Constructing algebraic varieties via commutative algebra

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Abstract Problems on the existence and moduli of abstract varieties in the classification of varieties can often be studied by embedding the variety X into projective space, preferably in terms of an intrinsically determined ample line bundle L such as the (anti-) canonical class or its submultiples. A comparatively modern twist on this old story is to study the graded coordinate ring

$$R(X, L) = \bigoplus_{n\geq 0} H^0(X, L^{\otimes n}),$$

which in interesting cases is a Gorenstein ring; this makes available theoretical and computations tools from commutative algebra and computer algebra. The varieties of interest are curves, surfaces, 3-folds, and historical results of Enriques, Fano and others are sometimes available to serve as a guide. This has been a prominent area of work within European algebraic geometry in recent decades, and the lecture will present the current state of knowledge, together with some recent examples.

EAGER

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EAGER objectives

(1) Classification of algebraic varieties

(2) Homological and categorical methods

(3) Moduli stacks of curves

(4) Moduli of vector bundles

(5) Abelian varieties and their moduli

(6) Hodge theory and algebraic cycles

(7) Toric methods and group actions

(8) Computer algebra

(9) Coding theory

(10) Computer Aided Geometric Design

Other Calabi–Yau manifolds and mirror symmetry. Topology of algebraic surfaces and 4-manifolds. Moduli spaces. Algebraic stacks and their Gromov–Witten invariants. Free resolutions, homological algebra and derived categories. Birational methods. Deformation theory. Analytic and differential geometric methods. Syzygies and homological methods, derived categories.

Today's lecture only treats a small fraction of the first topic, namely:

> Classification of algebraic varieties via commutative algebra methods.

Classification of varieties

The classification of surfaces goes back to the 19th century.

1846 Cayley and Salmon: 27 lines on $S_3 \subset \mathbf{P}^3$

1860s Riemann surfaces, Brill–Noether, RR theorem

1890–1910 Castelnuovo, Enriques and other: Birational classification of surfaces by their plurigenera

1930s Enriques and students: Surfaces of general type

1930s Fano: 3-folds $V_{2g-2} \subset \mathbf{P}^{g+1}$

1950s Kodaira: deformation theory, moduli, classification of complex analytic surfaces

1980s Mori theory, minimal models of 3-folds. The conclusion that classification is the division $K < 0$, $K = 0$, $K > 0$ plus fibrations, where K is the canonical class.

1980s Differentiable and symplectic 4-manifolds (Donaldson and others)

1990s Calabi–Yau 3-folds, orbifolds, mirror symmetry.

EAGERists are involved in all these topics (and many more, of course).

Any number of survey lectures could be made out of other EAGER topics.

Preliminary philosophical remarks

Surfaces In what follows the ultimate aim (not necessarily expressed) is the study of regular surfaces of general type, for example, the simply connected Godeaux surfaces (that is, canonical surfaces S with $p_g = 0$, $K_S^2 = 1$). This is a mature subject, that involves most other areas of geometry. To study S, it may be convenient to know a lot about curves $C \subset S$, possibly passing through singular points of S; or it may be convenient to express S as a hypersurface section of some higher dimensional "key variety", e.g., a Fano 3-fold or Fano 4-fold, possibly with orbifold singularities. Surprisingly, it turns out to be advantageous in some problems not to worry too much in advance what dimension of variety we are studying: taking a hypersurface section is a known operation.

Commutative algebra The geometric constructions of Enriques, Horikawa and others can often be interpreted in algebraic terms as constructions of rings by generators and relations. As samples:

(1) The hypersurface $X_d \subset \mathbf{P}^n$ defined by $f_d = 0$ has homogeneous coordinate ring the graded ring $\mathbf{C}[x_0, x_1, \dots, x_n]/(f_d)$.

(2) The geometric idea of *projection* corresponds algebraically to *elimination of variables.*

(3) "Key varieties" may have a homological or commutative algebra treatment, such as determinantal form of the equations.

Definition of graded ring

A *graded ring* $R = \oplus_{n\geq 0} R_n$ is a (commutative) ring with a grading such that multiplication does $R_i \times R_j \to R_{i+j}$.

