C}^2$. See [Rei02] for example. The quotient $X = \mathbb{C}^2/G$ is the hypersurface singularity

$$X\colon (u^2 - v^2 w + 4w^{n+1} = 0) \subset \mathbb{C}^3.$$

This has a unique minimal resolution $Y \to X$ with exceptional locus given by a chain of $\mathbb{P}^1$s shown by the Dynkin diagram D_{n+2}.

Let $\operatorname{Irr} G$ be the set of distinct irreducible representations of G. These are given as follows. If ρ is a 1 dimensional representation of the group then $\rho(g)^2 = 1$ and $\rho(\tau)^2 = \rho(g)^n$. Thus there are four 1 dimensional representations given by

$$L_\pm(g) = 1, \quad L_\pm(\tau) = \pm 1 \quad \text{and} \quad M_\pm(g) = -1, \quad M_\pm(\tau) = \pm i^n. \tag{2.1}$$

Clearly L_+ is the trivial representation.

For $j = 1, \dots, n-1$ define a 2 dimensional representation $V_j \cong \mathbb{C}^2$ as follows.

$$V_j(g) = \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix} \quad \text{and} \quad V_j(\tau) = \begin{pmatrix} 0 & 1 \\ (-1)^j & 0 \end{pmatrix} \tag{2.2}$$

Each of these representations is irreducible and $V_i \ncong V_j$ for $i \neq j$. Since $4n = |G| = \sum_{\rho \in \operatorname{Irr} G} (\deg \rho)^2 = 4 \cdot 1 + (n-1) \cdot 4$ these are all the irreducible representations and the regular representation of BD_{4n} is

$$\mathbb{C}[G] \;\cong\; L_+ \oplus L_- \oplus M_+ \oplus M_- \oplus \bigoplus_{j=1}^{n-1} V_j^{\oplus 2}.$$

2.2 G-graphs

Definition 2.1 A polynomial or pair of polynomials is said to belong to one of the irreducible representations of G if one of the following conditions hold.

- A polynomial $f \in \mathbb{C}[x, y]$ *belongs to* $L_\pm$ if $g(f) = f$ and $\tau(f) = \pm f$.

- A polynomial $f \in \mathbb{C}[x,y]$ *belongs to* $M_{\pm}$ if $g(f) = -f$ and $\tau(f) = \pm i^n f$.

- For a polynomial $f \in \mathbb{C}[x,y]$, the pair of polynomials $(f, \tau(f))$ *belong to* V_j if $g(f,\tau(f)) = (\varepsilon^j f, \varepsilon^{-j}\tau(f))$.

In the next section we will be interested in the following polynomials that belong to these representations.

L_+	$1,\ x^2y^2, x^{2n}+y^{2n}$
L_-	$xy,\ x^{2n}-y^{2n}$
M_+	$x^n - i^n y^n,\ xy\left(x^n + i^n y^n\right)$
M_-	$x^n + i^n y^n,\ xy\left(x^n - i^n y^n\right)$
V_j	$\left(x^j, y^j\right),\ \left(x^{j+1}y, -xy^{j+1}\right),$ $\left(y^{2n-j}, (-1)^j x^{2n-j}\right)$

We may sometimes abuse this notation and say $(f,g) \in V_j$ when we mean $(f,-g) \in V_j$. For example we say $(x^{j+1}y, xy^{j+1}) \in V_j$ where really $(x^{j+1}y, -xy^{j+1}) \in V_j$ or $(y^{2n-j}, x^{2n-j}) \in V_j$ where really $(y^{2n-j}, (-1)^j x^{2n-j}) \in V_j$. Notice that $\left(y\left(x^n \pm i^n y^n\right), x\left(x^n \pm i^n y^n\right)\right) \in V_{n-1}$.

Definition 2.2 A subset $\Gamma \subset \mathbb{C}[x,y]$ is called a *G-graph* if it contains precisely one polynomial belonging to each 1 dimensional representation $L_{\pm}$ and $M_{\pm}$ and two pairs of polynomials belonging to each of the irreducible 2 dimensional representations V_i.

An ideal I in $\mathbb{C}[x,y]$ defines a G-cluster if and only if $\mathbb{C}[x,y]/I \cong \mathbb{C}[G]$. Therefore we can choose a basis for the complex vector space $\mathbb{C}[x,y]/I$ which is a G-graph. We note that there will never be a unique choice of G-graph.

Definition 2.3 Let I be an ideal that defines a G-cluster. $\Gamma(I)$ is defined to be the set of G-graphs which form a basis for the complex vector space $\mathbb{C}[x,y]/I$.

Recall that $\mathrm{Soc}(I)$ is defined as the socle of $\mathbb{C}[x,y]/I$ viewed as a module over $\mathbb{C}[x,y]$ localised at the origin.

Theorem 2.4 *Let $G = \mathrm{BD}_{4n}$.*

1. *If an ideal I defines a G-cluster then one of the G-graphs given in the second list in Theorem 2.10 belongs to $\Gamma(I)$.*
2. $\mathrm{Soc}(I)$ *contains at most two irreducible representations of G.*
3. G-Hilb $\mathbb{C}^2$ *is a smooth rational variety.*

To show that G-Hilb $\mathbb{C}^2$ is a smooth rational variety I find a cover of G-Hilb $\mathbb{C}^2$ by smooth affines. In general these affine sets are smooth surfaces in some $\mathbb{C}^n$. However we find that G-Hilb $\mathbb{C}^2$ contains an open set isomorphic to an open subset of $\mathbb{C}^2$ therefore G-Hilb $\mathbb{C}^2$ is rational.

2.3 Preliminaries on G-clusters and G-graphs

Let I be an ideal that defines a G-cluster. Any 1 dimensional representation has multiplicity 1 in $\mathbb{C}[G]$. Thus for any polynomials g and h that belong to the same 1 dimensional representation there exists a polynomial

$$\alpha g - \beta h$$

in I for $\alpha, \beta \in \mathbb{C}$. If $g \in \Gamma$ then $\beta \neq 0$.

Let h be a polynomial belonging to a 1 dimensional representation. Provided that h is not zero in $\mathbb{C}[x,y]/I$ it can be chosen to be basic. In other

words the only reason we cannot choose h in Γ is if $h \in I$. In particular if $h = 1$ then 1 can always be chosen to be in Γ. This means that an ideal that defines a G-cluster always contains polynomials of the form

$$A := x^{2n} + y^{2n} - a \quad Q := x^2y^2 - q$$

for some $a, q \in \mathbb{C}$. For any $r \in \mathbb{Z}$ let

$$q_r := \left(x^2y^2\right)^{r-1} + q\left(x^2y^2\right)^{r-2} + \cdots + q^{r-2}\left(x^2y^2\right) + q^{r-1}. \tag{2.3}$$

Then

$$q_rQ = \left(x^2y^2\right)^r - q^r.$$

Lemma 2.5 *The submodule of $\mathbb{C}[x,y]$ consisting of polynomials belonging to L_- is generated by xy and $x^{2n} - y^{2n}$ as a $\mathbb{C}[x,y]^G$-module.*

Proof. First note that this submodule is contained in $\mathbb{C}[x,y]^A$ where $A \cong \mathbb{Z}_{2n}$ is the normal subgroup generated by g. The ring $\mathbb{C}[x,y]^A$ is generated by x^{2n}, xy, y^{2n}. If $f(x,y) \in L_-$ and $xy \mid f(x,y)$ then $f(x,y) = xyf'(x,y)$ for some $f'(x,y) \in \mathbb{C}[x,y]^G$.

Therefore we may assume that $f(x,y)$ has no monomial term which is divisible by xy and so $f(x,y) = P_1(x^{2n}) + P_2(y^{2n})$ where the P_i are polynomials in one variable. Therefore $\tau(f) = P_1(y^{2n}) + P_2(x^{2n})$. Since $f \in L_-$ it must be that $\tau(f) = -f$ so $P_1 = -P_2$ and $f = P_1(x^{2n}) - P_1(y^{2n})$. This implies that $(x^{2n} - y^{2n}) \mid f$ and so $f = (x^{2n} - y^{2n})\, f'(x,y)$ for some $f'(x,y) \in \mathbb{C}[x,y]^G$. □

Corollary 2.6 *If neither xy nor $x^{2n} - y^{2n}$ can be chosen to be basic then $\mathbb{C}[x,y]/I$ contains no copy of L_-.*

Proof. If neither xy nor $x^{2n} - y^{2n}$ can be chosen to be basic then

$$xy, x^{2n} - y^{2n} \in I.$$

Let f be any polynomial belonging to L_-. By Lemma 2.5

$$f = xyf' + \left(x^{2n} - y^{2n}\right) f''$$

for $f', f'' \in \mathbb{C}[x,y]^G$. Therefore $f \in I$. □

Lemma 2.7 *Let I be an ideal containing the polynomials A and Q. If neither $x^n + i^n y^n$ nor $xy(x^n - i^n y^n)$ is basic then $\mathbb{C}[x,y]/I$ contains no copy of M_-. If neither $x^n - i^n y^n$ nor $xy(x^n + i^n y^n)$ is basic then $\mathbb{C}[x,y]/I$ contains no copy of M_+.*

Proof. We prove the first assertion but the second is analogous. If $x^n + i^n y^n$ cannot be basic then $x^n + i^n y^n \in I$ and so

$$x^2(x^n + i^n y^n) - i^n y^{n-2} Q = x^{n+2} - i^n q y^{n-2} \in I.$$

Therefore only polynomials that have monomial terms of the form $x^\alpha y^\beta$ can be basic where $\alpha < n+2$ and $\beta < 2$ or $\beta < n+2$ and $\alpha < 2$. The only polynomials of this form that belong to M_- are $x^n + i^n y^n$ and $xy(x^n - i^n y^n)$. These polynomials are both contained in I so $\mathbb{C}[x,y]/I$ contains no copy of M_-. □

Suppose (f_1, f_2), (g_1, g_2) and (h_1, h_2) are pairs of polynomials that belong to some irreducible 2 dimensional representation V_i. Since V_i has multiplicity 2 in $\mathbb{C}[G]$ these three pairs cannot be linearly independent in $\mathbb{C}[x,y]/I \cong \mathbb{C}[G]$. Therefore there exist polynomials of the form

$$\alpha f_1 + \beta g_1 + \gamma h_1, \quad \alpha f_2 + \beta g_2 + \gamma h_2$$

in I for $\alpha, \beta, \gamma \in \mathbb{C}$. If (f_1, f_2) and (g_1, g_2) are chosen in a G-graph for I then γ is nonzero.

Lemma 2.8 *Let I be an ideal that defines a G-cluster. Then there exists a G-graph $\Gamma \in \Gamma(I)$ containing the set $\{1, (x,y), \ldots, (x^{n-1}, y^{n-1})\}$.*

PROOF. The G-graph Γ must contain $4n$ elements. If one of the pairs (x^i, y^i) for $0 < i < n$ cannot be chosen to be in Γ then both x^i and y^i belong to I. Therefore $x^j, y^j \in I$ for all $j \geq i$ and so Γ is contained in the set

$$\left\{1, x, \ldots, x^{i-1}, y, \ldots, y^{i-1}, xy, \ldots, x^{i-1}y, xy^2, \ldots, xy^{i-1}\right\}.$$

This has only $4(i-1) < 4n$ elements. Therefore the pairs (x^i, y^i) can always be chosen to be in Γ for $i = 1, \ldots, n-1$. □

We therefore consider only G-graphs containing these monomials.

Suppose $\{1, (x,y), \ldots, (x^{n-1}, y^{n-1})\} \subset \Gamma$ and $(h_1, h_2) \in V_i$. The only reason that this pair cannot be chosen to be in Γ is if there exist polynomials of the form

$$h_1 - \alpha x^i, \quad h_2 - \alpha y^i$$

in I.

Lemma 2.9 *Let I be an ideal containing the polynomials A and Q. Suppose neither of the pairs $\left(x^{k+1}y, xy^{k+1}\right)$ nor $\left(y^{2n-k}, x^{2n-k}\right)$ can be chosen as part of the basis of $\mathbb{C}[x,y]/I$. Then $\mathbb{C}[x,y]/I$ contains at most one copy of V_k.*

PROOF. If neither of the pairs $\left(x^{k+1}y, xy^{k+1}\right)$ nor $\left(y^{2n-k}, x^{2n-k}\right)$ can be chosen as part of the basis of $\mathbb{C}[x,y]/I$ then the four polynomials

$$C := x^{k+1}y - cx^k, \quad D := xy^{k+1} + cy^k,$$
$$E := y^{2n-k} - dx^k, \quad F := x^{2n-k} - (-1)^k dy^k$$

belong to I.

Suppose that there exists a pair $\left(x^\alpha y^\beta, x^\beta y^\alpha\right)$ in $\mathbb{C}[x,y]/I$ which belongs to V_k. We want to show that this pair cannot be linearly independent of

$\left(x^k, y^k\right)$ in $\mathbb{C}\left[x,y\right]/I$. In other words that there exist polynomials of the form

$$x^\alpha y^\beta - \gamma x^k, \quad x^\beta y^\alpha - (-1)^\beta y^k$$

in I.

Suppose $\alpha \geq 2$ and $\beta \geq 2$. The polynomial $Q \in I$ and so $x^{\alpha-2}y^{\beta-2}Q \in I$ and

$$x^{\alpha-2}y^{\beta-2}Q = x^\alpha y^\beta - qx^{\alpha-2}y^{\beta-2} \in I.$$

So $x^\alpha y^\beta$ is equal to a complex multiple of $x^{\alpha-2}y^{\beta-2}$ in $\mathbb{C}\left[x,y\right]/I$ so we may replace $x^\alpha y^\beta$ with $x^{\alpha-2}y^{\beta-2}$ in the basis. Continuing this way we may assume that either $\alpha < 2$ or $\beta < 2$. Assume that $\beta < 2$.

Again if $\alpha > 2n+1$ then the polynomial $A \in I$ can be used to replace $x^\alpha y^\beta$ with a polynomial having monomial terms of the form $x^\gamma y^\delta$ where $\gamma \leq 2n+1$ and $\delta < 2$ or $\gamma < 2$ and $\delta \leq 2n+1$.

More precisely

$$x^{\alpha-2n}y^\beta A = x^\alpha y^\beta + x^{\alpha-2n}y^{\beta+2n} - ax^{\alpha-2n}y^\beta \in I.$$

If $\alpha - 2n \geq 2$ then repeated use of the polynomial Q replaces $x^{\alpha-2n}y^{\beta+2n}$ with a monomial of the form $x^\gamma y^\delta$ where $\gamma < 2$ or $\delta < 2$. Therefore we may assume that $\alpha \leq 2n+1$ and $\beta < 2$ or $\alpha < 2$ and $\beta \leq 2n+1$.

However the only pairs $\left(x^\alpha y^\beta, y^\alpha x^\beta\right)$ in V_k where $\alpha \leq 2n+1$ and $\beta < 2$ or $\alpha < 2$ and $\beta \leq 2n+1$ are

$$\begin{gathered}\left(x^k, y^k\right), \qquad \left(x^{k+1}y, xy^{k+1}\right), \\ \left(y^{2n-k}, x^{2n-k}\right), \quad \left(xy^{2n-k+1}, x^{2n-k+1}y\right), \\ (x^{2n+1}, y^{2n+1}) \in V_1.\end{gathered}$$

Now we are assuming that $\left(x^{k+1}y, xy^{k+1}\right)$ and $\left(y^{2n-k}, x^{2n-k}\right)$ are not basic so the only other possibilities are $\left(xy^{2n-k+1}, x^{2n-k+1}y\right)$ and (x^{2n+1}, y^{2n+1}).

However

$$\left.\begin{aligned} xyE + dC &= xy^{2n-k+1} - cdx^k, \\ xyF + (-1)^i dD &= x^{2n-k+1}y + (-1)^i cdy^k \end{aligned}\right\} \in I$$

so $\left(xy^{2n-k+1}, x^{2n-k+1}y\right)$ cannot be basic. If k=1 then

$$\left.\begin{aligned} x^2F - dC &= x^{2n+1} + cdx \\ y^2E + dD &= y^{2n+1} + cdy \end{aligned}\right\} \in I$$

so (x^{2n+1}, y^{2n+1}) cannot be basic either. Therefore $\mathbb{C}[x,y]/I$ contains at most one copy of V_k. □

2.4 List of possible G-graphs

In this section we construct a list of G-graphs and show that every ideal that defines a G-cluster has a G-graph belonging to this list. We do this systematically by considering each representation and choosing a polynomial or pair of polynomials belonging to this representation.

Let I be an ideal that defines a G-cluster and Γ a G-graph in $\Gamma(I)$. Begin by considering L_+. By Lemma 2.8 we may always assume that $1 \in L_+$ belongs to Γ. Now $\mathbb{C}[x,y]/I$ contains only one copy of the trivial representation so I contains the polynomials

$$A := x^{2n} + y^{2n} - a, \quad Q := x^2y^2 - q.$$

By Lemma 2.8 we may assume that $(x,y), \ldots, (x^{n-1}, y^{n-1}) \in \Gamma$.

2.4.1 Case A

Next we consider L_-. If $xy \in I$ then $xy \notin \Gamma$. By Corollary 2.6 $x^{2n} - y^{2n} \notin I$ therefore take $x^{2n} - y^{2n} \in \Gamma$. Since $xy \in I$ all of the pairs $\left(x^{k+1}y, xy^{k+1}\right)$

belong to I and $xy\,(x^n \pm i^n y^n)$ belongs to I these cannot be in Γ. By Lemma 2.9 there exist no polynomials in I of the form

$$y^{2n-k} - cx^k, \quad x^{2n-k} - (-1)^k cy^k$$

for $1 \leq k \leq n-1$ and so $\left(y^{2n-k}, x^{2n-k}\right) \in \Gamma$. By Lemma 2.9 $x^n + i^n y^n \notin I$ and $x^n - i^n y^n \notin I$ and so we may take these in Γ.

Therefore $\Gamma_A \in \Gamma(I)$ where

$$\Gamma_A = \begin{array}{ccccccc} [\,\cdot\,] & & & & & & \\ y^{2n} & & & & & & \\ \vdots & & & & & & \\ y^2 & & & & & & \\ y & [\,\cdot\,] & & & & & \\ 1 & x & x^2 & \cdots & x^{2n} & [\,\cdot\,]. & \end{array}$$

In this G-graph we take $x^{2n} - y^{2n}$ but not $x^{2n} + y^{2n}$ and take $x^n + i^n y^n$ and $x^n - i^n y^n$ rather than x^n and y^n. Clearly this satisfies the definition of a G-graph.

2.4.2 Case B

From now on we assume that $xy \notin I$ and so we can take $xy \in \Gamma$. Since $(x, y) \in \Gamma$ it remains to consider the other pair in V_1. If

$$x^2 y - cx, \quad xy^2 + cy$$

belong to I then the polynomials

$$x^{k+1} y - cx^k, \quad xy^{k+1} + cy^k$$

belong to I so $\left(x^{k+1} y, xy^{k+1}\right) \notin \Gamma$ for all $k \geq 1$. By Lemma 2.9 the pair $\left(y^{2n-k}, x^{2n-k}\right) \in \Gamma$ for $1 \leq k \leq n-1$.

Also the polynomials

$$xy(x^n \pm i^n y^n) - c(x^n \mp i^n y^n) \in I.$$

If $x^n + i^n y^n$ or $x^n - i^n y^n$ is in I then $xy(x^n - i^n y^n)$ or $xy(x^n + i^n y^n)$ belong to I. By Lemma 2.7 this cannot happen so we may assume $x^n + i^n y^n$ and $x^n - i^n y^n$ can be chosen in Γ.

Therefore $\Gamma_B \in \Gamma(I)$ where

$$\Gamma_B := \begin{array}{ccccccc} [\,\cdot\,] & & & & & & \\ y^{2n-1} & & & & & & \\ \vdots & & & & & & \\ y^2 & [\,\cdot\,] & & & & & \\ y & xy & [\,\cdot\,] & & & & \\ 1 & x & x^2 & \cdots & x^{2n-1} & [\,\cdot\,]. \end{array}$$

Again $x^n + i^n y^n$ and $x^n - i^n y^n$ belong to Γ_B rather than x^n and y^n.

2.4.3 Case $\mathbf{C}_k$

Fix $k \in [2, n-1]$ and assume that k is the smallest integer so that we have the polynomials of the form

$$x^{k+1}y - cx^k, \quad xy^{k+1} + cy^k \in I.$$

Then we can take $(x^{j+1}y, xy^{j+1}) \in \Gamma$ for all $j < k$. Since $\left(x^k, y^k\right) \in \Gamma$ it remains to consider the other pair in V_k. The polynomials

$$x^{i+1}y - cx^i, \quad xy^{i+1} + cy^i \in I$$

for $i \geq k$. Therefore $(x^{i+1}y, xy^{i+1}) \notin \Gamma$ for $i \geq k$ and so we can take $(y^{2n-i}, x^{2n-i}) \in \Gamma$ for $k \leq i < n$. As in Case B $x^n + i^n y^n$ and $x^n - i^n y^n$ do not belong to I so can be taken in Γ.

Thus $\Gamma_{C_k} \in \Gamma(I)$ where

$$\Gamma_{C_k} := \begin{array}{ccccccccc} [\,\cdot\,] & & & & & & & & \\ y^{2n-k} & & & & & & & & \\ \vdots & & & & & & & & \\ y^{k+1} & [\,\cdot\,] & & & & & & & \\ y^k & xy^k & & & & & & & \\ \vdots & \vdots & & & & & & & \\ y^2 & xy^2 & [\,\cdot\,] & & & & & & \\ y & xy & x^2y & \cdots & x^ky & [\,\cdot\,] & & & \\ 1 & x & x^2 & \cdots & x^k & x^{k+1} & \cdots & x^{2n-k} & [\,\cdot\,]\,. \end{array}$$

Again $x^n + i^n y^n$ and $x^n - i^n y^n$ belong to Γ_{C_k} rather than x^n and y^n.

2.4.4 Case D

In each of the previous cases we were assuming that there exists some $j < n$ such that the polynomials

$$x^{j+1}y - cx^j, \quad xy^{j+1} + cy^j$$

belong to I. This implies that

$$xy\,(x^n - i^n y^n) - c(x^n + i^n y^n), \quad xy\,(x^n - i^n y^n) - c(x^n + i^n y^n)$$

belong to I. If $x^n \pm i^n y^n \in I$ then $xy\,(x^n \mp i^n y^n) \in I$ and by Lemma 2.7 $\mathbb{C}\,[x,y]\,/I$ contains no copy of $M_\mp$. For this reason in each of the previous cases we took $x^n \pm i^n y^n \in \Gamma$.

From now on we assume that xy, (x^j, y^j) and $(x^{j+1}y, xy^{j+1})$ belong to Γ for all $1 \leq j \leq n-1$. Consider the representation M_+. If $x^n - i^n y^n \in I$ then by Lemma 2.7 $xy(x^n + i^n y^n) \notin I$ and so $x^n + i^n y^n \notin I$. Therefore these can be

taken in Γ. Since $x^n - i^n y^n \in I$ it follows that $y\,(x^n - i^n y^n)\,, x\,(x^n - i^n y^n) \in I$. Therefore we can choose $(y(x^n + i^n y^n), x(x^n + i^n y^n))$ to be basic instead of $(x^n y, xy^n)$.

In this case $\Gamma_{D_+} \in \Gamma(I)$ where

$$
\begin{array}{cccccccc}
 & [\,\cdot\,] & & & & & & \\
 & y^{n+1} & xy^{n+1} & & & & & \\
 & y^n & xy^n & & & & & \\
\Gamma_{D_+} = & \vdots & \vdots & & & & & \\
 & y^2 & xy^2 & [\,\cdot\,] & & & & \\
 & y & xy & x^2y & \cdots & x^ny & x^{n+1}y & \\
 & 1 & x & x^2 & \cdots & x^n & x^{n+1} & [\,\cdot\,]\,.
\end{array}
$$

This time $x^n + i^n y^n$, $x\,(x^n + i^n y^n)$, $y\,(x^n + i^n y^n)$ and $xy\,(x^n + i^n y^n)$ are in Γ_{D_+} but $x^n - i^n y^n$, $x\,(x^n - i^n y^n)$, $y\,(x^n - i^n y^n)$ and $xy\,(x^n - i^n y^n)$ are not.

Similarly consider the representation M_-. If $x^n + i^n y^n \in I$ then by Lemma 2.7 $xy(x^n - i^n y^n) \notin I$ and so $x^n - i^n y^n \notin I$. Therefore these can be taken in Γ. Again the pair $(x(x^n - i^n y^n), y(x^n - i^n y^n))$ can be chosen to be basic rather than $(x^n y, xy^n)$.

In this case $\Gamma_{D_-} \in \Gamma(I)$ where Γ_{D_-} has the same picture as Γ_{D_+}. This time the polynomials $x^n - i^n y^n$, $x\,(x^n - i^n y^n)$, $y\,(x^n - i^n y^n)$ and $xy\,(x^n - i^n y^n)$ are in Γ_{D_-} but $x^n + i^n y^n$, $x\,(x^n + i^n y^n)$, $y\,(x^n + i^n y^n)$ and $xy\,(x^n + i^n y^n)$ are not.

2.4.5 Case E

Finally we may assume xy, $x^n + i^n y^n$, $x^n - i^n y^n$, (x^j, y^j) and $(x^{j+1}y, xy^{j+1})$ belong to Γ for $1 \leq j \leq n-1$. Then $\Gamma_{E_0} \in \Gamma(I)$ where

$$\Gamma_{E_0} := \begin{array}{ccccccc} [\,\cdot\,] & & & & & & \\ y^n & xy^n & & & & & \\ \vdots & \vdots & & & & & \\ y^2 & xy^2 & [\,\cdot\,] & & & & \\ y & xy & x^2y & \cdots & x^ny & & \\ 1 & x & x^2 & \cdots & x^n & [\,\cdot\,]. \end{array}$$

As in Case D, we may assume either the pair $(y(x^n + i^n y^n), x(x^n + i^n y^n))$ or $(y(x^n - i^n y^n), y(x^n - i^n y^n))$ is basic. If $(x(x^n + i^n y^n), y(x^n + i^n y^n))$ is basic then $\Gamma_{E_+} \in \Gamma(I)$ where

$$\Gamma_{E_+} = \begin{array}{cccccccc} [\,\cdot\,] & & & & & & & \\ y^{n+1} & [\,\cdot\,] & & & & & & \\ y^n & xy^n & & & & & & \\ \vdots & \vdots & & & & & & \\ y^2 & xy^2 & [\,\cdot\,] & & & & & \\ y & xy & x^2y & \cdots & x^ny & [\,\cdot\,] & & \\ 1 & x & x^2 & \cdots & x^n & x^{n+1} & [\,\cdot\,]. \end{array}$$

Here $x^n + i^n y^n$, $x^n - i^n y^n$, $y\,(x^n + i^n y^n)$ and $x(x^n + i^n y^n)$ are in Γ_{E_+} but $y(x^n - i^n y^n)$ and $x(x^n - i^n y^n)$ are not.

If $(x(x^n - i^n y^n), y(x^n - i^n y^n))$ is basic then $\Gamma_{E_-} \in \Gamma(I)$ where Γ_{E_-} has the same picture as Γ_{E_+} but $x^n + i^n y^n$, $x^n - i^n y^n$, $y\,(x^n - i^n y^n)$ and $x(x^n - i^n y^n)$ are in Γ_{E_-} but $y(x^n + i^n y^n)$ and $x(x^n + i^n y^n)$ are not.

2.4.6 Conclusion

Theorem 2.10 *Let I be an ideal which defines a G-cluster then there exist G-graphs Γ_1 and Γ_2 such that*

$$\Gamma_1 \in \left\{\Gamma_A,\ \Gamma_B,\ \Gamma_{C_2},\ \ldots,\ \Gamma_{C_{n-2}},\ \Gamma_{C_{n-1}},\ \Gamma_{D_+},\ \Gamma_{D_-},\ \Gamma_{E_0}\right\}$$
$$\Gamma_2 \in \left\{\Gamma_A,\ \Gamma_B,\ \Gamma_{C_2},\ \ldots,\ \Gamma_{C_{n-2}},\ \Gamma_{D_+},\ \Gamma_{D_-},\ \Gamma_{E_+},\ \Gamma_{E_-}\right\}$$

and $\Gamma_1, \Gamma_2 \in \Gamma(I)$.

Proof. We have shown that any ideal I which defines a G-cluster has $\Gamma \in \Gamma(I)$ where $\Gamma \in \left\{\Gamma_A, \Gamma_B, \Gamma_{C_2}, \ldots, \Gamma_{C_{n-1}}, \Gamma_{D_+}, \Gamma_{D_-}, \Gamma_{E_0}\right\}$. To prove the theorem it remains to show that if $\Gamma_{C_{n-1}} \in \Gamma(I)$ or $\Gamma_{E_0} \in \Gamma(I)$ then $\Gamma_{E_+} \in \Gamma(I)$ or $\Gamma_{E_-} \in \Gamma(I)$.

We prove that if $\Gamma_{C_{n-1}} \in \Gamma(I)$ then either Γ_{E_+} or Γ_{E_-} is in $\Gamma(I)$. The proof of the other case is identical. Let I be an ideal that defines a G-cluster which has $\Gamma_{C_{n-1}} \in \Gamma(I)$. If neither $(y(x^n + i^n y^n), x(x^n + i^n y^n))$ nor $(y(x^n - i^n y^n), x(x^n - i^n y^n))$ can be basic in $\mathbb{C}[x,y]/I$ then there exist polynomials of the form

$$C := y(x^n + i^n y^n) - cx^{n-1}, \quad D := x(x^n + i^n y^n) + i^n c y^{n-1}$$
$$E := y(x^n - i^n y^n) - dx^{n-1}, \quad F := x(x^n - i^n y^n) - i^n d y^{n-1}$$

in I. However

$$C - E = 2i^n y^{n+1} - (c-d)x^{n-1}$$
$$D + F = 2x^{n+1} + i^n (c-d) y^{n-1}$$

Therefore (y^{n+1}, x^{n+1}) cannot form part of a basis for $\mathbb{C}[x,y]/I$ which contradicts the assumption that $\Gamma_{C_{n-1}} \in I(\Gamma)$. □

2.5 Construction of open sets contained in G-Hilb $\mathbb{C}^2$

In this section we take a G-graph given in the previous section and list generators for any ideal I that defines a G-cluster and has this G-graph. This will give an open subset in G-Hilb $\mathbb{C}^2$ of points whose defining ideal I has $\Gamma \in \Gamma(I)$.

2.5.1 Case A

Let I be any ideal that defines a G-cluster such that $\Gamma_A \in \Gamma(I)$. Then I contains the polynomials

$$\begin{aligned} A &:= x^{2n} + y^{2n} - a, & B &:= xy - b\left(x^{2n} - y^{2n}\right), \\ C &:= x^{2n+1} - cx - dy^{2n-1}, & D &:= y^{2n+1} - cy + dx^{2n-1}, \\ E &:= xy^{2n} - ex - fy^{2n-1}, & F &:= x^{2n}y - ey + fx^{2n-1}, \\ & & Q &:= x^2y^2 - q \end{aligned}$$

for some $a, \ldots, f, q \in \mathbb{C}$.

Lemma 2.11 *Using only these polynomials every monomial in $\mathbb{C}[x,y]$ can be written as a linear combination of elements in Γ_A. In other words Γ_A spans $\mathbb{C}[x,y] / \langle A, B, C, D, E, F, Q\rangle$.*

Proof. Consider a monomial $x^\alpha y^\beta$. We want to show that this can be written as a linear combination of elements in Γ_A.

If $\alpha \geq 2$ and $\beta \geq 2$ then as in Lemma 2.9 we can use Q to write

$$x^\alpha y^\beta = q x^{\alpha-2} y^{\beta-2}.$$

Therefore it remains to prove the lemma for monomials of the form x^α, $x^\alpha y$, y^α and xy^β.

We show that x^α can be written as a linear combination of elements in Γ_A. If $\alpha \leq 2n$ then either $x^\alpha \in \Gamma_A$ or $\alpha = n, 2n$. However $x^n = ((x^n + i^n y^n) + (x^n - i^n y^n))/2$ and $(x^n + i^n y^n), (x^n - i^n y^n) \in \Gamma_A$. Now $x^{2n} = ((x^{2n} + y^{2n}) + (x^{2n} - y^{2n}))/2$ and $(x^{2n} - y^{2n}) \in \Gamma_A$ but $(x^{2n} + y^{2n}) \notin \Gamma_A$. The polynomial A allows us to replace $(x^{2n} + y^{2n})$ with $a \cdot 1$. So $x^{2n} = (a \cdot 1 + (x^{2n} - y^{2n}))/2$. Therefore the lemma holds for x^α when $\alpha \leq 2n$.

To prove the lemma holds for x^α when $\alpha > 2n$ we use induction on α. It holds for $\alpha = 2n + 1$ by the polynomial C.

Write $\alpha = \lambda 2n + \mu$ where $\lambda \geq 1$ and $0 \leq \mu \leq 2n - 1$. Assume that

$$x^\alpha = sx^\mu + ty^{2n-\mu}.$$

If $\mu \neq 0, n, 2n$ then $x^\mu, y^{2n-\mu} \in \Gamma_A$. If $\mu = 0$, n or $2n$ then as shown above x^n, y^n x^{2n} and y^{2n} can be written as a linear combinations of elements in Γ_A.

