THE KNITTING OF SURFACES

BY M. A. REID

Introduction As anyone who has worn a woolly will know, knitting is rather a nice way of representing some 2-dimensional manifolds. The question naturally presents itself: which of the two-manifolds can be knitted without seams? Unfortunately the answer to this is rather dull: they all can be, although not all of them very nicely. Since knitting has got an obvious 'grain', knitting a two manifold provides us with a combing of it. So the question that is interesting is the following: which of the 2-manifolds can be knitted without seams and with only respectable singularities?

For example, let us look at the object below, a cylinder dividing in two. If we were trying to knit this, the branch point would have to have a singularity of the type shown.

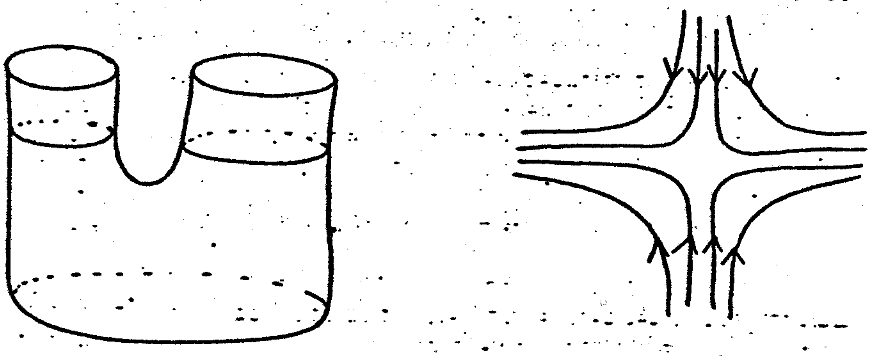


fig 1

This is something that cannot be very elegantly knitted: we will get a hole in the middle.

By contrast a singularity that does come off all right is the one that has been standard knitting practice since the invention of the bobble-cap. Starting with a cylinder we just decrease till there are only half-a-dozen stitches left, slip them all onto the thread, and pull tight.

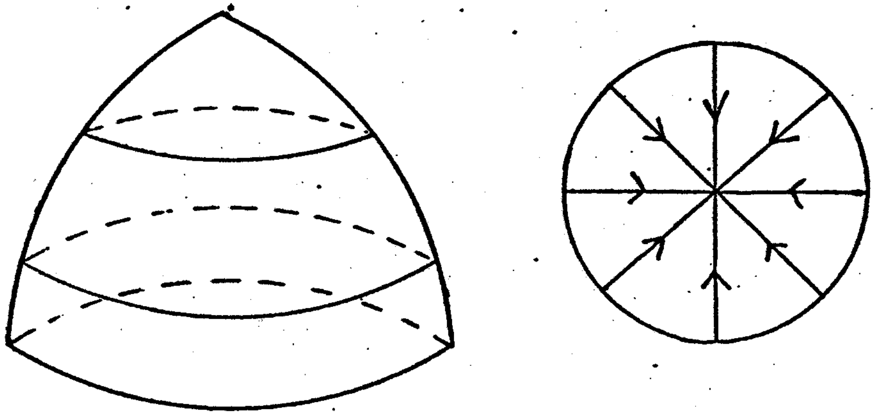


fig 2

This article provides patterns for the sphere, the torus, the klein bottle and one surface-with-boundary, the mobius strip. The torus and the klein bottle can be done with no singularity at all. The other surface than can be nicely combed is the real projective plane. I have a method for doing this but it is long and impossibly messy to describe. All the two-manifolds can now be knitted, by just taking connected sums of toruses and projective planes, using crude techniques, and of course the above untidy singularity.

Remark: the reader may wonder why I am always knitting on the round, rather than back and forth on rows. This is easy to explain: at any stage in the process, the piece of knitting is a 2-manifold with boundary. As is well known, the boundary now has to be a 1-manifold without boundary, i.e. a circle or collection of circles. This explains why I always start "cast on a cylinder of so-many stitches".

Technical Digression I require the use of three rather special techniques. Two of them are deduced from the appearance of a knitted cylinder. It is symmetric for reflections in a horizontal plane, and there is nothing to distinguish one row from any other. So one could have cast on a middle row first, and worked out; or alternatively, put the middle row in last of all.

The first is easy. Using spare (and different colour) wool, cast on a cylinder. Knit a couple of rows. Join in a main colour, and knit, say 10 rows. If the spare wool is now cut away, there is left a further row which can be slipped onto some further needles, and kept for future use.

The second is tricky, although a standard knitting technique (see the P. & B. booklet "Woolcraft" – the section on socks).

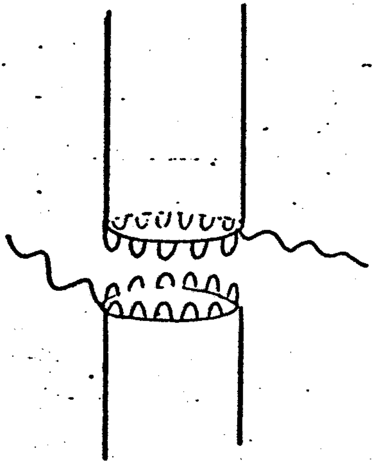
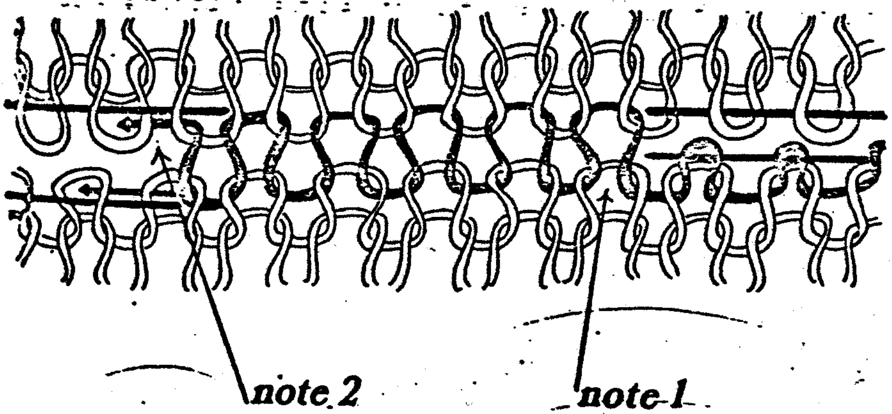


fig 3

As illustrated below, we have two cylinders with right side of work facing. The threads are at opposite sides of the cylinders, one of them being cut to a couple of yards, and threaded onto a bodkin. The process defies explanation, but I hope that the local diagram will make things clear.



In the local diagram note:

(1) The general stitch consists of one threading from front to back to front through the next stitch.

(2) The first stitch is perverse and confusing. The thread is passed through the first stitch from p. side to k. side of work.

I shall refer to this process as "grafting", since this is the standard terminology.

I do feel guilty that this is a cheat since it is "sewing up". I use the following remarks to satisfy my own conscience: a) it is standard b) it is dual to the two-sided casting on, which is irreproachable c) the purist who objects may say that I have cheated. but will not be able to say where, since the "grafting" row is in principle indistinguishable from any other row in which wool was joined, and is in practice indistinguishable, if the grafting was done carefully.

The third special technique is indispensable for making any of the non-orientable surfaces. Since they intersect themselves, we need a process for passing a cylinder through an already existing surface, a "wall". This is not very difficult, but requires a crochet hook. Slip the stitches onto a piece of spare wool, pull first the thread through a chosen hole, then each of the stitches, mounting them onto another piece of spare wool when they are through.

The Patterns requirements: set of four no.8 needles, two ounces of double knitting wool, a crochet hook, a few yards of a different colour scrap wool, kapok for stuffing (from Woolworth's).
The Sphere Cast on (both sides) a cylinder of 30 sts.

*k. 5 rounds, Decrease as follows.

Next round: (k.8 k.2 tog) 3 times
Next round: (k.7 k.2 tog) 3 times
Next round: (k.6 k.2 tog) 3 times
Next round: (k.5 k.2 tog) 3 times
Next round: (k.1 k.2 tog) 6 times
Next round: (k.2 tog) 6 times

Break off thread. Slip last sts onto thread and pull tight, leaving thread on wrong side of work.**

Join in thread at second side of casting on. Knit second hemisphere from *to**, stuffing firmly a few rounds before end if required.

The Torus (a) Cast on (both sides) a cylinder of 30 sts. Knit 80 rows, then half a round. Pick up sts from second side of casting on. Graft to finish, stuffing if required.

(b) Cast on (both sides) a cylinder of 32 sts. Knit one round.

Increase 8 sts in each of the next 8 rows as follows:
Inc in next st, k.3, inc in next st, k.3(5, .. 17) 4 times, knit 8 rounds, then *decrease* 8 sts in each of next 8 rows, as follows:
k.2 tog, k.3, k.2 tog, k.17(15 ... 3) 4 times, knit 7 rounds, knit half a round, graft off.