Extra assumptions The following are often in force:

(1) $R_0 = k$ is a field (often $k = \mathbf{C}$);

(2) The maximal ideal $m = \oplus_{n>0} R_n$ is finitely generated.

$$\Longrightarrow R = k[x_0, \ldots, x_n]/I_R,$$

where the generators $x_i \in R_{a_i}$ of m have $\operatorname{wt} x_i = a_i$, and I_R is the homogeneous ideal of relations.

(3) R is an integral domain.

Example The standard textbooks define a projective variety to be a closed subvariety $X \subset \mathbf{P}^n$ in "straight" projective space $\mathbf{P}^n$ (all the generators of degree 1, so $x_i \in R_1$). Write

$$I_X = \bigoplus_{d\geq 0} \{\text{forms of degree } d \text{ vanishing on } X\}.$$

Then I_X is a homogeneous ideal and $k[X] = k[x_0, \ldots, x_n]/I_X$ is the *coordinate ring* of X. Here R is generated by its elements of degree 1; we are usually interested in the more general case of varieties in weighted projective space.

For details, see my website + algebraic geometry links + surfaces + graded rings and homework.

The Proj construction $R \mapsto \operatorname{Proj} R$

As described in [EGA2] or [Hartshorne, Chap. II] or my notes (webloc. cit.), $X = \operatorname{Proj} R$ is defined as the quotient $(\operatorname{Spec} R \setminus 0)/\mathbf{C}^*$ of the variety $\operatorname{Spec} R = V(I) \subset \mathbf{C}^{n+1}$ by the action of the multiplicative group $\mathbf{C}^* = \mathbf{G}_m(k)$ induced by the grading.

In more detail, if $R = k[x_0, \dots, x_n]/I$ with $\operatorname{wt} x_i = a_i$ then $\lambda \in \mathbf{C}^*$ acts on R by multiplication by λ^n on R_n, that is,

$$\lambda\colon x_i \to \lambda^{a_i} x_i.$$

It therefore acts on the affine variety

$$\operatorname{Spec} R = V(I) \subset \mathbf{C}^{n+1}.$$

Note the philosophy: grading $=$ $\mathbf{C}^*$ action.

The origin $0 \in \mathbf{C}^{n+1}$ is in the closure of every orbit (because $(0, 0, \dots, 0) = \lim_{\lambda \to 0}(\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$; this uses the fact that the grading of $R = \bigoplus R_n$ is by $\mathbf{N}$ with $n > 0$, or $\operatorname{wt} x_i = a_i > 0$. Therefore we must exclude the *unstable* point 0 to be able to take a sensible quotient.

For all $f \in R_d$ homogeneous of degree $d > 0$, form the ring

$$\left(R\left[\frac{1}{f}\right]\right)^0 = \left\{ \frac{g}{f^e} \;\middle|\; \operatorname{wt} g = de \right\} \subset \operatorname{Frac} R \tag{1}$$

consisting of rational functions that are homogeneous of deg 0 with only f or its powers in the denominator. Then define

$$X_f = \operatorname{Spec}\left(R\left[\frac{1}{f}\right]\right)^0, \quad \text{and} \quad X = \bigcup_{f \in R_d} X_f.$$

In other words, on taking the quotient $(\operatorname{Spec} R \setminus 0)/\mathbf{C}^*$:

(1) The typical $\mathbf{C}^*$ invariant open set is $(f \neq 0)$ for $f \in R_d$.

(2) the ring (1) is the ring of all $\mathbf{C}^*$-invariant regular functions on this open.

Thus the quotient $\operatorname{Proj} R$ is the space of orbits of the $\mathbf{C}^*$ action, with all $\mathbf{C}^*$-invariant functions.

Remark $X = \operatorname{Proj} R$ is really a *stack*, and it is sometimes convenient to treat it as an orbifold. It is a projective scheme $X, \mathcal{O}_X$, but it has the extra structure of the sheaves $\mathcal{O}_X(k)$ for all $k \in \mathbf{Z}$, defined by

$$\Gamma(X_f, \mathcal{O}_X(k)) = \left\{ \frac{g}{f^e} \;\middle|\; \operatorname{wt} g = de + k \right\} \subset \operatorname{Frac} R.$$

Then $\oplus_{k \in \mathbf{Z}} \mathcal{O}_X(k)$ is a sheaf of graded algebras.