Then

$$x^{\alpha+1} = sx^{\mu+1} + txy^{2n-\mu}. \tag{2.4}$$

If $\mu < 2n - 1$ then $x^{\mu+1} \in \Gamma_A$. If $\mu = 2n - 1$ then as remarked previously $x^{\mu+1}$ can be written as linear combinations of elements in Γ_A. Consider the monomial $xy^{2n-\mu} \notin \Gamma_A$. Now

$$y^{2n-\mu-1}B = xy^{2n-\mu} - bx^{2n}y^{2n-\mu-1} + by^{4n-\mu-1}$$
$$y^{2n-\mu-1}A = y^{4n-\mu-1} + x^{2n}y^{2n-\mu-1} - ay^{2n-\mu-1}$$

Again $y^{2n-\mu-1} \in \Gamma_A$.

If μ is odd then

$$x^{\mu+1}q_{(2n-\mu-1)/2}Q = x^{2n}y^{2n-\mu-1} - q^{(2n-\mu-1)/2}x^{\mu+1}.$$

If μ is even then

$$y^{2n-\mu-2}F = x^{2n}y^{2n-\mu-1} - ey^{2n-\mu-1} + fx^{2n-1}y^{2n-\mu-2}$$
$$x^{\mu+1}q_{(2n-\mu-2)/2}Q = x^{2n-1}y^{2n-\mu-2} - q^{(2n-\mu-2)/2}x^{\mu+1}.$$

Substituting these values back into Equation 2.4 we have

$$x^{\alpha+1} = s'x^{\mu+1} + t'y^{2n-\mu-1}$$

where $x^{\mu+1}, y^{2n-\mu-1} \in \Gamma_A$ or can be written as linear combinations of elements in Γ_A. Thus the lemma is proved for x^α. The proof in the other three cases is identical. □

Consider a linear combination L of the polynomials $A, \ldots, F, Q$ with coefficients in $\mathbb{C}[x, y, a, b, c, d, e, f, q]$ which is equal to an expression $\sum a_i f_i$, where $f_i \in \Gamma_A$ and $a_i \in \mathbb{C}[a, b, c, d, e, f, q]$. Because $A, \ldots, F, Q$ belong to I it follows that $\sum a_i f_i \in I$. The elements of Γ_A are linearly independent in $\mathbb{C}[x, y]/I$ so we know that each of the coefficients a_i equals zero. Therefore $L = 0$ and the relation L is a *syzygy* between the generators. The syzygies correspond to the fact that the parameters $a, \ldots, f, q$ are not independent.

Lemma 2.12 *If Γ_A is linearly independent in $\mathbb{C}[x, y]/\langle A, B, C, D, E, F, Q\rangle$ then the parameters $a, \ldots, f, q$ must satisfy the following relations.*

$$e = a - c,\ f = -d,\ q = abd,\ d = 2bc - ab,\ c = a + 2bfq^{n-1} \tag{2.5}$$

Proof. Consider the relation

$$xyB - Q = -bxy\left(x^{2n} - y^{2n}\right) + q \tag{2.6}$$

in I. Now $xy\left(x^{2n} - y^{2n}\right) \in L_+$ and 1 is our element in Γ_A belonging to the representation L_+ so the polynomial $xy\left(x^{2n} - y^{2n}\right)$ can be written as a multiple of 1. Since

$$yC - xD = xy\left(x^{2n} - y^{2n}\right) - d\left(y^{2n} + x^{2n}\right)$$

and the polynomial A writes $(x^{2n}+y^{2n})$ as a multiple of 1 we have

$$xyB - Q + byC - bxD + bdA \;\; = \;\; q - abd \in I.$$

This relation implies that $(q-abd)1 \in I$. Now $1 \notin I$ if Γ_A are linearly independent in $\mathbb{C}[x,y]/I$ so $q = abd$. In fact this syzygy implies that Q is not needed as a generator of the ideal as $Q = xyB + byC - bxD + bdA$. However we continue to include it since it will simplify the other syzygies.

Next consider the relations

$$\begin{aligned} xA - C - E &= (c - a + e)x + (d+f)y^{2n-1} \\ yA - D - F &= (c-a+e)y - (d+f)y^{2n-1} \end{aligned} \tag{2.7}$$

in I. If x and y^{2n-1} are linearly independent in $\mathbb{C}[x,y]/I$ then the coefficient of x and of y^{2n-1} in Equation 2.7 must be zero. Therefore we have the relations $c = a - e$ and $d = -f$. Again these syzygies imply that we do not need the generators E and F since $E = xA - C$ and $F = yA - D$.

Again

$$xyA - yC - xD = -xya + 2cxy - d(x^{2n} - y^{2n})$$

belongs to I. Now $x^{2n} - y^{2n} \in L_-$ belongs to Γ_A but $xy \notin \Gamma_A$. The polynomial B allows xy to be replaced with $x^{2n} - y^{2n}$ and so

$$xyA - yC - xD + (a-2c)B = (-d - ab + 2bc)(x^{2n} - y^{2n}).$$

Again since $x^{2n} - y^{2n} \notin I$ it must be that $d = -ab + 2bc$.

Finally

$$xA - y^{2n-1}B - C = -ax + bx^{2n}y^{2n-1} - by^{4n-1} + cx + dy^{2n-1}.$$

Now

$$\begin{aligned} y^{2n-1}A &= x^{2n}y^{2n-1} + y^{4n-1} - ay^{2n-1}, \\ y^{2n-2}F &= x^{2n}y^{2n-1} - ey^{2n-1} + fx^{2n-1}y^{2n-2}, \\ xq_{n-1}Q &= x^{2n-1}y^{2n-2} - q^{n-1}x. \end{aligned}$$

Therefore

$$xA - y^{2n-1}B - C + by^{2n-1}A - 2by^{2n-2}F + 2bfq_{n-1}xQ = \\ \left(-a + c - 2bfq^{n-1}\right) x + (d - ab + 2be)\, y^{2n-1}.$$

In the same way the final syzygy we will consider

$$yA - x^{2n-1}B - D - bx^{2n-1}A + 2bx^{2n-2}E + 2bfyq_{n-1}Q = \\ \left(-a + c - 2bfq^{n-1}\right) y + (-d + ab - 2be)\, x^{2n-1}.$$

These syzygies can be written in a matrix M as follows.

$$\begin{pmatrix} bd & xy & by & -bx & 0 & 0 & -1 \\ x & 0 & -1 & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 & -1 & 0 \\ xy & a-2c & -y & -x & 0 & 0 & 0 \\ x + by^{2n-1} & -y^{2n-1} & -1 & 0 & 0 & -2by^{2n-2} & 2bfq_{n-1}x \\ y - bx^{2n-1} & -x^{2n-1} & 0 & -1 & 2bx^{2n-2} & 0 & 2bfq_{n-1}y \end{pmatrix}$$

This is called the matrix of syzygies and means that

$$M \begin{pmatrix} A & B & C & D & E & F & Q \end{pmatrix}^t = 0.$$

Together these syzygies correspond to the following relations between the parameters.

$$e = a - c,\ f = -d,\ q = abd,\ d = 2bc - ab,\ c = a + 2bfq^{n-1}$$

□

Proposition 2.13 *Let I be an ideal generated by $A, \ldots, F, Q$ where the parameters $a, \ldots, f, q$ are subject to the relations given in 2.5. Then Γ_A forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

PROOF. By Lemma 2.11 Γ_A spans $\mathbb{C}[x,y]/I$. It remains to show that Γ_A is linearly independent in $\mathbb{C}[x,y]/I$. Now suppose that Γ_A is not linearly independent. Then there exists a G-equivariant linear combination of the elements in Γ_A which belongs to I.

Suppose $x^{2n}-y^{2n}$ is not basic then $x^{2n}-y^{2n}, xy \in I$. Now $xE \in I$ which implies $ex^2 \in I$. If $e=0$ then $a=0$ or $b=0$ by Relations 2.5. If $a=0$ then

$$x^{2n}+y^{2n}, \quad xy-b(x^{2n}-y^{2n}), \quad x^{2n+1}, \quad y^{2n+1}$$

are generators for I. If $b=0$ then

$$x^{2n}+y^{2n}-a, \quad xy$$

are generators for I. In each case only one term in the generators divides $x^{2n}-y^{2n}$ and so these generators do not imply that $x^{2n}-y^{2n} \in I$. Therefore we can take $x^{2n}-y^{2n}$ in the basis.

Otherwise $x^2 \in I$ which implies $x^{2n}+y^{2n} \in I$ and so $a \in I$. We have already shown that if $a=0$ then $x^{2n}-y^{2n} \notin I$. If $a \neq 0$ then $1 \in I$. Again only one term in the generators A, B, C, D divides 1 and so the generators do not imply that $1 \in I$.

Suppose that (y^{2n-i}, x^{2n-i}) and (x^i, y^i) are not linearly independent. Now $y^{2n-i}, x^{2n-i}, x^i, y^i \notin I$ because $x^{2n}-y^{2n} \notin I$. Therefore there exist polynomials of the form

$$y^{2n-i}-\alpha x^i, \quad x^{2n-i}-(-1)^i \alpha y^i$$

in I where $\alpha \neq 0$. If i is even then these polynomials imply that $x^{2n}-y^{2n} \in I$. We have already shown this is not possible. If i is odd then as in the proof of Lemma 2.11

$$xy^{2n-i}-\alpha x^{i+1} = -aby^{2n-i-1} - \left(\alpha - 2bq^{(2n-i-1)/2}\right) x^{i+1}.$$

However we have shown that there exists no such relation when i is even so $ab = \alpha - 2bq^{(2n-i-1)/2} = 0$. If $ab = 0$ then $q = 0$ so $ab = \alpha - 2bq^{(2n-i-1)/2} = 0$ is impossible since $\alpha \neq 0$.

Therefore Γ_A forms a basis for $\mathbb{C}[x,y]/I$. □

Therefore this open subset of G-Hilb $\mathbb{C}^2$ is parametrised by a, b, c subject to the equation $a - c + 2a^{n-1}b^{2n}(2c-a)^n$. In other words this open subset of G-Hilb $\mathbb{C}^2$ is isomorphic to the smooth surface in $\mathbb{C}^3_{a,b,c}$ given by the equation

$$a - c + 2a^{n-1}b^{2n}(2c-a)^n.$$

Proposition 2.14 $\mathrm{Soc}(I)$ *contains at most the submodule of* $\mathbb{C}[x,y]/I$ *generated by* $\langle x^{2n} - y^{2n}\rangle \cong L_-$. *Moreover* $\langle x^{2n} - y^{2n}\rangle \subset \mathrm{Soc}(I)$ *if and only if* $a = 0$.

Proof. Since $\mathrm{Soc}(I)$ is isomorphic to a direct sum of G-representations we only need to check if submodules of $\mathbb{C}[x,y]/I$ isomorphic to any of the irreducible representations belong to $\mathrm{Soc}(I)$. Clearly the submodules generated by $1, x^n + i^n y^n, x^n - i^n y^n$ do not belong to $\mathrm{Soc}(I)$. Consider the submodule generated by $x^j + \lambda y^{2n-j}, y^j + (-1)^j \lambda x^{2n-j}$. For this to be contained in $\mathrm{Soc}(I)$ it must be that $x(x^j + \lambda y^{2n-j}) = 0$. Now

$$x(x^j + \lambda y^{2n-j}) = x^{j+1} + \lambda x y^{2n-j}.$$

If j is odd then as in the proof of Lemma 2.11

$$y^{2n-j-1}B - by^{2n-j-1}A + 2bx^{j+1}q_{(2n-j-1)/2}Q$$
$$= xy^{2n-j} - 2bq^{(2n-j-1)/2}x^{j+1} + aby^{2n-j-1}$$

and if j is even then

$$y^{2n-j-1}B - by^{2n-j-1}A + 2by^{2n-j-2}F - 2bfx^{j+1}q_{(2n-j-2)/2}Q$$
$$= xy^{2n-j} + 2bfq^{(2n-j-2)/2}x^{j+1} + (-2be + ab)y^{2n-j-1}.$$

Therefore if j is odd

$$x(x^j + \lambda y^{2n-j}) = x^{j+1} + 2bq^{(2n-j-1)/2}\lambda x^{j+1} - ab\lambda y^{2n-j-1}.$$

This is zero if and only if $\lambda ab = 0$ and $1 + 2bq^{(2n-j-1)/2}\lambda = 0$. However if $ab = 0$ then $q = 0$ and so $1 + 2bq^{(2n-j-1)/2} \neq 0$.

Similarly if j is even

$$x(x^j + \lambda y^{2n-j}) = x^{j+1} - 2bfq^{(2n-j-2)/2}\lambda x^{j+1} - (-2be + ab)\lambda y^{2n-j-1}.$$

For this to be zero $1 - 2bfq^{(2n-j-2)/2} = 0$ and $d = (-2be + ab) = 0$. If $d = 0$ then $q = 0$ and $1 + 2bq^{(2n-j-1)/2} \neq 0$. Therefore

$$\left\langle x^j + \lambda y^{2n-j}, y^j + (-1)^j \lambda x^{2n-j} \right\rangle \not\subset \mathrm{Soc}(I).$$

It only remains to see when $x^{2n} - y^{2n} \in \mathrm{Soc}(I)$. Using the relations C and E

$$x(x^{2n} - y^{2n}) = (c + e)x + (d + f)y^{2n-1}.$$

From Equation 2.5 $d + f = 0$ and $a = c + e$ and so $x^{2n} - y^{2n} \in \mathrm{Soc}(I)$ if and only if $a = 0$. □

Notice that $u = xy(x^{2n} - y^{2n}) = d(x^{2n} + y^{2n}) = ad$, $v = x^{2n} + y^{2n} = a$ and $w = x^2y^2 = abd$. The curve $a = 0$ defines the exceptional locus $u = v = w = 0$ in this affine subset. If $a = 0$ then $c = 0$ so b is the coordinate along this exceptional curve. Setting $a = 0$ and letting $b \to \infty$ we obtain the ideal which is described in the next section when the new parameters are all zero.

2.5.2 Case B

Let I be an ideal that defines a G-cluster such that $\Gamma_B \in \Gamma(I)$. Then I contains the polynomials

$$\begin{aligned} A &:= x^{2n} + y^{2n} - a, & B &:= x^{2n} - y^{2n} - bxy, \\ C &:= xy^2 - cx^{2n-1} - dy, & D &:= yx^2 - cy^{2n-1} + dx, \\ E &:= xy^3 - ex^{2n-2} - fy^2, & F &:= x^3y + ey^{2n-2} + fx^2 \end{aligned}$$
$$Q := x^2y^2 - q.$$

As in Case A we can show that using only these polynomials every monomial in $\mathbb{C}[x, y]$ can be written as a linear combination of elements in Γ_B.

To find the relations between the parameters we use syzygies. First consider

$$xC + yD + cA - 2Q \;=\; -ac + 2q \in I.$$

Again $1 \notin I$ so $q = ac/2$ and Q is not required as a generator.

Other syzygies are

$$-xC + yD + cB = -(cb + 2d)xy,$$

$$\begin{aligned} &yC - E + cx^{2n-4}F - cey^2q_{n-2}Q \\ &\qquad = (f - d - ceq^{n-2})y^2 + (e + cf)x^{2n-2}, \end{aligned}$$

$$\begin{aligned} &xD - F + cy^{2n-4}E + ey^2q_{n-2}Q \\ &\qquad = -(f - d - ceq^{n-2})x^2 - (e + cf)y^{2n-2}, \end{aligned}$$

$$\begin{aligned} &yA + yB - 2x^{2n-2}D - 2cyq_{n-1}Q + bC \\ &\qquad = \left(-a + 2cq^{n-1} - bd\right)y - (bc - 2d)x^{2n-1}, \end{aligned}$$

$$xA - xB - 2y^{2n-2}C - bD - 2cxq_{n-1}Q$$
$$= (-a - bd + 2cq^{n-1})x + (2d + bc)y^{2n-1}.$$

The matrix of syzygies is

$$\begin{pmatrix} c & 0 & x & y & 0 & 0 & -2 \\ 0 & c & x & -y & 0 & 0 & 0 \\ 0 & 0 & y & 0 & -1 & cx^{2n-4} & -cey^2q_{n-2} \\ 0 & 0 & 0 & x & cy^{2n-4} & -1 & cex^2q_{n-2} \\ y & y & b & -2x^{2n-2} & 0 & 0 & -2cq_{n-1}y \\ x & -x & -2y^{2n-2} & -b & 0 & 0 & -2cq_{n-1}x \end{pmatrix}$$

and these correspond to the following relations between the parameters

$$\begin{aligned} 2q = ac, \quad 2d = -bc, \quad e = -cf, \\ f - d - ceq^{n-2} = 0, \quad 0 = a + bd - 2cq^{n-1}. \end{aligned} \tag{2.8}$$

Proposition 2.15 *Let I be an ideal generated by $A, \ldots, F, Q$ where the parameters $a, \ldots, f, q$ are subject to the relations given in 2.8. Then Γ_B forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

Proof. The proof that Γ_B spans $\mathbb{C}[x,y]/I$ is the same as the proof of Lemma 2.11. That Γ_B is linearly independent in $\mathbb{C}[x,y]/I$ is proved using the techniques in the proofs of Lemma 2.11 and Proposition 2.13.

Suppose that (x,y) and (y^{2n-1}, x^{2n-1}) are not linearly independent in $\mathbb{C}[x,y]/I$. Then there exist polynomials of the form

$$\lambda x - \mu y^{2n-1}, \quad \lambda y + \mu x^{2n-1}$$

in I. Therefore $\mu(x^{2n} - y^{2n}) \in I$. If $\mu = 0$ then $x, y \in I$ and so $a = 0$ or $1 \in I$. As in Case A the generators $A, \ldots, F, Q$ do not imply $x, y \in I$ when $a = 0$ and do not imply $1 \in I$ in any case.

Otherwise $x^{2n} - y^{2n} \in I$ and so $xy \in I$ or $b = 0$. As in case A the generators $A, \ldots, F, Q$ do not imply a relation between x and y^{2n-1} when $b = 0$. If $xy \in I$ then relations C and D imply that $dx^2, dy^2 \in I$ and so either $d = 0$ or $x^2, y^2 \in I$. As above we can show that neither case is possible.

Finally we show by induction that (x^i, y^i) and (y^{2n-i}, x^{2n-i}) are linearly independent in $\mathbb{C}[x,y]/I$ for $1 \leq i \leq n$. We have shown that it is true when $i = 1$. Assume that it is not true for some $1 < i < n$. Then there exist polynomials of the form

$$y^{2n-i} - \alpha x^i, \quad x^{2n-i} - (-1)^i \alpha y^i$$

in I and $\alpha \neq 0$ by induction. These polynomials imply that

$$y^{2n-i+1} - \alpha x^i y \in I. \tag{2.9}$$

If i is even then

$$x^i y = c x^{i-2} y^{2n-1} - d x^{i-1} = c q^{(i-2)/2} y^{2n-i+1} - d x^{i-1}.$$

Substituting this back into Equation 2.9 implies

$$\alpha d x^{i-1} + (-\alpha c q^{(i-2)/2} + 1) y^{2n-i+1} \in I.$$

By induction there exists no relation between x^{i-1} and y^{2n-i+1} in I so $\alpha d = \alpha c q^{(i-2)/2} - 1 = 0$. This implies that $d = 0$. However if $d = 0$ then we can see that the ideal contains no polynomials of the form $y^{2n-i} - \alpha x^i$ and $x^{2n-i} - (-1)^i \alpha y^i$.

If i is odd then

$$x^i y = e x^{i-3} y^{2n-2} - f x^{i-1} = e q^{(i-3)/2} y^{2n-i+1} - f x^{i-1}.$$

As above we can show that this leads to a contradiction. □

Therefore these parameters define a smooth surface in $\mathbb{C}^4_{a,b,e,f}$ with equations

$$a - b^2c/2 - a^{n-1}c^n/2^{n-2} = f + bc/2 + a^{n-2}c^n f/2^{n-1} = 0$$

Proposition 2.16 $\mathrm{Soc}(I)$ *contains at most the submodules of* $\mathbb{C}[x,y]/I$ *generated by* $x^{2n} - y^{2n}$ *and* $\langle y^{2n-1}, x^{2n-1}\rangle$. *Moreover* $\langle x^{2n} - y^{2n}\rangle \cong L_- \subset \mathrm{Soc}(I)$ *if and only if* $c = 0$ *and* $\langle y^{2n-1}, x^{2n-1}\rangle \subset V_1 \in \mathrm{Soc}(I)$ *if and only if* $a = b = 0$.

Proof. The proof is similar to the proof of Proposition 2.14 □

Notice that $a = b = 0$ is a Cartier divisor since on the open set given by $c \neq 0$ it has the equation $a = 0$ and on the open set given by $1 - a^{n-2}c^n/2^{n-2} \neq 0$ it is given by the equation $b = 0$.

Again $u = abc/2$, $v = a$ and $w = ac/2$. Therefore $a = 0$ defines the exceptional locus. This decomposes into two prime divisors $a = b = 0$ and $c = 0$. If $c = 0$ then b is the coordinate along this exceptional curve. Setting $c = 0$ and letting $b \to \infty$ we obtain the ideal that was described in the previous section when the parameters were zero. If $a = b = 0$ then c is the coordinate along this exceptional curve. Setting $a = b = 0$ and letting $c \to \infty$ then we obtain the ideal which is described in the next section when $k = 2$ and the new parameters are all zero.

2.5.3 Case $\mathbf{C}_k$

Fix $k \in [2, n-2]$. Let I be an ideal that defines a G-cluster such that $\Gamma_{C_k} \in \Gamma(I)$. Then I contains the polynomials

$$\begin{aligned}
B &:= x^{2n-k+1} - bxy^k - cy^{k-1},\\
C &:= y^{2n-k+1} - (-1)^k bx^k y - (-1)^{j-1} cx^{k-1},\\
D &:= xy^{k+1} - dx^{2n-k} - ey^k,\\
E &:= x^{k+1}y - (-1)^{k+1} dy^{2n-k} + ex^k,\\
F &:= xy^{k+2} - fx^{2n-k-1} - gy^{k+1},\\
G &:= x^{k+2}y - (-1)^k fy^{2n-k-1} + gx^{k+1},\\
Q &:= x^2y^2 - q.
\end{aligned}$$

Again we can show that using only these polynomial every monomial in $\mathbb{C}[x, y]$ can be written as a linear combination of elements in Γ_{C_k}.

As in Case A we can construct a matrix of syzygies and so generate the relations between these parameters. The matrix of syzygies is

$$\begin{pmatrix}
d & 0 & x & 0 & 0 & 0 & -y^{k-1}\\
0 & (-1)^k d & 0 & y & 0 & 0 & -x^k\\
0 & 0 & 0 & -x & (-1)^k dy^{2n-2k-2} & 1 & (-1)^k df q_{n-i-1} x^{k+1}\\
0 & 0 & y & 0 & -1 & dx^{2n-2i-2} & (-1)^k df q_{n-i-1} y^{k+1}\\
y & 0 & b & -x^{2n-2k} & 0 & 0 & (-1)^k dq_{n-k} y^k\\
0 & x & -y^{2n-2k} & (-1)^k b & 0 & 0 & -dq_{n-k} x^k
\end{pmatrix}$$

and so the relations are

$$\begin{gathered}
e = -bd, \quad q = cd, \quad f = -dg,\\
g = e + (-1)^k df q^{n-k}, \quad 0 = (-1)^k dq^{n-k} + c + be.
\end{gathered} \tag{2.10}$$

Proposition 2.17 *Let I be an ideal generated by $B, \ldots, G, Q$ where the parameters $b, \ldots, g, q$ are subject to the relations given in 2.10. Then Γ_{C_k} forms a basis for $\mathbb{C}[x, y]/I$ and so I defines a G-cluster.*

PROOF. This is proved using the same techniques as the proof of Proposition 2.15. □

Again we have that this affine open set is isomorphic to a smooth surface in $\mathbb{C}^4_{b,c,d,g}$ given by the equations

$$c - b^2 d + (-1)^k d^{n-k+1} c^{n-k} = g + bd + (-1)^k g c^{n-k} d^{n-k+2} = 0.$$

Proposition 2.18 $\mathrm{Soc}(I)$ *contains at most the submodules* $\left\langle x^k y, xy^k \right\rangle$ *and* $\left\langle y^{2n-k}, x^{2n-k} \right\rangle$. *Moreover* $\left\langle x^k y, xy^k \right\rangle \cong V_{k-1} \subset \mathrm{Soc}(I)$ *if and only if* $d = 0$ *and* $\left\langle y^{2n-k}, x^{2n-k} \right\rangle \cong V_k \subset \mathrm{Soc}(I)$ *if and only if* $b = c = 0$.

PROOF. The proof is similar to the proof of Proposition 2.14. □

If k is odd then $u = 2b(cd)^{(k+1)/2}$, $v = 2c(cd)^{(k-1)/2}$ and $w = cd$. If k is even then $u = 2c(cd)^{k/2}$, $v = 2b(cd)^{k/2}$ and $w = cd$. Therefore $cd = 0$ defines the exceptional locus on this affine subset. This decomposes as the two prime divisor $d = 0$ and $b = c = 0$.

If $d = 0$ then b is the coordinate along this exceptional curve. Setting $d = 0$ and $b \to \infty$ gives the gives the ideal that is described in the previous case, Case C_{k-1} or Case B, when all the parameters are zero. If $b = c = 0$ then d is the coordinate along this exceptional curve. Setting $b = c = 0$ and $d \to \infty$ gives the gives the ideal which is described in the next case, Case C_{k+1}, when all the parameters are zero.

2.5.4 Case D

An ideal I which defines a G-cluster and $\Gamma_{D_+} \in \Gamma(I)$, is generated by the polynomials

$$\begin{aligned}
B &:= x^{n+2} - bxy^{n-1} - cy^{n-2}, \\
C &:= y^{n+2} - (-1)^{n+1}byx^{n-1} - (-1)^n cx^{n-2}, \\
D &:= (x^n - i^n y^n) - dxy\,(x^n + i^n y^n)\,, \\
E &:= x\,(x^n - i^n y^n) - ey^{n-1} - fx\,(x^n + i^n y^n)\,, \\
F &:= y\,(x^n - i^n y^n) - e(-i)^n x^{n-1} + fy\,(x^n + i^n y^n)\,, \\
Q &:= x^2y^2 - q
\end{aligned}$$

and the syzygy matrix is

$$\begin{pmatrix}
-dy & 0 & x & \alpha & 0 & i^n dy^{n-1} \\
0 & i^n dx & y & 0 & \alpha & dx^{n-1} \\
-1 + dxy & 0 & x^2 & 0 & 0 & (bd + i^n)y^{n-2} + i^n dxy^{n-1} \\
0 & i^n\,(1 + xyd) & y^2 & 0 & 0 & -((-i)^n bd + 1)x^{n-2} + dyx^{n-1} \\
0 & 0 & -(-i)^n e & y & x & 0
\end{pmatrix}$$

where $\alpha = -1 - (-i)^n bd/2$. This corresponds to the relations

$$\begin{aligned}
c &= i^n q + bdq, & f &= (-i)^n ed/2, \\
b &= dc + i^n dq, & 0 &= e - cd + (-i)^n bde/2 - i^n dq
\end{aligned} \tag{2.11}$$

between the parameters.

Proposition 2.19 *Let I be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.11. Then Γ_{D_+} forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

Proof. This is proved using the same techniques as the proof of Proposition 2.15. □

Thus this affine subset is the smooth surface in $\mathbb{C}^4_{b,d,e,q}$ given by the equations

$$b - 2i^n dq - bd^2 q = e - bd^2 q + (-i)^n bde/2 - 2i^n dq = 0.$$

Proposition 2.20 $\mathrm{Soc}(I)$ *contains at most the submodule* $\langle xy(x^n + i^n y^n)\rangle$. *Moreover* $\langle xy(x^n + i^n y^n)\rangle \cong M_+ \subset \mathrm{Soc}(I)$ *if and only if* $q = 0$.

Proof. The proof is similar to the proof of Proposition 2.14. □

If k is odd then $u = 2(i^n + bd)q^{(n+1)/2}$, $v = 2bq^{(n-1)/2}$ and $w = q$. If k is even then $u = 2bq^{n/2}$, $v = 2(i^n + bd)q^{n/2}$ and $w = q$. Therefore $q = 0$ defines the exceptional locus on this affine subset.

If $q = 0$ then d is the coordinate along this exceptional curve. Setting $q = 0$ and $d \to \infty$ gives the ideal that is described in Case E_+ when all the parameters are zero.

An ideal I that defines a G-cluster and $\Gamma_{D_-} \in \Gamma(I)$, is generated by the polynomials

$$\begin{aligned}
B &:= x^{n+2} - bxy^{n-1} - cy^{n-2},\\
C &:= y^{n+2} - (-1)^{n-1} bx^{n-1} y(-1)^n - c(-1)^n x^{n-2},\\
D &:= x^n + i^n y^n - dxy\left(x^n - i^n y^n\right),\\
E &:= x\left(x^n + i^n y^n\right) - ey^{n-1} - fx\left(x^n - i^n y^n\right),\\
F &:= y\left(x^n + i^n y^n\right) + e(-i)^n x^{n-1} + fy\left(x^n - i^n y^n\right),\\
Q &:= x^2 y^2 - q,
\end{aligned}$$

the syzygy matrix is

$$\begin{pmatrix}
dy & 0 & x & \alpha & 0 & i^n dy^{n-1}\\
0 & -i^n dx & y & 0 & \alpha & dx^{n-1}\\
-1 + dxy & 0 & x^2 & 0 & 0 & (bd - i^n)y^{n-2} - i^n dxy^{n-1}\\
0 & -i^n\left(1 + dxy\right) & y^2 & 0 & 0 & ((-i)^n bd - 1)x^{n-2} + dyx^{n-1}\\
0 & 0 & (-i)^n e & y & -x & 0
\end{pmatrix}$$

where $\alpha = -1 + (-i)^n bd/2$. The relations between the parameters are

$$\begin{aligned} c &= -i^n q + bdq, & b &= dc - i^n dq. \\ f &= -(-i)^n de/2 & 0 &= e - dc - (-i)^n bde/2 + i^n dq. \end{aligned} \tag{2.12}$$

Proposition 2.21 *Let I be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.12. Then Γ_{D_-} forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

PROOF. This is proved using the same techniques as the proof of Proposition 2.15. □

Thus this affine subset is the smooth surface in $\mathbb{C}^4_{b,d,e,q}$ given by the equations

$$b + 2i^n dq - bd^2 q = e - bd^2 q - (-i)^n bde/2 + 2i^n dq = 0.$$

Proposition 2.22 $\mathrm{Soc}(I)$ *contains at most the submodule $\langle xy(x^n - i^n y^n)\rangle$. Moreover $\langle xy(x^n - i^n y^n)\rangle \cong M_- \subset \mathrm{Soc}(I)$ if and only if $q = 0$.*

PROOF. The proof is similar to the proof of Proposition 2.14. □

If k is odd then $u = 2(-i^n + bd)q^{(n+1)/2}$, $v = 2b(q)^{(n-1)/2}$ and $w = q$. If k is even then $u = 2b(q)^{n/2}$, $v = 2(-i^n + bd)q^{n/2}$ and $w = q$. Therefore $q = 0$ defines the exceptional locus on this affine subset.

If $q = 0$ then d is the coordinate along this exceptional curve. Setting $q = 0$ and $d \to \infty$ gives the ideal that is described in Case E$_-$ when all the parameters are zero.

2.5.5 Case E

If I is an ideal that defines a G-cluster and $\Gamma_{E_+} \in \Gamma(I)$ then I is generated by the polynomials of the form

$$\begin{aligned}
B &:= x\left(x^n - i^n y^n\right) - by^{n-1} - cx\left(x^n + i^n y^n\right), \\
C &:= y\left(x^n - i^n y^n\right) - (-i)^n bx^{n-1} + cy\left(x^n + i^n y^n\right), \\
D &:= xy\left(x^n + i^n y^n\right) - d\left(x^n - i^n y^n\right), \\
E &:= x^{n+2} - ey^{n-2} - fxy^{n-1}, \\
F &:= y^{n+2} - (-1)^n ex^{n-2} - (-1)^{n-1} x^{n-1} y, \\
Q &:= x^2 y^2 - q.
\end{aligned}$$

The syzygy matrix is

$$\begin{pmatrix}
y & -x & 2c & 0 & 0 & 0 \\
y^2 & xy + 2d & 2y & 0 & 0 & -4x^{n-1} \\
0 & 0 & (-i)^n f & y^2 & -i^n x^2 & -(x^n - i^n y^n) \\
0 & y & 0 & 0 & i^n(1-c) & -(1+c)x^{n-2} \\
x & 0 & 0 & -(1-c) & 0 & i^n(1+c)y^{n-2}
\end{pmatrix}$$

and corresponds to the relations

$$\begin{aligned}
b = 2i^n cd, \qquad & q = cd^2 \\
0 = -e - i^n q + df \quad & 0 = -f + b + cf
\end{aligned} \tag{2.13}$$

between the parameters.