The Klein Bottle Cast on (both sides) a cylinder of 30 sts, knit 10 rounds. Increase 6 sts in every 4th row, 5 times, as follows:
1st (5th . . . 17th) row: k.4(5 . . . 8), inc in next sts six times (36, 42 . . 60 sts) others: knit. Knit three rounds.
Form a 'purl' window as follows:

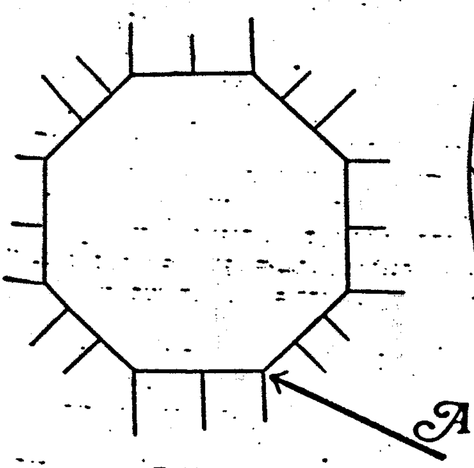
1st round: k.7, p.4, k. to end
2nd round: k.6, p.6, k. to end
3rd round: k.5, p.8, k. to end
4th–7th round: k.4, p.10, k. to end
8th round: as 3rd round
9th round: as 2nd round
10th round: as 1st round

K. 5 rounds, then first five sts of next round onto the end of the last needle, so that the round starts five sts later than previously.

Decrease 5sts every 4th round, 6 times, as follows:

1st (5th . . 21st) row: k.10, (k.8(7, . . 3) k.2 tog) 5 times (55,50 . . 30st) others: knit. K. 5 rounds, then first three stitches of next round. Pass thread through wall at "A" (in diagram) and pass cylinder through the wall at the outside edge of the purl window. K. 50 rounds. Graft off, stuffing.

The Mobius Strip This pattern requires a special weapon – a circular needle 32" long. The method is merely an improvement of "twosided" casting on such that both sides of the stitches can be used from the beginning.
Using spare wool and a pair of needles, cast on 90 sts.



Using main wool, knit these onto the circular needle. They now cover about 2/3 of the needle, the stitches at the far end from the thread lying on the plastic. The working end of the needle is bent round to the far end to pick these stitches up, and these are *not* slipped off the end as they are knitted. When the row is all picked up, the needle loops the work twice, and the effect is as of ordinary two-sided casting on except that the second row is held on the needles. Knit into this row again. Knit five rows, and cut away the spare row. Cast off. Obviously it is not satisfactory to do a Mobius strip in stocking stitch! However, two-sided casting on cannot possibly work for any rib! (try it and see)

THE ARCHIMEDEANS

The Archimedeans have had a good year, with the evening and tea meetings well attended. The evening meetings included talks by Professor E. C. Zeeman on "Catastrophe Machines" and Professor H. Laster, on "The Propagation of Cosmic Rays". There was also a meeting in the Easter term at which Professor Marshall Hall spoke on "Problems in Arrangements". Tea meetings were very successful, with Dr. J. H. Conway on "Hackenbush, Welter, Prune and other games" and Dr. A. F. W. Edwards' talk: "Probability Theory and Human Genetics".

There was the usual visit to Oxford to play games with the "Invariants" and the Problems Drive in which the "Invariants" visited Cambridge. The visit to the Rutherford High Energy Laboratory at Abington was very successful. In the Lent term a dinner was held in the Graduate Centre. Amongst the society's guests were Professor Sir Nevil Mott and Professor J. F. Adams. The Computer Group has had an active year and the Music group and the Bridge group have both met frequently, but the Puzzles and Games Ring died in the Lent Term and its future is uncertain. The Bookshop has continued to thrive.

Speakers for the coming year include Professor C. T. C. Wall on "How to Organise a Tournament". Lady Jeffreys, who will speak at a tea meeting, and an address from Dr. P. Neumann on "The Mathematical Analysis of "1066 and all That" " will be the first meeting of the academic year. There will be a Careers Meeting as usual and also a visit to Oxford.

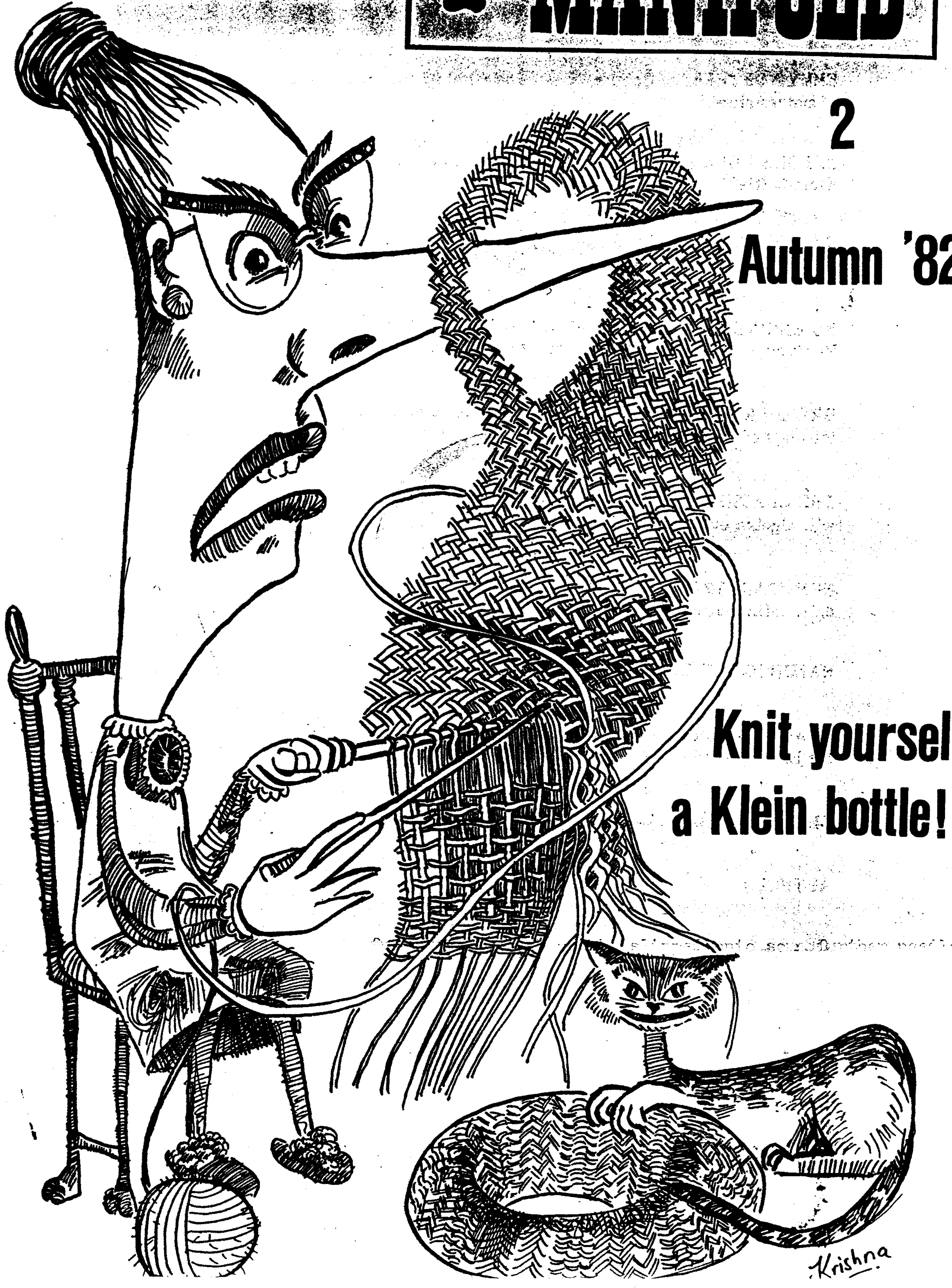
It is hoped that this year's programme will cater for all tastes. Suggestions for any change in the activities, or for speakers for future years would be most welcome; a book is kept in the Arts School for this purpose.

Simon Anscombe

2-MANIFOLD

2

Autumn '82



Knit yourself a Klein bottle!

Knitting 2-manifolds, 1

By Miles Reid.

The next issue of 2-manifold will contain a knitting pattern for the Boy's surface - an immersion in 3-space of the real projective plane $\mathbb{R}\mathbb{P}^2$. As a preliminary, I republish here extracts from my earlier paper (The knitting of surfaces, Eureka vol' - ' (I9)), written when I was an undergraduate. The patterns given here are very much easier to understand and to perform that that for the Boy's surface, and the techniques described here should be mastered first.