For *straight* projective space (that is, $\operatorname{wt} x_i = 1$ for all x_i), $\mathcal{O}_X(1)$ is an ample invertible sheaf, and

$$\mathcal{O}_X(k) = \mathcal{O}_X(1)^{\otimes k}.$$

But for $\mathrm{w}\mathbf{P}$ we must take $\oplus \mathcal{O}_X(k)$ as extra data. For example, if all the a_i have some common factor $q \mid a_i$ then $R_n = 0$ for all n not divisible by q, and so $\mathcal{O}_X(k) = 0$. In this case we say that X has *nontrivial orbifold structure in codim* 0.

Examples $C_{2g+2} \subset \mathbf{P}(1, 1, g+1)$ defined by $y^2 = f_{2g+2}(x_1, x_2)$ is a hyperelliptic curve of genus g.
$X_{10} \subset \mathbf{P}(1, 1, 2, 5)$ defined by $z^2 = f_{10}(x_1, x_2, y)$ is a famous example of Enriques and Kodaira of a canonical surface with $p_g = 2$, $K^2 = 1$.

Hilbert series

It follows from my assumptions on R that R_n if a finite dimensional vector space over $R_0 = \mathbf{C}$ for each n. Set

$$P_n(R) = \dim_k R_n \quad \text{and} \quad P_R(t) = \sum_{n\geq 0} P_n t^n.$$

The formal power series $P_R(t)$ is the *Hilbert series* of R. Under our assumptions it is a rational function in t; thus

$$R = k[x_0, \dots, x_n]/I_R \quad \text{with } \operatorname{wt} x_i = a_i$$

implies that $\prod_{i=0}^n (1 - t^{a_i}) \cdot P_R(t)$ is a polynomial in t, called the *Hilbert numerator*; it contains information and *hints* as to the homological algebra or commutative algebra properties of R.

Examples

(1) If $R = k[x_0, \dots, x_n]$ is the weighted polynomial ring then

$$P_R(t) = \frac{1}{\prod_{i=0}^n (1 - t^{a_i})}.$$

(2) If $R = k[x_0, \dots, x_n]/(f_d)$ is the ring of a weighted hypersurface of degree d in $\mathbf{P}(a_0, \dots, a_n)$ then

$$P_R(t) = \frac{1 - t^d}{\prod_{i=0}^n (1 - t^{a_i})}.$$

Likewise, a codim 2 complete intersection has Hilbert numerator $(1 - t^{d_1})(1 - t^{d_2})$.

See the homework sheet on webloc. cit. for more examples.

Hilbert series from orbifold RR

From now on, X is a projective variety, and $\mathcal{O}_X(k) = \mathcal{O}_X(kA)$ with A an ample $\mathbf{Q}$-divisor. So rA is an ample Cartier divisor for some $r > 0$. Assume that

$$R = R(X, A) = \bigoplus_{k \geq 0} H^0(X, \mathcal{O}_X(kA)).$$

(This is an extra assumption on R, akin to projective normality.)

Usually the terms of the Hilbert series

$$P_n(R) = h^0(X, \mathcal{O}_X(nA))$$

are given by RR and vanishing for $n \gg 0$, plus initial assumptions for small n. If A is $\mathbf{Q}$-Cartier, the form of RR we need is *orbifold RR* (also known as equivariant RR or the Atiyah–Singer Lefschetz formula). See [YPG, Chap. III] for details. A simple example gives the flavour.

Example C a curve, $A = D + \frac{a}{r}P$ with D an integral divisor, $r > 1$ and $a \in [1, \dots, r-1]$ coprime to r. Then $\mathcal{O}_C(nA) = \mathcal{O}_C([nA])$, where we round down the divisor nA to the nearest integer (because a meromorphic function has poles of integral order), so that RR takes the form

$$\chi(C, \mathcal{O}_C(nA)) = \chi(\mathcal{O}_C([nA])) = 1 - g + n \deg A - \left\{\frac{na}{r}\right\}.$$

Here the fractional part $\left\{\frac{na}{r}\right\}$ is the small change we lose on rounding down nA to $[nA]$.