Proposition 2.23 *Let I be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.13. Then Γ_{E_+} forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

Proof. This is proved using the same techniques as the proof of Proposition 2.15. □

This affine open set is isomorphic to a smooth surface in $\mathbb{C}^3_{c,d,f}$ given by the equation

$$-f(1-c)+2i^n cd=0.$$

Notice that on the open set given by $c \neq 1$ then f can be expressed as a regular function of c and d. Therefore G-Hilb $\mathbb{C}^2$ contains the open set given by $c \neq 1$ in $\mathbb{C}^2_{c,d}$. Thus G-Hilb $\mathbb{C}^2$ is rational.

Proposition 2.24 $\mathrm{Soc}(I)$ *contains at most the submodules isomorphic to* V_{n-2}, V_{n-1} *and* M_+. *Moreover* $V_{n-2} \subset \mathrm{Soc}(I)$ *if and only if* $c=1$, $V_{n-1} \subset \mathrm{Soc}(I)$ *if and only if* $d=0$ *and* $M_+ \subset \mathrm{Soc}(I)$ *if and only if* $c=0$.

Proof. The proof that $\mathrm{Soc}(I)$ contains at most the submodules isomorphic to V_{n-2}, V_{n-1} and M_+ is similar to the proof of Proposition 2.14.

If $d=0$ then it follows that $x^{n+2}(1+c)=y^{n+2}(1+c)=0$ and so if $c \neq -1$ the submodule $\langle y\left(x^n+i^n y^n\right), x\left(x^n+i^n y^n\right)\rangle \cong V_{n-1}$ is in $\mathrm{Soc}(I)$. If $c=-1$ then we have the polynomials $2x^{n+1}-by^{n-1}$ and $2y^{n+1}-b(-1)^n x^{n-1}$ in I and when $d=0$ the submodule $\langle yx^n, xy^n\rangle \cong V_{n-1}$ is in $\mathrm{Soc}(I)$. The point $(c,d)=(-1,0)$ is the ideal in Case E_0 given by setting every polynomial that is not in Γ_{E_0} equal to zero. Therefore if $d=0$ then V_{n-1} is in $\mathrm{Soc}(I)$.

If $c=0$ then the submodule $\langle x^n-i^n y^n\rangle \cong M_+$ is in $\mathrm{Soc}(I)$.

If $c=1$ then $d=b=q=0$. Therefore $x^2y^{n-1}=x^{n-1}y^2=xy^n=x^ny=0$ and so $\langle x^{n-1}y, xy^{n-1}\rangle \cong V_{n-2} \in \mathrm{Soc}(I)$. □

If n is even then $u=2c^{n/2}d^n e$, $v=2c^{(n-2)/2}d^{n-2}e$ and $w=cd^2$ and if n is odd $u=2c^{(n-1)/2}d^{n-1}e$, $v=2c^{(n-1)/2}d^{n-1}e$ and $w=cd^2$. Either way the exceptional locus on this affine subset is defined by cd^2. This is the union of the three prime divisors $c=0$, $d=0$ and $c=1$.

If $c=0$ then d is the coordinate along this exceptional curve. Setting $c=0$ and letting $d \to \infty$ gives the ideal in Case D_+ where all the parameters are

zero. If $d = 0$ then c is the coordinate along this exceptional curve. Setting $d = 0$ and letting $c \to \infty$ gives the ideal in Case E$_-$ when all of the parameters are zero. If $c = 1$ then f is the coordinate along this exceptional curve and letting $f \to \infty$ gives the ideal in Case C$_{n-2}$ when all of the parameters are zero.

An ideal I which defines a G-cluster and $\Gamma_{E_-} \in \Gamma(I)$, is generated by the polynomials

$$\begin{aligned}
B &:= x\left(x^n + i^n y^n\right) - by^{n-1} - cx\left(x^n - i^n y^n\right), \\
C &:= y\left(x^n + i^n y^n\right) + (-1)^n bx^{n-1} + cy\left(x^n - i^n y^n\right), \\
D &:= xy\left(x^n - i^n y^n\right) - d\left(x^n + i^n y^n\right), \\
E &:= x^{n+2} - ey^{n-2} - fxy^{n-1}, \\
F &:= y^{n+2} - (-1)^n ex^{n-2} - (-1)^{n-1} x^{n-1} y, \\
Q &:= x^2 y^2 - q.
\end{aligned}$$

The syzygy matrix is

$$\begin{pmatrix}
y & -x & 2c & 0 & 0 & 0 \\
y^2 & xy + 2d & 0 & 0 & 2y & -4x^{n-1} \\
0 & 0 & -(-i)^n f & y^2 & i^n x^2 & -(x^n + i^n y^n) \\
0 & y & 0 & 0 & -i^n(1-c) & (1+c)x^{n-2} \\
x & 0 & 0 & -(1-c) & 0 & -(1+c)i^n y^{n-2}
\end{pmatrix}$$

and corresponds to the relations

$$\begin{aligned}
b &= -2i^n cd, & q &= cd^2 \\
e &= i^n q + fd & 0 &= -f + b + cf.
\end{aligned} \tag{2.14}$$

Proposition 2.25 *Let I be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.14. Then Γ_{E_-} forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

PROOF. This is proved using the same techniques as the proof of Proposition 2.15. □

This affine open set is a smooth surface in $\mathbb{C}^3_{c,d,f}$ given by the equation

$$-f - 2i^n cd + cf = 0.$$

Notice that on the open set given by $c \neq 1$ f can be expressed as a regular function of c and d.

Proposition 2.26 $\mathrm{Soc}(I)$ *contains at most the submodules isomorphic to* V_{n-2}, V_{n-1} *and* M_-. *Moreover* $V_{n-2} \subset \mathrm{Soc}(I)$ *if and only if* $c = 1$, $V_{n-1} \subset \mathrm{Soc}(I)$ *if and only if* $d = 0$ *and* $M_- \subset \mathrm{Soc}(I)$ *if and only if* $c = 0$.

PROOF. The proof is similar to the proof of Proposition 2.24. □

If n is even then $u = 2c^{n/2}d^n e$, $v = 2c^{(n-2)/2}d^{n-2}e$ and $w = cd^2$ and if n is odd $u = 2c^{(n-1)/2}d^{n-1}e$, $v = 2ec^{(n-1)/2}d^{n-1}$ and $w = cd^2$. Either way the exceptional locus is given by $cd^2 = 0$ this is the union of the three prime divisors $c = 0$, $d = 0$ or $c = 1$.

If $c = 0$ then d is the coordinate along this exceptional curve. Setting $c = 0$ and letting $d \to \infty$ gives the ideal in Case D_- where all the parameters are zero. If $d = 0$ then c is the coordinate along this exceptional curve. Setting $d = 0$ and letting $c \to \infty$ gives the ideal in Case E_+ when all of the parameters are zero. If $c = 1$ then f is the coordinate along this exceptional curve and letting $f \to \infty$ gives the ideal in Case C_{n-2} when all of the parameters are zero.

2.5.6 Conclusion

We have constructed an affine cover of G-Hilb $\mathbb{C}^2$ where every open set is a smooth surface in $\mathbb{C}^4$. Two of the open sets contain open subsets of $\mathbb{C}^2$

and so G-Hilb $\mathbb{C}^2$ is rational. For any ideal I that defines a G-cluster $\mathrm{Soc}(I)$ contains at most two irreducible representations of G. Therefore we have proved Parts 2 and 3 of Theorem 2.4. Theorem 2.10 proves Part 1 and so Theorem 2.4 is proved.

Chapter 3

Explicit construction of G-Hilb $\mathbb{C}^3$ for trihedral groups

3.1 Classification of trihedral groups

Recall that a *trihedral group* is a finite subgroup of $\mathrm{SL}(3,\mathbb{C})$ of the form $G = A \rtimes \mathbb{Z}_3$, where A is a diagonal Abelian subgroup and $\mathbb{Z}_3$ is generated by

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We begin by classifying the possibilities for A.

Proposition 3.1 *If A is finite Abelian subgroup of $\mathrm{SL}(3,\mathbb{C})$. Then A can be written in the form $\mathbb{Z}_r \times \mathbb{Z}_R$ where $r \mid R$.*

Proof. Suppose A contains a subgroup $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{a_3}$. If a_i and a_j are coprime then $\mathbb{Z}_{a_i} \times \mathbb{Z}_{a_j} = \mathbb{Z}_{a_i a_j}$. So assume $\mathrm{hcf}(a_i, a_j) > 1$ for each pair i, j and write $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{a_3} \cong \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c$ where $a = \mathrm{lcm}(a_1, a_2)$, $b' = \mathrm{hcf}(a_1, a_2)$, $b = \mathrm{lcm}(b', a_3)$ and $c = \mathrm{hcf}(b', a_3)$. If $c = 1$ then $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{a_3} \cong$

$\mathbb{Z}_{b_1} \times \mathbb{Z}_{b_2}$ where $b_1 = \text{lcm}(a, b)$ and $b_2 = \text{hcf}(a, b)$. Otherwise A has a nontrivial subgroup of the form $\mathbb{Z}_c \times \mathbb{Z}_c \times \mathbb{Z}_c$.

Now compare the number of elements in A of order exactly c with the number of elements in the subset $D \subset \text{SL}(3, \mathbb{C})$ consisting of all diagonal elements of $\text{SL}(3, \mathbb{C})$ having order exactly c.

An element $g \in \text{SL}(3, \mathbb{C})$ belongs to D if $g = \frac{1}{c}(\alpha, \beta, \gamma)$ where

$$\text{hcf}(\alpha, \beta, c) = 1 \tag{3.1}$$

and $\gamma = \overline{c - \alpha - \beta}$.

Let $g \in \mathbb{Z}_a$, respectively $h \in \mathbb{Z}_b$, be a generator of a subgroup of $\mathbb{Z}_a$, respectively $\mathbb{Z}_b$, of order c and let $k \in \mathbb{Z}_c$ be a generator of the group. An element of A is of order c if it is of the form $g^p h^q k^s$ where

$$\text{hcf}(p, q, s, c) = 1, \tag{3.2}$$

Every element $\frac{1}{c}(\alpha, \beta, \gamma) \in D$ gives a pair (α, β) satisfying Equation 3.1. Every pair (α, β) satisfying Equation 3.1 gives c distinct triples (α, β, i) for $i = 0, \ldots, r-1$ satisfying Equation 3.2 which correspond to c elements $g^\alpha h^\beta k^i \in \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c$. Hence there are not enough elements in $\text{SL}(3, \mathbb{C})$ for A to have a subgroup of the form $\mathbb{Z}_c \times \mathbb{Z}_c \times \mathbb{Z}_c$. Therefore $\text{hcf}(a_i, a_j) = 1$ for at least one pair (a_i, a_j) and $A = \mathbb{Z}_{a_i a_j} \times \mathbb{Z}_{a_k}$. □

Proposition 3.2 *Let $A = \mathbb{Z}_R \times \mathbb{Z}_r$ be a finite Abelian subgroup of $\text{SL}(3, \mathbb{C})$ such that $A \ltimes \langle \tau \rangle$ is a trihedral group. Then A has a subgroup of the form $\mathbb{Z}_r \times \mathbb{Z}_r$ which can be generated by $\frac{1}{r}(0, 1, -1)$ and $\frac{1}{r}(1, -1, 0)$. It maybe that $r = 1$.*

Proof. By the previous proposition we may assume $r \mid R$. Thus A has a subgroup of the form $B = \mathbb{Z}_r \times \mathbb{Z}_r$. Let $g_1 = \frac{1}{r}(a, b, c)$ and $g_2 = \frac{1}{r}(d, e, f)$

be generators of the two groups $\mathbb{Z}_r$, and show that $h_1 = \frac{1}{r}(0,1,-1)$ and $h_2 = \frac{1}{r}(1,-1,0) \in \mathbb{Z}_r \times \mathbb{Z}_r$.

Claim 3.3 shows that $ae - bd$ and r are coprime and so $ae - bd$ has a multiplicative inverse modulo r. Hence

$$\begin{aligned} \lambda a - \mu d &\equiv 1 \pmod{r} \\ \lambda b - \mu e &\equiv r-1 \pmod{r} \\ \lambda c - \mu f &\equiv 0 \pmod{r} \end{aligned}$$

has a solution. The final equation is implied by the first two. Similarly there is a solution to

$$\begin{aligned} \lambda a - \mu d &\equiv 0 \pmod{r} \\ \lambda b - \mu e &\equiv 1 \pmod{r} \\ \lambda c - \mu f &\equiv r-1 \pmod{r}. \end{aligned}$$

Therefore $\langle h_1, h_2 \rangle \subset \langle g_1, g_2 \rangle$. However both groups are of order r^2 so must be equal. □

Claim 3.3 *If $\left\langle \frac{1}{r}(a,b,c) \right\rangle = B_1$ and $\left\langle \frac{1}{r}(d,e,f) \right\rangle = B_2$ such that $B_1 \times B_2 \cong \mathbb{Z}_r \times \mathbb{Z}_r$ then $ae - bd$ and r are coprime.*

Proof. Let $g_1 = \frac{1}{r}(a,b,c)$ and $g_2 = \frac{1}{r}(d,e,f)$ then the only solution to $g_1^\lambda = g_2^\mu$ is $r \mid \lambda$ and $r \mid \mu$ because $B_1 \cap B_2 = 1$. Thus the only solution to the following set of equations is $\lambda \equiv \mu \equiv 0 \pmod{r}$.

$$\begin{aligned} \lambda a - \mu d &\equiv 0 \pmod{r} \\ \lambda b - \mu e &\equiv 0 \pmod{r} \\ \lambda c - \mu f &\equiv 0 \pmod{r} \end{aligned} \tag{3.3}$$

If $\mathrm{hcf}(ae - bd, r) = s > 1$ then there exist ξ and η such that $\xi s = ad - be$ and $\eta s = r$. This implies that $\lambda = \eta d$ and $\mu = \eta a$ is a solution to Equation 3.3 and so is $\lambda = \eta e$ and $\mu = \eta b$. Thus r divides each of $\eta a, \eta b, \eta e$ and ηd and also ηc and ηf. If $r \mid \eta$ then $r = \eta$ which contradicts the assumption that $s > 1$. So $r = pq$ where $p \mid \eta$ and q divides a, b and c thus $\frac{1}{r}(a, b, c) = \frac{1}{p}(a/q, b/q, c/q)$ which contradicts the assumption that $\frac{1}{r}(a, b, c)$ generates a group of order r. □

Proposition 3.4 *Let G be a trihedral group with a cyclic normal subgroup $\mathbb{Z}_s$. Then $\mathbb{Z}_s$ is generated by $\frac{1}{s}(1, k, \overline{k^2})$ where $\overline{k^2}$ means take the smallest residue of $k^2 \pmod s$ and $s \mid 1 + k + k^2$.*

Proof. Let $\frac{1}{s}(a, b, c)$ be the generator of $\mathbb{Z}_s$. First show that a, b and c are coprime to s. Now $\mathbb{Z}_s \lhd G$ so $\frac{1}{s}(b, c, a), \frac{1}{s}(c, a, b) \in \mathbb{Z}_s$. That is there exists k such that $ak \equiv b,\ bk \equiv c$ and $ck \equiv a \pmod s$.

Now suppose that $\mathrm{hcf}(a, s) = t > 1$. Then $t \mid ak = b + \mu s$ so $t \mid b$ and similarly $t \mid c$. Therefore $\frac{1}{s}(a, b, c) = \frac{1}{s/t}(a/t, b/t, c/t)$ and the group is of order s/t. So a, b and c must be coprime to s and $\lambda a \equiv 1 \pmod s$ has a solution. Therefore $\frac{1}{s}(1, k, c')$ is a generator of $\mathbb{Z}_s$ where $k \equiv \lambda b$ and $c' \equiv \lambda c \equiv k^2 \pmod s$. Finally $s = 1 + k + c' \equiv 1 + k + k^2 \pmod s$ and so $s \mid 1 + k + k^2$. □

Proposition 3.5 *The order of A, the maximal Abelian normal subgroup of a trihedral group G, is not equal to 2 modulo 3.*

Proof. Compare with [Ito95, Prop 3.1]. The centre of G, denoted by $Z(G)$, is the subgroup $\left\langle \frac{1}{3}(1, 1, 1) \right\rangle \cap A$ of A. It has order either 1 or 3. If an element $g \in A$ is not in the centre of G then it belongs to a conjugacy class of G

containing precisely three distinct elements $g, \tau g \tau^{-1}, \tau^2 g \tau^{-2}$ of A. Therefore $|A| - |Z(G)|$ is divisible by 3 and so $|A| \equiv |Z(G)| \pmod 3$. □

Remark 3.6 This means that if $p \in \mathbb{Z}$ divides $1+k+k^2$ then $p \not\equiv 2 \pmod 3$. This is because if $p \mid 1+k+k^2$ then $\left\langle \frac{1}{p}(1, \overline{k}, \overline{k^2}) \right\rangle \ltimes \mathbb{Z}_3$ is a trihedral group. By this proposition $p \not\equiv 2 \pmod 3$.

3.1.1 Representation theory of G

First fix the following notation. Let $A = \mathbb{Z}_R \times \mathbb{Z}_r$ where $r \mid R$, and the possibilities that $r = R$ and $r = 1$ are allowed. Let $g = \frac{1}{R}(a, b, c)$ be the generator of $\mathbb{Z}_R$ and $h = \frac{1}{r}(1, -1, 0)$ the generator of $\mathbb{Z}_r$. Let ε be a primitive Rth root of unity, ξ a primitive rth root of unity and ω a primitive 3rd root of unity.

If $|A| \equiv 1 \pmod 3$, then G has three 1 dimensional representations and $(|A| - 1)/3$ irreducible 3 dimensional representations. In this case only the identity is in the centre of G. The remaining elements of A can be decomposed into conjugacy classes of G

$$\left(g^i h^j = \frac{1}{rt}(iar + jt, ibr - jt, icr), \tau g^i h^j \tau^{-1}, \tau^2 g^i h^j \tau^{-2} \right).$$

For each distinct conjugacy class define a 3 dimensional representation $V_{i,j}$ by the following formula

$$V_{i,j}(g) = \begin{pmatrix} \varepsilon^{ia}\xi^j & 0 & 0 \\ 0 & \varepsilon^{ib}\xi^{-j} & 0 \\ 0 & 0 & \varepsilon^{ic} \end{pmatrix} \quad \text{and} \quad V_{i,j}(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The three 1 dimensional representations L_i are given by

$$L_j(g) = 1 \quad \text{and} \quad L_j(\tau) = \omega^j.$$

If $|A| \equiv 0 \pmod 3$, then the centre of G is $Z(G) = \left\langle \frac{1}{3}(1,1,1) \right\rangle$. There are nine 1 dimensional representations and $(|A|-3)/3$ irreducible 3 dimensional representations. As above the conjugacy class of each element of A not in the centre defines a 3 dimensional representation $V_{i,j}$. The nine 1 dimensional representations are given by

$$L_{i,j}(g) = \omega^i \quad \text{and} \quad L_{i,j}(\tau) = \omega^j.$$

3.2 G-graphs

Definition 3.7 A polynomial or triple of polynomials is said to belong to one of the irreducible representations of G if one of the following conditions hold.

- A polynomial $f \in \mathbb{C}[x,y,z]$ *belongs to* $L_{i,j}$ if $g(f) = \omega^i f$ and $\tau(f) = \omega^j f$.
- For a polynomial $f \in \mathbb{C}[x,y]$, the triple of polynomials $(f, \tau(f), \tau^2(f))$ *belongs to* $V_{i,j}$ if $g\left(f, \tau(f), \tau^2(f)\right) = \left(\varepsilon^{ia}\xi^j f, \varepsilon^{ib}\xi^{-j}\tau\left(f\right), \varepsilon^{ic}\tau^2\left(f\right)\right)$.

Let $h \in \mathbb{C}[x,y,z]$ be a monomial such that $g(h) = \omega^i h$, $\tau(h) \neq h$ then $f = h + \omega^j\tau(h) + \omega^{2j}\tau^2(h) \in L_{i,j}$. If $j = 0$ then we do not need the condition that $\tau(h) \neq h$. In particular

$$\begin{gathered} xyz \in L_{0,0}, \\ x^R + \omega y^R + \omega z^R \in L_{0,j}, \\ x^{R/3} + \omega^j y^{R/3} + \omega^{2j} z^{R/3} \in L_{1,j}, \\ x^{2R/3} + \omega^j y^{2R/3} + \omega^{2j} z^{2R/3} \in L_{2,j}. \end{gathered}$$

Definition 3.8 A subset $\Gamma \subset \mathbb{C}[x,y,z]$ is called a *G-graph* if it contains precisely one polynomial belonging to each 1 dimensional representation and

three triples of polynomials belonging to each of the irreducible 3 dimensional representations.

If an ideal I in $\mathbb{C}[x,y,z]$ defines a G-cluster then $\mathbb{C}[x,y,z]/I \cong \mathbb{C}[G]$. Therefore we can choose a basis for the complex vector space $\mathbb{C}[x,y,z]/I$ which is a G-graph. We note that there will never be a unique choice of G-graph.

Definition 3.9 Let I be an ideal which defines a G-cluster. Denote by $\Gamma(I)$ the set of G-graphs which form a basis for the complex vector space $\mathbb{C}[x,y,z]/I$.

Suppose (f_1,f_2,f_3), (g_1,g_2,g_3), (h_1,h_2,h_3) and (k_1,k_2,k_3) are distinct triples of polynomials which belong to some representation $V_{i,j}$. Since $\mathbb{C}[G]$ contains $V_{i,j}$ with multiplicity 3 any ideal I which defines a G-cluster must contain three polynomials

$$\alpha f_1+\beta g_1+\gamma h_1+\delta k_1, \quad \alpha f_2+\beta g_2+\gamma h_2+\delta k_2, \quad \alpha f_3+\beta g_3+\gamma h_3+\delta k_3$$

for $\alpha,\beta,\gamma,\delta \in \mathbb{C}$. If (f_1,f_2,f_3), (g_1,g_2,g_3) and (h_1,h_2,h_3) belong to Γ then $\delta \neq 0$. Similarly for any polynomials g and h belonging to the same 1 dimensional representation there exists a polynomial of the form

$$\alpha g+\beta h$$

in I where $\alpha,\beta \in \mathbb{C}$. If $g \in \Gamma$ then $\beta \neq 0$.

Let h be a polynomial belonging to a 1 dimensional representation. Only if h is zero in $\mathbb{C}[x,y,z]/I$ can h not be chosen to be basic. In other words the only reason we cannot choose h in Γ is if $h \in I$. In particular this implies the following claim.

Claim 3.10 *For every ideal I which defines a G-cluster there exists a G-graph $\Gamma \in \Gamma(I)$ such that $1 \in \Gamma$.*

Therefore every ideal I defining a G-cluster must contain polynomials of the form

$$A := x^{|A|} + y^{|A|} + z^{|A|} - a, \quad Q := xyz - q$$

for some $a, q \in \mathbb{C}$.

Claim 3.11 *For every ideal I which defines a G-cluster there exists a G-graph $\Gamma \in \Gamma(I)$ that contains only polynomials which have monomial terms of the form $x^\lambda y^\mu z^\nu \in \Gamma$ where $\lambda\mu\nu = 0$.*

PROOF. Assume $\lambda \geq \mu \geq \nu > 0$ then repeated use of the polynomial Q shows that the polynomial $x^\lambda y^\mu z^\nu - q^\nu x^{\lambda-\nu} y^{\mu-\nu} \in I$. Therefore we may take $x^{\lambda-\nu} y^{\mu-\nu}$ to be basic instead of $x^\lambda y^\mu z^\nu$. □

Claim 3.12 *Let a polynomial h belong to some irreducible 3 dimensional representation. For every ideal I which defines a G-cluster there exists a G-graph $\Gamma \in \Gamma(I)$ such that if $h \in \Gamma$ then $\tau(h)$ and $\tau^2(h) \in \Gamma$.*

PROOF. If there is a polynomial

$$\tau(h) - \alpha_1 h_1 - \alpha_2 h_2 - \alpha_3 h_3 \tag{3.4}$$

where $h_1, h_2, h_3 \in \Gamma$. This implies there exists a polynomial $h - \alpha_1\tau^2(h_1) - \alpha_2\tau^2(h_2) - \alpha_3\tau^2(h_3)$ in I. Now h is basic and so one of the α_i is nonzero. Multiplying Equation 3.4 by α_i^{-1} means we can take $\tau(h)$ to be basic rather than h_i. □

Claim 3.13 *For every ideal I which defines a G-cluster there exists a G-graph $\Gamma \in \Gamma(I)$ such that the triple of polynomials belonging to a three dimensional representation in Γ is a triple of monomials.*

PROOF. Let $(f_1, \tau(f_1), \tau^2(f_1))$, $(f_2, \tau(f_2), \tau^2(f_2))$ and $(f_3, \tau(f_3), \tau^2(f_3))$ be the three copies of $V_{i,j}$ in some G-graph Γ. Consider a monomial term $\alpha_1 x^{\lambda_1} y^{\mu_1}$ of f_1. Only if $a_1 x^{\lambda_1} y^{\mu_1} + a_2 f_2 + a_3 f_3 \in I$ can $(f_1, \tau(f_1), \tau^2(f_1))$ not be replaced by $(x^{\lambda_1} y^{\mu_1}, y^{\lambda_1} z^{\mu_1}, z^{\lambda_1} x^{\mu_1})$ in the basis. If this is so then there exists no polynomial of the form $b_1(f_1 - \alpha_1 x^{\lambda} y^{\mu}) + b_2 f_2 + b_3 f_3$ in I. Continuing this way we will find a monomial term $\alpha_n x^{\lambda_n} y^{\mu_n}$ of f_1 such that $(x^{\lambda_n} y^{\mu_n}, y^{\lambda_n} z^{\mu_n}, z^{\lambda_n} x^{\mu_n})$, $(f_2, \tau(f_2), \tau^2(f_2))$ and $(f_3, \tau(f_3), \tau^2(f_3))$ are linearly independent in $\mathbb{C}[x, y, z]/I$. □

In fact as we will see later we will not always choose triples of monomials in the G-graph.

Claim 3.14 *For every ideal I which defines a G-cluster there exists a G-graph $\Gamma \in \Gamma(I)$ that satisfies the following conditions.*

1. *Let $x^{\alpha} y^{\beta} \in \Gamma$, $0 \leq i \leq \alpha$, and $0 \leq j \leq \beta$. If $x^i y^j$ belongs to some 3 dimensional representation then $x^i y^j$ is in Γ. Otherwise at least one of the three polynomials $x^i y^j + \omega^k y^i z^j + \omega^{2k} z^i x^j$ belongs to Γ.*

2. *Let $x^{\mu} y^{\nu} + \omega y^{\mu} z^{\nu} + \omega^2 z^{\mu} z^{\nu} \in \Gamma$, $0 \leq i \leq \mu$, $0 \leq j \leq \nu$ and $ij < \mu\nu$. If $x^i y^j$ belongs to some 3 dimensional representation then $x^i y^j \in \Gamma$. Otherwise the three polynomials $x^i y^j + \omega^k y^i z^j + \omega^{2k} z^i x^j$ can belongs to Γ.*

PROOF. We show how to prove Part 1. The proof of Part 2 is similar. Define a set $\Lambda_{M,N}$ as follows.

$$\Lambda_{M,N} := \left\{ x^n y^m, y^n z^m, z^n x^m \;\middle|\; m \geq M, n \geq N \right\}$$

Let $x^{\alpha} y^{\beta}$ be basic, $0 \leq i \leq \alpha$, $0 \leq j \leq \beta$ and $ij < \alpha\beta$. Suppose that $x^i y^j$ is not basic. Let $x^i y^j$ belong to some 3 dimensional representation. Then

there exists a polynomial

$$x^i y^j - a_1 f_1 - a_2 f_2$$

in I. If $a_1 = a_2 = 0$ then $x^i y^j \in I$ and $x^\alpha y^\beta \in I$. Therefore $x^\alpha y^\beta$ cannot be basic. From now on assume that $a_1 \neq 0$. If $f_1 \in \Lambda_{i,j}$ then replace $x^i y^j$ with f_1 in the basis. Therefore without loss of generality we may assume $a_1 \neq 0$ and $f_1 \notin \Lambda_{i,j}$. It follows that

$$x^\alpha y^\beta - a_1 f_1 x^{\alpha-i} y^{\beta-j} - a_2 f_2 x^{\alpha-i} y^{\beta-j}.$$

If $z \not| f_1$ then $f_1 x^{\alpha-i} y^{\beta-j} \notin \Lambda_{\alpha,\beta}$ and we may replace $x^\alpha y^\beta$ with $f_1 x^{\alpha-i} y^{\beta-j}$ in our basis. If $f_1 x^{\alpha-i} y^{\beta-j}$ is already basic then a_2 must also be nonzero and repeat this argument for f_2.

If $z \mid f_1$ then use the relation Q to replace $f_1 x^{\alpha-i} y^{\beta-j}$ a monomial of the form $x^\xi y^\eta z^\zeta$ where $\xi\eta\zeta = 0$. This is also not in $\Lambda_{\alpha,\beta}$ and we may replace $x^\alpha y^\beta$ with $x^\xi y^\eta z^\zeta$ in the basis.

Suppose $x^i y^j + \omega^k y^i z^j + \omega^{2k} z^i x^j$ belongs to one of the 1 dimensional representations. If it cannot be chosen to be basic for each $0 \leq k \leq 2$. Then $x^i y^j + \omega^k y^i z^j + \omega^{2k} z^i x^j \in I$ for all k and so $x^i y^j \in I$. Therefore $x^\alpha y^\beta \in I$ and so cannot be basic. □

Claim 3.15 *For every ideal I which defines a G-cluster there exists a G-graph $\Gamma \in \Gamma(I)$ that satisfies the following conditions.*

1. *If $x^\lambda y^\mu + \omega y^\lambda z^\mu + \omega^2 z^\lambda x^\mu \in L_{0,1}$ belongs to Γ then $x^\lambda y^\mu + \omega^2 y^\lambda z^\mu + \omega z^\lambda x^\mu \in L_{0,2}$ belongs to Γ.*

2. *Similarly if $x^\lambda y^\mu + \omega^j y^\lambda z^\mu + \omega^{2j} z^\lambda x^\mu \in L_{i,j}$ for $i = 1, 2$ and some j =1, 2 or 3 then $x^\lambda y^\mu + \omega^j y^\lambda z^\mu + \omega^{2j} z^\lambda x^\mu$ in Γ for all j.*

PROOF. Again we prove the first part. The proof of the second part is analogous. Note that $x^R + y^R + z^R \in L_{0,0}$ so does not belong to Γ we therefore assume $\lambda \leq R$ and $\mu < R$. If $\lambda = R$ then $\mu = 0$ because otherwise $x^R + y^R + z^R \in \Gamma$ by Claim 3.14. Note that $\mathbb{C}[x,y,z]^A \subset \mathbb{C}[x^r, y^r, z^r, xyz]$ and so $r \mid \lambda$ and $r \mid \mu$.

Suppose that

$$x^\lambda y^\mu + \omega y^\lambda z^\mu + \omega^2 z^\lambda x^\mu \in L_{0,1}$$

belongs to Γ but $x^\lambda y^\mu + \omega^2 y^\lambda z^\mu + \omega z^\lambda x^\mu \in L_{0,2}$ does not belong to Γ. This means that $x^\lambda y^\mu + \omega^2 y^\lambda z^\mu + \omega z^\lambda x^\mu = 0$. Thus

$$x^\eta y^\zeta + \omega^2 y^\eta z^\zeta + \omega z^\eta x^\zeta \in L_{0,2}$$

belongs to Γ. Without loss of generality we assume that $\mu < \zeta < R$ and $\eta < \lambda \leq R$.

If $\mu = 0$ then $\lambda = R$ and assume also that $r = R$. Then $\mathbb{C}[x,y,z]^A = \mathbb{C}[x^r, y^r, z^r, xyz]$ and so $x^r y^r \mid x^\eta y^\zeta$. Thus (x^r, y^r, z^r) must be contained in the vector space spanned by Γ by Claim 3.14. Therefore $x^r + y^r + z^r \in \Gamma$ and Γ contains two polynomials belonging to the trivial representation and so $\mathbb{C}[x,y,z]/I \ncong \mathbb{C}[G]$.

Now assume $\mu = 0$ and $R > r > 1$. Since $\mathbb{C}[x,y,z]^A \subset \mathbb{C}[x^r, y^r, z^r, xyz]$ it follows that $r \mid \eta$ and $r \mid \zeta$. The triples of monomials

$$\left(x^{R-r}, y^{R-r}, z^{R-r}\right), \quad \left(z^{R-kr}, x^{R-kr}, y^{R-kr}\right), \quad \left(y^{R-\overline{k^2}r}, z^{R-\overline{k^2}r}, x^{R-\overline{k^2}r}\right)$$

all belong to the same representation V. These three triples are distinct if $k \neq 1$ and are basic by Claim 3.14. If $k = 1$ the three polynomials

$$x^{R-r} + \omega^i y^{R-r} + \omega^{2i} z^{R-r}$$

belong to the representations $L_{2,i}$.