I require the use of three rather special techniques. Two of them are deducted from the appearance of a knitting cylinder. It is symmetrical for reflections in a horizontal plane, and there is nothing to distinguish one row from any other. So one could have cast on a middle row first, and worked out; or alternatively, put the middle row in last of all.

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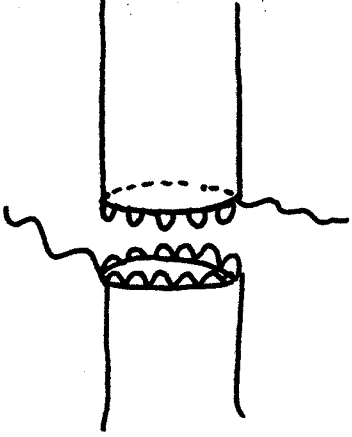

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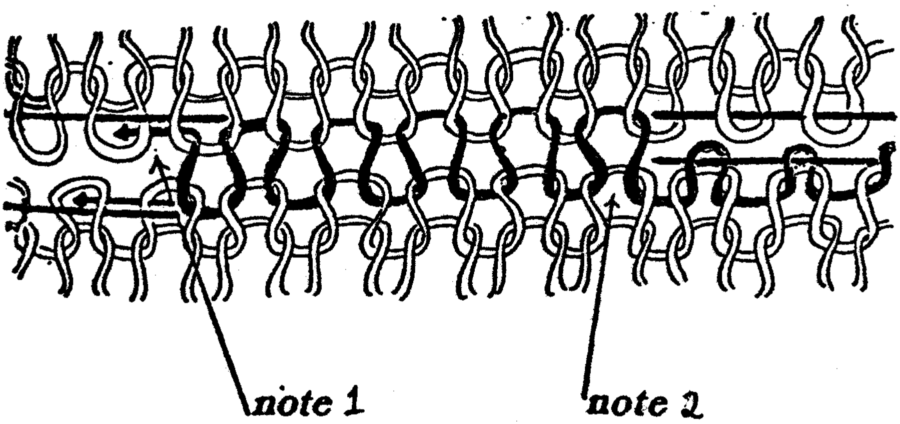

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Inc in next st K3, inc in next st K3 (5,..I7) 4 times, knit 8 rounds, then decrease 8 sts in each of the next 8 rows, as follows:
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Ist (5th..2Ist) row:K10, (K.8(7,..3) K.2tog) 5 times(55,50..30st) others: knit K.5 rounds, then first three stitches of next round. Pass thread through wall at "A" (in diagram) and pass cylinder through the wall at the outside edge of the purl window. K.50 rounds, Graft off, stuffing.

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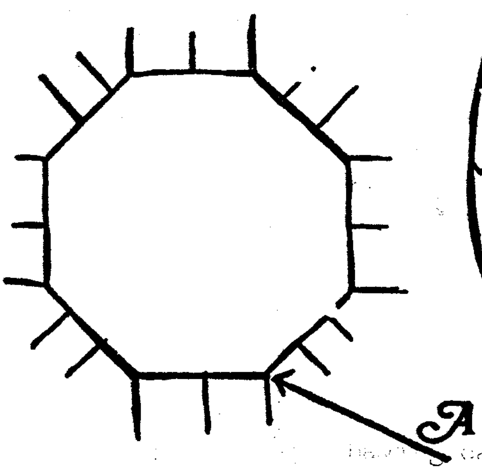

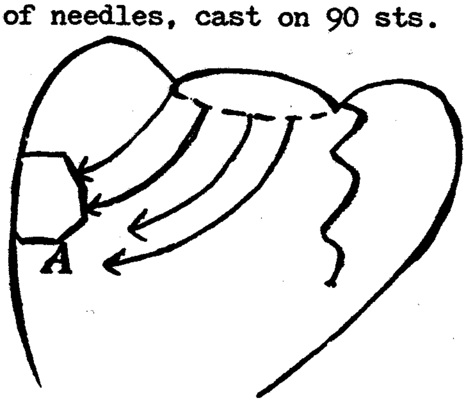

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On the Cardinality of God

By Boris

From an article by one Vox Fisher in Manifold 6 we have:

THEOREM (due to Anslen, Aquinas, and others).

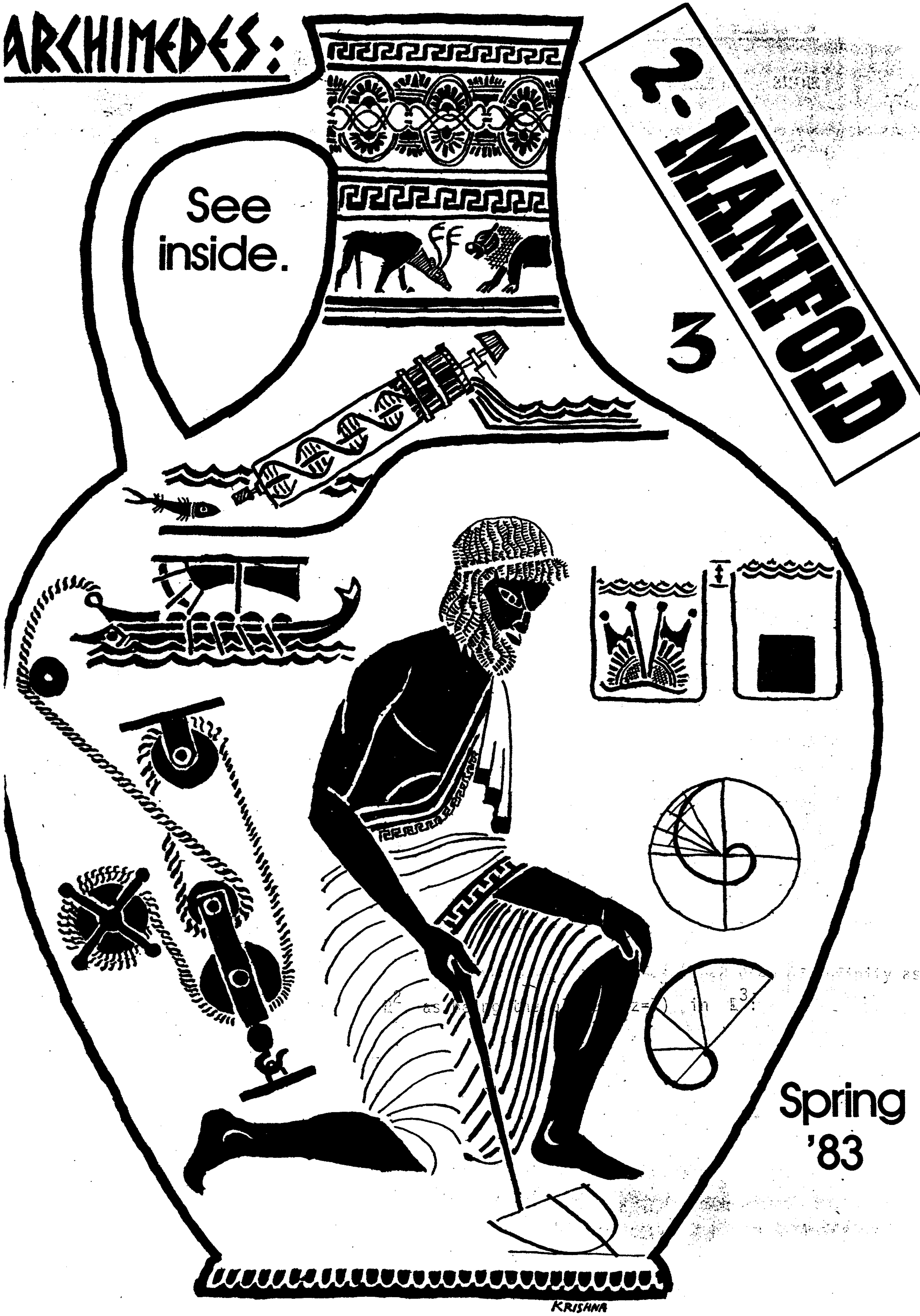
The axiom of choice is equivalent to the existence of a unique God.

Now since we have a link between Mathematics and Ontology, I feel that I would further this link by determining the cardinality of the above mentioned God.