This introduces the orbifold correction term

$$-\frac{1}{1-t^r}\cdot\sum_{i=1}^{r-1}\left\{\frac{ia}{r}\right\}t^i \tag{2}$$

into the Hilbert series. (The effect of multiplying by $\frac{1}{1-t^r} = 1 + t^r + t^{2r} + \cdots$ is just to repeat the rounding-down errors periodically.)

Remark Set $ab \equiv 1 \bmod r$ and let ε be a primitive rth root of 1 (for example, $\varepsilon = \exp(2\pi i/r)$). Then one checks that

$$\frac{1}{1-\varepsilon^b} = \sum_{i=1}^{r-1}\left\{\frac{ia}{r}\right\}\varepsilon^i$$

Thus the term (2) is "cyclotomic" in nature. Generalisations of this idea give very quick and convenient ways of calculating the orbifold contributions to RR. We are in fact close to the proof of the Atiyah–Singer equivariant Lefschetz formula: the denominator is the equivariant Todd class $\det(\varepsilon : T_{X,P})$. See [YPG, Chap. III].

Example [Bauer, Catanese, Pignatelli] C a curve of genus $g = 3$ with points $P, Q \in C$ such that $P + 3Q = K_C$. For example, $C = C_4 \subset \mathbf{P}^2$, with Q a flex and P the 4th point of intersection of the flex line with C.

I choose the divisor $A = \frac{1}{2}P + Q$. Then

$$h^0(nA) = \begin{cases} 1 & n = 0; \\ 1 & n = 1; \\ 2 & n = 2 \quad (P + 2Q = K_C - Q \text{ is a } g_3^1); \\ 3 & n = 3 \quad (3A = K_C + \frac{1}{2}P \text{ and } g = 3); \\ -2 + \frac{3n}{2} & \text{if } n \geq 4 \text{ even;} \\ -2 + \frac{3n-1}{2} & \text{if } n \geq 4 \text{ odd.} \end{cases}$$

Therefore

$$\begin{aligned} P_{C,A}(t) &= 1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 7t^6 + \cdots \\ (1-t^2)P_{C,A}(t) &= 1 + t + t^2 + 2t^3 + 2t^4 + 2t^5 + 3t^6 + \sum 3t^n \\ (1-t)(1-t^2)P_{C,A}(t) &= 1 + t^3 + t^6 \end{aligned}$$

Thus

$$P_{C,A}(t) = \frac{1 - t^9}{(1-t)(1-t^2)(1-t^3)}.$$

This gives $C_9 \subset \mathbf{P}(1, 2, 3)$ as a possible model for C. One checks that it works: C has a $\frac{1}{2}(1)$ orbifold singular points at $(0, 1, 0)$. The linear system $|2A| = P + 2Q$ is the g_3^1. $R(C, A)$ is a Gorenstein ring because $3A = K_C + \frac{1}{2}P$ is the orbifold canonical class of C.

Some classes of varieties to study

Regular surfaces of general type (Enriques) Assume that K_S is ample, and that $q = h^1(S, \mathcal{O}_S) = 0$. (We say that S is a regular surface; irregular surfaces with $q > 0$ are studied by different methods.)

$$P_n(S) = \begin{cases} 1 & k = 0; \\ p_g & k = 1 \quad \text{(the definition of } p_g\text{);} \\ 1 + p_g + \binom{n}{2} K^2 & k \geq 2 \quad \text{(by RR and vanishing).} \end{cases}$$

An easy calculation gives

$$p_S(t) = \frac{1 + (p_g - 3)t + (K^2 - 2p_g + 4)t^2 + (p_g - 3)t^3 + t^4}{(1-t)^3}.$$

About a dozen important cases were treated geometrically by Enriques, Kodaira, Horikawa and others. Algebraic treatment by Ciliberto, Catanese, Reid and others.