Now consider the triples

$$\left(x^{\eta-r}y^{\zeta}, y^{\eta-r}z^{\zeta}, z^{\eta-r}x^{\zeta}\right), \quad \left(z^{\eta}x^{\zeta-r}, x^{\eta}y^{\zeta-r}, y^{\eta}z^{\zeta-r}\right).$$

These also belong to the same representation V and are basic by Claim 3.14. At least one of these triples is distinct from the three given above unless $r = \eta = \zeta$. If $r = \eta = \zeta$ then $x^r y^r$ is invariant under A. In particular $x^r y^r$ is invariant under $\left\langle \frac{1}{R}(a,b,c) \right\rangle$ and so $\lambda(a+b) \cong 0$ for all $1 \leq \lambda \leq R$ which means $c = 0$. For A, which is generated by $\frac{1}{r}(1,-1,0)$ and $\frac{1}{R}(a,b,c)$, to be a normal subgroup it must contain the element $\frac{1}{R}(c,a,b)$. This is clearly impossible since $c = 0$.

Therefore if $k > 1$ we have at least four triples of monomials in Γ belonging to the same 3 dimensional representation. If $k = 1$ the polynomials

$$x^{\eta-r}y^{\zeta} + \omega^i y^{\eta-r}z^{\zeta} + \omega^{2i} z^{\eta-r}x^{\zeta}, z^{\eta}x^{\zeta-r} + \omega^i x^{\eta}y^{\zeta-r} + \omega^{2i} y^{\eta}z^{\zeta-r} \in L_{2,i}$$

and at least one of these polynomials is distinct from $x^{R-r} + \omega^i y^{R-r} + \omega^{2i} z^{R-r}$ so Γ contains two polynomials belonging to each of the 1 dimensional representations $L_{2,i}$. In either case $\mathbb{C}[x,y,z]/I \ncong \mathbb{C}[G]$.

Now assume $\mu = 0$ and $r = 1$. If $x^R + \omega y^R + \omega^2 z^R \in \Gamma$ then

$$\left\{1, (x,y,z), \ldots, \left(x^{R-1}, y^{R-1}, z^{R-1}\right), x^R + \omega y^R + \omega^2 z^R\right\} \subset \Gamma.$$

This set is missing only one polynomial belonging to $L_{0,2}$. By Claim 3.14 we are only able to add a polynomial which is a linear combination of the monomials xy, yz, zx or $x^R + \omega^2 y^R + \omega z^R \in \Gamma$. Since $xy, yz, zx \notin \mathbb{C}[x,y,z]^A$ no linear combination of these monomials can be in $L_{0,2}$ so $\eta = R$ and $\zeta = 0$.

Finally if $\mu > 0$ then by Claim 3.14 the four distinct triples of monomials

$$\left(x^{\lambda-1}y^{\mu}, y^{\lambda-1}z^{\mu}, z^{\lambda-1}x^{\mu}\right), \quad \left(z^{\lambda}x^{\mu-1}, x^{\lambda}y^{\mu-1}, y^{\lambda}z^{\mu-1}\right),$$
$$\left(x^{\eta-1}y^{\zeta}, y^{\eta-1}z^{\zeta}, z^{\eta-1}x^{\zeta}\right), \quad \left(z^{\eta}x^{\zeta-1}, x^{\eta}y^{\zeta-1}, y^{\eta}z^{\zeta-1}\right)$$

are all in Γ. Again these triples belong to the same representation. Therefore $\mathbb{C}[x,y,z]/I \ncong \mathbb{C}[G]$. □

In the next definition we construct set S and show how to construct from this set a G-graph. Let $T := \left\{x^i y^j \;\middle|\; i > 0 \; j \geq 0\right\}$ and $S \subset T$.

Definition 3.16 Define a set S as follows.

- For each irreducible 3 dimensional representation $V_{i,j}$ take three monomials in S which belong to $T \cap V_{i,j}$.
- Take one monomial f_0 in S belonging to T which is invariant under A.
- If $|A| \equiv 0 \pmod 3$ take one monomial f_1 in S belonging to T such that $g(f_1) = \omega f_1$ and one monomial f_2 in S belonging to T such that $g(f_2) = \omega^2 f_2$.

We say S is a *G-set* if it satisfies the following conditions

1. If $x^\eta y^\zeta \in S$ then $x^\zeta, x^i y^j \in S$ for all $1 \leq i \leq \eta$ and $0 \leq j \leq \zeta$.
2. Let $f_0 = x^\lambda y^\mu \in S$ be invariant under A. If $\mu \neq 0$ then no element of the form $x^\alpha y^\beta f_0 \in S$ for $\alpha\beta \neq 0$. Either take neither of the monomials $x^\alpha f_0, y^\alpha f_0$ or both. If we take both of them then count them with multiplicity $1/2$ in S.
3. If $f_0 = x^R \in S$ is invariant under A then take none, two or three of the elements in $x^{\alpha+R}, x^R y^\alpha, x^\alpha y^R$. If two of them then count them each with multiplicity a $1/2$ in S. If three of them then count each with multiplicity $2/3$ in S.

   For $i, j > 0$ either take neither of the monomials $x^{i+R}y^{j+1}, x^{i+1}y^{R+j}$ or both. If both then count each with multiplicity a $1/2$ in S.

We now show how S generates a G-graph. Fix a G-set S. In the following algorithm we define a G-graph, $\Gamma(S)$, and a set $\mathcal{B}(S) \supset \Gamma(S)$ which is a $\mathbb{C}$-basis for the vector space $\mathbb{C}[x, y, z]$.

Algorithm 3.17 *1. Take* $1 \in \Gamma(S)$.

2. For $f_0 \in S$ *take the polynomials*

$$F_j := f_0 + \omega^j \tau(f_0) + \omega^{2j} \tau^2(f_0) \in \Gamma(S)$$

for $j = 1, 2$ *and*

$$F_0 := f_0 + \tau(f_0) + \tau^2(f_0) \in \mathcal{B}(S).$$

These belong to the representation $L_{0,j}$.

3. If $|A| \equiv 0 \pmod 3$ *take the polynomials*

$$f_i + \omega^j \tau(f_i) + \omega^{2j} \tau^2(f_i) \in \Gamma(S)$$

for $i = 1, 2$ *and* $j = 0, 1, 2$. *These belong to* $L_{i,j}$.

4. For $f \in S$ *such that* f *belongs to one of the 3 dimensional representations,* $V_{i,j}$, *and* $f_0 \not| f$ *and* $y^R \not| f_0$ *take*

$$f, \tau(f), \tau^2(f) \in \Gamma(S).$$

These monomials base a copy of $V_{i,j}$.

5. For $f \in S$ *such that* f *belongs to one of the 3 dimensional representations,* $V_{i,j}$ *and* $f_0 = x^\lambda y^\mu \mid f$ *where* $\mu > 0$. *Then* $x^\alpha f_0$ *and* $y^\alpha f_0$ *belong to* S. *In this case take*

$$x^\alpha \left(f_0 - \tau^2(f_0)\right), y^\alpha \left(\tau(f_0) - f_0\right), z^\alpha \left(\tau^2(f_0) - \tau(f_0)\right) \in \Gamma(S)$$

and

$$\begin{matrix} x^\alpha\left(f_0+\tau^2(f_0)\right), y^\alpha\left(\tau(f_0)+f_0\right), z^\alpha\left(\tau^2(f_0)+\tau(f_0)\right) \\ x^\alpha\tau(f_0), y^\alpha\tau^2(f_0), z^\alpha f_0 \end{matrix} \in \mathcal{B}(S).$$

Each of these polynomials base a copy of $V_{i,j}$.

6. *For* $f \in S$ *such that* f *belongs to one of the 3 dimensional representations* $f_0 = x^R \mid f$ *or* $y^R \mid f$ *and* $f \in \left\{x^{R+\alpha}, x^R y^\alpha, x^\alpha y^R\right\}$. *If the three monomials in this set belong to* S *then take*

$$\begin{matrix} x^\alpha F_1, y^\alpha \omega^2 F_1, z^\alpha \omega F_1 \\ x^\alpha F_2, y^\alpha \omega F_2, z^\alpha \omega^2 F_2 \end{matrix} \in \Gamma(S)$$

and

$$x^\alpha F_0, y^\alpha F_0, z^\alpha F_0 \in \mathcal{B}(S).$$

If only two monomials in $\left\{x^{R+\alpha}, x^R y^\alpha, x^\alpha y^R\right\}$ *belong to* S *then take*

$$x^\alpha F_1, y^\alpha \omega^2 F_1, z^\alpha \omega F_1 \in \Gamma(S)$$

and

$$\begin{matrix} x^\alpha F_0, y^\alpha F_0, z^\alpha F_0 \\ x^\alpha F_2, y^\alpha \omega F_2, z^\alpha \omega^2 F_2 \end{matrix} \in \mathcal{B}(S).$$

7. *Finally for* $f \in S$ *such that* f *belongs to one of the 3 dimensional representations,* $f_0 = x^R$ *and* $f = x^\alpha y^\beta f_0$ *where* $\alpha\beta \neq 0$. *Then* $x^\alpha y^\beta y^R \in S$. *Take*

$$x^\alpha y^\beta\left(f_0-\tau(f_0)\right), y^\alpha z^\beta\left(\tau(f_0)-\tau^2(f_0)\right), z^\alpha x^\beta\left(\tau^2(f_0)-f_0\right) \in \Gamma(S)$$

$$\begin{matrix} x^\alpha y^\beta\left(f_0+\tau(f_0)\right), y^\alpha z^\beta\left(\tau(f_0)+\tau^2(f_0)\right), z^\alpha x^\beta\left(\tau^2(f_0)+f_0\right) \\ x^\alpha y^\beta \tau^2(f_0), y^\alpha z^\beta f_0, z^\alpha x^\beta \tau(f_0) \end{matrix} \in \mathcal{B}(S)$$

8. *Take every element in Γ_T in $\mathcal{B}(S)$. If m is any monomial in $\mathbb{C}[x,y,z]$ which does not belong to the vector space spanned by the elements already in $\mathcal{B}(S)$ then take $m \in \mathcal{B}(S)$.*

Denote by $I(S)$ or sometimes $I(\Gamma(S))$ the ideal generated by the elements in $\mathcal{B}(S) \setminus \Gamma(S)$.

Proposition 3.18 *An ideal I defines a G-cluster if and only if there exists a G-set S such that $\Gamma(S) \in \Gamma(I)$.*

Proof. Let S be a G-set. The set $\Gamma(S)$ contains three triples of monomials in each 3 dimensional representation and one polynomial in each 1 dimensional representation so $\Gamma(S)$ is a G-graph. By construction the ideal $I(S)$, generated by the elements in $\mathcal{B}(S) \setminus \Gamma(S)$, contains no element of $\Gamma(S)$. Therefore $\Gamma(S)$ is a basis for $\mathbb{C}[x,y,z]/I(S)$ and so $I(S)$ defines a G-cluster.

Now let I be an ideal defining a G-cluster and Γ its G-graph.

- By Claim 3.10 we can always take 1 in Γ.
- By Claim 3.11 we never have to take a polynomial containing a monomial of the form $x^\lambda y^\mu z^\nu$ in Γ for $\lambda\mu\nu \neq 0$.
- By Claim 3.12 if a monomial $x^\eta y^\zeta \in \Gamma$ then we can choose $y^\eta z^\zeta$ and $z^\eta x^\zeta$ in Γ.
- By Claim 3.13 we may assume that all the polynomials belonging to the 3 dimensional representations are monomial.
- By Claim 3.14 we may assume that if $x^\lambda y^\mu \in \Gamma$ or $x^\lambda y^\mu + \omega^j y^\lambda z^\mu + \omega^{2j} z^\lambda x^\mu$ then $x^i y^k \in \Gamma$ for all $i \leq \lambda, j \leq \mu$ and $ij < \lambda\mu$.
- By Claim 3.15 the polynomials belonging to $L_{0,1}$ and $L_{0,2}$ can be chosen to be $x^\lambda y^\mu + \omega^j y^\lambda z^\mu + \omega^{2j} z^\lambda x^\mu$ in Γ. We may also assume that the

polynomials belonging to $L_{1,j}$, can also be chosen to be of the form $x^\lambda y^\mu + \omega^j y^\lambda z^\mu + \omega^{2j} z^\lambda x^\mu$ for some $x^\lambda y^\mu$ such that $g(x^\lambda y^\mu) = \omega x^\lambda y^\mu$. Similarly for $L_{2,j}$.

Therefore we may assume that Γ is of the form $\Gamma = \Gamma(S)$ for some G-set S. □

Claim 3.19 *Let I be an ideal which defines a G-cluster then there exists $\Gamma \in \Gamma(I)$ such that the triples $\left(x^\lambda, y^\lambda, z^\lambda\right)$ are in Γ for $0 < \lambda \leq (|A|)^{1/2}$.*

PROOF. Now Γ must contain $3\,|A|$ elements. If one of the triples $\left(x^\lambda, y^\lambda, z^\lambda\right)$ for $0 < \lambda \leq (|A|)^{1/2}$ cannot be chosen to be in Γ then $x^\lambda, y^\lambda, z^\lambda \in I$ and therefore $x^j, y^j, z^j \in I$ for all $j \geq \lambda$. Thus Γ is contained in the set

$$\left\{x^i y^j, y^i z^j, z^i x^j \;\middle|\; 0 \leq i, j < \lambda\right\}.$$

This has only $1+3(\lambda-1)\lambda < 3\,|A|$ elements. Therefore the triples $\left(x^\lambda, y^\lambda, z^\lambda\right)$ can always be chosen to be in Γ for $0 < \lambda \leq (|A|)^{1/2}$. □

Definition 3.20 If S is a G-set then $f \in S$ is called a *horn* of S if xf, yf, zf do not belong to the corresponding G-graph $\Gamma(S)$.

If $f = x^i y^j$ for any $j > 0$ then automatically $zf \notin \Gamma$ so it only needs to be checked that $xf, yf \notin S$. For $f = x^i$ to be a horn we also need to check $zf \notin \Gamma$. This is equivalent to xy^i not being in S. The horns of S generate $\mathrm{Soc}(I(\Gamma))$ in the same way that a monomial $f \in S$ generates a subset of $\Gamma(S)$ in Algorithm 3.17. This subset defines a submodule of $\mathbb{C}[x, y, z]/I(S)$ which belongs to $\mathrm{Soc}(I(S))$. For example if $f \in V_{i,j}$ is a horn of S and $f_0 \not| f$ then the submodule generated by $f, \tau(f), \tau^2(f)$ belongs to $\mathrm{Soc}(I(S))$.

Conjecture 3.21 *Let G be any trihedral group then G-Hilb $\mathbb{C}^3$ is a smooth rational 3 dimensional variety. Then there exists a G-set S such that $\Gamma(S) \in \Gamma(I)$ and S has at most 5 horns.*

To prove this I try to compute an explicit cover of G-Hilb $\mathbb{C}^3$ as in Chapter 2. However the cover depends on the particular family to which the trihedral group belongs and this method does not allow us to construct the general case. It does compute G-Hilb $\mathbb{C}^3$ for interesting families of trihedral groups.

3.3 Classification of G-graphs for the group $\left\langle \frac{1}{31}(1,5,25) \ltimes \mathbb{Z}_3 \right\rangle$

In this section we classify G-graphs when the maximal normal Abelian subgroup A of G is generated by $g = \frac{1}{31}(1,5,25)$. We begin by computing the irreducible representations of G. For $i = 0, \ldots, 30$ define a 3 dimensional representation V_i by the following formula.

$$V_i(g) = \begin{pmatrix} \varepsilon^i & 0 & 0 \\ 0 & \varepsilon^{5i} & 0 \\ 0 & 0 & \varepsilon^{25i} \end{pmatrix} \quad \text{and} \quad V_i(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The representation V_0 decomposes as the sum of three 1 dimensional representations L_0, the trivial representation, L_1 and L_2 which are defined as follows.

$$L_i(g) = 1 \quad \text{and} \quad L_i(\tau) = \omega^i.$$

The remaining representations are irreducible but not distinct. For example $V_1 \cong V_5$. Two representations are isomorphic if and only if they have the same character so the subset

$$\left\{ V_i \;\middle|\; \text{for } i \in \Lambda = \{0,1,2,3,4,6,8,11,12,16,17\} \right\}$$

is a set of distinct G-representations and only the representation V_0 is reducible. Thus

$$\mathbb{C}[G] = \bigoplus_{i \in \Lambda \setminus 0} V_i^{\oplus 3} \oplus \bigoplus_{i=0}^{2} L_i.$$

Figure 3.1 shows to which representation V_k the monomial $x^i y^j$ belongs for $i > 0$ and $j \geq 0$. For example $x^5 y \in V_2$ and $x^6 y^5 \in V_0$. Note that if $xy^6 \in S_0$ then $xy^6 + yz^6 + zx^6 - a \in I$ and so $x^2 y^7 \notin S$ by Claim 3.14.

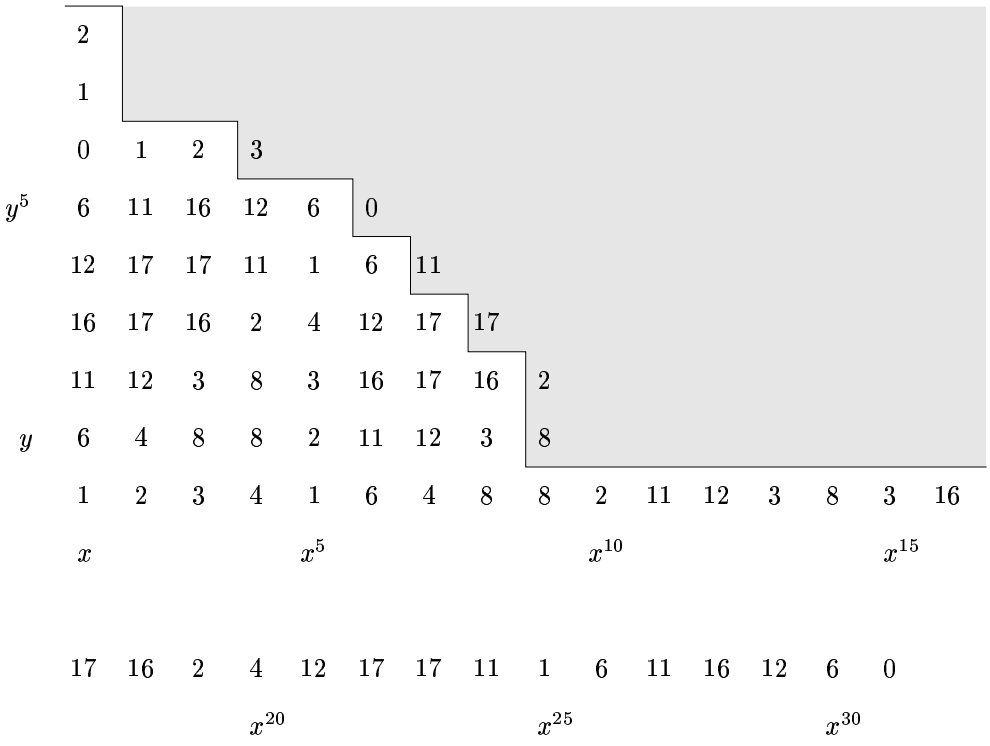


Figure 3.1: Decomposition of the monomials in T .

For $k \in \Lambda$ let S_k be the subset of S defined by $S_k := \{f | f \in V_k \cap S\}$. Note that $x^6 y^5$ cannot be the element in S_0 because if $x^6 y^5 \in S$ then each of the monomials $x^6, x^5 y^5, x^6 y^4, xy \in S$ by Claim 3.14 and therefore there would be 4 monomials in S_6. For this reason the only possible monomials in S_0 are x^{31} and xy^6. Similarly any monomials in the shaded area of Figure 3.1 cannot be in S as there would be too many monomials in some S_i.

3.3.1 Step 0

If x^{31} is in S_0 then $S_A = \{x, \ldots, x^{31}\}$ and so the graph which S_A generates is

$$\Gamma_A = \{1, x, \ldots, x^{30}, y, \ldots, y^{30}, z, \ldots, z^{30}, \\ x^{31} + \omega y^{31} + \omega^2 z^{31}, x^{31} + \omega^2 y^{31} + \omega z^{31}\}.$$

The ideal $I(\Gamma_A)$, is as follows.

$$I(\Gamma_A) = \left\langle x^{31} + y^{31} + z^{31}, xy, yz, zx \right\rangle \tag{3.5}$$

Only x^{31} is a horn of S_A and since x^{31} is invariant under A it generates the two representations L_1 and L_2 in $\Gamma(S_A)$.

Let I be any ideal with $\Gamma(S_A) \in \Gamma(I)$ as its G-graph. Then

$$A := x^{31} + y^{31} + z^{31} - a$$

is in I for $a \in \mathbb{C}$. The triple xy, yz, zx is a basis of V_6. Now in Γ there are three copies of V_6 with bases $\{x^6, y^6, z^6\}, \{y^{26}, z^{26}, x^{26}\}$ and $\{z^{30}, x^{30}, y^{30}\}$. Therefore the following three polynomials

$$B := xy - b_1 x^6 - c_1 y^{26} - d_1 z^{30},$$
$$C := yz - b_1 y^6 - c_1 z^{26} - d_1 x^{30},$$
$$D := zx - b_1 z^6 - c_1 x^{26} - d_1 y^{30}$$

belong to I. In order to define an ideal we need polynomials which express every monomial $x^i y, y^i z, z^i x, xy^i, yz^i, zx^i$ for $i < 31$ in terms of the elements in $\Gamma(S_A)$. As in Chapter 2 we can show that these polynomials generate an ideal which defines a G-cluster. In other words these polynomials allow any element in $\mathcal{B}(S_A) \setminus \Gamma(S_A)$ to be written as a linear combination of the polynomials in $\Gamma(S_A)$.

3.3.2 Step 1

If $x^{31} \notin S$ then $xy^6 \in S$. By Claim 3.14 $xy, \ldots, x^5y \in S$ and so $x^{26}, \ldots, x^{30} \notin S$. If $x^{25} \in S$ then

$$S_B = \begin{array}{ccc} xy^6 & & \\ \vdots & & \\ xy & & \\ x & \cdots & x^{25} \end{array}$$

and $I(\Gamma_B) = \langle xy^6 + yz^6 + zx^6, x^{26}, y^{26}, z^{26}, x^2y, y^2z, z^2x, xyz \rangle$.

Let I be any ideal with $\Gamma(S_B) \in \Gamma(I)$. Again $xy^6 + yz^6 + zx^6$ and xyz belong to L_0 so I contains polynomials of the form

$$xy^6 + yz^6 + zx^6 - a, \quad xyz - h.$$

The triple $\{y^{26}, z^{26}, x^{26}\}$ is a basis of V_6. In Γ the three copies of V_6 are based by $\{x^6, y^6, z^6\}, \{xy, yz, zx\}$ and $\{yz^5, zx^5, xy^5\}$. Therefore I contains

$$\begin{aligned} y^{26} &- bx^6 - cxy - dyz^5, \\ z^{26} &- by^6 - cyz - dzx^5, \\ x^{26} &- bz^6 - czx - dxy^5. \end{aligned}$$

The triple $\{y^2z, z^2x, x^2y\}$ are a basis for V_4. In Γ the three copies of V_4 are based by $\{x^4, y^4, z^4\}, \{y^7, z^7, x^7\}$ and $\{z^{20}, x^{20}, y^{20}\}$. Therefore

$$\begin{aligned} y^2z &- e_1x^4 - f_1y^7 - g_1z^{20}, \\ z^2x &- e_1y^4 - f_1z^7 - g_1x^{20}, \\ x^2y &- e_1z^4 - f_1x^7 - g_1y^{20} \end{aligned}$$

belong to I. To generate an ideal we also need polynomials that express each monomial x^iy, y^iz, z^ix for $i < 26$ in terms of monomials in $\Gamma(S_B)$.

The horns of S_B are xy^6 and x^{25}. Again xy^6 is invariant under A so defines two subsets of $\Gamma(S_B)$ corresponding to 1 dimensional representations. The monomial x^{25} defines one subset of $\Gamma(S_B)$ corresponding to a 3 dimensional representation. Thus the horns of S_B correspond to

$$L_1 \oplus L_2 \oplus V_1.$$

Now assume that $x^{25} \notin S$. There must still be three elements in S_1. The monomials $x, x^5 \in S$ by Claim 3.19. Figure 3.1 shows that the remaining element is x^5y^4, xy^7 or x^6y^2. By Proposition 3.18 the last two monomials will only correspond to one 3 dimensional representation

$$\left(x\left(xy^6 - zx^6\right), y\left(yz^6 - xy^6\right), z\left(zx^6 - yz^6\right)\right) \in \Gamma.$$

If $x^5y^4 \in S_1$ then all of the elements contained in the shaded area of Figure 3.2 are in S.

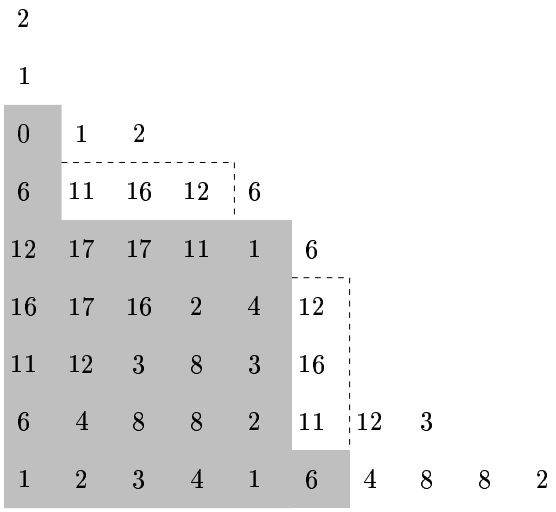


Figure 3.2: The set obtained by choosing x^5y^4 in S

For S to be a G-set there only needs to be one more element in each of the sets S_{11}, S_{12} and S_{16}. Figure 3.2 shows that there are two ways of doing

this. Firstly

$$S_C = \begin{array}{cccccc} xy^6 & & & & & \\ xy^5 & x^2y^5 & x^3y^5 & x^4y^5 & & \\ xy^4 & x^2y^4 & x^3y^4 & x^4y^4 & x^5y^4 & \\ xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & x^5y^3 & \\ xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & x^5y^2 & \\ xy & x^2y & x^3y & x^4y & x^5y & \\ x & x^2 & x^3 & x^4 & x^5 & x^6 \end{array}$$

then

$$I(\Gamma_C) = \left\langle xy^6 + yz^6 + zx^6, x^7, y^7, z^7, x^6y, y^6z, z^6x, x^5y^5, y^5z^5, z^5x^5, xyz \right\rangle .$$

Any ideal I_C having $\Gamma(S_C) \in \Gamma(I_C)$ is generated by polynomials that express each of the generators of $I(\Gamma_C)$ as a linear combination of elements in $\Gamma(S_C)$. In other words the generators of I_C are

$$\begin{array}{cc} xy^6 + yz^6 + zx^6 - a, & \\ x^7 - bz^4 - cx^2y - dy^5z^3, & x^6y - ey^2x - fz^5y^2 - gx^4z^4, \\ y^7 - bx^4 - cy^2z - dz^5x^3, & y^6z - ez^2y - fx^5z^2 - gy^4x^4 \\ z^7 - by^4 - cz^2x - dx^5y^3, & z^6x - ex^2z - fy^5x^2 - gz^4y^4 \\ x^5y^5 - hy^6 - iyz - jx^5z, & \\ y^5z^5 - hz^6 - izx - jy^5x, & \\ z^5x^5 - hx^6 - ixy - jz^5y, & \\ xyz - k & \end{array} . \tag{3.6}$$

Note that x^6 is not a horn of Γ_C since $xy^6 \in \Gamma_C$ and so $x^6z \neq 0$. The only horns are $xy^6 \in S_0$, $x^4y^5 \in S_{12}$ and $x^5y^4 \in S_1$. These correspond to the representations

$$L_1 \oplus L_2 \oplus V_{12} \oplus V_1.$$

Alternatively if

$$S_D = \begin{array}{cccccc} xy^6 & & & & & \\ xy^5 & & & & & \\ xy^4 & x^2y^4 & x^3y^4 & x^4y^4 & x^5y^4 & \\ xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & x^5y^3 & x^6y^3 \\ xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & x^5y^2 & x^6y^2 \\ xy & x^2y & x^3y & x^4y & x^5y & x^6y \\ x & x^2 & x^3 & x^4 & x^5 & x^6, \end{array}$$

then the ideal is

$$I(\Gamma_D) = \left\langle xy^6 + yz^6 + zx^6, x^7, y^7, z^7, x^6y^4, y^6z^4, z^6x^4, x^2y^5, y^2z^5, z^2x^5, xyz \right\rangle.$$

As with $\Gamma(S_C)$ any ideal I_D having $\Gamma(S_D) \in \Gamma(I_D$ is generated by polynomials that express each of the generators of $I(\Gamma_D)$ as a linear combination of elements in $\Gamma(S_D)$.

The four horns of S_D correspond to the representations

$$L_1 \oplus L_2 \oplus V_1 \oplus V_6 \oplus V_{12}.$$

3.3.3 Step 2

All the remaining clusters must have x^7y and x^6y^2 in S_1. Now consider the space S_2, the monomial $x^2 \in S_2$. If $x^{19} \in S$ then

$$S_E = \begin{array}{cccc} xy^7 & & & \\ xy^6 & x^2y^6 & & \\ \vdots & \vdots & & \\ xy & x^2y & & \\ x & x^2 & \cdots & x^{19}. \end{array}$$

Then

$$I(\Gamma_E) = \left\langle xy^6 + yz^6 + zx^6, x^{20}, y^{20}, z^{20}, x^3y, y^3z, z^3x, xyz \right\rangle.$$

As with $\Gamma(S_A)$ and $\Gamma(S_B)$ a general ideal will have the generators of $I(\Gamma_E)$ as well as the monomials x^iy, y^iz, z^ix for $i < 20$ expressed as a linear combination of the elements of $\Gamma(S_E)$.

The two horns of S_E correspond to the representations

$$V_1 \oplus V_2.$$

If $x^{19} \notin S_2$ then consider the possibilities for the remaining two elements in S_2. From Figure 3.3 it follows that xy^8, x^6y^3, x^{10}, x^4y^3 and x^5y are the other possibilities for monomials in S_2. As before the first two monomials

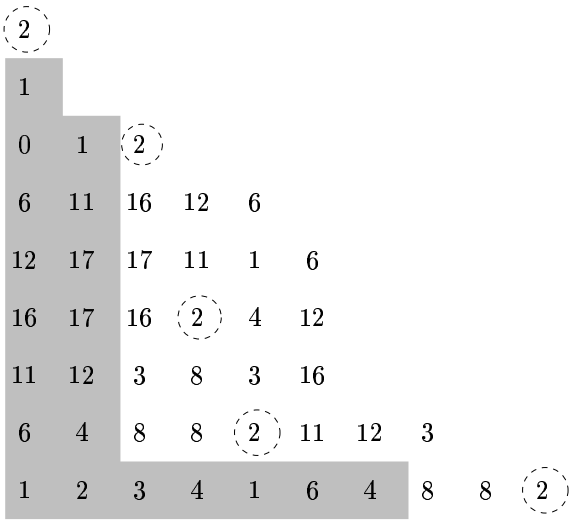


Figure 3.3: If $x^2y^6 \in S$ and $x^{19} \notin S$

count as one element in S. The following pairs cannot be contained in S as it would mean having too many elements in S_8: $\{x^5y, x^{10}\}$, $\{x^4y^3, x^{10}\}$ or $\{x^4y^3, xy^8, x^3y^6\}$.

If the monomials $xy^8, x^6y^3, x^{10} \in S_2$ then

$$S_F = \begin{array}{ccccc} xy^8 & & & & \\ xy^7 & & & & \\ xy^6 & x^2y^6 & x^3y^6 & & \\ \vdots & \vdots & \vdots & & \\ xy & x^2y & x^3y & & \\ x & x^2 & x^3 & \cdots & x^{13}, \end{array}$$

$$I(\Gamma_F) = \left\langle xy^6 + yz^6 + zx^6, x^{14}, y^{14}, z^{14}, x^4y, y^4z, z^4x, xyz \right\rangle.$$

Any ideal having $\Gamma(S_F)$ for its G-graph is generated by polynomials that express each of the generators of $I(\Gamma_F)$ as a linear combination of elements in $\Gamma(S_F)$. The two horns of S_F correspond to the representations

$$V_2 \oplus V_3.$$

Let $x^5y, x^4y^3 \in S_2$ then the shaded region in Figure 3.4 is in S.

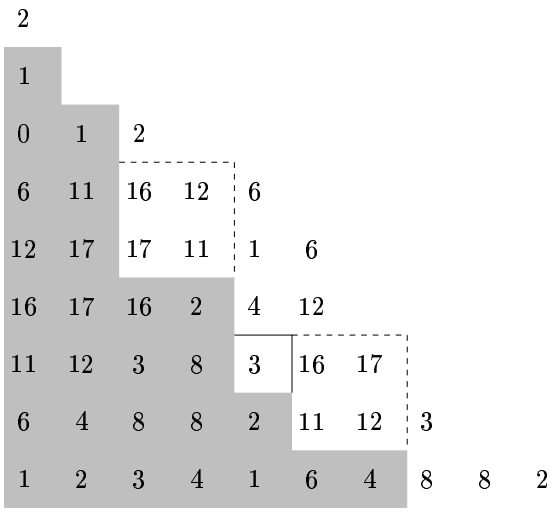


Figure 3.4: If $x^5y, x^4y^3 \in S$

Note that there are already three monomials in S_8 so $x^8 \notin S$. Similarly there are already three monomials in S_4 so $x^5y^3 \notin S$. To make this into a G-set there must be one more monomial in each of the spaces S_3, S_{11}, S_{12}, S_{16} and S_{17}. The only choice for a monomial in S_3 is x^3y^2 and then there are four possible ways of adding the others.