Consider god as the set determined by Hir potential, this set having cardinality C, Clearly C must be transfinite by (omnipotence proof above), so C $\geq \aleph_0$

Since we have that "God moves in mysterious ways" (1) Hir potential must be beyond countability, therefore C $> \aleph_0$

(1) "God moves in mysterious ways, his wonders to perform" William Cooper I73I - I800


ARCHIMEDES:
2-MANIFOLD
3
See inside.
Spring '83
KRISHNA

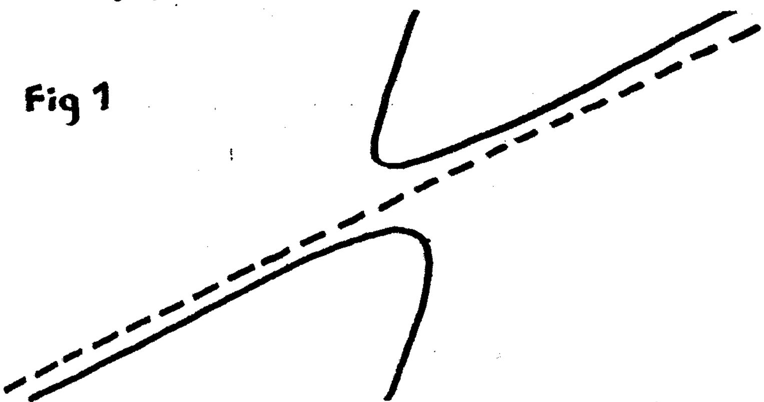
Needlework Section:

Knitting 2 - Manifolds 2: the Boy's Surface

by Miles Reid

(a) The projective plane. Doing geometry in $\mathbb{R}^2$, it often happens that a curve "goes off to infinity", and one needs to take note of its asymptotic directions:

Fig 1

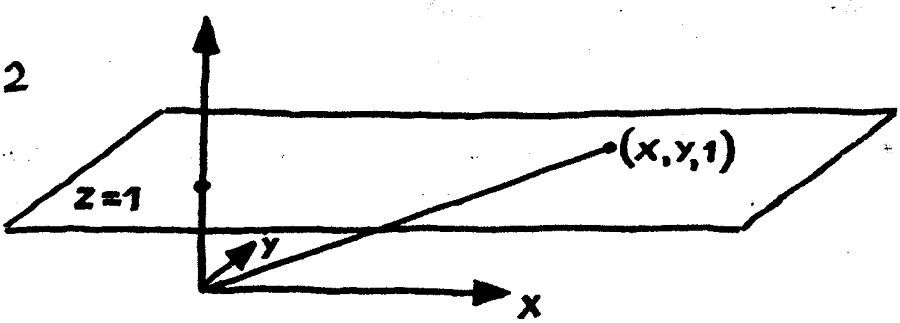


A typical phenomenon that arises for parametrised curves is that a curve will go off to infinity along an asymptotic line ℓ, to reappear on the same line ℓ, but "at the other end".

To follow the behaviour at infinity of curves, one introduces the projective plane, which first arises as $\mathbb{R}^2 \cup \{\text{pts at } \infty\}$, where there is one point at infinity for every pencil of parallel lines; a curve passes through a point at infinity if its asymptotic tangent direction belongs to the corresponding pencil.

You get a less prejudiced view of infinity as follows: think of $\mathbb{R}^2$ as being the plane $(z=1)$ in $\mathbb{R}^3$:

Fig 2



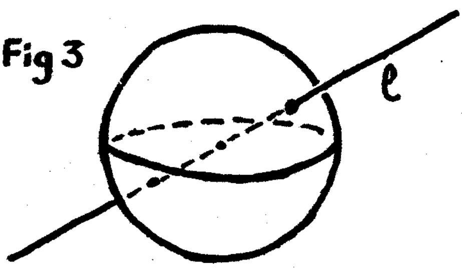
Then points of $(z=1)$ are in bijection with lines through 0 in $\mathbb{R}^3$, except that the lines of $\mathbb{R}^3$ in $(z=0)$ are parallel to $(z=1)$, so never actually meet it; they do however, determine the pencil of parallels, so correspond to points at infinity of $\mathbb{R}^2$. If you imagine a 1-parameter family of lines through 0 in $\mathbb{R}^3$, this will from time to time cross over the plane $(z=0)$, and it is easy to see that the lines cut $(z=1)$ in points which wander off to infinity, reappearing at the opposite end of $\mathbb{R}^2$.

<u>Definition</u>. $\mathbb{P}^2_{\mathbb{R}} = \{\text{lines } \ell \subset \mathbb{R}^3 \text{ thro' } 0\}$.

Any point $\underline{x} \in \mathbb{R}^3 - 0$ determines a unique line $\ell = \mathbb{R}.\underline{x}$, and two points $\underline{x}$, $\underline{y}$ determine the same line if and only if $\underline{x} = \lambda\underline{y}$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Thus equivalently $\mathbb{P}^2_{\mathbb{R}} = (\mathbb{R}^3-0)/\sim$, where $\sim$ is the equivalence relation "is a scalar multiple of". The coordinates (x_1, x_2, x_3) of $\underline{x} \in \mathbb{R}^3$ are <u>homogeneous coordinates</u> of a point of $\mathbb{P}^2_{\mathbb{R}}$, a notion essential for the purposes of studying curves in algebraic geometry. Many other beautiful and important results about the projective plane, and projective geometry, are taught in the Warwick 2nd year geometry course.

To get closer to a topological representation, note that every line ℓ through 0 in $\mathbb{R}^3$ meets the unit sphere S^2: $(x_1^2+x_2^2+x_3^2=1)$ in a pair of antipodal points:

Fig 3



(a point is said to be <u>antipodal</u> if it goes round saying "Gudday, yer Poms").

Hence $\mathbb{P}^2_{\mathbb{R}}$ can also be considered as $S^2/\sim$, that is, S^2 with pairs of antipodal points identified. Since every pair of antipodal points has a representative on the equator or in the N.hemisphere, $\mathbb{P}^2_{\mathbb{R}}$ is the same thing as a disc with opposite points of the circumference identified (Fig.4). To understand a neighbourhood of the glued equator, consider what happens to a strip around a half-circumference: A gets glued onto ∀ with a half-twist:

Fig 4

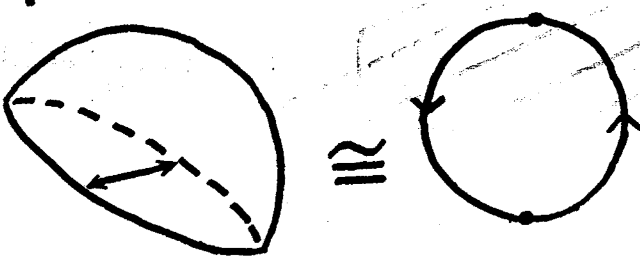
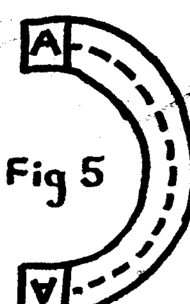
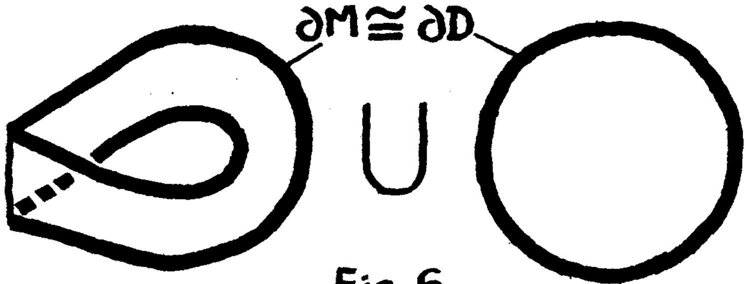


Fig 5



Hence a construction of $\mathbb{P}^2_{\mathbb{R}}$ (as an abstract topological space) can be given as follows: take a Möbius strip M and a disc D; these both have boundaries homeomorphic to a circle S^1: $\partial D \cong \partial M \cong S^1$; hence we can attach D to M along the boundary circle:



$\cong \mathbb{P}^2_{\mathbb{R}}$.

Fig 6

This sounds simple, but it can't be done in $\mathbb{R}^3$, because the boundary circle ∂M is knotted, that is, cannot span an embedded disc.

If on reading the sequel the reader is appalled at the contortions one has to go to in order to have a representation of $\mathbb{P}^2_{\mathbb{R}}$ in $\mathbb{R}^3$, I hope that he will not be misled into thinking that $\mathbb{P}^2_{\mathbb{R}}$ is in any way nasty, complicated or pathological. It's $\mathbb{R}^3$ that's entirely to blame.

(b) Boy's immersion. To get a representation in $\mathbb{R}^3$ one has to allow multiple points, which I now discuss: a subset $X \subset \mathbb{R}^3$ is called an immersed surface if for every $x \in X$ there exists a small neighbourhood N of x such that $N \cap X$ is a union of finitely many smooth surfaces S_i . X itself is singular, but if you think of the S_i as being made out of transparent material of different colours, and put on the special 5-dimensional tinted spectacles included free in the TV Times, the branches S_i jump out of $\mathbb{R}^3$ so that they occupy different positions of $\mathbb{R}^5$.