Examples $p_g = 4$, $K^2 = 6$. The first possible case suggested by the Hilbert series is $S_{3,4} \subset \mathbf{P}(1,1,1,1,2)$. This really works. There are lots of degenerate cases studied by Horikawa, and recently by [Bauer, Catanese and Pignatelli]; see below. The situation for $p_g = 3$, $K^2 = 2, 3, 4$ or for $p_g = 2$, $K^2 = 1, 2, 3$ is similar. Beyond these initial cases, the calculations get very difficult.

Fano 3-folds Nonsingular 3-folds V with $-K_V$ ample, usually anticanonically embedded as $V_{2g-2} \subset \mathbf{P}^{g+1}$. These were studied by Fano in the 1930s and Iskovskikh from 1970s, later Mori and Mukai.

Q-Fano 3-folds 3-folds V with terminal singularities and $-K_V$ ample (Mori, Reid and others, 1990s). In studying 3-folds, terminal singularities are unavoidable; the most important and interesting singularities are the cyclic quotient singularities $\frac{1}{r}(1,a,r-a)$ with $r \geq 2$ and $a \in [1, r-1]$ coprime to r. Several hundred families of **Q**-Fano 3-folds are known, for example the "famous 95" Fano hypersurfaces studied in [Corti, Pukhlikov, Reid]. See [DB].

Q-K3s These are surfaces X with quotient singularities and $K_X = \mathcal{O}_X$, $H^1(\mathcal{O}_X) = 0$ polarised by a **Q**-divisor. They appear naturally as anticanonical surfaces $X \in |-K_V|$ on a **Q**-Fano 3-fold V.

Remark It can happen that a surface of general type S is contained in a **Q**-Fano 3-fold V, for example:

(1) $S \in |-2K_V|$, so adjunction gives $K_S = K_{V|S}$;

(2) or V is a **Q**-Fano 3-fold of index 2 with $-K_V = 2A$ and $S \in |3A|$, so that $K_S = A_{|S}$.

A striking fact: the basket of singularities of V (giving the fractional contributions to its Hilbert series) is then already determined by S: in the two cases above

(1) V has basket $(K^2 - 4p_g + 12) \times \frac{1}{2}(1,1,1)$. So for example, if S has $p_g = 1$, $K^2 = 1$ then V has $9 \times \frac{1}{2}(1,1,1)$ points, whereas if S has $p_g = 1$, $K^2 = 2$ then V has $10 \times \frac{1}{2}(1,1,1)$ points. We really meet these cases below.

(2) V has basket $(K^2 - 3p_g + 6) \times \frac{1}{3}(1,2,2)$.

This follows automatically from orbifold RR!

Appendix: Cohen–Macaulay and Gorenstein

I omit the definitions and treatment by homological algebra, which are standard and not very difficult. In practice, we want R to be Cohen–Macaulay and (better) Gorenstein; otherwise the ring and the variety are very difficult to construct.

Criterion *Let $R = R(X, A)$. Then*

R is Cohen–Macaulay if and only if $H^i(X, \mathcal{O}_X(kA)) = 0$ for all i with $0 < i < \dim X$ and all k, for $i = 0$ and $k < 0$, and for $i = \dim X$ and $k \gg 0$.

R is Gorenstein if and only if it is Cohen–Macaulay and $K_X = kA$ for some $k \in \mathbf{Z}$.

Examples These conditions hold in most of our cases:

(1) X is a K3 surface with quotient singularities and A an ample Weil divisor;

(2) X is a regular surface of general type and $A = K_X$. Then $H^1(K_X) = 0$ follows from regularity and Serre duality, and $H^1(nK_X) = 0$ for $n \geq 2$ from Kodaira vanishing;

(3) V is a Q-Fano 3-fold of Fano index f and $-K_V = fA$;

(4) C is an orbifold curve (with a point $\frac{1}{r}P$), and we interpret K_C in the criterion as orbi-$K_C = K_C + \frac{r-1}{r}P$.

The cone over a projectively embedded Abelian surfaces is a simple example of a geometrically interesting variety that is not Cohen–Macaulay.