The first possibility is that

$$S_G = \begin{array}{ccccccc} xy^7 & & & & & & \\ xy^6 & x^2y^6 & & & & & \\ xy^5 & x^2y^5 & x^3y^5 & x^4y^5 & & & \\ xy^4 & x^2y^4 & x^3y^4 & x^4y^4 & & & \\ xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & & & \\ xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & x^5y^2 & & \\ xy & x^2y & x^3y & x^4y & x^5y & & \\ x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7, \end{array}$$

so

$$I(\Gamma_G) = \left\langle xy^6 + yz^6 + zx^6, x^8, y^8, z^8, x^6y, y^6z, z^6x, x^5y^3, y^5z^3, z^5x^3, xyz \right\rangle$$

Any ideal having $\Gamma(S_G)$ for its G-graph is generated by polynomials that express each of the generators of $I(\Gamma_G)$ as a linear combination of elements in $\Gamma(S_G)$. The three horns of S_G correspond to the representations

$$V_1 \oplus V_3 \oplus V_{12}.$$

If

$$S_H = \begin{array}{ccccccc} xy^7 & & & & & & \\ xy^6 & x^2y^6 & & & & & \\ xy^5 & x^2y^5 & x^3y^5 & & & & \\ xy^4 & x^2y^4 & x^3y^4 & & & & \\ xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & & & \\ xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & x^5y^2 & & \\ xy & x^2y & x^3y & x^4y & x^5y & x^6y & x^7y \\ x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7, \end{array}$$

then

$$I(\Gamma_H) = \left\langle \begin{array}{c} xy^6 + yz^6 + zx^6, y^2x^6, z^2y^6, x^2z^6, x^8, y^8, z^8, \\ x^5y^3, y^5z^3, z^5x^3, x^4y^4, y^4z^4, z^4x^4, xyz \end{array} \right\rangle.$$

Any ideal having $\Gamma(S_H)$ for its G-graph is generated by polynomials that express each of the generators of $I(\Gamma_H)$ as a linear combination of elements in $\Gamma(S_H)$. The five horns of S_H correspond to the representations

$$V_1 \oplus V_2 \oplus V_3 \oplus V_{12} \oplus V_{16}.$$

If

$$S_I = \begin{array}{ccccccc} xy^7 & & & & & & \\ xy^6 & x^2y^6 & & & & & \\ xy^5 & x^2y^5 & & & & & \\ xy^4 & x^2y^4 & x^3y^4 & & & & \\ xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & & & \\ xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & x^5y^2 & x^6y^2 & \\ xy & x^2y & x^3y & x^4y & x^5y & x^6y & x^7y \\ x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7, \end{array}$$

then

$$I(\Gamma_I) = \left\langle \begin{array}{c} xy^6 + yz^6 + zx^6, x^8, y^8, z^8, y^2x^7, z^2y^7, x^2z^7 \\ , x^5y^3, y^5z^3, z^5x^3, x^4y^4, y^4z^4, z^4x^4, y^5x^3, z^5y^3, x^5z^3, xyz \end{array} \right\rangle.$$

Any ideal having $\Gamma(S_I)$ for its G-graph is generated by polynomials that express each of the generators of $I(\Gamma_I)$ as a linear combination of elements in $\Gamma(S_I)$. The five horns of S_I correspond to the representations

$$V_1 \oplus V_2 \oplus V_{12} \oplus V_{16} \oplus V_{17}.$$

Finally if

$$S_J = \begin{array}{ccccccc} xy^7 & & & & & & \\ xy^6 & x^2y^6 & & & & & \\ xy^5 & x^2y^5 & & & & & \\ xy^4 & x^2y^4 & & & & & \\ xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & & & \\ xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & x^5y^2 & x^6y^2 & x^7y^2 \\ xy & x^2y & x^3y & x^4y & x^5y & x^6y & x^7y \\ x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7. \end{array}$$

Then

$$I_(\Gamma_J) = \left\langle xy^6 + yz^6 + zx^6, x^8, y^8, z^8, x^5y^3, y^5z^3, z^5x^3, y^4x^3, z^4y^3, x^4z^3, xyz \right\rangle.$$

Any ideal having $\Gamma(S_J)$ for its G-graph is generated by polynomials that express each of the generators of $I(\Gamma_J)$ as a linear combination of elements in $\Gamma(S_J)$. The three horns of S_J correspond to the representations

$$V_1 \oplus V_2 \oplus V_{17}.$$

Finally if $xy^8, x^6y^3, x^5y \in S_2$ and a monomial must be added to each of the spaces S_3, S_{11} and S_{12}. Figure 3.3 shows that there is only one way of

doing this.

$$S_K = \begin{array}{cccccccc} xy^8 & & & & & & & \\ xy^7 & & & & & & & \\ xy^6 & x^2y^6 & x^3y^6 & & & & & \\ xy^5 & x^2y^5 & x^3y^5 & & & & & \\ xy^4 & x^2y^4 & x^3y^4 & & & & & \\ xy^3 & x^2y^3 & x^3y^3 & & & & & \\ xy^2 & x^2y^2 & x^3y^2 & & & & & \\ xy & x^2y & x^3y & x^4y & x^5y & x^6y & x^7y & x^8y \\ x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \end{array}$$

Then

$$I(\Gamma_K) = \left\langle xy^6 + yz^6 + zx^6, x^9, y^9, z^9, x^4y^2, z^2y^4, x^2z^4, xyz \right\rangle.$$

Any ideal having $\Gamma(S_K)$ for its G-graph is generated by polynomials that express each of the generators of $I(\Gamma_K)$ as a linear combination of elements in $\Gamma(S_K)$. The two horns of S_K correspond to the representations

$$V_2 \oplus V_3.$$

Thus by Proposition 3.18 we have classified the G-graphs. For any ideal I defining a G-cluster there exists $\Gamma \in \Gamma(I) \cap \{\Gamma_A, \ldots, \Gamma_K\}$.

3.4 Syzygies

To compute the relations between the generators we have to use two different methods. The first method uses syzygies and is the same as the method used in the previous chapter. The second method uses A-Hilb $\mathbb{C}^3$.

3.4.1 Using syzygies

We show how to do this in two examples. Let I_G be any ideal which defines a G-cluster and has $\Gamma(S_G) \in \Gamma(I_G)$. Then I_G is generated by

$$\begin{aligned}
A &= xy^6 + yz^6 + zx^6 - a, \\
B &= x^6y - bxy^2 - cz^4x^4 - dz^5y^2, \\
C &= y^6z - byz^2 - cx^4y^4 - dx^5z^2, \\
D &= z^6x - bzx^2 - cy^4z^4 - dy^5x^2, \\
E &= x^8 - ex^3y - fxz^4 - gy^4z^2, \\
F &= y^8 - ey^3z - fyx^4 - gz^4x^2, \\
G &= z^8 - ez^3x - fzy^4 - gx^4y^2, \\
H &= x^5y^3 - hy^4 - iz^7 - jz^2x, \\
I &= y^5z^3 - hz^4 - ix^7 - jx^2y, \\
J &= z^5x^3 - hx^4 - iy^7 - jy^2z, \\
K &= xyz - k.
\end{aligned}$$

To find the syzygies between the generator of this ideal first find syzygies for the generators of the ideal $I(\Gamma_G)$. For example the syzygy

$$0 = xE - y^2B$$

of $I(\Gamma_G)$ becomes

$$0 = xE - y^2B = -xhy^4 - xiz^7 - jz^2x^2 + y^4bx + y^2cz^4x^4 + y^4dz^5$$

for the general I_G. This can be written as a linear combination of the monomials in Γ_G, that is of xy^4, z^2x^2 and z^5y^4 by using the above equations to remove the unwanted terms.

$$\begin{aligned}
0 &= xE - y^2B + izD - (cx^3yz^3 - ckx^2z^2 + idxy^4)K \\
&= (-h + b - idk)\,xy^4 + \left(-j - bi + ck^2\right)x^2z^2 + (d - ci)\,y^4z^5
\end{aligned}$$

Now since xy^4, z^2x^2, z^5y^4 form part of a basis for $\mathbb{C}[x,y,z]/I_G$ the coefficients of each of these monomials must be zero so

$$\begin{aligned} -h+b-idk &= 0 \\ -j-bi+ck^2 &= 0 \\ d-ci &= 0. \end{aligned}$$

Consider three other syzygies

$$\begin{aligned} 0 &= -x^4y^2K+zH+iG \\ &= kx^4y^2-igx^4y^2-hzy^4-ifzy^4-jz^3x-iez^3x, \end{aligned}$$

$$\begin{aligned} 0 &= x^3z^5K+x^2zE+y^3H-x^4A+iy^2zA+(k+f)J+ \\ &\quad (ex^4-iy^7+jy^2z-ix^5yz+gxy^3z^2)K-(ikx^4-gky^2z)K \\ &= -fiy^7-hy^7+ax^4-fhx^4-khx^4-ekx^4+ik^2x^4- \\ &\quad iay^2z-fjy^2z-2kjy^2z-gk^2y^2z \end{aligned}$$

and

$$\begin{aligned} 0 &= yE-x^2B-cz^3A+cyzG+ \\ &\quad (-dxyz^4+(f+ce-dk)z^3+cz^2y^5+cgx^3y^2)K \\ &= -fkz^3+acz^3+dk^2z^3-cekz^3 \\ &\quad +bx^3y^2-ex^3y^2-cgkx^3y^2-gy^5z^2-cfy^5z^2-ckz^2y^5. \end{aligned}$$

These imply the following relations between the generators.

$$\begin{array}{llllll} a=fh+kh+ek-ik^2 & b=h+idk & d=ci & e=b-cgk & & \\ h=-if & & j=-ie & k=ig & 0=-g-cf-ck & \end{array}$$

Thus this affine open set is the smooth three fold in $\mathbb{C}^4_{c,f,g,i}$ defined by the equation

$$g+cgi+cf.$$

Let I_C be any ideal which defines a G-cluster and has $\Gamma(S_C) \in \Gamma(I_C)$. Then the generators for I_C are shown in Equation 3.6.

The syzygies

$$x^4y^4Q - zK - hH = (-q + hg)x^4y^4 + (i + eh)yz^2 + (j + hf)x^5z^2$$

$$(-xy^5 - z^6)Q - z^2B + xzA - dL - cxzQ = \\ (q + dj)xy^5 + (q + b + dh)z^6 + (-a + di + gq)xz$$

$$-x^5N + zG + fyA - fxC + ((e - fc)y - fx^5)Q) = \\ (q + fb + fq)x^5 + (-g + fd)x^4z^5 + (-af - eq + fcq)y$$

$$-yB + xG - dz^2H + fz^4yN - x^2y^2dg(xyz + q)Q = \\ (b + ed - fq)yz^4 + (c - e + dgq^2)x^2y^2 + (-g + df)x^5z^4$$

$$-y^4G + xK - yfL + i(1 - f)Q - g(x^3y^3z^3 + qx^2y^2z^2 + q^2xyz + q^3)Q - hfA = \\ (e - h + fj - hf)y^6x + (-j - hf)x^6z - iq + ifq + gq^4 + hfa$$

give the relations

$$q = hg \quad i = -he \quad j = -hf \quad b = -q - hd$$
$$a = di + cq \quad g = fd \quad c = e - dgq^2 \quad e = (h + hf + f^2h).$$

Hence this affine piece is a copy of $\mathbb{C}^3$ with coordinates d, h, f.

For the ideals I_C, I_H and I_I calculations of this type determine show that this affine piece is a copy $\mathbb{C}^3$. For the remaining ideals this method determines only on threefolds in $\mathbb{C}^n$.

3.5 Proof of Conjecture 3.21

We prove Conjecture 3.21 when $A = \left\langle \frac{1}{1+k+k^2}(1, k, k^2) \right\rangle$. In Section 3.6 we develop an algorithm which constructs sets S, in Section 3.7 we prove that these sets are G-sets and that these are the only G-sets and in Section 3.9 we show that the G-sets S have at most 5 horns.

The Hirzebruch continued fraction

$$\frac{1+k+k^2}{k^2+1} = 2 - \frac{1}{2 - \ldots \frac{1}{2-\frac{1}{k+1}}} = [2, 2, \ldots, 2, k+1]$$

can be used to compute the generators of the A-invariant monomials contained in T. The generators are therefore $x^{1+k+k^2}, x^{k^2+1}y, \ \ldots, x^{k+1}y^k, xy^{k+1}$. The monomials 1 and $x^{k+1}z$ also need to be considered.

For $i = 1, \ldots, k-1$ a monomial is in the ith representation space of G if it is given by multiplying an invariant monomial by x^i, y^i or z^i. Notice $x^m y^n z^i$ and $x^{m-i}y^{n-i}$ belong to the same representation space if $m \geq i, n \geq i$. An invariant multiplied by x^{ik} and $x^{\overline{ik^2}} = x^{k^2-(i-1)(k+1)}$ will also be in V_i but this would mean taking too many elements in S unless the invariant was 1.

If $x^n y^m \in S$ then $nm \leq 1+k+k^2$. The only monomials which can be in S_0 are

$$a_0 := xy^{k+1} \quad e_0 := x^{1+k+k^2}.$$

For $i < k$ the only monomials which can be in S_i are

$$\begin{aligned}
a_i &:= xy^{k+i+1} \left(= y^i xy^{k+1}\right) & a_i' &:= x^{i+1}y^{k+1} \left(= x^i xy^{k+1}\right) \\
b_i &:= x^{k-i+1}y^{k-i} \left(= z^i x^{k+1} y^k\right) & c_i &:= x^k y^{i-1} \left(= y^i x^{k+1} z\right) \\
d_i &:= x^{ik} & e_i &:= x^{k^2-(i-1)(k+1)} \\
f_i &:= x^{2k+1-i}y^{k-1-i} \left(= z^i x^{2k+1} y^{k-1}\right) & g_i &:= x^i.
\end{aligned}$$

Let $i = \lfloor (k-1)/2 \rfloor$. If $f_i \in S$ then $b_{i+1}, c_{i+1}, f_{i+1}, g_{i+1} \in S$. This is too many monomials in S_{i+1} so we may assume that $f_j \notin S$ for all $j \leq (k-1)/2$.

Figure 3.5 shows the positioning of these monomials in T. The monomials a_i and a_i' count as one element of S by Proposition 3.18. By Condition 1 of the same proposition this can only happen if $x^{k+i+1} \in S$.

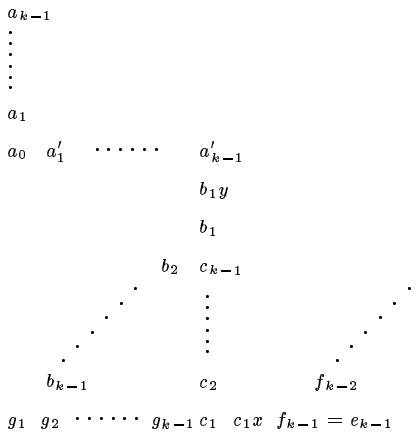


Figure 3.5: The monomials in T

Eigenspace will describe the representation space of A and representation space the representation space of G to which a monomial belongs. The monomial $x^i y^j$ belongs to the $(i + jk)$th eigenspace of A. The ith and jth eigenspaces of A are the same representation space of G if $i = j$ or, in the case when $3i \not\equiv 0 \pmod{1 + k + k^2}$, also if $i \equiv jk$ or $jk^2 \pmod{1 + k + k^2}$.

3.6 Construction of possible sets S

3.6.1 Step Zero

First choose a monomial in S_0. If $e_0 \in S_0$ then the set

$$S(e_0) = x, \ldots, x^{1+k+k^2} = e_0$$

contains $1 + k + k^2$ elements. If $k = 1$ then this is the only possibility. From now on assume that $k > 1$. If $e_0 \notin S$ then $a_0 = xy^{k+1} \in S_0$ and also $x^{k+1} \in S$

by Claim 3.14.

3.6.2 Step One

Now consider the possible monomials in S_1. Since $x,\ x^k \in S_1$, we only choose one monomial and there are three possibilities a_1, b_1 and e_1. If $e_1 \in \Gamma$ then this determines a set

$$S(a_0, e_1) = \begin{array}{cccc} xy^{k+1} & & & \\ \vdots & & & \\ xy & & & \\ x & x^2 & \cdots & x^{k^2}. \end{array}$$

If $b_1 \in S_1$ then $a_1 \notin S$ and every monomial below and to the left of b_1 must be in S, that is, the shaded area of Figure 3.6 must be contained in S. The monomial $x^{k+2} = f_{k-1}$, which is marked with one of the crosses, cannot be in S_{k-1} because the monomials b_{k-1}, c_{k-1} and g_{k-1} are all in the shaded area. A G-set is given by choosing one of the two blocks marked in Figure 3.6. Write $S(b_1)$ for a set of this type.

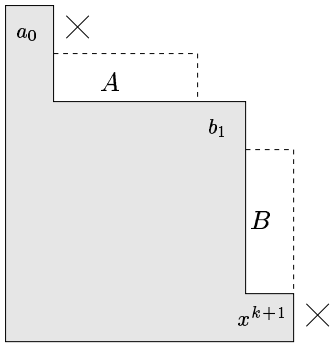


Figure 3.6: $S(b_1)$

If $b_1 \notin S$ then $a_1 \in S_1$ and so $f_{k-1} = x^{k+2} \in S$. If $k = 2$ then a_1 cannot be in S_1 and the first three choices determine all the possible G-sets. Also

if $k = 3$ then choosing a_1 uniquely determines a G-set and so from now on assume that $k > 3$.

3.6.3 Step Two

The only monomial that must be in S_2 is x^2. Defining a G-set involves choosing two more monomials in S_2. If $e_2 \in S$ then $d_2 \in S$ and

$$S(e_2, a_1) = \begin{array}{cccc} xy^{k+2} & & & \\ xy^{k+1} & x^2y^{k+1} & & \\ \vdots & \vdots & & \\ x & x^2 & \cdots & x^{k^2-k-1}. \end{array}$$

If $b_2 \in S$ then $c_2 \in S$. This is because if $b_2 \in S$ then the monomials $x^{k-1}y^{k-3}, x^3y, x^{k-1}y \in S$ and so $x^{k+3} \notin S$ since these monomials all belong to the same representation spaces. Therefore $a_2, d_2 \notin S$.

If $x^{k+3} \notin S$ then $f_{k-2}, a_{k-2} \notin S$ and so $c_{k-2} \in S$. Thus S contains the shaded region shown in Figure 3.7. The crosses show monomials that cannot be in S. Now $c_{k-1} \notin S$ because $g_{k-1}, b_{k-1}, f_{k-1} \in S$.

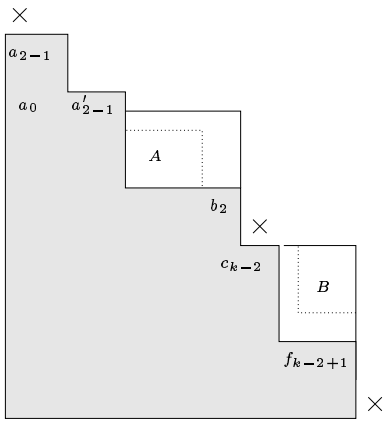


Figure 3.7: $S(b_i, c_i)$

These crosses imply that the only monomials that can be added to S must be from the Blocks A or B. In what follows count the columns of A from the right and the rows of A from the top but count the columns of B from the left and the rows of B from the bottom. This is because monomials can only be removed from the top row or right hand column of A and they must be added to B starting from the bottom row or left hand column.

As we will prove later monomials can only be taken from these blocks in the following way. For the first set take all of Block A. The next set is given by removing the first column of A and replacing it with the first row of B. Next remove what remains of the first row of A and replace it with the first column of B. Continue this way until all of A has been replaced with B. In other words take sets that are made up of a rectangle in A and an L shape in the B where the ends of the L shape reach the edges of B and the number of rows of the L shape minus the number of columns is equal to 0 or 1. When the rectangle and the L shape are put together they must form one of the larger rectangles. This is shown by the dotted lines in Figure 3.7. Write $S(b_2, c_2)$ for such a set containing (b_2, c_2).

If $k = 4$ then choosing a_2 is impossible and there are no more possibilities for G-sets. From now on assume that $k > 4$.

If $b_2, e_2 \notin S$ then a_2 must be one of the monomials in S_2. If d_2 is the other monomial in S_2 then cannot have $x^{k-1}y \in S$ since there are already three monomials in this representation space and so $c_{k-j}, f_{k-j}, a_{k-2} \notin S$ for $j > 1$. Thus $e_{k-2} \in S$ and two monomials from $a_{k-j}, b_{k-j}, e_{k-j}, d_{k-j}$ must be chosen for $j = 3, \ldots, k-3$. Set $i = 3$.

If $e_i \in S$ then $d_i \in S$ and call this set $S(e_i, a_{i-1})$. If $b_i \in S$ then $a_i \in S$. This is because there are already 3 elements in the $(k+i+1)$th representation space so $x^{ik-1} \notin S$ and $d_i \notin S$. One more element in the $(k-i)$th representa-

tion space is required. It cannot be d_{k-i} or e_{k-i} because $d_i \notin S$ so $a_{k-i} \in S$. This set is denoted by $S(e_{k-i+1}, a_{k-i})$. If $i = k/2$ and $a_i \in S$ then $b_i \in S$ and this is the final possibility. Else a_i and d_i are in S and so $e_{k-i+1} \in S$. If $i = (k-1)/2$ then $e_{i+1} \in S$ and there are no more possibilities. This set is called $S(e_{i+1}, a_i)$. Otherwise let $i \mapsto i+1$ and repeat this paragraph. Thus

$$S(e_j, a_{j-1}) = \begin{array}{ccccc} xy^{k+j} & & & & \\ \vdots & & & & \\ xy^{k+1} & \cdots & x^j y^{k+1} & & \\ \vdots & & & & \\ xy & \cdots & x^j y & & \\ x & \cdots & x^j & \cdots & x^{k^2-j(k+1)}. \end{array}$$

If $d_2 \notin S_2$ then $a_2, c_2 \in S_2$ and set $i = 2$.

3.6.4 Step $i+1$

The monomial $g_{i+1} \in S_{i+1}$ by Claim 3.19 so it remains to choose two monomials in S_{i+1}. There are two basic types of sets depending on whether S is determined by the fact it contains a_j and b_j or b_j and c_j for some j. The details are sketched below.

If $2i+1 < k$ then let $i \mapsto i+1$. In order for S to be a G-set we need to choose two monomials in S_i. We have already shown that $f_j \notin S$ for $2j \leq k-1$. Therefore $f_i \notin S$. Now $d_2 \notin S$ and since $d_2 \mid d_i$ and $d_2 \mid e_i$ it follows that d_i and e_i do not belong to S by Claim 3.14. Therefore we must choose two monomials from the set $\{a_i, b_i, c_i\}$.

If $b_i, c_i \in S_i$ then this is the case shown in Figure 3.7 where 2 is replaced with j. We construct G-sets $S(b_i, c_i)$ in the same that the sets $S(b_2, c_2)$ were determined.

If $a_i, b_i \in S_i$ then the shaded area shown in Figure 3.8 is in S. This is because $c_i \notin S$ so $f_{k-i}, c_{k-i} \notin S$ and so $a_{k-i} \in S$. Also because $b_{i-1}, c_i \notin S$ so $a_{k-i+1}, c_{k-i+1} \notin S$ and $f_{k-i+1} \in S$. The crosses are due to the monomials $x^{2k-i+2}, x^{k+i+1}y^{i-2}, x^{k-1}y^{i-1}$ and $x^{k-i+2}y^{k-i}$ that cannot be in S since the shaded area has enough monomials in these representation spaces.

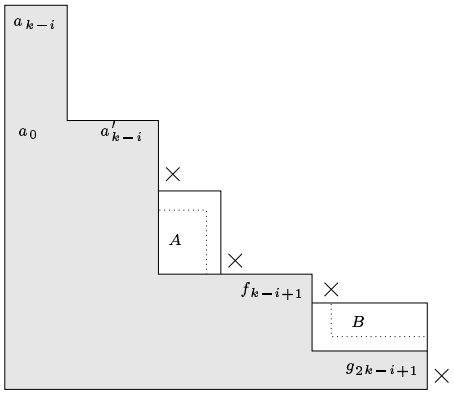


Figure 3.8: $S(a_i, b_i)$

We construct G-sets in a similar manner to the case $S(b_2, c_2)$. First add Block A to the shaded area in Figure 3.8. Then from this set remove the first row of A and replace it with the first column of B. From this remove what remains of the first column of A and replace it with the first row of B. Continue this process until all of A has been replaced with B.

If b_i does not belong to S then it must be that a_i, c_i belong to S_i. In this case return to the start of Step $i+1$ and repeat this process.

If $2i+1=k$ then there exists a unique set $S(\text{end})$ shown in Figure 3.9. This is because $x^{i+2}y^i, x^{k+i+2} \notin S$.

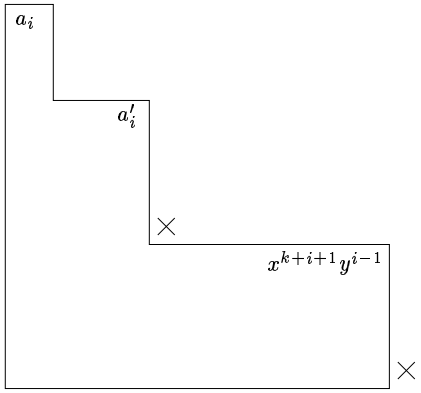


Figure 3.9: $S(\text{end})$

Finally when $2i+1>k$ there are no more cases.

3.7 Proof that each of these sets is a G-set

I will now show that each of these sets is a G-set and that these are the only ones. There are two standard domains S' and S'' in T which contains all the monomials necessary for a G-set. These are

$$S' := \left\{x, \ldots, x^{1+k+k^2}\right\} \quad \text{and} \quad S'' := \left\{xy^{k+1}, x^i y^j \;\middle|\; 0 \leq i \leq k, 0 < j \leq k\right\}.$$

Now S'' is not a G-set because it contains xy^{k+1} but not x^{k+1}.

First show that a set $S(e_j, a_{j-1})$ is a G-set. It is obtained from S' by replacing the blocks

$$A_0 := \left\{x^{k^2+1}, \ldots, x^{k^2+k+1}\right\}, \cdots, A_{j-1} := \left\{x^{k^2-j(k+1)+1}, \ldots, x^{k^2-(j-1)(k+1)}\right\}$$

with the blocks

$$B_0 := \left\{xy, \ldots, xy^{k+1}\right\}, \cdots, B_{j-1} := \left\{x^{j+1}y, \ldots, x^{j+1}y^{k+1}\right\}.$$

We claim that the monomials in Block A_i belong to the same representation spaces as those in Block B_i and that they occur in the same order. The nth monomial in A_i is $x^{k^2-(i+1)(k+1)+n}$ which is in the $(k^2-(i+1)(k+1)+n) \equiv (n+(i+2)k^2)$th eigenspace of A. The nth monomial in B_i is $x^{i+2}y^n$ which is in the $(i+2+nk)$th eigenspace of A. These belong to the same representation space of G since $(n+(i+2)\,k^2)\,k \equiv i+2+nk \pmod{1+k+k^2}$.

The first part of this claim proves that the set given by replacing A_i with B_i still has all the characters. The fact that they occur in the same order means that part of some Block A_i cannot be replaced with part of B_i.

Consider the two sets $S(b_1)$. Take the set containing Block A then this is obtained from S'' by replacing $x^k y^k$ with x^{k+1} which is in the same representation space. This set is a G-set. Block A has monomials

$$xy^k, \ldots, x^{k-1}y^k$$

which belong to the eigenspaces

$$2+k^2, 3+k^2, \ldots, k-1+k^2.$$

Block B has monomials

$$x^{k+1}y, \ldots, x^{k+1}y^{k-2}$$

which belong to the eigenspaces

$$k+1+k \equiv k(2+k^2), k+1+2k \equiv k(3+k^2), \ldots, k+1+k(k-2) \equiv k(k-1+k^2).$$

So replacing A with B is still a G-set. Since the representation spaces in Block A occur in the same order as those in Block B and are distinct, part of Block A cannot be replaced with part of Block B. Hence these are the only two sets containing b_1 which are G-sets.

Next consider sets of the form $S(b_i, c_i)$. First show that the set which contains all of Block A is a G-set. The shaded area in Figure 3.10 shows the set $S(b_i, c_i)$ where all of Block A has been chosen. If the Blocks C and E are replaced with D in this set then the set S'' is obtained. This is a G-set if the monomials in C and E are in the same representation spaces as those in D. This is true since $x^{k-j}y^{k-j-1} \in D$ belongs to the same representation space as $x^{j+2}y^{k+1} \in C$ for $j = 0, \ldots, i-1$. Also $x^{k-m-n}y^{k-m} \in D$ belongs to the same representation space as $x^{k+1+n}y^{n+m} \in E$ for $m = 0, \ldots, i-1$ and $n = 0, \ldots, i-1-m$ and $x^{k+2+n+m}y^{n} \in D$ belongs to the same representation space as $x^{k-m}y^{k-2-n-m} \in E$ for $m = 0, \ldots, i-1$ and $n = 0, \ldots, i-1-m$. Thus we have constructed a bijection $\theta\colon D \to C \cup E$ such that $f \in D$ belongs to the same representation as $\theta(f)$.

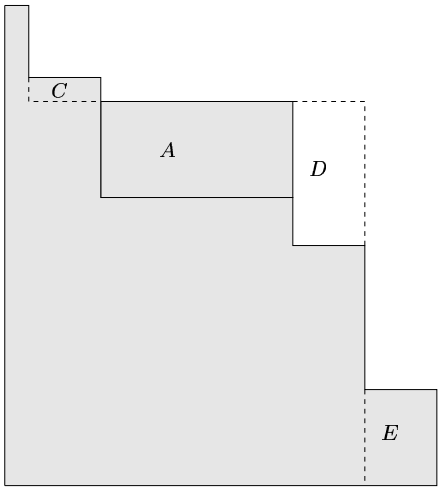


Figure 3.10: $S(b_i, c_i)$

Then show that any of the sets $S(b_i, c_i)$ are G-sets. Block A contains the

monomials

$$\begin{array}{ccccc} x^{i+2}y^k & x^{i+3}y^k & x^{i+4}y^k & \cdots & x^{k-i+1}y^k \\ \vdots & \vdots & \vdots & & \vdots \\ x^{i+2}y^{k-i+1} & x^{i+3}y^{k-i+1} & x^{i+4}y^{k-i+1} & \cdots & x^{k-i+1}y^{k-i+1} \end{array}$$

and Block B contains the monomials

$$\begin{array}{ccccc} x^{k+1}y^{k-i-1} & x^{k+2}y^{k-i-1} & \cdots & x^{i+k}y^{k-i-1} \\ \vdots & \vdots & & \vdots \\ x^{k+1}y^{i-1} & x^{k+2}y^{i-1} & \cdots & x^{i+k}y^{i-1}. \end{array}$$

Once n rows and n columns have been removed from A and added to B, the monomials that remain in the $(n+1)$th column of A are

$$\begin{array}{c} x^{k-i-n+1}y^{k-n} \\ \vdots \\ x^{k-i-n+1}y^{k-i-1}. \end{array}$$

These monomials belong to the same representation spaces as those monomials

$$x^{k+n+1}y^{i+n-1}\cdots x^{i+k}y^{i+n-1}$$

that will be added to the $(n+1)$th row of B. This is checked as in the case of $S(b_1)$. The same method shows that the monomials remaining in the $(n+1)$th row of A belong to the same representation space as those that will be added to the $(n+1)$th column of B.

Take monomials $x^\lambda y^\mu$ and $x^{\lambda+i}y^{\mu+j}$ in A. Now $\lambda+\mu k \not\equiv \lambda+i+(\mu+j)k$, $\lambda k+\mu k^2 \not\equiv \lambda+i+(\mu+j)k$ and $\lambda k^2+\mu k^3 \not\equiv \lambda+i+(\mu+j)k$ so $x^\lambda y^\mu$ and $x^{\lambda+i}y^{\mu+j}$ in Block A belong to distinct representations. This is the only way to move monomials from A to B and these are the only sets containing b_i and c_i which are G-sets.

Now consider the sets of the form $S(a_i, b_i)$. Again start by showing that the set which contains all of Block A is a G-set. This set is the shaded area in Figure 3.11. Replacing Blocks C, D and E with Block F gives S''. As in the previous example there is a one-to-one correspondence between the representation space to which monomials in $C \cup D \cup E$ belongs and the representation space to which monomials in F belongs. So this set is a G-set

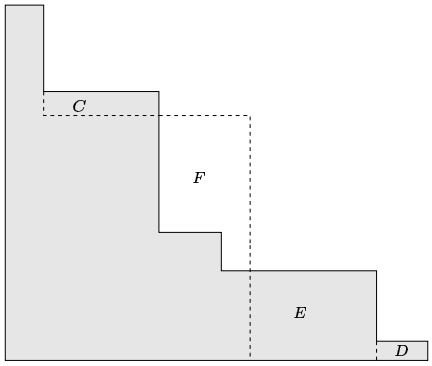


Figure 3.11: $S(a_i, b_i)$

The proof that all other sets $S(a_i, b_i)$ are G-sets is the same as the proof given for all other sets $S(b_i, c_i)$. Again the monomials in A belong to distinct representation spaces so the sets listed above are the only possible G-sets containing a_i and b_i.