In fact I will only discuss two rather simple kinds of singularities: (i) double curve: this looks like the locus given by $xy=0$ in a neighbourhood of the z-axis; the two surfaces S_1, S_2 are the planes $x=0$ and $y=0$; (ii) triple point: this looks like the locus $xyz = 0$ in a neighbourhood of the origin; the 3 coordinate planes pass through 0, with the 3 coordinate axes as double curves:

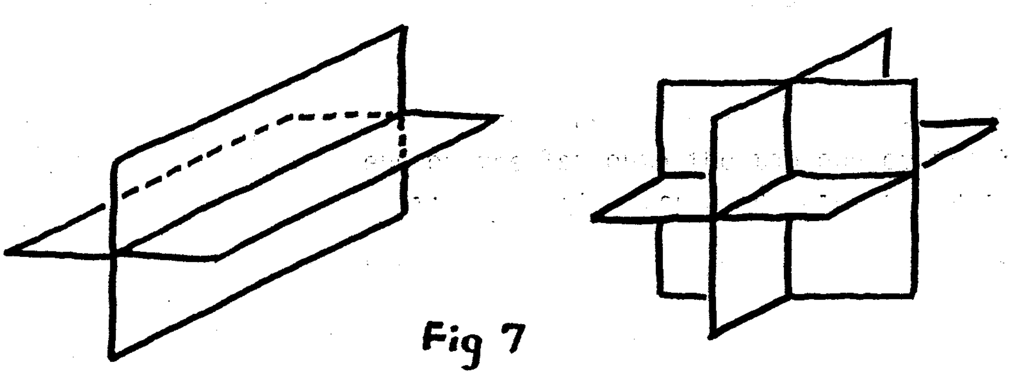


Fig 7

Boy's immersion is a method of putting a Möbius strip M into $\mathbb{R}^3$, with only self-crossings of this type, and in such a way that the boundary ∂M goes into an unknotted S^1 of $\mathbb{R}^3$; the immersed surface X can then be capped off with a disc, giving an immersed $\mathbb{P}^2_{\mathbb{R}}$. I explain first how to make this immersed surface out of cardboard. Start with 3 strips cut to the following shape and size:

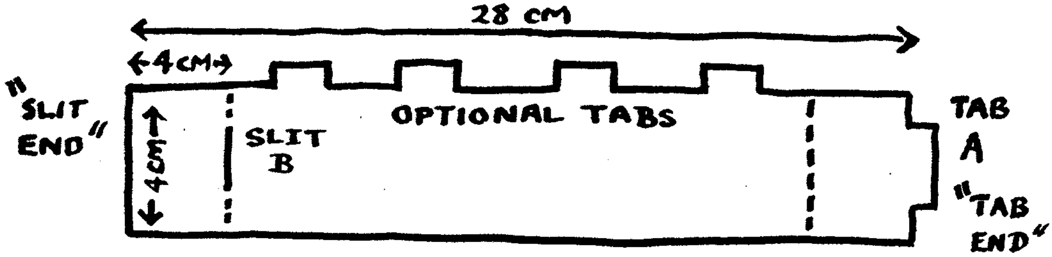
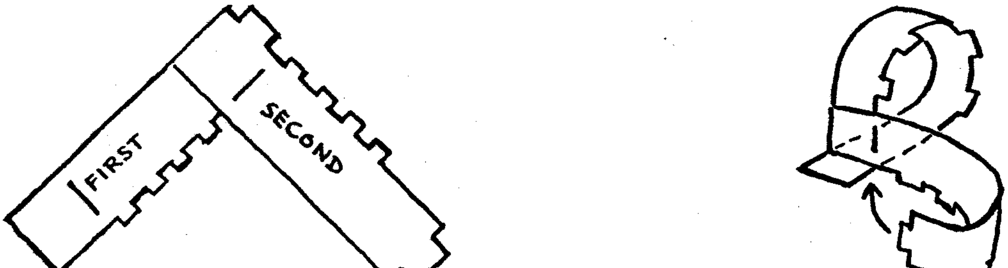


Fig 8

Each strip can be bent over so that its Tab A inserts into its own Slot B is a rather autoerogenous manner. Now glue the 3 strips together as follows the 4 x 4 square at the slit end of the 2nd strip glues onto the 4 x 4 square at the tab end of the 1st, so that the strips are at right angles:

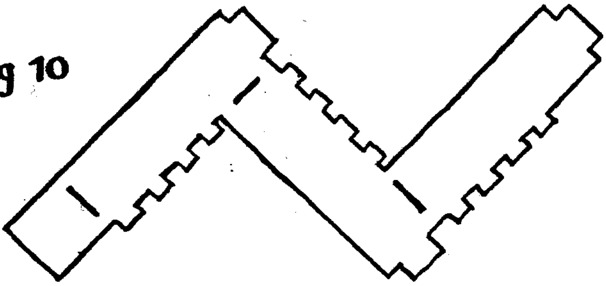


laid flat   assembled

Fig.9

(if you're making this out of a corn-flake packet, and make all 3 strips the same, you'll want to glues the inside face of the 1st strip to the inside face of the 2nd, so that the orientation changes at the corner.) Now glue the 3rd strip into the 2nd in the same way; this can still be laid flat:

Fig 10



If all 3 strips are now assembled, there is only one way to glue the slit end of the 1st onto the tab end of the 3rd. The construction has 3-fold cyclic symmetry; after the glue has dried one can pull the tabs out of the slots again, and verify that the ring formed by the 3 strips is a Möbius band with 3 half-twists (Fig.11).

When assembled, each tab now emerges onto the top end of a cylinder (optionally crenellated) made up of one loop of the Möbius strip (Fig.12), and the next step is to make a slightly overhanging lid for each cylinder (Fig.13). Suitable dimensions for the lid, and the required slotting to allow the crenellations of the cylinder interpenetrate the lid convincingly are safely left to the reader.