Application 1

Horikawa's study of surfaces with $p_g = 4$, $K^2 = 6$ divides them into several cases, and solves many problems, but leaves the existence of degenerations between cases II and III_b as an open question. [Bauer, Catanese, Pignatelli] have recently proved that such a degeneration does occur.

II The case assumption is that $|K_X|$ is a free linear system and defines a 3-to-1 morphism $\varphi_{K_X}\colon X \to Q \subset \mathbf{P}^3$, where Q is the quadric cone $x_1x_3 = x_2^2$. In this case pulling back the pencil of the quadric cone provides a pencil $|A|$ on the canonical model X with $2A = K_X$. In general X has an orbifold point of type $\frac{1}{2}(1,1)$ over the vertex of Q. Restricting A to a general $C \in |A|$ gives rise to the example treated above of a curve of genus 3 and an orbifold divisor $A = \frac{1}{2}P + Q$, so that $2A = P + 2Q$ is a g^1_3.

It follows that $X = X_9 \subset \mathbf{P}(1,1,2,3)$. This has all the required properties, and every surface in II is given by this construction.

$\mathbf{III}_b$ The case assumption is that $|K_X|$ has a double point as its base locus on the canonical model (or a -2-curve as base component on the minimal model), and $\varphi_K\colon \widetilde{X} \to Q \subset \mathbf{P}^3$ is a 2-to-1 morphism to the quadric cone. Then again $K_X = 2A$ with $A^2 = 3/2$. At the level of a general curve $C \in |A|$, the curve C is a nonsingular hyperelliptic curve of genus 3, and the restriction $A_{|C}$ is $\frac{3}{2}P$, where P is a Weierstrass point. (Thus $2A = P + g^1_2$ can be viewed as a g^1_3 with a fixed point.)

[BCP2] (and also [Coughlan]) calculate $R(C,A)$ and $R(X,K_X)$ in case III_b:

$$R\left(C,\frac{1}{2}P\right) = k[a,b,c]/(c^2 - f_7(a^4,b)) \quad \text{with } \operatorname{wt} a,b,c = 1,4,14,$$

giving $C = C_{28} \subset \mathbf{P}(1,4,14)$. Then $R(C,A) = R(C,\frac{3}{2}P)$ is the third Veronese embedding: it needs *generators*

$$x = a^3, \quad y = a^2b, \quad z = ab^2, \quad t = b^3, \quad u = ac, \quad v = bc$$

with $\operatorname{wt} x,y,z,t,u,v = 1,2,3,4,5,6$. And *relations*

$$\operatorname{rank}\begin{pmatrix} x & y & z & u \\ y & z & t & v \end{pmatrix} \le 1 \tag{3}$$

(meaning the 2×2 minors $= 0$, which gives 6 equations); and 3 further equations derived from $c^2 = f_7$, of the form

$$u^2 = [a^2f], \quad uv = [abf], \quad v^2 = [b^2f],$$

where $[a^2f]$ means that we write out the terms $a^{30}, a^{26}b, \dots, a^2b^7$ of a^2f in terms of x,y,z,t. If we group together the terms in f as

$$f = a^{28} + a^{24}b + \cdots + a^4b^6 + b^7 = aA + b^4B$$

with

$$A = A_9, B = B_4 \in k[x,y,z,t]$$

then the 3 final equations become

$$u^2 = xA + z^2B, \quad uv = yA + ztB, \quad v^2 = zA + t^2B. \tag{4}$$

This is the "rolling factors" format of [Dicks]: you go from one relation to the next by replacing an entry in the top row of the matrix of (3) by an entry in the bottom.