The same methods show that $S(\text{end})$ is a G-set.

3.8 The ideals

Let $\Gamma(e_j, a_{j-1})$ be the G-graph generated by $S(e_j, a_{j-1})$. If $j = 1$ then the generators of $I(\Gamma(e_j, a_{j-1}))$ consist of polynomials

$$\begin{aligned}
&x^{1+k+k+2} + y^{1+k+k+2} + z^{1+k+k+2} - a,\\
&x^i y + b_{i,1} x^{\overline{i+k}} + c_{i,1} y^{\overline{ik^2+1}} + d_{i,1} z^{\overline{ik+k^2}},\\
&y^i z + b_{i,1} y^{\overline{i+k}} + c_{i,1} z^{\overline{ik^2+1}} + d_{i,1} x^{\overline{ik+k^2}},\\
&z^i x + b_{i,1} z^{\overline{i+k}} + c_{i,1} x^{\overline{ik^2+1}} + d_{i,1} y^{\overline{ik+k^2}},\\
&xy^i + b_{1,i} x^{\overline{1+ik}} + c_{1,i} y^{\overline{i+k^2}} + d_{1,i} z^{\overline{ik^2+k}},\\
&yz^i + b_{1,i} y^{\overline{1+ik}} + c_{1,i} z^{\overline{i+k^2}} + d_{1,i} x^{\overline{ik^2+k}},\\
&zx^i + b_{1,i} z^{\overline{1+ik}} + c_{1,i} x^{\overline{i+k^2}} + d_{1,i} y^{\overline{ik^2+k}},
\end{aligned}$$

for $i < 1 + k + k^2$ where $b_{r,s}, c_{r,s}, d_{r,s} \in \mathbb{C}$.

If $j > 1$ Then the corresponding ideal, $I(\Gamma(e_j, a_{j-1}))$ has generators

$$\begin{aligned}
&xy^{k+1} + yz^{k+1} + zx^{k+1} - a,\\
&x^{k^2-(j-1)(k+1)+1} - bz^{j+1}\left(zx^{k+1} - yz^{k+1}\right) - cz^{j+1} - dy^{(j+1)k},\\
&y^{k^2-(j-1)(k+1)+1} - bx^{j+1}\left(xy^{k+1} - zx^{k+1}\right) - cx^{j+1} - dz^{(j+1)k},\\
&z^{k^2-(j-1)(k+1)+1} - by^{j+1}\left(yz^{k+1} - xy^{k+1}\right) - cy^{j+1} - dx^{(j+1)k},\\
&x^i y + b_i x^{\overline{i+k}} + c_i y^{\overline{ik^2+1}} + d_i z^{\overline{ik+k^2}},\\
&y^i z + b_i y^{\overline{i+k}} + c_i z^{\overline{ik^2+1}} + d_i x^{\overline{ik+k^2}},\\
&z^i x + b_i z^{\overline{i+k}} + c_{i,1} x^{\overline{ik^2+1}} + d_{i,1} y^{\overline{ik+k^2}},\\
&xyz - q
\end{aligned}$$

for $i < k^2 - (j-1)(k+1) + 1$ where $b_i, c_i, d_i \in \mathbb{C}$.

Let $\Gamma(b_1)$ be the G-graph which is generated by $S(b_1)$ where Block A is chosen. The corresponding ideal $I(\Gamma(b_1))$ has generators

$$xy^{k+1} + yz^{k+1} + zx^{k+1} - a,$$

$$
\begin{aligned}
&x^{k+2} - bx^2y - cy^kz^{k-2} - dz^{k-1},\\
&y^{k+2} - by^2z - cz^kx^{k-2} - dx^{k-1},\\
&z^{k+2} - bz^2x - cx^ky^{k-2} - dy^{k-1},\\
&x^{k-1}y - exy^2 - fx^{k-1}z^{k-1} - gy^2z^k,\\
&y^{k-1}z - eyz^2 - fy^{k-1}x^{k-1} - gz^2x^k,\\
&z^{k-1}x - ezx^2 - fz^{k-1}y^{k-1} - gx^2y^k,\\
&x^ky^k - lyz - mz^{k+1} - nx^kz,\\
&y^kz^k - lzx - mx^{k+1} - ny^kx,\\
&z^kx^k - lxy - my^{k+1} - nz^ky,\\
&xyz - q.
\end{aligned}
$$

If Block B is chosen swap the triple $\left(x^{k-1}y, y^{k-1}z, z^{k-1}x\right)$ in the middle three generators with $\left(y^2z^k, z^2x^k, x^2y^k\right)$ and replace the triple $\left(x^ky^k, y^kz^k, z^kx^k\right)$ with $\left(z^{k+1}x^{k-1}, x^{k+1}y^{k-1}, y^{k+1}z^{k-1}\right)$ in the last triple of polynomials.

Let $\Gamma(b_j, c_j)$ be the G-graph generated by $S(b_jc_j)$ where i_1 columns and i_2 rows of Block A are chosen and $i_1 = i_2$ or $i_1 = i_2 + 1$. Then the ideal $I(\Gamma(b_j, c_j))$ has generators

$$
\begin{aligned}
&xy^{k+1} + yz^{k+1} + zx^{k+1} - a,\\
&x^{k+j+1} - bx^{j+1}y - cy^{k-j+1}z^{k-j-1} - dx^{j-1}z^{k-1},\\
&y^{k+j+1} - by^{j+1}z - cz^{k-j+1}x^{k-j-1} - dy^{j-1}x^{k-1},\\
&z^{k+j+1} - bz^{j+1}x - cx^{k-j+1}y^{k-j-1} - dz^{j-1}y^{k-1},\\
&x^{k-j+2}y^{k-j} - ez^{k+j} - fxz^j - gy^{k-1}z^{j-2},\\
&y^{k-j+2}z^{k-j} - ex^{k+j} - fyx^j - gz^{k-1}x^{j-2},\\
&z^{k-j+2}x^{k-j} - ey^{k+j} - fzy^j - gx^{k-1}y^{j-2},\\
&x^{k-j+2-i_1}y^{k-j+1} - ly^{k+i_1}z^{j-2+i_1} - my^{i_1}z^{j-1+i_1} - nx^{k+1-i_1}z^j,\\
&y^{k-j+2-i_1}z^{k-j+1} - lz^{k+i_1}x^{j-2+i_1} - mz^{i_1}x^{j-1+i_1} - ny^{k+1-i_1}x^j,\\
&z^{k-j+2-i_1}x^{k-j+1} - lx^{k+i_1}y^{j-2+i_1} - mx^{i_1}y^{j-1+i_1} - nz^{k+1-i_1}y^j,
\end{aligned}
$$

$$
\begin{aligned}
&x^{j+1}y^{k+1-i_2} - ox^{j+i_2-1}z^{k+i_2} - py^{k-j+1-i_2}z^{k-j} - rx^{j+i_2}z^{i_2},\\
&y^{j+1}z^{k+1-i_2} - oy^{j+i_2-1}x^{k+i_2} - pz^{k-j+1-i_2}x^{k-j} - ry^{j+i_2}x^{i_2},\\
&z^{j+1}x^{k+1-i_2} - oz^{j+i_2-1}y^{k+i_2} - px^{k-j+1-i_2}y^{k-j} - rz^{j+i_2}y^{i_2},\\
&x^{k+1+i_2}y^{j-1+i_1} - sx^{1+i_2}y^{j+i_2} - tx^{k-j+1}z^{k-j+1-i_1} - uy^jz^{k-i_1},\\
&y^{k+1+i_2}z^{j-1+i_1} - sy^{1+i_2}z^{j+i_2} - ty^{k-j+1}x^{k-j+1-i_1} - uz^jx^{k-i_1},\\
&z^{k+1+i_2}x^{j-1+i_1} - sz^{1+i_2}x^{j+i_2} - tz^{k-j+1}y^{k-j+1-i_1} - ux^jy^{k-i_1},\\
&xyz - q
\end{aligned}
$$

if $i_1 = i_2$. Otherwise replace the final triple of polynomials with

$$
\begin{aligned}
&x^{k+1+i_2}y^{j-1+i_1} - sx^{i_1}y^{j+i_1} - ty^{j+1}z^{k-i_2} - uz^{k-j-i_2}x^{k-j},\\
&y^{k+1+i_2}z^{j-1+i_1} - sy^{i_1}z^{j+i_1} - tz^{j+1}x^{k-i_2} - ux^{k-j-i_2}y^{k-j},\\
&z^{k+1+i_2}x^{j-1+i_1} - sz^{1+i_2}x^{j+i_2} - tx^{j+1}y^{k-i_2} - uy^{k-j-i_2}z^{k-j}.
\end{aligned}
$$

Let $\Gamma(a_j, b_j)$ be the G-graph generated by $S(a_j, b_j)$ where i_1 rows are taken from Block A and i_2 rows from Block B. Then ideals $I(\Gamma(a_j, b_j))$ have generators

$$
\begin{aligned}
&xy^{k+1} + yz^{k+1} + zx^{k+1} - a,\\
&x^{2k-j+2} - bx^{k-j+2}y - cy^iz^{j-2} - dy^{k+j}z^{j-3},\\
&y^{2k-j+2} - by^{k-j+2}z - cz^ix^{j-2} - dz^{k+j}x^{j-3},\\
&z^{2k-j+2} - bz^{k-j+2}x - cx^iy^{j-2} - dx^{k+j}y^{j-3},\\
&x^{k-j+2}y^{k-j-i_1} - ex^{i_1}z^{k+j+i_1} - fx^{1+i_1}z^{i_1+j} - gy^{k-1-i_1}z^{j-2},\\
&y^{k-j+2}z^{k-j-i_1} - ey^{i_1}x^{k+j+i_1} - fy^{1+i_1}x^{i_1+j} - gz^{k-1-i_1}x^{j-2},\\
&z^{k-j+2}x^{k-j-i_1} - ez^{i_1}y^{k+j+i_1} - fz^{1+i_1}y^{i_1+j} - gx^{k-1-i_1}y^{j-2},\\
&x^{k-1-i_2}y^{j-1} - ly^{k+j+1+i_2}z^{i_2} - my^{j+1+i_2}z^{1+i_2} - nx^{k-j-i_2-1}z^{k-j+1},\\
&y^{k-1-i_2}z^{j-1} - lz^{k+j+1+i_2}x^{i_2} - mz^{j+1+i_2}x^{1+i_2} - ny^{k-j-i_2-1}x^{k-j+1},\\
&z^{k-1-i_2}x^{j-1} - lx^{k+j+1+i_2}y^{i_2} - mx^{j+1+i_2}y^{1+i_2} - nz^{k-j-i_2-1}y^{k-j+1},
\end{aligned}
$$

$$x^{k+j+1+i_1}y^{i_2+1} - ox^{j+1+i_1}y^{2+i_2} - px^{j-2+i_1-i_2}z^{k-2-i_2} - ry^{k-j+2-i_1+i_2}z^{k-j-1-i_1},$$
$$y^{k+j+1+i_1}z^{i_2+1} - oy^{j+1+i_1}z^{2+i_2} - py^{j-2+i_1-i_2}x^{k-2-i_2} - rz^{k-j+2-i_1+i_2}x^{k-j-1-i_1},$$
$$z^{k+j+1+i_1}x^{i_2+1} - oz^{j+1+i_1}x^{2+i_2} - pz^{j-2+i_1-i_2}y^{k-2-i_2} - rx^{k-j+2-i_1+i_2}y^{k-j-1-i_1},$$
$$x^{k+j+1}y^{i-2} - sz^{k-j+1}x - tz^{2k-j+1} - ux^{j+1}y^{j-1},$$
$$y^{k+j+1}z^{i-2} - sx^{k-j+1}y - tx^{2k-j+1} - uy^{j+1}z^{j-1},$$
$$z^{k+j+1}x^{i-2} - sy^{k-j+1}z - ty^{2k-j+1} - uz^{j+1}x^{j-1},$$
$$xyz - q.$$

The ideals of the form $I(\Gamma(\text{end}))$ have generators

$$xy^{k+1} + yz^{k+1} + zx^{k+1} - a,$$
$$x^{(3k+3)/2} - bx^{(k+3)/2}y - cy^{(k+1)/2}z^{(k-3)/2} - dy^{(3k+1)/2}z^{(k-5)/2},$$
$$y^{(3k+3)/2} - by^{(k+3)/2}z - cz^{(k+1)/2}x^{(k-3)/2} - dz^{(3k+1)/2}x^{(k-5)/2},$$
$$z^{(3k+3)/2} - bz^{(k+3)/2}x - cx^{(k+1)/2}y^{(k-3)/2} - dx^{(3k+1)/2}y^{(k-5)/2},$$
$$x^{(k+3)/2}y^{(k-1)/2} - ez^{(3k+1)/2} - fxz^{(k+1)/2} - gy^{k-1}z^{(k-3)/2},$$
$$y^{(k+3)/2}z^{(k-1)/2} - ex^{(3k+1)/2} - fyx^{(k+1)/2} - gz^{k-1}x^{(k-3)/2},$$
$$z^{(k+3)/2}x^{(k-1)/2} - ey^{(3k+1)/2} - fzy^{(k+1)/2} - gx^{k-1}y^{(k-3)/2},$$
$$xyz - q.$$

In each case syzygies between these generators can be computed using the methods outlined in Section 3.4.

3.9 Conclusion of the proof

The set $S(e_0)$ has only one horn, this is invariant. A set $S(e_i, a_{i-1})$ has two horns one of which is invariant if $i = 1$ Both of the sets $S(b_1)$ have three horns and one of these horns is invariant. Note that x^{k+1} is not a horn since $xy^{k+1} \in S$. The set $S(\text{end})$ has only two horns and neither of these are invariant.

The set given by choosing Block A to be in $S(b_i, c_i)$ has four horns. Removing the first column of A and replacing it with the first row of B adds one horn. All the other moves leave the number of horns unchanged, except the final move which reduces the number of horns by 2. Thus a set $S(b_i, c_i)$ has at most five horns.

The set $S(a_i, b_i)$ given by choosing Block A has three horns. Again note that x^{2k-i+1} is not a horn. Removing the first row from A increases the number of horns by one so this set has four horns. Removing the first column increases the number of horns by one more. The other moves will not alter the number of horns so all the sets of this type will have five horns. Except the final move which will decrease the number of horns by two. Thus a set $S(a_i, b_i)$ has at most five horns.

Thus a G-set S has at most five horns.

3.10 Conclusion

It is therefore possible to give an explicit description of an affine cover of G-Hilb $\mathbb{C}^3$. This shows that G-Hilb $\mathbb{C}^3$ is again a smooth rational variety. It has an affine cover but again most of the affines are not isomorphic to $\mathbb{C}^3$.

The same methods as those used in the proof of Conjecture 3.21 when $A = \left\langle \frac{1}{1+k+k^2}(1, k, k^2) \right\rangle$ can be used to prove Conjecture 3.21 for other families of trihedral groups. For example when A is a cyclic group generated by an element of the form $\frac{1}{3k^2+3k+1}(1, 3k+1, \overline{(3k+1)^2})$ or $A = \mathbb{Z}_r \times \mathbb{Z}_r$. We conclude this chapter by making some remarks about these cases.

Consider the group A generated by $\frac{1}{1+k+k^2}(1, k, k^2)$. When $k \equiv 1 \pmod 3$ the group A has a subgroup A' generated by $\frac{1}{3k'^2+3k'+1}\left(1, 3k'+1, 3k'^2-1\right)$ where $k = 3k' + 1$. The group $A' \rtimes \mathbb{Z}_3$ is a trihedral group. When G is a

trihedral group of this type there are three choices for the invariant monomial in S these are xy^{3k+2}, $x^{2k+1}y^k$ and $x^{1+3k+3k^2}$. When G is a trihedral group of type $\mathbb{Z}_r \times \mathbb{Z}_r \rtimes \mathbb{Z}_3$ then x^r must be invariant monomial in S we can show that any G-set S has 1, 2 or 3 horns.

Chapter 4

Equivariant and Orbifold Riemann–Roch

4.1 Introduction

4.1.1 Notation

Fix the following notation throughout this chapter. Let G be a finite group acting on a smooth projective variety Z. For a nontrivial element $g \in G$ of order r say, let $H := \langle g \rangle$ be the subgroup generated by g. Fix ε a primitive rth root of unity. Let $\pi\colon Z \to X = Z/H$ be the quotient morphism. Then X is normal and π is a finite morphism.

For $h \in H$ let $Z^h = \left\{ z \in Z \;\middle|\; hz = z \right\}$ and define Z^H by the following formula $Z^H = \bigcup_{h \in H \smallsetminus 1} Z^h$. The action is *coprime* if $Z^H = Z^h$ for any $h \neq 1 \in H$.

If D is a prime divisor and D_1 any $\mathbb{Q}$-Cartier divisor we write $\lambda D \in \operatorname{Supp} D_1$ to mean $D \in \operatorname{Supp} D_1$ but $D \notin \operatorname{Supp}(D_1 - \lambda D)$. We also write $P \in \operatorname{Supp} D_1$ to mean $P \in D$ and $D \in \operatorname{Supp} D_1$.

Let $\mathcal{F}$ be a coherent sheaf on a variety X then denote by $\mathcal{F}^*$ the dual $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ of the sheaf $\mathcal{F}$.

4.1.2 Background to the equivariant Riemann–Roch theorem

Let x be an irreducible component of Z^g and $\mathcal{N}_x = \mathcal{N}_{x\big|Z}$ the normal sheaf of x in Z which is a locally free sheaf of rank $\operatorname{codim}(x, Z)$ on the trivial G-variety x. By Proposition 1.18 this decomposes into eigensheaves. We denote this decomposition by

$$\mathcal{N}_x = \bigoplus_i \mathcal{N}_{xi} \otimes \chi_i,$$

where χ_i is the character of $\langle g \rangle$ given by $\chi_i(g) = \varepsilon^i$. This gives a decomposition of $\mathcal{N}^g$, the normal sheaf of $Z^g \in Z$, as follows, $\mathcal{N}^g = \bigoplus_i \mathcal{N}_i^g \otimes \chi_i$.

For a locally free G-sheaf $\mathcal{F}$ on a trivial G-space X write $\lambda_{-1}(\mathcal{F})$ for the element $\bigoplus(-1)^i \Lambda^i(\mathcal{F})$ in $K^G(X)$ and write $\operatorname{td}(X)$ for the Todd class of X.

Let G be a finite group acting on a smooth projective variety Z and $\mathcal{F}$ a G-sheaf. The vector space $\mathrm{H}^i(Z, \mathcal{F})$ is a representation of G so the character of any element g acting on the virtual representation $\bigoplus_{i\geq 0}(-1)^i \mathrm{H}^i(Z, \mathcal{F})$ can be defined. Equivariant Riemann–Roch gives an expression for this in terms of the irreducible components of the fixed locus.

Theorem 4.1 (Equivariant Riemann–Roch [AS68]) *Let Z be a non-singular projective variety, G a finite group acting on Z and $\mathcal{F}$ a locally free G-sheaf. Then*

$$\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \mathrm{H}^p(Z, \mathcal{F})\right) = \int_{Z^g} \frac{\operatorname{ch}(\mathcal{F}_{|Z^g})(g) \cdot \operatorname{td}(Z^g)}{\operatorname{ch}\lambda_{-1}((\mathcal{N}^g)^*)(g)}. \qquad (4.1)$$

In this chapter we give an explicit proof of this theorem when Z is a curve or a surface and $\mathcal{F}$ is an invertible G-sheaf. We then use these results to generate a Riemann–Roch theorem for an orbifold of dimension n having only isolated cyclic quotient singularities.

4.1.3 Technical preliminaries on higher direct image sheaves

We begin by recalling some properties of higher direct image sheaves. See [Har77, III.8] for details. The results in this section are not new but some proofs are included here.

Definition 4.2 Let $f\colon Y \to X$ be a continuous map of topological spaces, $i \geq 0$ and $\mathcal{F}$ a sheaf of Abelian groups on Y. The *ith higher direct image* of $\mathcal{F}$ is the sheaf, $\mathcal{R}^i f_*(\mathcal{F})$, associated to the presheaf

$$V \mapsto \mathrm{H}^i\left(f^{-1}V, \mathcal{F}(f_{-1}V)\right)$$

on X. Notice that $\mathcal{R}^0 f_*(\mathcal{F}) = f_*\mathcal{F}$.

Lemma 4.3 *If $f\colon Y \to X$ is an affine morphism and $\mathcal{F}$ a coherent $\mathcal{O}_Y$-module then $\mathcal{R}^i f_*(\mathcal{F}) = 0$ for all $i > 0$.*

Proof. If $V \subset X$ is affine then $f^{-1}(V)$ is affine and so $\mathrm{H}^i\left(f^{-1}V, \mathcal{F}|_{f^{-1}V}\right) = 0$. Thus $\mathcal{R}^i f_*(\mathcal{F}) = 0$ for all $i > 0$. □

The Leray spectral sequence relates the cohomology of the sheaf $\mathcal{F}$ with the cohomology of the sheaves $\mathcal{R}^i f_*(\mathcal{F})$.

Theorem 4.4 (Leray spectral sequence [God73]) *Let $f\colon Y \to X$ be a morphism between ringed spaces and $\mathcal{F}$ an $\mathcal{O}_Y$-module. Then there exists a*

spectral sequence with

$$E_\infty \implies \mathrm{H}^*(Y, \mathcal{F}) \text{ where } E_2^{p,q} = \mathrm{H}^p(X, \mathcal{R}^q f_* \mathcal{F}).$$

Taking alternating sums in the Leray spectral sequence shows that

$$\chi(Y, \mathcal{F}) = \sum_q (-1)^q \chi(X, \mathcal{R}^q f_* \mathcal{F}). \tag{4.2}$$

It follows immediately from Theorem 4.4 that if $\mathcal{R}^i f_*(\mathcal{F}) = 0$ for all $i > 0$ then $\mathrm{H}^i(Y, \mathcal{F}) \cong \mathrm{H}^i(X, f_*\mathcal{F})$ and so $\chi(Y, \mathcal{F}) = \chi(X, f_*\mathcal{F})$. In particular this is true if f is an finite morphism by Lemma 4.3.

It could be hoped that if $f: Y \to X$ is a resolution of rational singularities on X and $\mathcal{M} = f^*\mathcal{L}/(\text{torsion})$ then $\chi(Y, \mathcal{M}) = \chi(X, f_*\mathcal{M})$. However this is only true when $\dim Y \leq 2$ as we will see in Lemma 4.5. In higher dimensional cases we will also have terms $\chi(X, \mathcal{R}^i f_*(\mathcal{M}))$, where $\mathcal{R}^i f_*(\mathcal{M})$ is a sheaf supported only over the singular locus of X. This will be proved in the following lemma.

Lemma 4.5 *Let X be a projective variety having at worst rational singularities and $f: Y \to X$ be a resolution of X. Let $\mathcal{L}$ be a divisorial sheaf on X. Then $\mathcal{M} = (f^*\mathcal{L})/(\text{torsion})$ is a rank one torsion free sheaf on Y and*

$$\chi(Y, \mathcal{M}) = \chi(X, f_*\mathcal{M}) + \sum_{i=1}^{n-2} (-1)^i \chi(X, \mathcal{R}^i f_* \mathcal{M}).$$

If $i > 0$ and X has only isolated singularities then

$$\chi(X, \mathcal{R}^i f_* \mathcal{M}) = h^0(X, \mathcal{R}^i f_* \mathcal{M}).$$

PROOF. For any coherent sheaf $\mathcal{L}$ on a projective variety there exists a surjective morphism

$$\mathcal{O}_X^N(-n) \to \mathcal{L} \to 0$$

for some $n, N \in \mathbb{Z}$. See [Har77, II.5 Cor 5.18]. Since $f^*\mathcal{O}_X = \mathcal{O}_Y$ this gives a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-n) \to \mathcal{M} \to 0$$

where $\mathcal{M} = (f^*\mathcal{L})/(\text{torsion})$ as in the definition and $\mathcal{K}$ is the defined as the kernel of the surjective morphism $\mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-n) \to \mathcal{M}$. Applying f_* gives the following long exact sequence.

$$\begin{aligned} 0 \longrightarrow \quad f_*\mathcal{K} \quad \longrightarrow \quad f_*(\mathcal{O}_Y^N \otimes f^*(\mathcal{O}_X(-n)) \quad \longrightarrow \quad f_*\mathcal{M} \quad \longrightarrow \\ \to \ \mathcal{R}^1 f_*(\mathcal{K}) \ \to \ \mathcal{R}^1 f_*(\mathcal{O}_Y^N \otimes f^*(\mathcal{O}_X(-n)) \ \to \ \mathcal{R}^1 f_*(\mathcal{M}) \ \to \ \cdots \end{aligned}$$

By the Projection Formula [Har77, III Ex 8.3] $\mathcal{R}^i f_*(\mathcal{O}_Y^N \otimes f^*(\mathcal{O}_X(-n)) = \mathcal{R}^i f_*(\mathcal{O}_Y^N) \otimes \mathcal{O}_X(-n)$ and if $i > 0$ then this is zero by definition of rational singularities. Thus $\mathcal{R}^i f_*(\mathcal{M}) \cong \mathcal{R}^{i+1} f_*(\mathcal{K})$. Since $\mathcal{K}$ is supported on the fibres of f, the sheaves $\mathcal{R}^i f_*(\mathcal{M}) = 0$ for $i \geq n-1$.

Now $f_*\mathcal{M} = \mathcal{L}$ since $\mathcal{M}$ is generated by the sections of $\mathcal{L}$. Thus by Equation 4.2

$$\chi(Y, \mathcal{M}) = \sum_{i=0}^{n-2} (-1)^i \chi(X, \mathcal{R}^i f_*(\mathcal{M})) = \chi(X, \mathcal{L}) + \sum_{i=1}^{n-2} (-1)^i \chi(X, \mathcal{R}^i f_*(\mathcal{M})).$$

For $i > 0$ the sheaves $\mathcal{R}^i f_*(\mathcal{M})$ are supported only over the singularities of X. If the singularities are all isolated then the sheaves $\mathcal{R}^i f_*(\mathcal{M})$ are supported only over points so all its higher cohomology groups vanish. □

Lemma 4.6 *In notation of Lemma 4.5 if X is 2 dimensional then $\mathcal{M}$ is invertible and $\mathcal{R}^i f_*(\mathcal{M}) = 0$ for all $i > 0$ so*

$$\chi(X, \mathcal{L}) = \chi(Y, \mathcal{M}).$$

PROOF. The fact that $\mathcal{R}^i f_*(\mathcal{M}) = 0$ follows immediately from Lemma 4.5. Since $\mathcal{M}$ is torsion free there exists a short exact sequence

$$0 \to \mathcal{M} \to \mathcal{M}^{**} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined to be the cokernel of the embedding $\mathcal{M} \to \mathcal{M}^{**}$. Applying f_* gives the long exact sequence

$$\begin{aligned} 0 \longrightarrow\ & f_*\mathcal{M} \longrightarrow f_*(\mathcal{M}^{**}) \longrightarrow f_*\mathcal{Q} \longrightarrow \\ & \longrightarrow \mathcal{R}^1 f_*(\mathcal{M}) = 0. \end{aligned}$$

As in the proof of Lemma 4.5 $f_*\mathcal{M} = \mathcal{L}$. The sheaf $f_*(\mathcal{M}^{**})$ is torsion free, contains the reflexive sheaf $\mathcal{L}$ and $\mathrm{Supp}(f_*(\mathcal{M}^{**})/\mathcal{L})$ is contained in the singular locus of X. Therefore by Proposition 1.9 $f_*(\mathcal{M}^{**}) = \mathcal{L}$. This implies that $f_*\mathcal{Q} = 0$ and so $\mathcal{Q} = 0$. Hence $\mathcal{M} \cong \mathcal{M}^{**}$ and so $\mathcal{M}$ is reflexive. A reflexive rank 1 sheaf on a smooth variety is invertible so $\mathcal{M}$ is an invertible sheaf. □

In particular the next proposition implies that if $\pi\colon Z \to X$ is the quotient morphism and $\mathcal{F}$ a locally free sheaf then $\pi_*\mathcal{F}$ is locally free away from the singularities.

Proposition 4.7 (II.5 Cor 2 [Mum85]) *Let $f\colon Z \to X$ be a proper morphism of Noetherian schemes, X integral and $\mathcal{F}$ a coherent sheaf on Z which flat over X. If for some $i \geq 0$ the function $x \mapsto h^i(Z_x, \mathcal{F}_x)$ is constant then the sheaf $\mathcal{R}^i f_*(\mathcal{F})$ is locally free of rank $h^i(Z_x, \mathcal{F}_x)$.*

4.1.4 Roots of unity

It will also be useful to recall the following properties of roots of unity. For any nontrivial rth root of unity, ε, we have

$$1 + \varepsilon + \varepsilon^2 + \cdots + \varepsilon^{r-1} = 0 \tag{4.3}$$

$$\frac{-\varepsilon - 2\varepsilon^2 - \cdots - (r-1)\varepsilon^{r-1}}{r} = \frac{1}{1-\varepsilon}. \tag{4.4}$$

4.2 Results on G-sheaves

The proof of Theorem 4.1 also uses the following results.

Lemma 4.8 *Let Z be a smooth projective variety, G a finite cyclic group acting on Z such that only the trivial element of G acts trivially on Z and $\mathcal{F}$ an invertible G-sheaf on Z. Let $\pi\colon Z \to X = Z/G$ be the quotient morphism. Then*

$$\pi_*\mathcal{F} = \bigoplus_i \mathcal{F}_i$$

and each $\mathcal{F}_i$ is a divisorial sheaf which is invertible away from the singularities of X.

Proof. Let $\mathcal{F}_i$ be the torsion free sheaf $\mathcal{H}om(V_i, \pi_*\mathcal{F})^G$. We show that this is invertible away from the singularities. Since X is normal and the singular locus has codimension ≥ 2 this means that the $\mathcal{F}_i$ are divisorial by Proposition 1.13 and therefore $\mathcal{F}_i = \mathcal{O}(D_i)$ for some $\mathbb{Q}$-Cartier divisor D_i.

Let X^0 be the nonsingular locus of X and $Z^0 = \pi^{-1}(X^0)$. Then the restriction of the morphism π to Z^0, is proper and $\mathcal{F}_{|Z^0}$ is flat over X^0. The function $x \mapsto h^0(Z_x, \mathcal{F}_x)$ is equal to $|G|$ for all $x \in X^0$. Therefore by Proposition 4.7 the sheaf $\pi_*(\mathcal{F}_{|X^0})$ is locally free of rank $|G|$. Since X^0 is a trivial G-space $\pi_*\left(\mathcal{F}_{|X^0}\right)$ decomposes into eigensheaves $\mathcal{F}_i^0$ which are locally free by Proposition 1.18 and $\mathcal{F}_i^0 \cong \mathcal{F}_{i|X^0}$. □

Lemma 4.9 *Let G be a finite group acting on a smooth projective variety Z such that only the trivial element of G acts trivially on Z and $\mathcal{F}$ an invertible G-sheaf. Then $\mathcal{F}$ is isomorphic to a subsheaf of $k(Z)$ as G-sheaves.*

PROOF. Let $\mathcal{F}$ be an invertible G-sheaf. As in [Har77, II Prop 6.13] we know that $\mathcal{F} \otimes k(Z)$ is 1 dimensional vector space over $k(Z)$. The compatibility conditions of λ_g imply that the map $\mathcal{F} \otimes k(Z) \to g^*\mathcal{F} \otimes k(Z)$ is quasilinear. Then by [Sha94, Appendix 3 Prop 1] we may assume that $\mathcal{F} \otimes k(Z)$ is based by an invariant element of $\mathcal{F} \otimes k(Z)$ over $k(Z)$. Thus $\mathcal{F} \subset \mathcal{F} \otimes k(Z) \cong k(Z)$ as G-sheaves and so $\mathcal{F}$ is isomorphic to a subsheaf of $k(Z)$ as G-sheaves. □

Lemma 4.10 ("Hilbert's Theorem 90") *Let G be a cyclic group, $\mathcal{F}$ an invertible G-subsheaf of $k(Z)$ and assume that the map $\lambda_g \colon \mathcal{F} \to g^*\mathcal{F}$ is induced by the action of g on $k(Z)$. Then $\mathcal{F} \cong \mathcal{O}(D)$ where D is G-invariant.*

PROOF. Compare with [Lan93, proof of VI Thm 6.1]. Let G be a cyclic group of order r and g a generator of G. Let $\mathcal{F}$ be an invertible G-sheaf and f a generator of $\mathcal{F}_P$ where P is a fixed point. Now $\mathcal{F}_P = (g^*\mathcal{F})_P$ and since λ_g is an isomorphism $\lambda_g(f)$ is also a generator of the stalk $\mathcal{F}_P$. Therefore $\beta = \lambda_g(f)/f \in \mathcal{O}_P^*$.

We want to find $\gamma \in \mathcal{O}_P^*$ such that $\lambda_g(\gamma)/\gamma = \varepsilon^{-i}\beta$ and then f/γ is a generator of $\mathcal{F}_P$ and $\lambda_g(f/\gamma) = \varepsilon^i f/\gamma$.