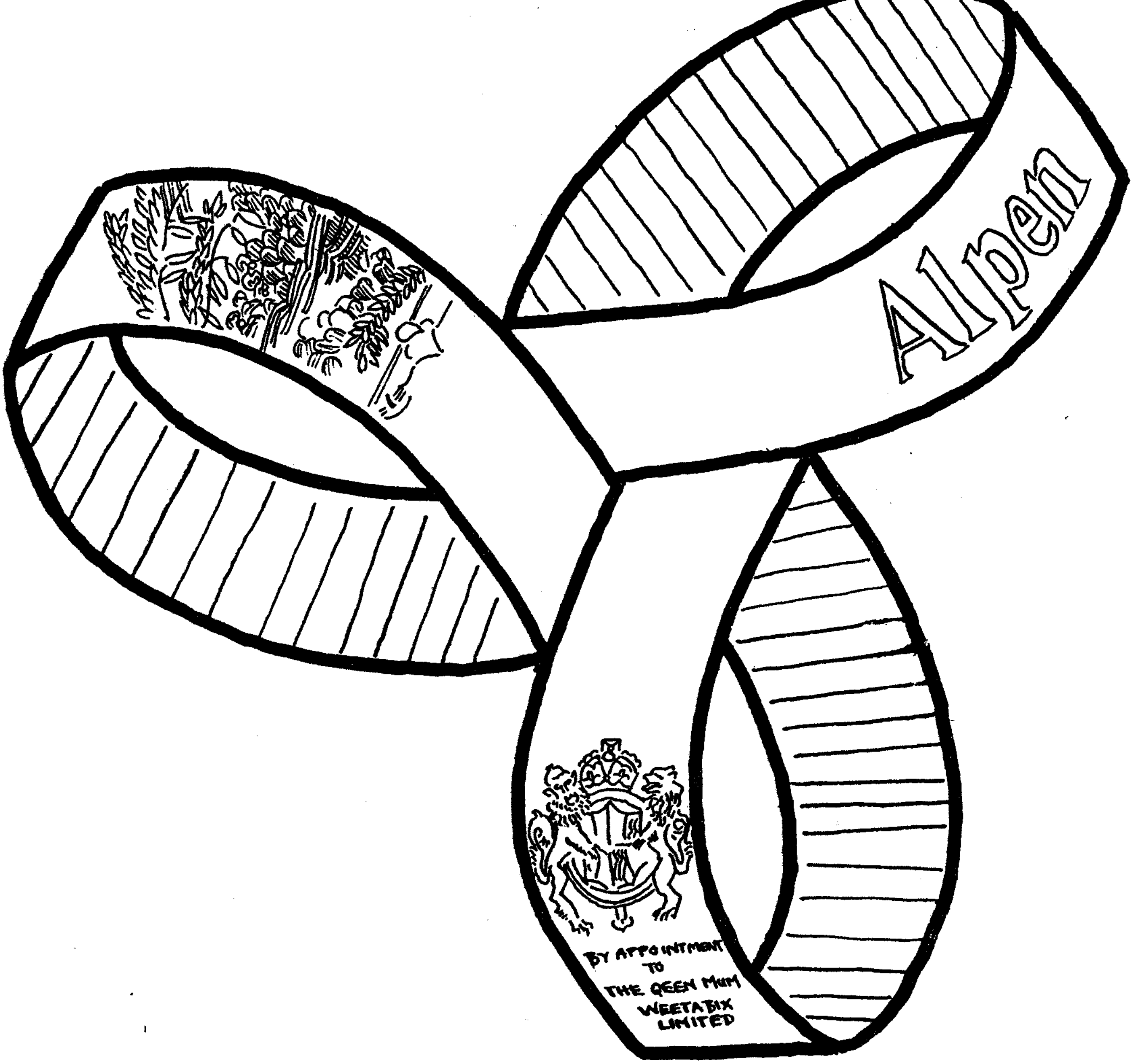


Fig 11

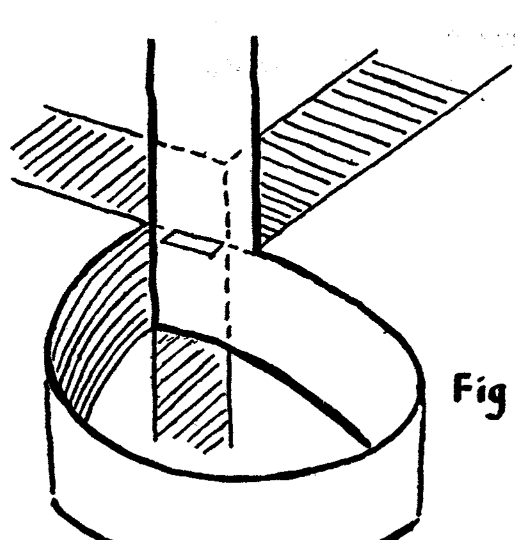


Fig 12

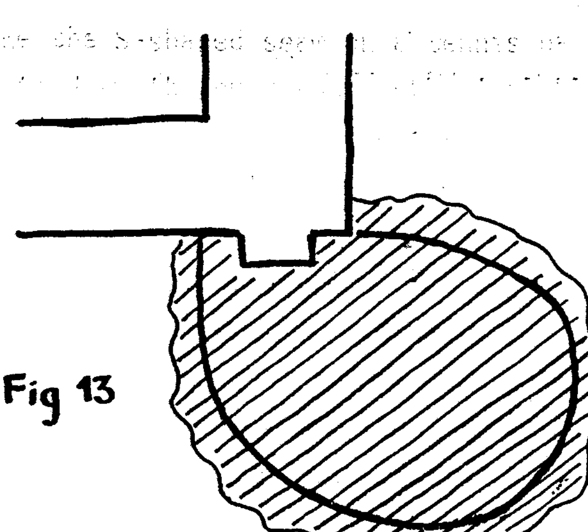


Fig 13

I now consider the different branches of the surface as passing through each other, giving an immersed surface with double curve and one triple point. After making the cardboard model, it is possible to verify that the boundary curve is now unknotted. Since this is implicitly proved in the course of the knitting pattern, I do not explain this directly, but give a slightly more mathematical description of the construction, from which it is clear how to find a disc spanning the boundary curve.

Construction. Choose ε such that $0 < \varepsilon < 1/\sqrt{2}$ (for example, $\varepsilon = \frac{1}{4}$). x_1, x_2, x_3 are coordinates in $\mathbb{R}^3$. Take X_1 to be the subset obtained by deleting from the unit disc in the $x_1=0$ plane two discs of radius ε around the poles $(0,-1,0)$ and $(0,0,1)$ of the unit sphere S^2. Take Y_1 to be the set of points inside the unit ball, in the quarter-space $x_2 \leq 0$, $x_1 \geq 0$, and at a distance ε from the arc of great circle running down from the N.pole $(1,0,0)$ to the E.pole $(0,-1,0)$; Y_1 is a kind of gutter running down from around the N.pole to fit exactly onto the boundary of the disc deleted from X_1 around the E.pole. ($X_1 \cup Y_1$ has a ridge along this arc of circle, but is a topological 2-manifold there.)

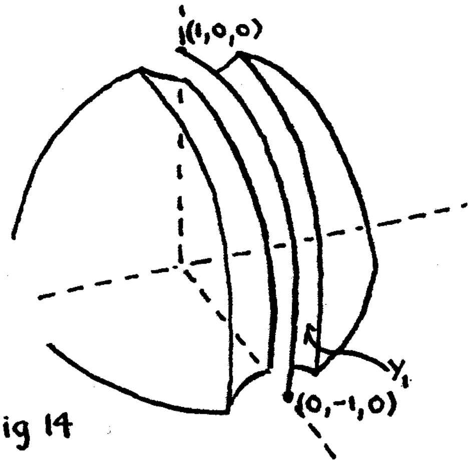


Fig 14

Now make X_2, X_3 and Y_2, Y_3 in the same way with cyclic permutations of the coordinates x_1, x_2, x_3, and set $X = X_1 \cup X_2 \cup X_3 \cup Y_1 \cup Y_2 \cup Y_3$. It is clear that X is an ideal version of the above cardboard model; it is an immersed Möbius strip with 3 half-twists; furthermore, the boundary curve ∂X is connected, and is a simple closed curve lying in S^2. By the Jordan curve theorem, X divides S^2 into two regions which are topological discs (a bit like the S-shaped seam on a tennis ball, only much more so; it's a good exercise to draw the curve on a spherical blackboard, or on a plastic ball, and colour in the two regions). Choosing one of the two regions, I get a disc D embedded in S^2 (and hence in $\mathbb{R}^3$) such that $\partial D = \partial X$, hence X can be capped off to an immersed $\mathbb{P}^2_{\mathbb{R}}$.

(This X has 6-fold symmetry, since in addition to cyclic permutations of the coordinates, a rotation through 180° about an axis through (1,-1,0) preserves X, taking Y_1 into itself and intechanging Y_2 and Y_3. This further symmetry, however, interchanges the two connected components of $S^2 - \partial X$, so that $X \cup D$, the immersed projective plane, has only 3-fold symmetry. A topologist may prefer to think of X, the immersed Möbius strip with unknotted boundary, as being the essential part of the Boy's surface, with the actual choice of the disc for the capping off as being less important, and just making things more complicated. There are several reasons why this is not very satisfactory from the point of view of a knitted object, among them the problem of the choice of stich (see below), and more important the advantage of having an object which can be stuffed, becoming a cuddly toy.)

Acknowledgements, references. I received considerable help in understanding the Boy's surface, and in preparing the first prototype for the knitting pattern from Dr Anthony Smith (at the time my graduate supervisor). The X appearing in the mathematical construction above seems to be due to Peter Eccles and Reg Wood.

There are several beautiful pictures of the Boy's surface in P.Eccles, Multiple points of codimension 1 immersions, in Topology Symposium Siegen 1979, Springer Lecture Notes in Math 788, esp. p.25. The original reference is W.Boy, Über die Curvatura integra und die Topologie geschlossener Flächen, Math Ann. 57 (1903), 151-184.

(c) The pattern. You require: a circular needle, size 3 mm. (old No.11's), length 100 cms; a pair of straight 3 mm needls; a set of 4 short double-ended 3 mm needles; a 3 mm crochet hook; a bodkin or darning needle.

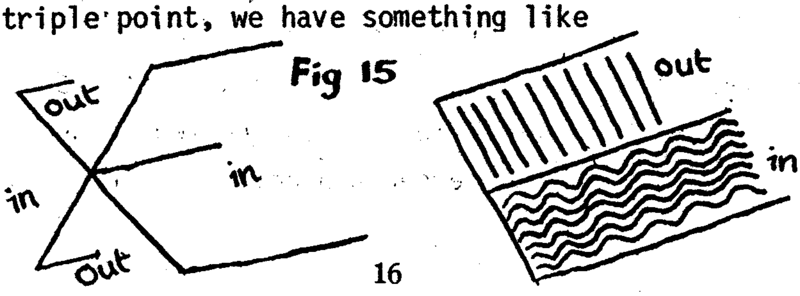
50 g of blue double knitting wool (or pink wool if you wish to make a Girl's surface); several lengths of spare wool of a distinctive colour (wool with nylon or acrylic is preferable, since the thread has to stand up to a certain amount of manhandling).

Kapok or other suitable stuffing material; a supply of aspirin or other analgesic.

The pattern uses standard notation, except that I use an exponent $(\ldots)^n$ to indicate that the instruction in the brackets are to be carried out n times. In rounds 1-12 the instructions from * to * are to be carried out whole 3 times (the pattern has 3-fold cyclic symmetry). Technical footnotes i) follow the main pattern.