(3–4) are 9 equations with 16 syzygies defining a codim 4 Gorenstein ring. They can be written as the 4×4 Pfaffians of the following *extrasymmetric* matrix:

$$M = \begin{pmatrix} 0 & z & x & y & u \\ & t & y & z & v \\ & & u & v & A \\ & & & 0 & Bz \\ -\mathrm{sym} & & & & Bt \end{pmatrix} \quad \text{of weights} \quad M = \begin{pmatrix} 0 & 3 & 1 & 2 & 5 \\ & 4 & 2 & 3 & 6 \\ & & 5 & 6 & 9 \\ & & & 4 & 7 \\ & & & & 8 \end{pmatrix}$$

The matrix M is skew, with the following extra symmetry: the top right 3×3 block is symmetric, and the bottom right 3×3 block is B times the top left. Thus instead of 15 independent entries it only has 9, and likewise, only 9 independent 4×4 Pfaffians. The format relates closely to the Segre embedding of $\mathbf{P}^2 \times \mathbf{P}^2$ as a (nongeneric) linear section of $\mathrm{Grass}(2,6)$.

This format is *flexible*: it carries its own syzygies with it, so that we can vary the entries as we like and obtain a flat deformation. Replacing by

$$M = \begin{pmatrix} \lambda & z & x & y & u \\ & t & y & z & v \\ & & u & v & A \\ & & & B\lambda & Bz \\ -\mathrm{sym} & & & & Bt \end{pmatrix}$$

with a constant $\lambda \neq 0$ deforms the hyperelliptic curve to a nonhyperelliptic trigonal curve. Similarly (but with some more work), one can prove that the surfaces in case III_b have small deformations in II.

Appendix: All about Pfaffians

Let $M_0 = \{m_{ij}\}$ be a $2k \times 2k$ skew matrix. Its Pfaffian is

$$\operatorname{Pf} M_0 = \sum{}' \operatorname{sign}(\sigma) \prod_{i=1}^{k} m_{\sigma(2i-1)\sigma(2i)};$$

(sum over the symmetric group S_{2k}), and $\sum'$ means that we only take 1 occurrence of each repeated factor. Skewsymmetry causes each term to occur $2^k \cdot k!$ times, so the Pfaffian consists of

$$\frac{2k!}{2^k \cdot k!} = 1 \cdot 3 \cdots (2k-1)$$

terms. For example, a 4×4 Pfaffian is of the form

$$\operatorname{Pf}_{12.34} = m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}$$

which is familiar as the Plücker equations of $\operatorname{Grass}(2, n)$.

In fact $\det M_0 = (\operatorname{Pf} M_0)^2$. The Pfaffian is a *skew determinant*, and every aspect of the theory of determinants extends to Pfaffians. For example, it follows from the definition that a Pfaffian can be expanded along any row exactly like a determinant: thus a 6×6 Pfaffian is

$$\operatorname{Pf}_{12.34.56} = m_{12} \cdot \operatorname{Pf}_{34.56} - m_{13} \cdot \operatorname{Pf}_{24.56} + \cdots.$$

If M is a $(2k+1) \times (2k+1)$ skew matrix, write

$$\operatorname{Pf}_i = (-1)^i \operatorname{Pf} M_i,$$

where M_i is the skew $2k \times 2k$ matrix obtained by deleting the ith row and column from M. Then the adjoint matrix of M (matrix of $2k \times 2k$ cofactors) is the matrix of rank 1 (or 0)

$$\operatorname{adj} M = \operatorname{Pf} \cdot {}^t \operatorname{Pf}, \quad \text{where } \operatorname{Pf} = (\operatorname{Pf}_1, \dots, \operatorname{Pf}_{2k+1})$$

Since $\det M = 0$ we get $\operatorname{Pf} \cdot M = 0$, and if M has rank $2k$ then Pf generates $\ker M$ (skew Cramer's rule).

Application 2

Surfaces with $p_g = 1$, $K^2 = 2$ were studied in [Catanese and Debarre], following Enriques; an alternative construction as a section of a higher dimensional variety was given by Jan Stevens in 1995 (but as far as I know not written down).