By Artin's Theorem on characters [Lan93, VI Thm 4.1] the map Θ on $k(Z)$ given by

$$\Theta(\xi) = \mathrm{id}^*(\xi) + \beta^{-1}g^*(\xi) + \cdots + \beta^{-1}g^*\left(\beta^{-1}\right)\cdots(g^{r-2})^*\left(\beta^{-1}\right)g^{r-1*}(\xi)$$

for $\xi \in k(Z)$ is not identically zero. So there exists $\eta \in k(Z)$ such that $\gamma = \Theta(\eta) \neq 0$ and then $\lambda_g(\gamma)/\gamma = \beta$.

Now $\Theta(\eta)$ may have a zero or pole at P. Define a map Θ_i by

$$\Theta_i(\xi) = \mathrm{id}^*(\xi) + \varepsilon^i\beta^{-1}g^*(\xi) + \cdots + \varepsilon^{-i}\beta^{-1}g^*\left(\beta^{-1}\right)\cdots(g^{r-2})^*\left(\beta^{-1}\right)(g^{r-1})^*(\xi)$$

for $\xi \in k(Z)$. Then $\lambda_g(\Theta_i(\eta))/\Theta_i(\eta) = \varepsilon^{-i}\beta$. If $\Theta_i(\eta)$ has a zero at P for all i then η has a zero at P. If η has a zero at P then $\eta + n$ does not for any $n \in \mathbb{Z}$. At least one of $\Theta_i(\eta + n)$ or $\Theta_i(n)$ is nonzero and for some i this has no zero at P. Set γ equal to $\Theta_i(\eta + n)$ or $\Theta_i(n)$. Then γ does not have a zero at P and $\lambda_g(\gamma)/\gamma = \varepsilon^{-i}\beta$. Similarly we can find γ such that γ has no pole at P.

Consider an open set $U \subset Z$ on which G acts freely. By [Mum85, 2.7 Prop 2] $\mathcal{F}_{|U} \cong \pi^*\left((\pi_*\mathcal{F})^G\right)_{|U}$. Thus for any $P \in Z \smallsetminus Z^g$ the stalks $\mathcal{F}_P$ and $\left(\pi^*(\pi_*\mathcal{F})^G\right)_P$ are isomorphic as G-modules over $\mathcal{O}_P$. Therefore we can choose a generator of $\mathcal{F}_P$ which is G-invariant.

We have shown that for all $P \in Z$ we can find a generator of $\mathcal{F}_P$ such that $\lambda_P(f) = \varepsilon^i f$. Thus F is a G-invariant divisor. □

4.3 Proof of Theorem 4.1 on a smooth projective curve

Let $Z = C$ be a smooth projective curve. Theorem 4.1 can be restated as follows.

Theorem 4.11 *Let G be a finite group, C a smooth curve on which G acts and $\mathcal{F}$ an invertible G-sheaf on C. Assume that only the trivial element of G acts trivially on C. Then for all nontrivial $g \in G$*

$$\mathrm{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i\,\mathrm{H}^i(C,\mathcal{F})\right) \;=\; \sum_{P\in C^g}\frac{\mathrm{Tr}\left(g\colon \mathcal{F}_{|P}\right)}{1-\mathrm{d}g_P} \tag{4.5}$$

where $\mathrm{d}g_P\colon T^*_P C \to T^*_P C$ *is the induced map on the cotangent space and* $\mathcal{F}_{|P} \in K_G(P) = R(G)$.

The trace of the identity acting on a virtual representation is its dimension. Thus, by Hirzebruch Riemann–Roch for curves,

$$\begin{aligned} \operatorname{Tr}\left(1\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{F})\right) &= \chi(C,\mathcal{F}) \\ &= \deg \mathcal{F} + 1 - \mathrm{p_a}(C). \end{aligned} \tag{4.6}$$

Recall the notation, $g \in G$ is of order r, $H = \langle g\rangle$ and $\pi\colon C \to X = C/H$ is the quotient morphism.

Let $\pi_*\mathcal{F} = \bigoplus_{i=0}^{r-1} \mathcal{F}_i$ be the decomposition of $\pi_*\mathcal{F}$ into eigensheaves. Now X is smooth and H is cyclic so each of the sheaves $\mathcal{F}_i$ is invertible by Lemma 4.8. If $\mathcal{F}$ is the trivial G-sheaf then denote the decomposition of $\pi_*\mathcal{O}_C$ into eigensheaves by $\pi_*\mathcal{O}_C = \bigoplus_i \mathcal{L}_i$ and write $\mathcal{L}_i = \mathcal{O}_X(D_i)$. Since the sheaves $\mathcal{L}_i$ are invertible $\mathcal{L}_i \otimes \mathcal{L}_j \cong \mathcal{O}(D_i + D_j)$.

Using the previous results

$$\begin{aligned} &\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{F})\right) \\ &\quad= \operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(X,\pi_*\mathcal{F})\right) && \text{by Lemma 4.3} \\ &\quad= \sum_{k=0}^{r-1} \operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i H^i(X,\mathcal{F}_k)\right) && \text{by Lemma 4.8} \\ &\quad= \sum_{k=0}^{r-1} \varepsilon^k\big(\deg \mathcal{F}_k + 1 - \mathrm{p_a}(X)\big) && \text{by Equation 4.6} \\ &\quad= \sum_{k=0}^{r-1} \varepsilon^k \deg \mathcal{F}_k && \text{by Equation 4.3.} \end{aligned}$$

So it remains to calculate the degrees of the invertible sheaves $\mathcal{F}_k$.

4.3.1 Proof of Theorem 4.1 for the trivial G-sheaf when the action is coprime.

Example 4.12 Let the action of $G = \mathbb{Z}_r$ on $C = \mathbb{P}^1$ be generated by $g = \frac{1}{r}(0,1)$ and let $\mathcal{F}$ be the trivial G-sheaf. The vector space $\mathrm{H}^0(C, \mathcal{O}_C) \cong \mathbb{C}$ and g acts trivially on it so

$$\mathrm{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \mathrm{H}^i(C,\mathcal{O}_C)\right) = 1.$$

The fixed locus of g acting on C is $C^g = \{P_0 = (1:0), P_1 = (0:1)\}$. At P_0 the local coordinate is x_1/x_0 and so the induced action on the cotangent space is $\mathrm{d}g_P\mathrm{d}(x_1/x_0) = \varepsilon \mathrm{d}(x_1/x_0)$. At P_1 the coordinate is x_0/x_1 and so the induced action on the cotangent space is $\mathrm{d}g_P\mathrm{d}(x_0/x_1) = \varepsilon^{-1}\mathrm{d}(x_0/x_1)$. Thus

$$\sum_{P\in C^g} \frac{\mathrm{Tr}(g\colon \mathcal{O}_{|P})}{1-\mathrm{d}g_P} = \frac{1}{1-\varepsilon} + \frac{1}{1-\varepsilon^{-1}} = 1.$$

The ith eigensheaf $\mathcal{L}_i$ of $\pi_*(\mathcal{O}_Z)$ is not a subsheaf of $k(X)$ but the sheaf $\mathcal{L}_i^{\otimes r}$ is. Therefore its Cartier divisor can be recovered along with its embedding into $k(X)$ [Har77, II Prop 6.13]. In what follows we show how to construct the divisor corresponding to rD_i.

Let P be any point in C, $Q = \pi(P)$ and u_i a local generator of $\mathcal{L}_{iQ}$. If P is not fixed by g, then $u_i^r \in \mathcal{O}_Q$ is a unit. Therefore $\mathcal{L}_{iQ}^{\otimes r}$ is generated by 1 as an $\mathcal{O}_Q$-module and so $Q \notin \mathrm{Supp}\, rD_i$.

If P is fixed by g then the quotient morphism $\pi\colon C \to X$ is ramified with order r at P. If t is a local coordinate of C at P then $s = t^r$ is a local coordinate of X at Q. The element g induces an action on the cotangent space of C at P which is given by $\mathrm{d}g_P(\mathrm{d}t) = \varepsilon^a \mathrm{d}t$ for some a coprime to r.

Let u_i be a local generator of $\mathcal{L}_{iQ}$ then $u_i^r \in \mathcal{O}_Q$ is no longer a unit so $u_i^r = (\text{unit})s^\lambda$. Since $\mathrm{d}g\mathrm{d}t = \varepsilon^a \mathrm{d}t$ it follows that $u_i = t^{b_i}$ where $b_i a \equiv i \pmod r$. Therefore u_i^r has a zero of order b_i at Q and $-b_i Q \in \operatorname{Supp} rD_i$.

Continuing this process over all $P \in C$ we obtain

$$\mathcal{L}^{\otimes r} = \mathcal{O}_X\left(-\sum_Q b_{i,Q} Q\right)$$

where $0 \leq b_{i,Q} \leq r-1$ is defined as follows. If Q is not a branch point then $b_{i,Q} = 0$. If Q is a branch point and the induced action on the cotangent space to the point $P = \pi^{-1}(Q)$ is $\mathrm{d}g_P(\mathrm{d}t) = \varepsilon^{a_P}\mathrm{d}t$ then $b_{i,Q}$ is defined such that $b_{i,Q}a_P \equiv i \pmod r$. Hence

$$\deg(D_i) = -\sum_Q \frac{b_{i,Q}}{r}.$$

The divisor D_i is Cartier so $\deg D_i$ is an integer and the following corollary is immediate.

Corollary 4.13 *The order of g divides $\sum_Q b_{i,Q}$.*

This corollary restricts the way a group may act on a curve. For example if $\mathbb{Z}_r$ acts on a curve then any $g \in \mathbb{Z}_r$ cannot fix only one point. If g fixes exactly two points then g acts by ε^a at one point and ε^{-a} at the other.

Example 4.14 Continuing with the above example. Let $Q_0 = \pi(P_0)$. Now x_1/x_0 is a local coordinate of P_0, $g(x_1/x_0) = \varepsilon x_1/x_0$ and $(x_1/x_0)^r$ is a local coordinate at Q_0. If u_i is the generator of $\mathcal{L}_{iQ_0}$ then u_i^r is invariant so $u_i^r = (x_1/x_0)^{r\lambda}$ and since $u_i = (x_1/x_0)^\lambda$ it follows that $\lambda = i$. Similarly if $Q_1 = \pi(P_1)$ and u_i is the generator of $\mathcal{L}_{iQ_1}$ then $u_i = (x_1/x_0)^{r-i}$. Thus

$$\deg \mathcal{L}_i = -\frac{i}{r} - \frac{r-i}{r} = -1$$

and

$$\sum_i \varepsilon^i \deg \mathcal{L}_i = -\varepsilon - \varepsilon^2 - \cdots - \varepsilon^{r-1} = 1.$$

So far Equation 4.5 has been shown to equal

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{O}_C)\right) = \sum_{k=0}^{r-1} \varepsilon^k \deg \mathcal{L}_k = -\sum_{k=0}^{r-1}\sum_{Q} \frac{b_{k,Q}}{r}\varepsilon^k \qquad (4.7)$$

where Q runs through all the branch points of π. Now fix a branch point Q, the contribution this point makes to $\sum_i \varepsilon^i \deg \mathcal{L}_i$ is

$$-\sum_{k=0}^{r-1} \frac{b_{k,Q}}{r}\varepsilon^k = -\sum_{i=1}^{r-1} \frac{i\varepsilon^{ia_P}}{r}$$
$$= \frac{-\varepsilon^{a_p} - 2\varepsilon^{2a_P} - \cdots - (r-1)\varepsilon^{(r-1)a_P}}{r} = \frac{1}{1-\varepsilon^{a_P}}$$

by Equation 4.4. Substituting this into Equation 4.7 gives

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{O}_C)\right) = \sum_{Q} \frac{1}{1-\mathrm{d}g_P} = \sum_{P\in C^g} \frac{1}{1-\mathrm{d}g_P}.$$

□

4.3.2 Proof of Theorem 4.1 for any invertible G-sheaf

Example 4.15 Let C and G be as above. Let $\mathcal{F} = \mathcal{O}_C(\mu P_0)$ and assume $\mu > 0$ so that $\mathcal{F}_{P_1}$ is generated by 1 and $\mathcal{F}_{P_0}$ is generated by $(x_0/x_1)^\mu$. Let the action $\mathcal{F} \to g^*\mathcal{F}$ be induced by the action of g on $k(C)$. Then isomorphism $\mathcal{F}_{|P_1} \to \mathcal{F}_{|P_1}$ is multiplication by 1 and the isomorphism $\mathcal{F}_{|P_0} \to \mathcal{F}_{|P_0}$ is multiplication by ε^μ. The rational functions $1, x_0/x_1, \ldots, (x_0/x_1)^\mu$ generate the global sections of the sheaf $\mathcal{F}$ and so

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{F})\right) = 1 + \varepsilon^{-1} + \cdots + \varepsilon^{-\mu}.$$

At the fixed points $\mathrm{Tr}\left(g\colon \mathcal{F}_{|P_0}\right) = \varepsilon^{-\mu}$ and $\mathrm{Tr}\left(g\colon \mathcal{F}_{|P_1}\right) = 1$ so

$$\sum_{P\in C^g} \frac{\mathrm{Tr}\left(g\colon \mathcal{F}_{|P}\right)}{1-\mathrm{d}g_P} = \frac{\varepsilon^{-\mu}}{1-\varepsilon} + \frac{1}{1-\varepsilon^{-1}} = 1+\varepsilon^{-1}+\cdots+\varepsilon^{-\mu}.$$

Now let $\mathcal{F} = \mathcal{O}(F)$ be an invertible G-sheaf and so $\mathcal{F}$ is also a H-sheaf. By Lemma 4.10 we may assume F is H-invariant so $F = \sum_S \lambda_S S + \sum_R \mu_R R$ where the S are isolated fixed points and the R are free H-orbits. Let $\mathcal{F}_i$ be the ith eigensheaf of $\pi_*(\mathcal{F})$. As in the case when $\mathcal{F}$ was the trivial G-sheaf the theorem is proved by computing the degree of the sheaves $\mathcal{F}_i$. Let f be the generator of $\mathcal{F}_P$.

If $P \notin \mathrm{Supp}\, F$ then $f = 1$ and so if f_i is a generator of $\mathcal{F}_{iQ}$ then $f_i = u_i$. If $\lambda P \in \mathrm{Supp}\, F$ and $P \notin C^g$ then $f_i = u_i s^{-\lambda}$ where s is the local coordinate at $Q = \pi(P)$. If $P \in C^g$ then $f_{ia} = u_{ia} s^{-\lfloor \frac{i+\lambda}{r} \rfloor}$. If $\mathcal{L}_i = \mathcal{O}_X\left(-\sum_Q \frac{b_{i,Q}}{r} Q\right)$ then

$$\mathcal{F}_i = \mathcal{O}_X\left(-\sum_Q \frac{b_{i,Q}}{r} Q + \sum_S \left\lfloor \frac{i+\lambda_S}{r} \right\rfloor \pi(S) + \sum_R \mu_R \pi(R)\right).$$

Now $\pi(R)$ makes the same contribution to each of the sheaves $\mathcal{F}_i$ so makes no contribution to $\sum \varepsilon^i \deg \mathcal{F}_i$ by Equation 4.3. If P is fixed by g but $P \notin \mathrm{Supp}\, F$ then P makes the same contribution $\sum \varepsilon^i \deg \mathcal{F}_i$ as it made to $\sum \varepsilon^i \deg \mathcal{L}_i$ and since $\mathrm{Tr}(g\colon \mathcal{F}_{|P}) = \mathrm{Tr}(g\colon \mathcal{O}_{|P})$ the theorem holds.

If $P \in C^g$ and $\lambda P \in \mathrm{Supp}\, F$ then P contributes

$$\sum_{i=0}^{r-1} \left(\left\lfloor \frac{i+\lambda}{r} \right\rfloor - \frac{i}{r} \right) \varepsilon^{ia}$$

to $\sum \varepsilon^i \deg \mathcal{F}_i$.

Now there exists $0 \leq \alpha < r$ such that $\alpha \equiv -\lambda \pmod{r}$. Thus

$$\begin{aligned}
&\sum_{i=0}^{r-1} \left(\left\lfloor \frac{i+\lambda}{r} \right\rfloor - \frac{i}{r} \right) \varepsilon^{ia} \\
&\quad = \left(\frac{\lambda+\alpha}{r} - \frac{\alpha}{r} \right) \varepsilon^{\alpha a} + \cdots + \left(\frac{\lambda+\alpha}{r} - \frac{\alpha+r-1}{r} \right) \varepsilon^{(\alpha+r-1)a} \\
&\quad\quad = \frac{\lambda}{r}\varepsilon^{\alpha a} + \frac{\lambda-1}{r}\varepsilon^{(\alpha+1)a} + \cdots + \frac{\lambda-(r-1)}{r}\varepsilon^{(\alpha+r-1)a} \\
&\quad\quad\quad = \frac{-\varepsilon^{\alpha a}}{r}\left(\varepsilon^a + 2\varepsilon^{2a} + \cdots + (r-1)\varepsilon^{(r-1)a}\right) = \frac{\operatorname{Tr}(g\colon \mathcal{F}_P)}{1-\mathrm{d}g_P}
\end{aligned}$$

□

4.3.3 Proof of Theorem 4.1 when the action is not coprime

In this case we show that the points in $C^H \smallsetminus C^g$ will contribute to $\deg \mathcal{F}_i$ but these terms will cancel in the summation $\sum_i \varepsilon^i \deg \mathcal{F}_i$. Assume for simplicity of notation that $\mathcal{F} = \mathcal{O}_C$ but the proof of the general case is the same.

Take a point $P \in C^H \smallsetminus C^g$. Then $\operatorname{Stab}_H P$ is generated by some element $g^a \in H$ and let a be the least such a. Now $g^a t_P = \varepsilon^{\lambda a} t_P$ and assume for simplicity that $\lambda = 1$. Let $a' = r/a$ and consider the short exact sequence

$$0 \to \mathbb{Z}_{a'} \to \mathbb{Z}_r \to \mathbb{Z}_a \to 1.$$

Then consider the chain of quotients

$$C \to C_1 = C/\mathbb{Z}_{a'} \to C_1/\mathbb{Z}_a = X.$$

Let $\pi_1\colon C \to C_1$ and $\pi_1(P) = Q_1$ then $\pi_{1*}\mathcal{O}_C = \bigoplus_{i=0}^{a'-1} \mathcal{L}'_i$. The morphism $\pi_2\colon C_1 \to X$ is unramified at Q_1 and $\pi_{2*}\mathcal{L}'_i = \bigoplus_{j=0}^{a-1} \mathcal{L}_{i+ja'}$. Then as in the previous subsection, for fixed i, the sheaf $\mathcal{L}_{i+ja'}^{\otimes r}$ has the same degree b_i at

$Q = \pi_2(Q_1)$ for all j. Thus this point contributes

$$\sum_{i=0}^{a'-1}\sum_{j=0}^{a-1} b_i \varepsilon^{i+ja'} = \sum_{i=0}^{a-1} b_i \left(\varepsilon^i + \varepsilon^{i+a'} + \cdots + \varepsilon^{i+a'(a-1)}\right) = 0$$

to $\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{F})\right)$. The same argument holds for any invertible G-sheaf $\mathcal{F}$ on C.

Hence for any nontrivial $g \in G$

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{F})\right) \;=\; \sum_{P\in C^g} \frac{\operatorname{Tr}\left(g\colon \mathcal{F}_{|P}\right)}{(1-\operatorname{d}g_P)}.$$

□

4.4 Proof of Theorem 4.1 on a smooth projective surface.

Let $Z = S$ a smooth projective surface then Theorem 4.1 can be rewritten in the following way.

Theorem 4.16 *Let G be a finite group acting on S and $\mathcal{F}$ an invertible G-sheaf on S. Assume that only the trivial element of G acts trivially on S. Then for any nontrivial $g \in G$*

$$\begin{aligned}\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{F})\right) &= \sum_{P\in S^g} \frac{\operatorname{Tr}\left(g\colon \mathcal{F}_{|P}\right)}{\det(I-\operatorname{d}g_P)} \\ &+ \sum_{C\subset S^g} \operatorname{Tr}\left(g\colon \mathcal{F}_{|C}\right)\left(\frac{1-\operatorname{p_a}(C)+\operatorname{c}_1(\mathcal{F}_{|C})}{\det(I-\operatorname{d}g_C)} - \frac{\operatorname{d}g_C C^2}{\det(I-\operatorname{d}g_C)^2}\right). \quad (4.8)\end{aligned}$$

The first sum runs through the isolated fixed points of g and the second sum runs though prime divisors C fixed by g. The genus of the curve C is denoted by $\operatorname{p_a}(C)$ and $\operatorname{d}g_C\colon N^*_{C/S} \to N^*_{C/S}$ is the induced action of g on the conormal space to $C \subset S$.

As in Theorem 4.11 if $g = 1$ then by Hirzebruch Riemann–Roch

$$\operatorname{Tr}\left(1\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{F})\right) = \chi(S,\mathcal{F}) = \frac{c_1(\mathcal{F})\,(c_1(\mathcal{F}) - K_S)}{2} + 1 + \operatorname{p_a}(S).$$

To prove this theorem first show that there exists a formula of the form $\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{O}_S)\right) = \sum_{F\subset S^g} \operatorname{RR}(F)$ and then compute the terms $\operatorname{RR}(F)$ for each irreducible $F \subset S^g$.

4.4.1 Notation

Let $S^g = \{C_1, \ldots, C_n, P_1, \ldots, P_m\}$, where C_i are divisors and P_i are isolated points. The C_i are disjoint smooth curves and $P_j \notin C_i$ for all i, j. Let $X = S/H$ and $\pi\colon S \to X$ be the quotient morphism. Let $B_i = \pi(C_i)$ and $Q_i = \pi(P_i)$.

The sheaf $\pi_*\mathcal{F}$ on X decomposes as a direct sum of eigensheaves $\bigoplus_{i=0}^{r-1}\mathcal{F}_i$. Each of the $\mathcal{F}_i$ are divisorial sheaves on X which are invertible away from the singularities of X by Lemma 4.8. Again the decomposition of $\pi_*\mathcal{O}_S$ into eigensheaves is denoted by $\bigoplus_{i=0}^{r-1}\mathcal{L}_i$ when $\mathcal{O}_S$ is the trivial G-sheaf. Denote the $\mathbb{Q}$-Cartier divisor corresponding to $\mathcal{L}_i$ by D_i.

Let $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ be reflexive sheaves on X and let X^0 be the non-singular locus of X. It may be that $\mathcal{O}(D_1)\otimes\mathcal{O}(D_2) \neq \mathcal{O}(D_1+D_2)$ but $(\mathcal{O}(D_1)\otimes\mathcal{O}(D_2))^{**} = \mathcal{O}(D_1+D_2)$. Since $\operatorname{codim}(X\smallsetminus X^0, X) = 2$ the sheaf $(\mathcal{O}(D_1)\otimes\mathcal{O}(D_2))^{**}$ is the unique reflexive sheaf on X such that

$$(\mathcal{O}(D_1)\otimes\mathcal{O}(D_2))^{**}_{|X^0} = \mathcal{O}(D_1)_{|X^0}\otimes\mathcal{O}(D_2)_{|X^0}.$$

See [KM98] for details. We denote by $\mathcal{L}^{[r]}$ the reflexive sheaf $(\mathcal{L}^{\otimes r})^{**}$.

Let $f\colon Y \to X$ be a resolution of X such that f is a log resolution of each pair (X, D_i). Let $\mathcal{M}_i = (f^*\mathcal{L}_i)/(\text{torsion})$. Then by Lemma 4.6 $\mathcal{M}_i$ is

an invertible sheaf. Write M_i for the corresponding divisor. Let $A_i = f^*B_i$ and $E_i = \sum_j E_{i,j} = f^{-1}Q_i$ where each $E_{i,j}$ is an irreducible curve. Lemma 4.5 implies that

$$\chi(\mathcal{M}_i) \;=\; \chi(\mathcal{L}_i). \tag{4.9}$$

As in the case of curves

$$\begin{aligned}
\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{F})\right) &= \sum_{i=0}^{r-1}\varepsilon^i\chi(\mathcal{L}_i) = \sum_{i=0}^{r-1}\varepsilon^i\chi(\mathcal{M}_i)\\
&= \sum_{i=0}^{r-1}\varepsilon^i\left(\chi(\mathcal{O}_Y)+\frac{1}{2}M_i(M_i-K_Y)\right)\\
&= \sum_{i=0}^{r-1}\frac{\varepsilon^i}{2}M_i(M_i-K_Y).
\end{aligned}$$

Theorem 4.16 is proved in three steps. First assume that the action is coprime and that the invertible sheaf is trivial, then prove the theorem for any invertible G-sheaf and finally prove it for any action.

4.4.2 Proof of Theorem 4.1 for the trivial G-sheaf when the action is coprime

If $S^H = S^g$ then the points Q_i are the only singularities of X.

Example 4.17 Let $\mathbb{Z}_r$ act on $S = \mathbb{P}^2$ by $g = \frac{1}{r}(0,0,1)$. Let $L_2\colon (x_2 = 0)$ and $P_2 = (0:0:1)$ then $S^g = \{L_2, P_2\}$. Again let $\mathcal{F}$ be the trivial G-sheaf so

$$\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{O}_S)\right) = 1.$$

The coordinate on the conormal space to L_2 in S is $\mathrm{d}\,(x_2/x_0)$ and so $\mathrm{d}g_{L_2} = \varepsilon$. The coordinates on the cotangent space to P in S are $\mathrm{d}(x_0/x_2)$

and $\mathrm{d}(x_1/x_2)$ so $\mathrm{d}g_P = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$. Therefore

$$\sum_{C\subset S^g} \mathrm{Tr}\left(g\colon \mathcal{O}_{|L_2}\right)\left(\frac{1-\mathrm{p_a}(L_2)+\mathrm{c}_1(\mathcal{O}_{|L_2})}{\det(I-\mathrm{d}g_{L_2})} - \frac{\mathrm{d}g_{L_2}L_2^2}{\det(I-\mathrm{d}g_{L_2})^2}\right) + \sum_{P\in S^g} \frac{\mathrm{Tr}\left(g\colon \mathcal{O}_{|P}\right)}{\det(I-\mathrm{d}g_P)} = \frac{1-0}{1-\varepsilon} - \frac{\varepsilon}{(1-\varepsilon)^2} + \frac{1}{(1-\varepsilon^{-1})^2} = 1.$$

The theorem will be proved by constructing the divisor M_i and computing M_i^2 and M_iK_Y. Let P be any point on S which is not fixed by g so $Q=\pi(P)$ is a nonsingular point of X and let $u_{i,Q}$ be the generator of the free $\mathcal{O}_Q$-module $\mathcal{L}_{iQ}$. As in the case of curves $\mathcal{L}_{iQ}^{[r]}$ is generated by 1 as an $\mathcal{O}_Q$-module. In a neighbourhood of $\widetilde{Q}=f^{-1}(Q)$, the resolution Y is isomorphic to X so $u_{i,Q}$ is also a generator for $\mathcal{M}_{i\widetilde{Q}}$.

Next let $P\in C$, one of the fixed divisors, again $Q=\pi(P)$ is a nonsingular point of X and $\mathcal{L}_{iQ}$ is a free $\mathcal{O}_Q$-module generated by $u_{i,Q}$. Now $\mathcal{L}_{iQ}$ is a submodule of $\mathcal{O}_P$ and choose $s,t\in m_P/m_P^2$ such that s is the local equation for C and $g(s,t)=(\varepsilon^a s,t)$. Therefore $u_{i,Q}=(\mathrm{unit})s^{b_i}$ where $b_ia\equiv i \pmod r$. Now $u_{i,Q}^r = s^{rb_i}\in\mathcal{O}_Q$ and s^r is locally the equation of $B=\pi(C)$. Thus $-b_iB\in \mathrm{Supp}\, rD_i$.



Figure 4.1: Exceptional locus for the blowup of Q

Finally let $P=P_j$ be an isolated fixed point. Then $s,t\in m_P/m_P^2$ can be chosen so that $g(s,t)=(\varepsilon^a s,\varepsilon^b t)$ and, since we are assuming the action is

coprime, a and b are coprime to r. Now $Q = \pi(P)$ is a singular point of X. Figure 4.1 shows the exceptional locus of Y in a neighbourhood of $f^{-1}Q$.

Let $\widetilde{Q}$ be any point in $f^{-1}Q$. At $\widetilde{Q}$ the local coordinates are s_k and s_{k+1} where s_k is the local coordinate along the line E_k. Now $s_k = s^{\alpha_k}/t^{\beta_k}$ and $s_{k+1} = t^{\beta_{k+1}}/s^{\alpha_{k+1}}$ where α_i and β_i depend only on the local analytic nature of the singularity Q, that is only on how g acts in a local analytic neighbourhood of P.

Now $\mathcal{M}_{i\widetilde{Q}}$ is a free $\mathcal{O}_{\widetilde{Q}}$-module generated by some $u_i \in \mathcal{L}_{iQ}$. Since $u_i^r \in \mathcal{O}_{\widetilde{Q}}$ it follows that $u_i^r = (\text{unit}) s_k^{c_{i,k+1}} s_{k+1}^{c_{i,k}}$ where $c_{i,k+1}$ and $c_{i,k}$ are defined such that

$$ac_{i,k+1}(\alpha_k - \alpha_{k+1}) + bc_{i,k}(\beta_{k+1} - \beta_k) \equiv i \pmod{r}.$$

Thus they depend only on a, b, r, s_k and s_{k+1}.

Let L_0 be the strict transform of the curve $(s^r = 0)$ shown in Figure 4.1 and $\widetilde{Q}$ a point in a neighbourhood of $L_0 \cap E_1$. Now $s_0 = t^r$ and $s_1 = s/t^\mu$ are local coordinates for $\widetilde{Q}$ for some μ. Since $t^\lambda \in \mathcal{M}_{i\widetilde{Q}}$ for some $1 \leq \lambda \leq r-1$ it follows that $u_i^r = s_0^\lambda$ and $L_0 \notin \operatorname{Supp} rM_i$. Similarly $L_1 \notin \operatorname{Supp} rM_i$.

Thus the local generator of the sheaf $\mathcal{M}_{i\widetilde{Q}}$ depends only on how g acts on a local analytic neighbourhood of the fixed point.

By continuing this process over all $\widetilde{Q} \in Y$ it follows that

$$rM_j = \sum_i -b_{i,j}A_i + \sum_{i,k} -c_{i,k,j}E_{i,k} = f^*(rD_i) + \sum_{i,k} -c_{i,k,j}E_{i,k}.$$

The terms $c_{i,k,j}$ depend only on how g acts on a local analytic neighbourhood of the points P_i.

Also $rK_Y = f^*(rK_X) - \sum d_{i,k}E_{i,k}$ and $d_{i,k}$ depends on only the local analytic nature of the singularites of X. By the projection formula $f^*A \cdot B =$

$A \cdot f_* B$ and so if B is exceptional then $f^*A \cdot B = 0$. Therefore

$$\begin{aligned}
\chi(\mathcal{M}_j) &= \chi(\mathcal{O}_Y) + \frac{1}{2}(M_j^2 - M_j K_Y) \\
&= \chi(\mathcal{O}_Y) + \sum_i \frac{b_{i,j}^2 A_i^2 - r b_{i,j} A_i f^* K_X}{2r^2} \\
&\quad + \sum_i \frac{\left(\sum_k c_{i,k,j} E_{i,k}\right)^2 - \left(\sum_k c_{i,k,j} E_{i,k}\right)\left(\sum_k d_{i,k} E_{i,k}\right)}{2r^2} \\
&= \chi(\mathcal{O}_Y) + \sum_{F \subset S^g} \mathrm{RR}(F, j)
\end{aligned}$$

where the sum runs through all irreducible components of the fixed locus. Again $E_{i,j}E_{i,k}$ depends only on the local analytic nature of the singularites. Hence if F is an isolated fixed point then $\mathrm{RR}(F, j)$ depends only on how g acts on a local analytic neighbourhood of the singularity.

Define $\mathrm{RR}(F) := \sum_j \varepsilon^j \mathrm{RR}(F, j)$. We complete the proof by computing these terms for each $F \subset S^g$.

If $F = C_i \subset S$ is a fixed curve then as shown above

$$\mathrm{RR}(C_i, j) = \frac{b_{i,j}^2}{2r^2} A_i^2 - \frac{b_{i,j}}{2r} A_i K_Y = \frac{b_{i,j}^2}{2r^2} B_i^2 - \frac{b_{i,j}}{2r} B_i K_X$$

where $b_{i,j}$ is defined such that $b_{i,j} a_i \equiv j \pmod{r}$.

For ease of notation assume $a_i = 1$ and so $b_{i,1} = 1$. If instead $b_{i,\lambda} = 1$ then replace ε with ε^λ in what follows. Thus

$$\mathrm{RR}(C_i) = \sum_{j=0}^{r-1} \frac{\varepsilon^j}{2r} \left(\frac{j^2}{r} B_i^2 + j B_i K_X \right).$$

By the ramification formula $K_S = \pi^* K_X + (r-1) \sum C_i$. For any divisors D_1 and D_2 in X such that D_1 does not pass through the singularities of X then $r D_1 D_2 = \pi^*(D_1)\pi^*(D_2)$. Thus

$$\begin{aligned}
\mathrm{RR}(C_i) &= \sum_{j=0}^{r-1} \frac{\varepsilon^j}{2r} \left(j^2 C_i^2 + j K_S C_i - j(r-1) C_i^2 \right) \\
&= -\frac{\varepsilon C_i^2}{(1-\varepsilon)^2} + \frac{1 - \mathrm{p_a}(C_i)}{1-\varepsilon}. \qquad (4.10)
\end{aligned}$$

The final equality holds since

$$\sum_{i=1}^{r-1} i^2 \varepsilon^i = \frac{-r^2}{1-\varepsilon} + \frac{-2r\varepsilon}{(1-\varepsilon)^2}.$$

Let $F = P_i$ be an isolated fixed point in S. Choose local analytic coordinates at P_i such that g acts by $g(s,t) = (\varepsilon^a s, \varepsilon^b t)$. We have shown above that $\mathrm{RR}(P_i)$ depends only on the action of g on a local analytic neighbourhood of P_i. Thus $\mathrm{RR}(P_i)$ can be computed for a point $P_i \in C \times C$ where C is a smooth curve acted on by $\mathbb{Z}_r$.