Orientation and the choice of stich. The immersion of $\mathbb{P}^2_{\mathbb{R}}$ divides $\mathbb{R}^3$ into two components, an inside (to be stuffed with Kapok) and an outside, and X has no internal walls; that is, every small disc in the non-singular part of X is a boundary surface between in and out. Along a double curve, and away from the triple point, we have something like

Fig 15



so that at the double curve each branch of the surface has a divide with the property that as you cross, the inside and outside swop status.

The idea behind the choice of stich in the course of the pattern is that the finished object will be entirely in stocking stich, with the knit side out. At intermediate stages the work will consist of blocks of stocking stich, with the knit or purl side up alternately; a divide where a block knit side up is adjacent to a block purl side up occurs precisely to mark the line along which another branch of the surface has been, or will be pulled through to form a double curve. The words block and divide will be used in this meaning during the pattern.

* * * *

Cast on a Möbius strip with 3 half-twists, as follows: using spare wool and pair of straight needls, cast on 180 sts., work 2 rows in st.st., break off spare wool. Join in the main wool, and knit the row of 180 sts. onto the circular needle; this row is to be the central line of the Möbius strip. It should cover approx. one half the length of the circular needle (Fig.16); bring the working end round to the far end of the work (that is, the middle of the plastic wire) and pick up and work[1)] the row of stiches (without discarding from the plastic wire) as follows: k.60, p.60, k.60. The circular needle now loops around the work twice; the stiches at the beginning of the rd. can now be eased up onto the far end of the circular needle, becoming a row of stiches in the main wool, with the middle of the plastic wire underneath, and the rows in the spare wool generally getting in the way. The spare wool is to be cut away, but not before it has assisted in the crucial task of ensuring that you have 3 right-hand overhand half-twists:

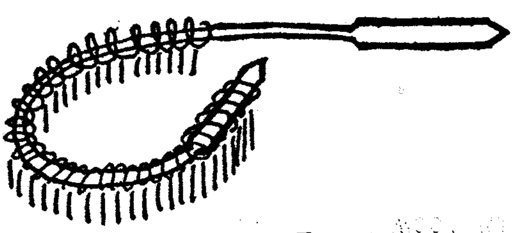


Fig 16

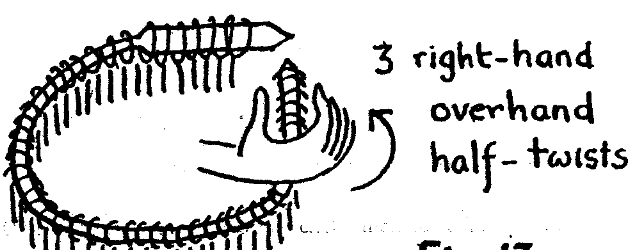


Fig 17

The rows of spare wool hanging down vertically all round would correspond to an (untwisted) cylinder; holding your right hand around the needle at the working end, with thumb towards the point, twist the work so that the flat formed by the rows of spare wool has 3 half-twists in the direction indicated by the fingers, relative to the untwisted cylinder (Fig.17). You can now begin to cut away the spare wool and work the 1st rd. Note that from now on a "round" goes the whole way round the boundary of the Möbius strip, that is twice round the homology generator (360 sts.).

1st rd.: * p.51, p.2 tog.tbl.[2)], p.1, p.2 tog., p.4, k.52, inc., k.1, inc., k.5 * (in the course of this rd., please check that the number of sts. in each block is worked exactly as stated - the 1st or last st. will be singular;[3)] at the end of the 3rd block, as you pass the "tail" where the work began, you may need to discard a "shadow st.", that is a loop of wool over the needle which may have arisen in the course of the twisting of the Möbius strip; finally, it may happen that the final block of p. at the end of this rd. is out by 1 st., in which case inc. or dec. in the last st. to compensate).

2nd (3rd, 4th) rd.: * p. to last 8 (7, 6) of block, p.2 tog.tbl., p.1, p.2 tog., p.3 (2, 1), k. to last 9 (10, 11) of block, inc., k.1, inc., k.6 (7, 8) * .

5th rd.: * p. to last 5 of block, p.2 tog.tbl., p.1, p.2 tog., form "hood" over next block as follows:k.22, k.2 tog.tbl., k.11, (k.2 tog.)^2, k.1, turn;

(a) sl.1, p.14, p.2 tog., p.1, turn;

sl.1, k.1, k.2 tog.tbl., k.9, k.2 tog., k.1, k.2 tog. tbl., k.1, turn;

repeat row (a);

sl.1, k.2, k.2 tog.tbl., k.7, k.2 tog., k.2, k.2 tog.tbl, k.1, turn;

repeat row (a);

sl.1, k.15, k.2 tog.tbl., k.1, turn;

(b) sl.1, p.16, p.2 tog., p.1, turn;

(c) sl.1, k.3, k.2 tog., k.7, k.2 tog.tbl., k.3, k.2 tog.tbl., k.1, turn;

repeat (b) and (c) 4 times more, omitting the final turn; (the present block now has 5 sts. up to the little gap, 18 more sts. on working needle, 12 sts. on l.h. needle), inc., k.1, inc., k.9 *.

Before starting on 6th rd., prepare a piece of spare wool 2 m in length, and thread onto bodkin; the next round is to be worked on a double-ended needle, the sts. being progressively threaded onto the spare wool; at the end of the rd. nothing will be left on the circular needle, and the spare wool will both hold the sts. and mark the path of the rd. around quite a complicated maze.

6th rd.: * p. to last 4 of block, p.2 tog.tbl., p.2 tog., k.4, k.2 tog., k.18, inc., k.1, inc., k.10, slip all but last 13 sts onto spare wool. Fold the fina k. block in half, with right side out, bringing the "tab" formed by the last 13 sts. opposite the "slit" formed by the immediately preceding divide; using crochet hook, pull these sts. through the fabric of the divide (there should be exactly 13 suitable holes, including one immediately under the spare wool/plastic wire at top and bottom, and one at the singular st. in the middle), and mount onto needle. Using pair of straight needles, make these sts. into a "heel" as follows: (turn), k.1, inc., k.8, inc., k.2 (15 sts.) work 20 rows in st.st. on these 15 sts., ending on k. row. With right side of work facing[4] use a crochet hook to pick up and k.16 sts. from the 21 rows of the side of the heel towards the spare wool on the bodkin, taking care that the last st. is picked up as close as possible to the last st. on the spare wool (pick up approx. 3 sts. for every 4 rows); turn, sl.1, p.30. slip these sts. onto spare wool; with right side of work facing, pick up and k.17 sts. from the 22 rows down the other side of the heel, taking care that the last st. picked up is as close as possible to the first st. on the continuing round *.

<u>Explanation before rd.7</u>: the work has now reached the triple point, and can be described schematically as in Fig.18 (a); the round goes 1) along the base of a hooded cylinder (48 sts.); 1') around the rim of a hood (25 sts.); 1") all the way round the perimeter of a heel (48 sts.); then on around two similar loops. The rows of sts. 1) and 1"), 2)and 2") and 3)and 3") are to be pulled through each other as described in Fig.18:

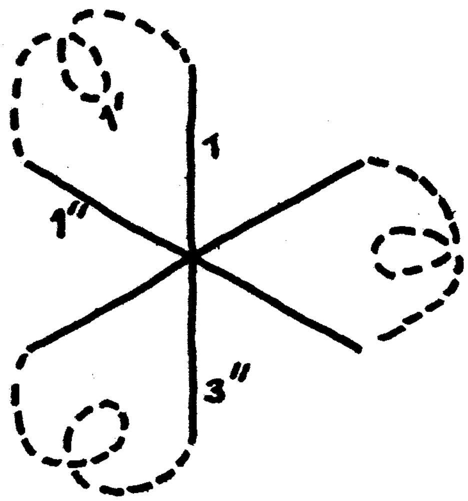


Fig 18(a)

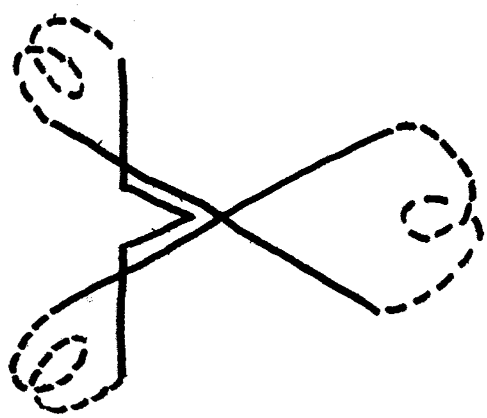


Fig 18(b)