I start from the graded ring over the canonical curve $C \in |K_S|$: a reasonably general 4×4 symmetric matrix M of linear forms on $\mathbf{P}^2_{y_1,y_2,y_3}$ defines an invertible sheaf $\mathcal{O}_C(A)$ on the plane quartic $C = C_4 : (\det M = 0) \subset \mathbf{P}^2$, with the resolution

$$\mathcal{O}_C(A) \leftarrow 4\mathcal{O}_{\mathbf{P}^2}(-1) \xleftarrow{M} 4\mathcal{O}_{\mathbf{P}^2}(-2) \leftarrow 0, \tag{5}$$

and satisfying $\mathcal{O}_C(2A) = K_C$ (in other words, A is an ineffective theta characteristic on C). The corresponding graded ring

$$R(C, A) = k[y_1, y_2, y_3, z_1, z_2, z_3, z_4]/I_C$$

is generated by $y_1, y_2, y_3 \in H^0(\mathcal{O}_C(2A))$ and $z_1, \dots, z_4 \in H^0(\mathcal{O}_C(3A)) = \mathcal{O}_C(A)(1)$ with relations $(z_1, \dots, z_4)M = 0$ from (5) and $z_i z_j = M_{ij}$ (the ijth maximal minor of M. These equations define a codim 5 embedding $C \subset \mathbf{P}(2^3, 3^4)$ with Hilbert numerator

$$1 - 4t^5 - 10t^6 + 15t^8 + 20t^9 - 20t^{11} - \cdots$$

The same construction starting from a 4×4 symmetric matrix M over $\mathbf{P}^3$ leads to a quartic K3 surface $X_4 \subset \mathbf{P}^3$ carrying an ineffective Weil divisor A_X with a resolution similar to (5), and $R(X, A)$ embeds X into $\mathbf{P}(2^4, 3^4)$. However, now X has 10 nodes at points where $\operatorname{rank} M = 2$. These are $\frac{1}{2}(1,1)$ orbifold points at which $\mathcal{O}_X(A_X)$ is the odd eigensheaf.

The problem is to deform the graded ring $R(C,A)$ or $R(X,A_X)$ with new generators of degree 1. First project X from a chosen node to $X'_{6,6} \subset \mathbf{P}(2,2,2,3,3)$; the exceptional curve of this projection is $\mathbf{P}^1 = \mathbf{P}(1,1)$ embedded into $\mathbf{P}(2,2,2,3,3)$. Since $\mathbf{P}(2,2,2,3,3)$ has no forms of degree 1, this embedding is *not projectively normal*; in coordinates it is

$$(v,w) \mapsto (v^2, vw, w^2, v^3 + \alpha v^2 w, \beta v w^2 + w^3)$$

with $1 + \alpha\beta \neq 0$.

The following result is joint work with Grzegorz Kapustka and Michal Kapustka (who held an EAGER visiting studentship at Warwick in spring 2004).

Claim *General forms of degree* $1,2,2,2,3,3$ *define an embedding* $\mathbf{P}^2 \cong \Pi \subset \mathbf{P}(1,2,2,2,3,3)$ *with image* Π *contained in* 3 *sextics. The complete intersection of two general sextics through* Π *is a* $\mathbf{Q}$*-Fano* 3*-fold* $V'_{6,6}$ *with* $9 \times \frac{1}{2}(1,1,1)$ *orbifold points on* $\mathbf{P}^2_{y_1,y_2,y_3}$, 24 *ordinary nodes on* Π*, and nonsingular otherwise.*

The 24 *nodes of* $V'_{6,6}$ *on* Π *are resolved by the (small) blowup* $V'' \to V'_{6,6}$ *of* Π*, and the birational image* $E \subset V''$ *of* Π *has* $E \cong \mathbf{P}^2$, $\mathcal{O}_E(-E) \cong \mathcal{O}_{\mathbf{P}^2}(2)$*; it contracts to a tenth orbifold point* $\frac{1}{2}(1,1,1)$ *on a Fano* $V \subset \mathbf{P}(1,2^4,3^4)$.

The proof is a calculation in computer algebra. According to results of Jan Stevens, V actually extends to a Fano 6-fold $W \subset \mathbf{P}(1^4,2^4,3^4)$ of Fano index 4 having 10 isolated orbifold points of type $\frac{1}{2}(1,\dots,1)$. (It can be obtained by an immersion $\mathbf{P}^5 \to \mathbf{P}(1^4,2^3,3^2)$ contained in two sextics, but the computation is quite bulky.)

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