Let $\mathbb{Z}_r$ act on $\mathbb{P}^2$ by $h = \frac{1}{r}(1,1,0)$. We showed in Example 4.17 that $\mathbb{P}^{2h} = \{L_2, P_2\}$. Let C_r be a smooth $\mathbb{Z}_r$-invariant curve of degree r in $\mathbb{P}^2$ which does not pass through the point $P_2 = (0:0:1)$. For example let $C_r\colon (x_0^r + x_1^r + x_2^r = 0)$. Then h fixes the r points where C_r meets the line $L_2\colon (x_2 = 0)$. At each of these points the induced action on the cotangent space is $\mathrm{d}h\mathrm{d}t_P = \varepsilon \mathrm{d}t_P$. Thus by equivariant Riemann–Roch for curves

$$\begin{aligned} \mathrm{Tr}\left(h^a\colon \bigoplus_{j\geq 0} (-1)^j \,\mathrm{H}^j(C_r, \mathcal{O}_{C_r})\right) &= \sum_{P \in C_r^h} \frac{1}{1-\mathrm{d}h_P^a} \\ &= \frac{r}{1-\varepsilon^a}. \end{aligned}$$

Let $g' = (h^a, h^b) \in \mathbb{Z}_r \times \mathbb{Z}_r$ act on $C_r \times C_r$ so that the induced action on the cotangent space is given by $\mathrm{d}g(\mathrm{d}s, \mathrm{d}t) = (\varepsilon^a \mathrm{d}s, \varepsilon^b \mathrm{d}t)$. This action fixes r^2 points in $C_r \times C_r$. The Künneth formula says that

$$\mathrm{H}^i(C_r \times C_r, \mathcal{O}_{C_r \times C_r}) = \sum_{j+k=i} \mathrm{H}^j(C_r, \mathcal{O}_{C_r}) \otimes \mathrm{H}^k(C_r, \mathcal{O}_{C_r}).$$

Thus

$$\begin{aligned} &\mathrm{Tr}\left(g\colon \bigoplus_{i\geq 0} (-1)^i \,\mathrm{H}^i(C_r \times C_r, \mathcal{O}_{C_r \times C_r})\right) \\ &= \mathrm{Tr}\left(h^a\colon \bigoplus_{j\geq 0} (-1)^j \,\mathrm{H}^j(C_r, \mathcal{O}_{C_r})\right) \mathrm{Tr}\left(h^b\colon \bigoplus_{k\geq 0} (-1)^k \,\mathrm{H}^k(C_r, \mathcal{O}_{C_r})\right) \\ &= \frac{r}{1-\varepsilon^a}\frac{r}{1-\varepsilon^b}. \end{aligned}$$

Since there are r^2 fixed points in $C_r \times C_r$ each point contributes $\frac{1}{(1-\varepsilon^a)(1-\varepsilon^b)}$ to $\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C_r \times C_r, \mathcal{O}_{C_r\times C_r})\right)$ and so

$$\operatorname{RR}(P_i) = \frac{1}{(1-\varepsilon^a)(1-\varepsilon^b)} = \frac{1}{\det(I - \operatorname{d}g_{P_i})}. \tag{4.11}$$

Together Equations 4.10 and 4.11 imply

$$\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{O}_S)\right) = \sum_{i=0}^{r-1} \frac{\varepsilon^i}{2} M_i(M_i - K_Y) = \sum_{F\subset S^g} \operatorname{RR}(F)$$
$$= \sum_{P\in S^g} \frac{1}{\det(I-\operatorname{d}g_P)} + \sum_{C\subset S^g} \frac{1-\operatorname{p_a}(C)}{\det(I-\operatorname{d}g_C)} - \frac{\operatorname{d}g_C C^2}{\det(I-\operatorname{d}g_C)^2}.$$

□

4.4.3 Proof of Theorem 4.1 for any invertible G-sheaf

Let $\mathcal{F}$ be an invertible G-sheaf on S. By Lemma 4.9 we may assume that $\mathcal{F} \subset k(S)$ and so $\mathcal{F} = \mathcal{O}_S(F)$. Then $\mathcal{F}$ is also an H-sheaf and so F is an H-invariant divisor by Lemma 4.10. If P, an isolated fixed point, belongs to $\operatorname{Supp} F$ then we can replace F by a linearly equivalent H-invariant divisor F' where $P \notin \operatorname{Supp} F'$. Then $\mathcal{F} \cong \mathcal{F}' \otimes V_i$ where $\mathcal{F}' = \mathcal{O}(F')$ and V_i is an irreducible representation of H. Since

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(S,\mathcal{F})\right) = V_i(g)\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(S,\mathcal{F}')\right)$$

and for any fixed point P

$$\operatorname{Tr}(g\colon \mathcal{F}_P) = V_i(g)\operatorname{Tr}(g\colon \mathcal{F}'_P)$$

it is sufficient to prove the theorem when $\mathcal{F} = \mathcal{O}_S(F)$ and no isolated fixed point belongs to the support of F.

Let $\pi_*\mathcal{F} = \bigoplus \mathcal{F}_i$ and $\mathcal{E}_i = f^*\mathcal{F}_i/(\text{torsion})$. As before let $\pi_*\mathcal{O}_S = \bigoplus \mathcal{L}_i$ and $\mathcal{M}_i = f_*\mathcal{L}_i/(\text{torsion})$. We prove Theorem 4.1 when F is an irreducible

G-divisor this includes the case that $F = \sum_{i=1}^{r} F_i$ where each F_i is irreducible and $F_{i+1} = gF_i$.

Let f be the generator of $\mathcal{F}_P$ as an $\mathcal{O}_P$-module and u_i a generator of $\mathcal{M}_{i\widetilde{Q}}$ as an $\mathcal{O}_{\widetilde{Q}}$-module for some $\widetilde{Q} \in \pi^{-1}(P)$. If $P \notin \operatorname{Supp} F$ then $\mathcal{E}_{i\widetilde{Q}}$ is also generated by u_i. In particular this implies that $\mathrm{RR}_{\mathcal{F}}(P) = \mathrm{RR}_{\mathcal{O}}(P)$ for any of the isolated fixed points.

Let $P \in \operatorname{Supp} F$. By Lemma 4.10 we can choose a generator of $\mathcal{F}_P$ such that $g^*f = \varepsilon^{\lambda} f$ so $\pi_*\mathcal{F}_Q$ is generated by $fu_1, \ldots, fu_r$ as an $\mathcal{O}_Q$-module. If $\lambda \equiv 0$ then $f \in k(X)$ and so $\mathcal{E}_i = \mathcal{M}_i\left(f^*\pi(F)\right)$. In this case

$$\mathrm{RR}_{\mathcal{F}}(C_j) = \sum_{k=0}^{r-1} \frac{\varepsilon^k}{2r}\left(\frac{k^2}{r}A_j^2 - kA_jf^*\pi(F) + r\left(f^*\pi(F)\right)^2 + kA_jK_Y - rf^*\pi(F)K_Y\right) = \mathrm{RR}_{\mathcal{O}_S}(C_j) + \frac{C_jF}{1-\varepsilon}.$$

If $\lambda \not\equiv 0$ then $f \notin k(X)$ but $f^r \in k(X)$ and so $fu_{i-\lambda} \in \mathcal{F}_{iQ}$. In this case $\mathcal{E}_i^{\otimes r} = \mathcal{O}_Y\left(f^*\pi(F) + r\sum_j c_{i,j}A_j\right)$ where $c_{i,j} = b_{i,j}$ if A_j does not meet $f^*\pi(F)$ and $c_{i,j} = b_{i-\lambda,j}$ otherwise. Again if A_j does not meet $f^*\pi(F)$ then $\mathrm{RR}_{\mathcal{F}}(C_j) = \mathrm{RR}_{\mathcal{O}_S}(C_j)$. If $f^*\pi(F)$ meets some A_j then

$$\begin{aligned}\mathrm{RR}_{\mathcal{F}}(C_j) &= \sum_{k=0}^{r-1} \frac{\varepsilon^k}{2r}\left(\frac{(k-\lambda)^2}{r}A_j^2 - \frac{k-\lambda}{r}A_jf^*\pi(F) + \left(f^*\pi(F)\right)^2 + kA_jK_Y - f^*\pi(F)K_Y\right)\\ &= \varepsilon^{\lambda}\sum_{k=0}^{r-1} \frac{\varepsilon^k}{2r}\left(\frac{k^2}{r}A_j^2 - \frac{k}{r}A_jf^*\pi(F) + f^*\pi(F)^2 + kA_jK_Y - f^*\pi(F)K_Y\right)\\ &= \varepsilon^{\lambda}\left(\mathrm{RR}_{\mathcal{O}_S}(C_j) + \frac{C_jF}{1-\varepsilon}\right) = \mathrm{Tr}\left(g\colon \mathcal{F}_{|C}\right)\left(\mathrm{RR}_{\mathcal{O}_S}(C_j) + \frac{C_jF}{1-\varepsilon}\right).\end{aligned}$$

Note this shows that if $f^*\pi(F)$ does not meet any of the A_j, or equivalently F does not meet any of the fixed curves then, F does not affect $\mathrm{Tr}\left(g\colon \bigoplus(-1)^p \mathrm{H}^p\left(S, \mathcal{F}\right)\right)$.

The argument for any invertible G-sheaf is identical. □

4.4.4 Proof of Theorem 4.1 when the action is not coprime

In this case there will be curves and points which are fixed by some subgroup of $\langle g \rangle = H$. There are the following five possibilities.

1. P is a point which is not fixed by any element of the group.
2. P is a point on a curve C which is fixed by some nontrivial subgroup of H and $\mathrm{Stab}_H C = \mathrm{Stab}_H P$.
3. P is a point in the intersection of two curves C_1 and C_2 which are fixed by nontrivial subgroups of H. Now $\mathrm{Stab}_H C_1 \cap \mathrm{Stab}_H C_2 = 1$. Suppose $\mathrm{Stab}_H C_1 \times \mathrm{Stab}_H C_2 = \mathrm{Stab}_H P$.
4. As in Case 2 but $\mathrm{Stab}_H C \neq \mathrm{Stab}_H P$.
5. As in Case 3 but $\mathrm{Stab}_H C_1 \times \mathrm{Stab}_H C_2 \neq \mathrm{Stab}_H P$.

In the first three cases the point $\pi(P) = Q$ is smooth.

We will only show the proof when $\mathcal{F} = \mathcal{O}_S$ but the case of a general invertible sheaf follows easily. First compute the generator of $\mathcal{M}_{i\widetilde{Q}}$ for $\widetilde{Q} \in f^{-1}Q$ in each of the above cases.

In Case 1 it follows as before that u_i^r is a unit and so $P \notin \mathrm{Supp}\, rD_i$.

Case 2 is analogous to the case of curves when $C^g \neq C^H$. We show that $\mathrm{RR}(C) = 0$. Let C be a curve fixed by g^a say, where a is the least such a, $\pi(C) = B$ and $a' = r/a$. Consider the short exact sequence

$$0 \to \mathbb{Z}_{a'} \to H \to \mathbb{Z}_a \to 1.$$

Decompose the morphism $\pi\colon S \to X$ as follows:

$$S \to S_1 := S/\mathbb{Z}_{a'} \to X = S_1/\mathbb{Z}_a.$$

Now $\pi_1\colon S \to S_1$ is ramified along C and $\pi_{1*}\mathcal{O}_S = \bigoplus_{i=0}^{a'-1} \mathcal{L}_i'$. The morphism $\pi_2\colon S_1 \to X$ is unramified along $\pi_1(C)$. Let u be a generator of $\mathcal{L}_{i+ja'Q}$ and v a generator of $\mathcal{L}_{i+j'a'Q}$. Then $u^r = v^r$ so if $a'|i-i'$ then $\mathrm{RR}(C,i) = \mathrm{RR}(C,i')$ and so

$$\begin{aligned}
\mathrm{RR}(C) &= \sum_i \varepsilon^i \,\mathrm{RR}(C,i) \\
&= \left(1 + \varepsilon^{a'} + \cdots + \varepsilon^{(a-1)a'}\right) \mathrm{RR}(C,0) \\
&\quad + \left(\varepsilon + \varepsilon^{1+a'} + \cdots + \varepsilon^{1+(a-1)a'}\right) \mathrm{RR}(C,1) \\
&\quad + \cdots \\
&\quad + \left(\varepsilon^{a'-1} + \varepsilon^{a'-1+a'} + \cdots + \varepsilon^{a'-1+(a-1)a'}\right) \mathrm{RR}(C,a-1) \\
&= 0.
\end{aligned}$$

Simplify Case 3 by assuming that P is fixed by H. Again this assumption is to simplify the notation. Let g^a be the generator of $\mathrm{Stab}_H C_1$ and g^b the generator of $\mathrm{Stab}_H C_2$. This case is similar to the above but $\pi_2\colon S_1 \to X$ is ramified over $\pi_1(C_2)$. Let $a' = r/a$ and $b' = r/b$. Choose local coordinates for the point so that $g(s,t) = (\varepsilon^{a'\lambda}s, \varepsilon^{b'\mu}t)$ and assume, for ease of notation, that $\lambda = \mu = 1$. Thus

$$\begin{aligned}
&\mathrm{RR}(P) \\
&= \frac{1}{2}\sum_{j=0}^{a-1}\sum_{k=0}^{b-1} \frac{\varepsilon^{a'j+b'k}}{r^2}\left(j^2A_1^2 + 2jkA_1A_2 + k^2A_2^2 + jrA_1K_Y + krA_2K_Y\right) \\
&= \left(\sum_{j=0}^{a-1} \frac{j}{r}\varepsilon^{a'j}\right)\left(\sum_{k=0}^{b-1} \frac{k}{r}\varepsilon^{b'k}\right) \\
&= \frac{1}{1-\varepsilon^a}\frac{1}{1-\varepsilon^b}.
\end{aligned}$$

Also simplify Case 4 by assuming that the point P is fixed by g. Let $\mathrm{Stab}_H C$ be generated by g^a and simplify further by assuming that $g(s,t) = (\varepsilon^{a'}s, \varepsilon t)$ where s and t are the local coordinates at the point P. Let $A = f^*\pi(C)$.



Figure 4.2: Exceptional locus for the blowup of Q

Figure 4.2 shows the exceptional locus of Y in a neighbourhood of $f^{-1}(Q)$. If $\widetilde{Q} \in E_1$ then as in the prime case $u_i = t^i$ and so $L_0 \notin \mathrm{Supp}\, M_i$. If $\widetilde{Q} \in E_n$, then $u_i^r = \left(t^l/s^m\right)^{\lambda_i} (s^a)^{\mu_i}$ and again if $a'|i-i'$ then $\lambda_i = \lambda_{i'}$.

Again $rK_Y = f^*(rK_X) - \sum d_{i,k}E_{i,k}$. Therefore

$$\begin{aligned}
\mathrm{RR}(P) &= \sum_i \frac{\varepsilon^i}{2r^2}\left(\left(\sum_{j=1}^n a_{i,j}E_j + \lambda_i A\right)^2 + r\sum_{j=1}^n a_j E_j K_Y + r\lambda_i A K_Y\right) \\
&= \sum_i \frac{\varepsilon^i}{2r^2}\left(\left(\sum_{j=1}^n a_{i,j}E_j\right)^2 + 2\lambda_i a_{n,i} - \sum a_{i,j}E_j \sum d_k E_k\right)
\end{aligned}$$

where $a_{i,j}$, λ_i, E_jE_k and d_j depend only on the local analytic nature of the singularity $Q = \pi(P)$. Thus $\mathrm{RR}(P)$ is determined only by how g acts in a local analytic neighbourhood of the point P.

Let C_r be the curve in $\mathbb{P}^2$ of degree r defined previously. Let the action of $\mathbb{Z}_r$ on $\mathbb{P}^2$ be generated by $h = \frac{1}{r}(0,0,1)$ and $g = (h, h^a)$ acts on $C_r \times C_r$. Then $g^{a'}$ fixes the one of the curves C_r and g fixes r^2 points on $C_r \times C_r$. One of these fixed points P' contributes

$$\mathrm{RR}(P') = \frac{1}{1-\varepsilon^a}\frac{1}{1-\varepsilon}.$$

to $\operatorname{Tr}\left(g\colon \bigoplus(-1)^i \operatorname{H}^i(C_r \times C_r, \mathcal{O}_{C_r \times C_r})\right)$. Since $\operatorname{RR}(P)$ depends only on how g acts in a local analytic neighbourhood of the point P it follows that

$$\operatorname{RR}(P) = \operatorname{RR}(P') = \frac{1}{1-\varepsilon^a}\frac{1}{1-\varepsilon}.$$

Exactly as in the previous case, for the final possibility we can show that $\operatorname{RR}(P)$ depends only on how g acts on the local analytic neighbourhood of P. So compute $\operatorname{RR}(P)$ for $P \in C_r \times C_r$ and $g = \left(h^a, h^b\right)$. Thus

$$\operatorname{RR}(P) \quad = \quad \frac{1}{1-\varepsilon^a}\frac{1}{1-\varepsilon^b}.$$

□

4.5 Orbifold Riemann–Roch

Let X be a orbifold with at worst isolated cyclic quotient singularities. Let P be a singular point of X such that there exists a neighbourhood U of P such that $U \cong V/\mathbb{Z}_r$ where $\mathbb{Z}_r$ is generated by $\frac{1}{r}(a_1, \ldots, a_n)$ with a_i coprime to r. Let $\mathcal{L} = \mathcal{O}(D)$ be a divisorial sheaf on X such that $\mathcal{L}_{|U}$ is isomorphic to the ith eigensheaf of $\pi_*\mathcal{O}_V$. In this case we say P with the sheaf $\mathcal{L}$ is of type $i\left(\frac{1}{r}(a_1, \ldots, a_n)\right)$.

Let $f\colon Y \to X$ be a log resolution of the pair (X, D). The sheaf $\mathcal{M} = f^*\mathcal{L}/(\text{torsion})$ is the torsion free rank one sheaf on Y such that $f_*\mathcal{M} = \mathcal{L}$ by Lemma 4.5. The sheaf $\mathcal{M}^{**}$ is reflexive and hence invertible. The proof of Lemma 4.6 implies that $f_*(\mathcal{M}^{**}) = f_*(\mathcal{M})$. Hence from now on we replace $\mathcal{M}$ with $\mathcal{M}^{**}$.

Now $\mathcal{M} = \mathcal{O}(f^*D + \sum b_i E_i)$ and $K_Y = f^*K_X + \sum a_i E_i$. Thus

$$\chi(Y, \mathcal{M}) = \int_Y \operatorname{ch}(f^*D + \sum b_i E_i)\operatorname{td}(Y) = \int_X \operatorname{ch}(D)\operatorname{td}(X) + \sum_P c'_P(D)$$

where $c'_P(D)$ depends only on the local analytic nature of the singularity. The expression $\int_X \mathrm{ch}(D)\,\mathrm{td}(X)$ is interpreted as follows. Take $\mathrm{td}(X)$ on the nonsingular locus of X and intersect with D. This makes sense because some multiple of D is Cartier so can be moved away from the singularities.

By Lemma 4.5 the sheaves $\mathcal{R}^i f_*(\mathcal{M})$ are supported only on the singularities so its space of global sections and hence $h^0(X, \mathcal{R}^i f_*(\mathcal{M}))$ depends only on the analytic nature of the singularities. Thus

$$\chi(X, \mathcal{L}) = \int_X \mathrm{ch}(D)\,\mathrm{td}(X) + \sum_P c_{i,P}$$

where $c_{i,P} = c'_{iP} + \sum_{i=1}^{n-2}(-1)^{i+1}\chi(P, \mathcal{R}^i f_*(\mathcal{M})_{|P})$.

To compute the term $c_{i,P}$ for a singularity of type $i(\frac{1}{r}(a_1, \ldots, a_n))$ with $a_1, \ldots, a_n$ coprime to r calculate them for a global orbifold Z/G having only singularities of this type.

We may assume that $\mathcal{F} \cong \mathcal{F}' \otimes V$ where $\mathcal{F}'$ is not supported at P and V is an irreducible representation of $\mathbb{Z}_r$. The contribution that P makes to $\mathrm{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p\,\mathrm{H}^p(Z, \mathcal{F})\right)$ depends only on how g acts on a local analytic neighbourhood of P. Therefore the contribution that P makes to $\mathrm{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p\,\mathrm{H}^p(Z, \mathcal{F})\right)$ equals the contribution a point $P' \in C_1 \times \cdots \times C_n$ makes to $\mathrm{Tr}\left(g'\colon \bigoplus_{p\geq 0}(-1)^p\,\mathrm{H}^p\left(C_1 \times \cdots \times C_n, V \otimes \mathcal{O}_{C_1\times\cdots\times C_n}\right)\right)$ for suitable g'.

As in the case of a surface each curve can be taken to be $C_r \subset \mathbb{P}^2$ a curve of degree r invariant under the action of $h = \frac{1}{r}(0, 0, 1)$ not passing through the point $(0 : 0 : 1)$. This curve has induced action $\frac{1}{r}(1)$ at each of the fixed points.

The Künneth formula implies that

$$\mathrm{H}^p(C_r \times \cdots \times C_r, \mathcal{O}_{C_1\times\cdots\times C_n}) = \bigoplus_{p_1+\cdots+p_n=p} \mathrm{H}^{p_1}(C_1, \mathcal{O}_{C_1}) \otimes \cdots \otimes \mathrm{H}^{p_n}(C_n, \mathcal{O}_{C_n}).$$

Let $g = (h^{a_1}, \dots, h^{a_n})$ act on $C_r \times \cdots \times C_r$. This fixes r^n points in $C_r \times \cdots \times C_r$. Since each of the a_i is coprime to r it follows that $C_r^h = C_r^H$. Then Theorem 4.11 gives

$$\begin{aligned}
&\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(C_1 \times \cdots \times C_n, V \otimes \mathcal{O}_{C_1\times\cdots\times C_n}\right)\\
&= \operatorname{Tr}(h^{a_1}\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(C_r, V\otimes \mathcal{O}_{C_r}))\prod_{i=2}^{n}\operatorname{Tr}(h^{a_i}\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(C_r, \mathcal{O}_{C_r}))\\
&= \left(\sum_{P\in C_r^h}\frac{\operatorname{Tr}\left(h^{a_1}\colon V\otimes\mathcal{O}_{C_r|P}\right)}{1-\operatorname{d}h_P^{a_1}}\right)\prod_{i=2}^{n}\left(\sum_{P\in C_r^h}\frac{\operatorname{Tr}\left(h^{a_i}\colon \mathcal{O}_{C_r|P}\right)}{1-\operatorname{d}h_P^{a_i}}\right)\\
&= r^n\frac{\operatorname{Tr}\left(h^{a_1}\colon V\otimes\mathcal{O}_{C_r|P}\right)}{1-\operatorname{d}h_P^{a_1}}\prod_{i=2}^{n}\left(\frac{\operatorname{Tr}\left(h^{a_i}\colon \mathcal{O}_{C_r|P}\right)}{1-\operatorname{d}h_P^{a_i}}\right)\\
&= r^n\frac{V(g)}{1-\varepsilon^{a_1}}\frac{1}{1-\varepsilon^{a_2}}\cdots\frac{1}{1-\varepsilon^{a_n}}.
\end{aligned}$$

Therefore $P\in Z$ contributes

$$\frac{V(g)}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})} \text{ to } \operatorname{Tr}\left(g\colon \bigoplus(-1)^p \operatorname{H}^p(Z,\mathcal{F})\right).$$

Thus if G is a cyclic group acting on a smooth variety Z of any dimension and having only isolated fixed points then

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & \varepsilon & \varepsilon^2 & \cdots & \varepsilon^{r-1}\\ \vdots & \vdots & \vdots & & \\ 1 & \varepsilon^{r-1} & \varepsilon^{r-2} & \cdots & \varepsilon \end{pmatrix}\begin{pmatrix} \chi(X,\mathcal{L}_0)\\ \chi(X,\mathcal{L}_1)\\ \vdots\\ \chi(X,\mathcal{L}_{r-1})\end{pmatrix} = \begin{pmatrix} \chi(Z,\mathcal{F})\\ \sum_{P\in Z^g}\frac{\operatorname{Tr}(g\colon(\mathcal{F}_P))}{\det(I-\operatorname{d}g_P)}\\ \vdots\\ \sum_{P\in Z^g}\frac{\operatorname{Tr}(g^{r-1}\colon(\mathcal{F}_P))}{\det(I-\operatorname{d}g_P^{r-1})}\end{pmatrix}. \quad (4.12)$$

Inverting the Vandermonde matrix gives a Riemann–Roch expression for a sheaf $\mathcal{L}_i$ on a global orbifold X.

$$\frac{1}{r}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & \varepsilon^{r-1} & \varepsilon^{r-2} & \cdots & \varepsilon\\ \vdots & \vdots & \vdots & & \\ 1 & \varepsilon & \varepsilon^2 & \cdots & \varepsilon^{r-1} \end{pmatrix}\begin{pmatrix} \chi(Z,\mathcal{F})\\ \sum_{P\in Z^g}\frac{\operatorname{Tr}(g\colon(\mathcal{F}_P))}{\det(I-\operatorname{d}g_P)}\\ \vdots\\ \sum_{P\in Z^g}\frac{\operatorname{Tr}(g^{r-1}\colon(\mathcal{F}_P))}{\det(I-\operatorname{d}g_P^{r-1})}\end{pmatrix} = \begin{pmatrix} \chi(X,\mathcal{L}_0)\\ \chi(X,\mathcal{L}_1)\\ \vdots\\ \chi(X,\mathcal{L}_{r-1})\end{pmatrix}$$

This proves the following theorem.

Theorem 4.18 (Orbifold Riemann–Roch) *Let X be a cyclic orbifold having at worst isolated quotient singularities. Let $\mathcal{L}$ be a divisorial sheaf such that $\mathcal{L}_{|U_P}$ is isomorphic to the ith eigensheaf of $\pi_*(\mathcal{O}_V)$. Then*

$$\chi(X,\mathcal{L}) = \int_X \operatorname{ch}(D)\operatorname{td}(X) + \sum_{P\in \operatorname{Sing} X} c_{i,P}.$$

If P is a point of type $i\left(\frac{1}{r}(a_1,a_2,\ldots,a_n)\right)$ for $i = 0,\ldots r-1$ then

$$c_{i,P} = \frac{1}{r}\sum_{l=1}^{r-1} \frac{\varepsilon^{-il}}{(1-\varepsilon^{la_1})\cdots(1-\varepsilon^{la_n})}.$$

By Equation 4.12 when $X = Z/G$ then

$$\int_X \operatorname{ch}(D)\operatorname{td}(X) = \frac{1}{r}\int_Z \operatorname{ch}(D')\operatorname{td}(Z)$$

where $D' = \pi^* D$.

This expression can be simplified to show that

$$\begin{aligned} c_{i,P} &= (-1)^{n+1}\left(\frac{(r-1)^n}{2^n r} - \sum_{(\lambda_1,\ldots,\lambda_n)\in\Lambda_{i,P}} \frac{\lambda_1\cdots\lambda_n}{r^n}\right), \qquad (4.13)\\ \Lambda_{i,P} &:= \left\{(\lambda_1,\ldots,\lambda_n) \;\middle|\; \sum a_j\lambda_j \equiv i \pmod r,\ 0\le\lambda_k\le r-1\right\}. \end{aligned}$$

Remark 4.19 The following two observations follow immediately from this definition.

- If $(\lambda_1,\ldots,\lambda_n)\in\Lambda_{i,P}$ then $(r-\lambda_1,\ldots,r-\lambda_n)\in\Lambda_{-i,P}$.
- $\sum_i c_{i,P} = (-1)^{n+1}\left(r(r-1)^n/(2^n r) - (1+\cdots+r-1)^n/r^n\right) = 0$.

If $n = 1$ then there are no singular points but we can still compute $c_{i,P}$. In Section 4.3 we computed the degree of the sheaves $\mathcal{L}_j$ by calculating the degree of the divisors D_j as $\mathbb{Q}$-divisors. If we let $\mathcal{L}'_j = \mathcal{O}(\lfloor D_j\rfloor)$ then the

difference between $\chi(X, \mathcal{L}_j)$ and $\chi(X, \mathcal{L}'_j)$ is the sum of the terms $c_{j,P}$ for all P appearing in $\operatorname{Supp} D_j$. For P a point of type $i\left(\frac{1}{r}(1)\right)$, Equation 4.13 implies

$$c_{i,P} = \frac{r-1}{2r} - \frac{i}{r}.$$

Let P be a singularity of type $i\left(\frac{1}{r}(1,-1)\right)$. Then

$$\Lambda_0 = \{(1,1),(2,2),\ldots,(r-1,r-1)\}$$

and

$$\begin{aligned} c_{0,P} &= \frac{1+4+\cdots+(r-1)^2}{r^2} - \frac{(r-1)^2}{4r} \\ &= \frac{r(r-1)(2r-1)}{6r^2} - \frac{(r-1)^2}{4r} = \frac{(r-1)(r+1)}{12r}. \end{aligned} \tag{4.14}$$

The following result is taken from Reid's "Young person's guide to canonical singularities" [Rei87]

$$c_{i,P}\left(\frac{1}{r}(a_1,\ldots,a_n)\right) - c_{i-a_n,P}\left(\frac{1}{r}(a_1,\ldots,a_n)\right) = c_{i,P}\left(\frac{1}{r}(a_1,\ldots,a_{n-1})\right).$$

Together with Equation 4.14 this implies

$$c_{i,P} = \frac{r^2-1}{12r} - \frac{i(r-i)}{2r}.$$

This agrees with [Rei87].

If P is a singularity of type $i\left(\frac{1}{r}(1,1)\right)$ then

$$\Lambda_{0,P} = \{(1,r-1),(2,r-2),\ldots,(r-1,1)\}$$

and in the same way it follows that

$$c_{i,P} = \frac{(r-1)(5-r)}{12r} + \frac{i(r-i-2)}{2r}.$$

Let G be any finite group such that Z^g is zero dimensional for all $g \in G$. Then we will obtain a similar matrix expression to Equation 4.12 where the square matrix will be the character table of G. In the same way inverting this matrix will imply an orbifold Riemann–Roch formula for any orbifold having isolated singularities.

Bibliography

[AS68] M. Atiyah and I. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87: pages 546–604, 1968.

[Ati67] M. Atiyah. *K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[BKR01] T. Bridgeland, A. King, and M. Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14: pages 535–554, 2001.

[Bou65] N. Bourbaki. *Algèbre commutative.* Hermann, 1965.

[CR99] A. Craw and M. Reid. How to calculate A-Hilb $\mathbb{C}^3$. Preprint math.AG/9909085, 1999.

[Cra01] A. Craw. *The Mckay correspondence and representations of the McKay quiver.* PhD thesis, University of Warwick, 2001.

[God73] R. Godement. *Topologie algébrique et théorie des faisceaux.* Hermann, 1973.

[GSV83] G. Gonzalez-Sprinberg and J. Verdier. Construction géométrique de la correspondance de McKay. *Ann. Sci. École Norm. Sup.*, 16: pages 409–449, 1983.

[Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, 1977.

[IN99] Y. Ito and I. Nakamura. Hilbert schemes and simple singularities. In *New trends in algebraic geometry (Warwick, 1996)*, pages 151–233. Cambridge Univ. Press, 1999.

[Ito95] Y. Ito. Crepant resolution of trihedral singularities and the orbifold Euler characteristic. *International Journal of Mathematics*, 6(1): pages 33–43, 1995.

[KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*. Cambridge University Press, 1998.

[Lan93] S. Lang. *Algebra*. Addison-Wesley Publishing Company, third edition, 1993.

[Mum85] D. Mumford. *Abelian varieties*. Oxford University Press, 1985.

[Nak01] I. Nakamura. Hilbert schemes of Abelian group orbits. *J. Alg. Geom.*, 10, 2001.

[Rei80] M. Reid. Canonical 3-folds. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, 1980.

[Rei87] M. Reid. Young person's guide to canonical singularities. In *Proceedings of Symposia in Pure Mathematics*, volume 46, pages 345–414. Amer. Math. Soc., 1987.

[Rei97] M. Reid. Mckay correspondence. *Proc. of algebraic geometry symposium (Kinosaki, Nov 1996)*, pages 14–41, 1997.

BIBLIOGRAPHY

[Rei02] M. Reid. La correspondance de McKay. *Astérisque*, 276: pages 53–72, 2002.

[Sha94] I. R. Shafarevich. *Basic algebraic geometry 1.* Springer-Verlag, 1994.