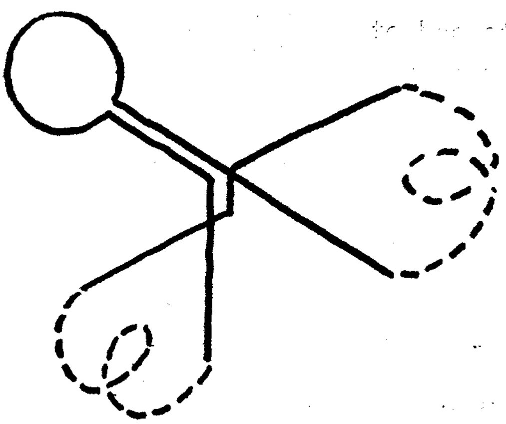


Fig 18(c)

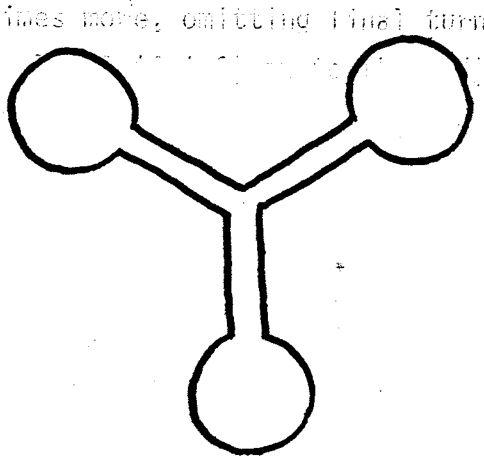


Fig 18(d)

Rd.7: Hold work with the triple point upwards as in Fig.18 (a), and lay together rows 3" and 3 (these are respectively the final row of sts. worked in rd.6, and the row immediately following in clockwise order); use the crochet hook to pull the final 10 sts. of row 3" through row 3 (that is pull the sts. one by one under the spare wool and through the gaps between the sts. of row 3), and mount on double-ended needle; in the following round the index i runs from 1 to 3, with minor variations for the final repeat.

* Now pull the first 12 sts of the following base of cylinder i) (on the free end of the spare wool; this is unambiguous) through heel i"), and mount onto a new double-ended needle; work back and forth on these 22 sts. as follows: k.1, k.2 tog. tbl., k.3, turn;

(x) sl.1, p.7, p.2 tog., p.1, turn;

(y) sl.1, k.1, k.2 tog., k.2, (k.2 tog.tbl., k.1)^2, turn;

repeat (x, y) twice more, omitting the final turn. There are 2 stiches between the sets of decreasings just worked, and the gap between these marks the beginning and end of the present loop. Now pull through and k. the remaining 36 sts. of base of cylinder i) through heel i"), slipping all but the last 3 sts. onto spare wool. Continue onto rim of hood i'), dividing among the 4 double-ended needles, working as follows: k.7, k.2 tog., new needle, k.7, new needle, k.2 tog.tbl., k.6, turn; (there are now 49 sts. on to the end of the loop (43 if i = 3));

(a) sl.1, p.18, p.2 tog., p.1, turn;

(there are 41 sts. back to the beginning of loop)

(b) sl.1, k.4, k.2 tog., k.7, k.2 tog. tbl., k.4, k.2 tog.tbl, k.1, turn;

repeat (a, b) 8 times more, picking up sts. from rows i) and i") on the spare wool a few at a time as convenient; (there are now 25 sts. back to beginning of loop, and 31 (25 if i = 3) on to the end);

(c) sl.1, p.18, p.2 tog., turn;

(d) sl.1, k.18, k.2 tog. tbl., turn;

repeat (c, d) 19 times more, omitting final turn; (this leaves 5 sts. back to beg.of loop, and 11 (5 if i=3) on to the end); k.11 (5 if i = 3) onto end of loop; slip all but the last 10 sts. onto spare wool *.

Mounting the entire round onto the 4 double-ended needles, proceed as follows:

Rd.8: * k.4, k.2 tog., k.3, k.2 tog., k.4, new needle, k.3, k.2 tog. tbl., k.3, (k.2 tog.)^2, k.3 *.

Rd.9 (11): * k.1, k.2 tog. tbl., k.19 (15), k.2 tog., k.1 *.

Rd.10 (12): * k.6 (4), k.2 tog., k.7, k.2 tog.tbl., k.6 (4) *.

Slip the entire round back onto spare wool, and stuff the inside firmly with kapok. Mount back onto 4 needles; decrease 6 sts. evenly every round for 7 rounds until 9 sts. remain; thread off, stuffing firmly before pulling tight.

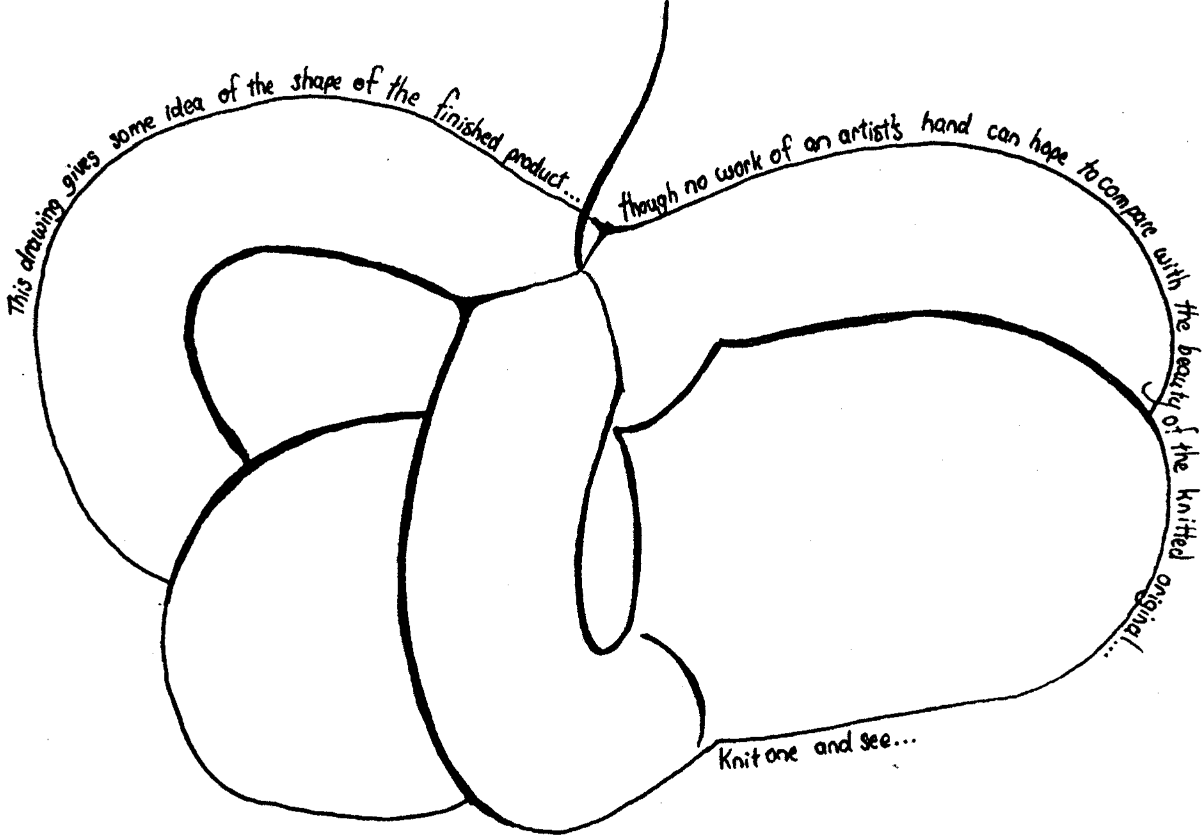
Technical footnotes:

1) "Pick up and work": when working a row of sts. which is mounted either on the middle of the plastic wire or on spare wool, it is most convenient to pick up the sts. a few at a time onto a spare double-ended needle and work from there.

2) "tbl." (= through back of loop). Ignore this if you're not sure what it means. This is purely a cosmetic device to ensure a nice rib when decreasing is done in the same place in successive rows (compare the sleeve shapings of most sweaters). Better than simply "work 2 tog.tbl." is to first turn the 2 sts. concerned back-to-front by slipping onto r.h. needle knitwise, then slipping back purlwise, then work 2 tog. tbl.;(this has several advantages over the traditional sl.1, k.1., p.s.s.o.).

3) "singular". The st. will appear to be half k. and half p., an effect whic can only be produced by two-sided casting on.

4) "right side of work facing". This means just the k. side of st.st., and is a warning that an unsightly seam is about to be produced on one side of the work; while trying to keep both sides of the work nice and clean, it is normal practice to keep anything nasty that crops up on the reverse or unseen side; this applies to other branches of needlework, politics, etc.


This drawing gives some idea of the shape of the finished product...
though no work of an artist's hand can hope to compare with the beauty of the knitted original...
Knit one and see...